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► **To cite this version:**

Hichem Lassoued. Dimension preserving resolutions of singularities of Poisson structures. 2017. hal-01487059v2

HAL Id: hal-01487059

<https://hal.science/hal-01487059v2>

Preprint submitted on 28 Apr 2019

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DIMENSION PRESERVING RESOLUTIONS OF SINGULAR POISSON STRUCTURES

HICHEM LASSOUED^{1,2}

ABSTRACT. We give examples of Poisson structures that admit symplectic resolutions of the same dimension. We also give a simple condition under which proper in the smooth case or semi-connected symplectic resolutions in the real analytic and holomorphic case can not exist: open symplectic leaves have to be dense and the singular locus can not be of codimension one.

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INTRODUCTION

Poisson manifolds make sense in smooth, real analytic and holomorphic geometries. In these three contexts, Poisson manifolds of dimension n are known to admit symplectic realizations of dimension $2n$ (see [4], [7], [9], [17] for the smooth case and [3] for the real analytic and holomorphic cases). Recall that a *symplectic realization* of a Poisson manifold (M, π_M) is a triple $(\Sigma, \Pi_\Sigma, \varphi)$ where (Σ, Π_Σ) is a symplectic¹ Poisson manifold and $\varphi : \Sigma \rightarrow M$ is a surjective submersion which is also a Poisson morphism.

It is obviously impossible to find a symplectic realization $(\Sigma, \Pi_\Sigma, \varphi)$ of a Poisson manifold (M, π_M) such that $\dim(\Sigma) = \dim(M)$, unless M is itself symplectic. But it is possible to modify the concept of realization and to impose only that φ is a surjective map, but not necessarily a submersion. More precisely, we define symplectic resolutions as follows:

Definition 0.1. *Let (M, π_M) be a Poisson manifold of dimension n . We call symplectic resolution a triple $(\Sigma, \pi_\Sigma, \varphi)$ where (Σ, Π_Σ) is a symplectic manifold of dimension n and $\varphi : \Sigma \rightarrow M$ is a surjective Poisson morphism.*

Under the mild assumption that all manifolds are second-countable, we shall see that the only real or complex Poisson manifolds that may admit a symplectic resolution are those that admit open symplectic leaves (see Proposition 1.2). Definition 0.1 therefore only makes sense when n is even.

Date: First submission: March 10th, 2017. This version March 20th, 2019.

¹Symplectic manifolds are Poisson manifolds for which the Poisson bivector field is invertible.

The term symplectic resolution has been coined in the context of algebraic geometry, see, e.g. Arnaud Beauville [1], Bao-Hua Fu [8] or Gwym Bellamy - Travis Schedler [2]. We refer to [2] for a list of examples. The concept that we have introduced above is consistent with this previously given meaning. It can not be compared in a rigorous manner: We work within differential geometry and they work in algebraic geometry. Moreover, for these authors, singular points are those where the variety is singular and for us singular points are those where the Poisson structure is singular. But the Poisson structures that algebraic geometers resolve are symplectic at regular points, and their resolutions are birational symplectomorphisms at these points. In our context, at regular points, resolution are also local smooth, real analytic or holomorphic symplectomorphisms. It is therefore justified to use the same name. Also, Definition 0.1 matches the definition of symplectic resolution given in [11], Section 4 (with minor adaptations).

Several examples of smooth symplectic resolutions exist. For instance, the Poisson structure on \mathbb{R}^2 given by:

$$\{x, y\}_{\mathbb{R}^2} := x^2 + y^2$$

is shown in [11] to admit symplectic resolutions. More non-trivial Examples are given in Section 1. However, we give in this article a simple example of a smooth Poisson structure of dimension 2 which does not have *proper symplectic resolution*, namely the Poisson structure on $M = \mathbb{R}^2$ given by:

$$\{x, y\}_M := x.$$

We also show that its complexification does not admit connected symplectic resolutions in the holomorphic case. In fact, we show the following result: For (M, π_M) a manifold of dimension 2, whose singular locus contains a curve, there exists no proper symplectic resolutions in the smooth case and no connected symplectic resolutions in the real analytic or holomorphic case, see Theorems 2.6 and 2.12.

We then extend these results to Poisson manifolds of arbitrary dimension. In the smooth case, we show in Theorem 2.21 and 2.25 that, if a Poisson manifold admits a proper symplectic resolution, then it admits an open symplectic leaf and do not admit a sub-manifold of codimension 1 in its singular locus. In the real analytic case, we show that if a Poisson manifold admits a connected symplectic resolution, then it admits an open symplectic leaf and do not admit a sub-manifold of codimension 1 in their singular locus. In the holomorphic setting, we deduce from this result a much stronger result: connected symplectic resolutions do not exist for non-symplectic Poisson manifolds, so that the theory of holomorphic symplectic resolutions is essentially empty, see Theorem 2.26.

Acknowledgment: The comments of the referee have greatly helped to improve both the mathematical content and the presentation of the article. She or he really did an exceptional contribution and I would like to thank her or him for her or his kindness and professionalism. Of course, I thank my advisors Camille Laurent-Gengoux and Zoubida Benhamadi. I had also interesting comments from Martin Bordemann, Mohamed Boucetta, Aziz El Kacimi and Salah Mehdi.

Conventions: Throughout this article, unless otherwise specified, all manifolds are second countable. We denote a Poisson structure on a smooth, real analytic or complex manifold M by π_M or $\{\cdot, \cdot\}_M$ indifferently. We write π_M when we see it as a section of $\wedge^2 TM$ and by $\{\cdot, \cdot\}_M$ when we see it as a skew-symmetric biderivation

of the sheaf of smooth, real analytic or holomorphic functions on M . As we shall in general not consider two Poisson structures on the same manifold, this notation will not be ambiguous. For us, a symplectic structure on a manifold Σ is a Poisson bivector field which is non-degenerate at all point $\sigma \in \Sigma$. It shall be denoted by a capital Greek letter Π_Σ . A symplectic point of a Poisson manifold (M, π_M) is a point where π_M is non-degenerate.

We denote by \mathbf{M}_{sing} the set of singular points of a Poisson structure π_M . In particular, for (M, π_M) a Poisson manifold of dimension $2n$:

$$\mathbf{M}_{sing} := \left\{ m \in M, \text{Rk} \left(\pi_M^\#|_m \right) \neq 2n \right\}.$$

1. EXAMPLES OF SMOOTH SYMPLECTIC RESOLUTIONS

We give examples of smooth and complex manifolds of dimension 2 that admit symplectic resolutions. Taking products of two-dimensional Poisson structures and products of their corresponding symplectic resolutions, we obtain examples of arbitrary even dimension.

First, let us define some notions that we will use throughout this article.

A *smooth Poisson structure* [12] on a manifold M is a section π_M of $\wedge^2 TM \rightarrow M$ that commutes with itself with respect to the Schouten-Nijenhuis bracket:

$$[\pi_M, \pi_M] = 0.$$

Poisson structures are in one-to-one correspondence with Lie algebra brackets $\{.,.\}_M$ on $C^\infty(M, \mathbb{R})$ which are derivations in each variables (referred to as *Poisson brackets*). *Real analytic and holomorphic Poisson structures* on a real analytic or complex manifold M are also defined as being real analytic or holomorphic bivector fields π_M on M satisfying $[\pi_M, \pi_M] = 0$. For every $U \subset M$, a real analytic or holomorphic Poisson structure on M induces a Poisson bracket on the algebra of holomorphic functions defined on U . When U is an open subset of \mathbb{R}^n or \mathbb{C}^n , Poisson structures on U are in one-to-one correspondence with Poisson brackets on real analytic or holomorphic functions on U , see e.g. [10].

Since, in dimension 2, every bivector field is Poisson, the Jacobi identity does not play any role. But the dimension 2 case is an interesting example because there exist open symplectic leaves, and they are dense (unless we choose a bivector field which is equal to zero on an open subset). Smooth Poisson structures on $M := \mathbb{R}^2$ (or an open subset of \mathbb{R}^2) are given by:

$$(1.1) \quad \{x, y\}_M = f(x, y),$$

where x, y are the canonical coordinates on M , and $f(x, y)$ is a smooth function. We suppose that the Poisson manifold (M, π_M) described by (1.1) admits a symplectic resolution $(\Sigma, \Pi_\Sigma, \varphi)$, with Π_Σ the symplectic structure on Σ . Since $M = \mathbb{R}^2$, the map φ is of the form $\varphi = (u, v)$, where u and v are two smooth functions on Σ with values in \mathbb{R} . For every choice (p, q) of local coordinates on (Σ, Π_Σ) , the functions u and v are functions of the variables p and q .

Proposition 1.2. *Let Σ be a smooth manifold, Π_Σ a symplectic structure on Σ , and $\varphi : \Sigma \rightarrow M = \mathbb{R}^2$ a surjective smooth map, the pair $(\Sigma, \Pi_\Sigma, \varphi)$ is a symplectic resolution of the Poisson manifold (M, π_M) which is given by (1.1) if and only if*

$$(1.3) \quad \{p, q\}_\Sigma \left(\frac{\partial u}{\partial p} \frac{\partial v}{\partial q} - \frac{\partial u}{\partial q} \frac{\partial v}{\partial p} \right) = f(u(p, q), v(p, q)),$$

where p, q are local coordinates on (Σ, Π_Σ) and $\varphi = (u, v)$.

Proof. The map φ is Poisson if and only if $\{\varphi^*x, \varphi^*y\}_\Sigma = \varphi^*\{x, y\}_M = \varphi^*f(x, y)$, with (x, y) the coordinates on $M = \mathbb{R}^2$. As $u = \varphi^*x$ et $v = \varphi^*y$, this condition is equivalent to $\{u, v\}_\Sigma = f(u(p, q), v(p, q))$, and the result is obtained by writing in the coordinates (p, q) the bracket $\{u, v\}_\Sigma$. \square

Remark 1.4. *Proposition 1.2 extends in an obvious manner to real analytic and holomorphic Poisson structure.*

Example 1.5. This example has appeared in [11], Section 6. We construct a symplectic resolution of the smooth Poisson structure on $M := \mathbb{R}^2$ given by

$$(1.6) \quad \{x, y\}_M = x^2 + y^2,$$

where x, y denote the canonical coordinates of $M = \mathbb{R}^2$. Our candidate of symplectic resolution is given by $\Sigma := \mathbb{R}^2$ equipped with the canonical symplectic bracket $\{p, q\}_\Sigma = 1$, with (p, q) the coordinates on $\Sigma = \mathbb{R}^2$. We define a map φ from Σ to M by:

$$(1.7) \quad \begin{array}{ccc} \varphi : & \Sigma & \rightarrow M \\ & (p, q) & \mapsto (q \sin(pq), q \cos(pq)). \end{array}$$

It is easy to check that φ is surjective. We are left with the task of showing that this map φ is a Poisson morphism, for which it suffices to check that the condition given by Equation (1.3) is satisfied. This is done by direct computation. For $u(p, q) = q \sin(pq)$ and $v(p, q) = q \cos(pq)$. We have on the one hand

$$f(u(p, q), v(p, q)) = (q \sin(pq))^2 + (q \cos(pq))^2 = q^2,$$

and on the other hand:

$$\begin{aligned} \frac{\partial u}{\partial p} \frac{\partial v}{\partial q} - \frac{\partial u}{\partial q} \frac{\partial v}{\partial p} &= \frac{\partial q \sin(pq)}{\partial p} \frac{\partial q \cos(pq)}{\partial q} - \frac{\partial q \sin(pq)}{\partial q} \frac{\partial q \cos(pq)}{\partial p} \\ &= q^2(\cos^2(pq) + \sin^2(pq)) = q^2. \end{aligned}$$

Since $\{p, q\}_\Sigma = 1$ and $f(u(p, q), v(p, q)) = q^2$. This proves the claim.

Remark 1.8. The symplectic resolution of Example 1.5 is a real analytic symplectic resolution. But it is not a holomorphic symplectic resolution. The map φ of Example 1.5 extends naturally to a holomorphic Poisson map from $\Sigma_{\mathbb{C}} = \mathbb{C}^2$ to $M_{\mathbb{C}} = \mathbb{C}^2$, equipped with the natural holomorphic extensions of the Poisson structures that appear in Example 1.5. This map is still Poisson and $\Sigma_{\mathbb{C}}$ is still symplectic. But φ , as defined in (1.7), is not surjective from $\Sigma_{\mathbb{C}}$ to $M_{\mathbb{C}}$, so it does not define a holomorphic symplectic resolution.

We construct a symplectic resolution of a smooth and real analytic Poisson structure more general than the one given in Example 1.5.

Example 1.9. We equip $\mathbb{R}^2 := M$ with the Poisson structure defined by

$$\{x, y\}_M = x^{2n} + y^{2m}, \text{ with } n \geq m \geq 1,$$

where x, y denote the canonical coordinates of $M = \mathbb{R}^2$. Our candidate of symplectic resolution is given by $\Sigma := \mathbb{R}^2$, equipped with the Poisson bracket

$$(1.10) \quad \{p, q\}_\Sigma = q^{2n-2m} \sin^{2n}(pq^{2m-1}) + \cos^{2m}(pq^{2m-1}),$$

where p, q denote the canonical coordinates of $\Sigma = \mathbb{R}^2$. This Poisson bracket is symplectic, because $q^{2n-2m} \sin^{2n}(pq^{2m-1}) + \cos^{2m}(pq^{2m-1})$ is strictly positive for all $p, q \in \mathbb{R}$. We define a map φ from Σ to M by:

$$\begin{aligned} \varphi : \quad \Sigma &\rightarrow M \\ (p, q) &\mapsto (q \sin(pq^{2m-1}), q \cos(pq^{2m-1})). \end{aligned}$$

To show that (Σ, φ) is a symplectic resolution of M , it is sufficient to check that Equation (1.3) is satisfied. In this case, we have $u(p, q) = q \sin(pq^{2m-1})$ and $v(p, q) = q \cos(pq^{2m-1})$ and

$$(1.11) \quad f(u(p, q), v(p, q)) = (q \sin(pq^{2m-1}))^{2n} + (q \cos(pq^{2m-1}))^{2m}.$$

Now, a direct computation gives:

$$\begin{aligned} \frac{\partial u}{\partial p} \frac{\partial v}{\partial q} - \frac{\partial u}{\partial q} \frac{\partial v}{\partial p} &= \frac{\partial q \sin(pq^{2m-1})}{\partial p} \frac{\partial q \cos(pq^{2m-1})}{\partial q} \\ &\quad - \frac{\partial q \sin(pq^{2m-1})}{\partial q} \frac{\partial q \cos(pq^{2m-1})}{\partial p} \\ &= q^{2m} (\cos^2(pq^{2m-1}) + \sin^2(pq^{2m-1})) = q^{2m}. \end{aligned}$$

The explicit values of $\frac{\partial u}{\partial p} \frac{\partial v}{\partial q} - \frac{\partial u}{\partial q} \frac{\partial v}{\partial p}$, $\{p, q\}_\Sigma$ and $f(u(p, q), v(p, q))$ given respectively in the previous equation, in Equation (1.10) and in Equation (1.11) satisfy Equation (1.3). This proves that the triple $(\Sigma, \Pi_\Sigma, \varphi)$ is a symplectic resolution.

More generally, for any Poisson structure of the form

$$\{x, y\}_M = \kappa(x, y)(x^{2n} + y^{2m}),$$

with $\kappa(x, y)$ a strictly positive function on M , the triple $(\Sigma, (\varphi^* \kappa) \Pi_\Sigma, \varphi)$ is a symplectic resolution.

Example 1.12. Examples 1.5 and 1.9 are Poisson structures with isolated singularities. We want to introduce an example with non-isolated singularities. We equip $M := \mathbb{R}^2$ with the bracket given in the canonical coordinates by $\{x, y\}_M = x$. We define $\Sigma := \mathbb{R}^2 \amalg \mathbb{R}^2 \amalg \mathbb{R}^2$ to be the disjoint union of three copies of \mathbb{R}^2 . We equip each copy of \mathbb{R}^2 with the canonical symplectic structure.

Consider the map $\varphi : \Sigma \rightarrow M$ defined by :

$$\varphi(p, q) := \begin{cases} (\exp(p), q) & \text{on the first copy} \\ (-\exp(p), q) & \text{on the second copy} \\ (0, q) & \text{on the third copy.} \end{cases}$$

A direct computation using Condition (1.3) implies that the restriction of φ to each of the three copies is a Poisson morphism. As a consequence, φ is a Poisson morphism. The map φ is surjective, making the couple (Σ, φ) a symplectic resolution of (M, π) . Note that the resolution $(\Sigma, \Pi_\Sigma, \varphi)$ is *not* semi-connected (see Definition 2.4) and not proper.

Here is a non-trivial example in dimension 4 of a *Poisson structure* which admits a symplectic resolution.

Example 1.13. Let (M, π_M, E_M) be a Jacobi variety [6] of dimension 3 whose Jacobi structure is given by the following vector field and bivector field:

$$E_M = 2 \frac{\partial}{\partial z} \quad \pi_M = (x^4 + y^4) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \wedge \left[x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right].$$

It is routine to check that this structure is Jacobi:

$$[\pi_M, \pi_M] = 2E_M \wedge \pi; \quad [E, \pi] = 0.$$

We consider the Poisson structure on $P = M \times \mathbb{R}$ induced by the Jacobi structure (the "Poissonization" of the previous Jacobi structure see [6], page 113). It is defined by

$$\pi_P = e^{-t}(\pi + Z \wedge E).$$

Here, $Z = \frac{\partial}{\partial t}$, and t is the canonical coordinate on \mathbb{R} .

A symplectic resolution of the Poisson manifold (P, π_P) is the triple $(\Sigma, \Pi_\Sigma, \varphi)$ where Σ is a symplectic manifold whose symplectic structure is given by the non-degenerate Poisson bivector field $\Pi_\Sigma = e^{-\frac{q_2}{2}}(\cos^4(p_1^3 q_1) + \sin^4(p_1^3 q_1)) \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q_1} + 4e^{-\frac{3q_2}{4}} \frac{\partial}{\partial p_2} \wedge \frac{\partial}{\partial q_2}$. It defines a symplectic structure on Π_Σ . The application $\varphi : \Sigma \rightarrow P$ defined by

$$\varphi = (f_1, f_2, f_3, f_4),$$

with $f_1 = e^{-\frac{q_2}{4}} p_1 \cos(p_1^3 q_1)$, $f_2 = e^{-\frac{q_2}{4}} p_1 \sin(p_1^3 q_1)$, $f_3 = p_2$, $f_4 = q_2$ is a Poisson morphism. This is a routine computation. It also surjective. It gives a symplectic resolution of the 4-dimensional Poisson manifold (P, π_P)

Example 1.14. So far, all manifolds were assumed to be second countable. If relax this condition. There is a general but unsatisfactory construction for making a symplectic resolution of any Poisson manifold of even dimension. Let (M, π_M) be a Poisson manifold of dimension $2d$, which can be assumed to be smooth, real analytic or holomorphic. Below, $\mathbb{K} = \mathbb{R}$ or \mathbb{C} depending on the context.

For a symplectic leaf \mathcal{S} of π_M , consider the direct product $\Sigma_{\mathcal{S}} := \mathcal{S} \times \mathbb{K}^{2d-2s}$ where $2s$ is the dimension of the leaf \mathcal{S} . Let us equip $\Sigma_{\mathcal{S}}$ with the direct product of the symplectic structure $\Pi_{\Sigma_{\mathcal{S}}}$ of the symplectic leaf \mathcal{S} with the canonical symplectic structure on the vector space of even dimension $\mathbb{K}^{2(d-s)}$. This Poisson structure is symplectic by construction. The natural map $\varphi_{\mathcal{S}}$ obtained by, first, projecting $\mathcal{S} \times \mathbb{K}^{2d-2s}$ onto \mathcal{S} , then by including \mathcal{S} into M , is a Poisson map.

Now, let S be the set of all symplectic leaves. Let Σ be the disjoint union $\Sigma := \coprod_{\mathcal{S} \in S} \Sigma_{\mathcal{S}}$. All connected components of this manifold have dimension $2d$ and are symplectic manifolds. This implies that Σ has dimension $2d$ and is symplectic. The map $\varphi : \Sigma \rightarrow M$, whose restriction to $\mathcal{S} \in S$ is $\varphi_{\mathcal{S}}$, is surjective. It is a Poisson map because its restriction to each connected component $\Sigma_{\mathcal{S}}$ is a Poisson map. Hence, $(\Sigma, \Pi_\Sigma, \varphi)$ defines a symplectic resolution.

Using this construction, we see that any Poisson manifold of even dimension admits a symplectic resolution. However, this construction is not satisfactory. In general, the symplectic leaves form a non-countable family that has the cardinality of \mathbb{R} . Hence Σ may not admit a dense countable subset, i.e. be second countable. Even when there are finitely many symplectic leaves, the resolution described in this example is not semi-connected (see Definition 2.4). This example, therefore, is not convincing. Also, it is in general not proper.

2. OF POISSON STRUCTURES THAT DO NOT ADMIT PROPER OR CONNECTED SYMPLECTIC RESOLUTIONS

We describe in this section broad classes of Poisson manifolds that do not admit reasonable symplectic resolutions. As shown in Example 1.14, it is reasonable to assume that the symplectic resolutions are second countable, since without such a condition symplectic resolutions always exist, but they are not second countable. In this section, as in the previous one, some of the statements are about holomorphic and real analytic Poisson manifolds, some are about the smooth case. The context will be stated clearly where relevant.

Several proofs shall require Sard's theorem (see e.g. [16]). Let us state the version of this theorem which is required here. For Σ, M smooth real analytic or holomorphic manifolds and $\varphi : \Sigma \rightarrow M$ a surjective smooth, real analytic or holomorphic map, a point $m \in M$ is said to be a *regular value* if $T_\sigma\varphi$ is surjective for all $\sigma \in \Sigma$ with $\varphi(\sigma) = m$. A *critical value* of a map $\varphi : \Sigma \rightarrow M$ is a point $m \in M$ for which there exists $\sigma \in \Sigma$ such that $\varphi(\sigma) = m$ and $T_\sigma\varphi : T_\sigma\Sigma \rightarrow T_mM$ is not surjective. For a surjective map $\varphi : \Sigma \rightarrow M$, the set of critical values and the set of regular values form a partition of M .

Theorem 2.1. [16] [*Sard's theorem.*] *Let P, N be second countable smooth, real analytic or complex manifolds and $\varphi : P \rightarrow N$ a surjective smooth, real analytic or holomorphic map. The set of all critical values has Lebesgue measure zero. In particular, the set of regular values is dense in N .*

Let us now find conditions that Poisson manifolds need to satisfy in order to admit a second countable symplectic resolution. Recall that a point m in a Poisson manifold (M, π_M) is said to be symplectic when $\pi_M|_m$ is invertible. It will be said *non-symplectic* if $\pi_M|_m$ is not invertible

Proposition 2.2. *Let $(\Sigma, \Pi_\Sigma, \varphi)$ be a smooth, real analytic or complex symplectic resolution of the smooth, real analytic or complex Poisson manifold (M, π_M) . A point $m \in M$ is a symplectic point of π_M if and only if it is a regular value of φ .*

Proof. The map φ is a Poisson map if and only if the following diagram is commutative for all $m \in M$ and all $\sigma \in \Sigma$ with $\varphi(\sigma) = m$:

$$\begin{array}{ccc} T_\sigma^*\Sigma & \xrightarrow{\Pi_{\Sigma|_\sigma}^\#} & T_\sigma\Sigma \\ T_\sigma^*\varphi \uparrow & & \downarrow T_\sigma\varphi \\ T_m^*M & \xrightarrow{\pi_M^\#|_m} & T_mM. \end{array}$$

Since all vector spaces in this diagram have the same dimension and since $\Pi_{\Sigma|_\sigma}^\#$ is invertible, the commutativity of this diagram implies that $\pi_M^\#|_m$ is invertible if and only if vertical arrows are invertible, i.e. if and only if $T_\sigma\varphi$ is invertible. This proves the result. \square

Proposition 2.3. *A Poisson manifold (M, π_M) that admits a second countable symplectic resolution is symplectic on an open dense subset.*

Proof. By the theorem of Sard (Theorem 2.1), regular values are dense in M . By Proposition 2.2, regular values of φ are symplectic points of π_M . This proves the claim. \square

It follows from Proposition 2.2 that a point $m \in M$ is

- (1) a symplectic point for the Poisson structure π_M if and only if $T_\sigma\varphi$ is bijective for every point $\sigma \in \varphi^{-1}(m)$,
- (2) a non-symplectic point for the Poisson structure π_M if and only if $T_\sigma\varphi$ is non-bijective for every point $\sigma \in \varphi^{-1}(m)$.

The existence of the symplectic resolution given in Example 1.12 of the Poisson structure on \mathbb{R}^2 given by $\{x, y\} = x$ means that we should impose more conditions on symplectic resolutions to obtain an interesting theory. Even if it is second countable, we feel that having singular and regular parts that have distinct inverse images is not acceptable, and would give symplectic resolutions which are useless. Also, it does not match resolutions of singularities as they appear in algebraic geometry.

There are two manners to avoid symplectic resolutions as described in Example 1.12: we can impose that the map $\varphi : \Sigma \rightarrow M$ is a proper map, i.e. the inverse image under φ of a compact subset of M is a compact subset of Σ . A second manner to avoid such resolutions is to impose Σ to be a connected manifold, at least when M is itself connected. More precisely:

Definition 2.4. A symplectic resolution $(\Sigma, \Pi_\Sigma, \varphi)$ of a Poisson manifold (M, π_M) is said to be

- (1) proper when the map ϕ is a proper map,
- (2) semi-connected if the image of a connected component of Σ by a map φ can not be included in the singular locus of π_M .

Remark 2.5. A symplectic resolution $(\Sigma, \Pi_\Sigma, \varphi)$ with Σ connected is also semi-connected.

Imposing symplectic resolutions to be proper is the relevant assumption for the smooth case, and semi-connected is relevant in the real analytic and holomorphic cases, as we shall see.

In order not to confuse the smooth case with the holomorphic (or real analytic) case, we will distinguish them by giving two theorems. We will start with the smooth case.

Theorem 2.6. A smooth Poisson structure on $M := \mathbb{R}^2$ given by

$$(2.7) \quad \{x, y\}_M = f(x, y),$$

with $f(x, y)$ a smooth function that vanishes if $x = 0$ does not admit a proper smooth symplectic resolution.

Remark 2.8. The following problem is still open: In the smooth case, does the Poisson manifold (M, π) described by (2.7) admit a connected symplectic resolution?

Let us prove Theorem 2.6.

Proof. Let $(\Sigma, \Pi_\Sigma, \varphi)$ be a proper symplectic resolution of (M, π_M) . Since $M = \mathbb{R}^2$, we have $\varphi = (u, v)$ with u, v smooth real valued functions on Σ . The theorem of Sard (Theorem 2.1) implies that the complement of the set of critical values of the differentiable function $v : \Sigma \rightarrow \mathbb{R}$, which is surjective since φ is surjective, is dense in \mathbb{R} .

Let $v_0 \in \mathbb{R}$ be a regular value of v . Since the dimension of Σ is 2 the inverse image $v^{-1}(v_0)$ by $v : \Sigma \rightarrow \mathbb{R}$ of v_0 is a union $(\mathcal{C}_i)_{i \in I}$ of smooth curves.

Consider the point $(0, v_0) \in M = \mathbb{R}^2$. This point is a singular point of π_M by the definition (2.7) thereof. Consider a point $\sigma \in \varphi^{-1}(0, v_0)$. This point σ belongs to a curve \mathcal{C}_{i_0} for a certain $i_0 \in I$. We are going to show that the image of \mathcal{C}_{i_0} by φ is reduced to the point $(0, v_0)$, or, equivalently, that u identically vanishes along the curve \mathcal{C}_{i_0} .

Since the function v has a differential that does not vanish at the point σ , there exists another local function p , defined on a neighborhood of σ , such that the pair (p, v) form local Darboux coordinates on a open neighborhood $U_\sigma \subset \Sigma$. On $U_\sigma \subset \Sigma$ the map φ reads $\varphi : (p, v) \rightarrow (u(p, v), v)$. Also, upon shrink U_σ if necessary, the restriction to $U_\sigma \subset \Sigma$ of the curve \mathcal{C}_{i_0} is given by $v = v_0$. By Proposition 1.2, since the map φ is a Poisson morphism, the following differential equation in the variable p holds:

$$(2.9) \quad \frac{\partial u}{\partial p}(p, v) = f(u(p, v), v).$$

In particular, for $v = v_0$, we obtain the differential equation:

$$(2.10) \quad \begin{cases} \frac{\partial u}{\partial p}(p, v_0) = f(u(p, v_0), v_0) \\ \text{and } u(p_0, v_0) = 0. \end{cases}$$

where (p_0, v_0) are the coordinates of the point σ . Since $f(0, v) = 0$ and in particular $f(0, v_0) = 0$, the Cauchy-Lipschitz theorem implies that the differential equation (2.10) admits for unique solution the zero function, i.e. $u(p, v_0) = 0$ for all p in a neighborhood of p_0 . As a consequence, the restriction of u to \mathcal{C}_{i_0} is a function which is zero in a neighborhood of any point where it is zero. Since \mathcal{C}_{i_0} is connected, the function u vanishes identically on the whole curve \mathcal{C}_{i_0} if and only if it vanishes at some point of \mathcal{C}_{i_0} .

The conclusion of the previous lines is that there are two types of curves in $v^{-1}(v_0)$, those on which the restriction of the function u is never equal to 0 (curves that we call curves of the *first type*) and those where u vanishes identically, (curves that we call curves of the *second type*). Since the map φ is a surjection, there is necessarily at least one curve of each type.

Since the unions of all curves of the second type is the inverse image $\varphi^{-1}(0, v_0)$, this union is a compact set. (Notice that also, each one of these curves is compact, and therefore is a circle).

Let K be the inverse image under φ of $[-1, 1] \times \{v_0\} \subset M$. Since φ is proper, the set K is compact. Let us now say that a curve is *good*, those that intersect K . Curves of the second type are also of the good type.

By definition of v_0 , $T_\sigma v : T_\sigma \Sigma \rightarrow T_\sigma \mathbb{R}$ is surjective for every $\sigma \in v^{-1}(v_0)$. Hence, the map v is a submersion in a neighborhood of all point in $v^{-1}(v_0)$, and the curves $(\mathcal{C}_i)_{i \in I}$ whose union forms $v^{-1}(v_0)$ can be splitted, that is to say there are open sets $(U_i)_{i \in I}$, with U_i containing \mathcal{C}_i for all indices $i \in I$, such that $U_i \cap U_j \cap v^{-1}(v_0) = \emptyset$ for all distinct $i, j \in I$.

The open subsets $(U_i \cap K)_{i \in I}$ form a partition of K . Now, in a partition of a compact set by open subsets, only finitely many can be non-empty. Said otherwise, there are finitely many good curves.

The finite subset of curves of the good type can be splitted in two-subsets: those which are of the second type and those which are of the first type. Since both subsets of curves are finite, and since for any curve in the first subset and any curve in the second subset, there exists non-intersecting open subsets containing them,

there exists non-intersecting open subsets V and W such that good curves of the first type are in V and good curve of the second type are in W .

We now consider a sequence $(y_n)_{n \in \mathbb{N}} \in \Sigma$ such that $\varphi(y_n) = (\frac{1}{n}, v_0)$. Since the map φ is proper, out of the sequence y_n , we can extract a convergent subsequence. Let $\tilde{y} \in \Sigma$ be its limit. By construction, $\varphi(\tilde{y}) = (0, v_0)$. For any $n \in \mathbb{N}$, the element y_n belongs to a curve of the first type and of the good type, and therefore is in V , but its limit has to belong to a curve of the good type and of the second type, and is therefore in W . This contradicts the assumption that $V \cap W = \emptyset$, and completes the proof. \square

We now look at the real analytic or holomorphic case. As in Equation (2.7), we consider the real analytic (resp, holomorphic) Poisson structure on an open subset of \mathbb{R}^2 (resp, \mathbb{C}^2) containing at least one point of the vertical axis $\{x = 0\}$. Such a Poisson structure is given by:

$$(2.11) \quad \{x, y\}_M = f(x, y),$$

with $f(x, y)$ a real analytic (resp. holomorphic) function that vanishes on the vertical axis $\{x = 0\}$.

Theorem 2.12. *The real analytic (resp. holomorphic) Poisson manifold (M, π_M) described in Equation (2.11) does not admit a real analytic (resp. holomorphic) semi-connected symplectic resolution.*

Let $(\Sigma, \Pi_\Sigma, \varphi)$ be a real analytic (resp. holomorphic) semi-connected symplectic resolution of the Poisson structure in (2.11). We write $\varphi = (u, v)$ with u, v real analytic (resp. holomorphic) functions from Σ to \mathbb{R} or \mathbb{C} . Let us consider $\Gamma = u^{-1}(0) \subset \Sigma$. The set Γ is a closed subset of Σ which is the disjoint union of

$$\Gamma_{sing} = \{\sigma \in \Sigma \mid u(\sigma) = 0 \text{ and } d_\sigma v = 0\}$$

and

$$\Gamma_{reg} = \{\sigma \in \Sigma \mid u(\sigma) = 0 \text{ and } d_\sigma v \neq 0\}.$$

We start with a lemma:

Lemma 2.13. *Let $(\Sigma, \Pi_\Sigma, \varphi)$ be a real analytic (resp. holomorphic) semi-connected symplectic resolution of the real analytic (resp. holomorphic) Poisson manifold (M, π_M) described in Equation (2.11).*

For any point $\sigma \in \Gamma_{reg}$, there exists a neighborhood U_σ of σ in Σ such that $v(U_\sigma \cap \Gamma_{reg})$ is a point.

We prove the lemma in the holomorphic case (the real analytic case is along the same lines).

Proof. For any point $\sigma \in \Gamma_{reg}$, there exists a neighborhood U_σ of σ in Σ and a function p defined on U_σ such that the pair (p, v) are local Darboux coordinates on Σ . The map φ reads in these coordinates as:

$$\begin{aligned} \varphi : U_\sigma \subset \Sigma &\rightarrow M \\ (p, v) &\mapsto (u(p, v), v). \end{aligned}$$

Proposition 1.2 implies that

$$(2.14) \quad \frac{\partial u}{\partial p}(p, v) = f(u(p, v), v).$$

Let (p_0, v_0) be the coordinates of $\sigma \in \Sigma$ and consider the function $h : v \rightarrow u(p_0, v)$. Since the function h is holomorphic, and h has a zero at $v = v_0$, there are two possibilities: either this function is identically equal to zero or it has an isolated zero at v_0 .

Cauchy-Lipschitz theorem and the assumption $f(0, v) = 0$ imply that solutions $p \rightarrow u(p, v)$ of (2.14) are identically zero if $h(v) = 0$. Hence if h is identically equal to 0 on a neighborhood of v_0 , the function u vanishes at all points of U_σ . Since the function u is holomorphic, this implies that $u = 0$ on the whole connected component Σ_σ of σ in the manifold Σ . But this is impossible because $(\Sigma, \Pi_\Sigma, \varphi)$ is semi-connected: $\varphi(\Sigma_\sigma)$ can not be included in the singular locus of π_M . Hence v_0 is necessarily an isolated zero of h . Upon shrinking U_σ if necessary, we can assume it is the only zero of the function h . Cauchy-Lipschitz theorem and the assumption $f(0, v) = 0$ imply then that $u(p, v) \neq 0$ for all $(p, v) \in U_\sigma$ that satisfy $v \neq v_0$. Hence $U_\sigma \cap \Gamma_{reg}$ is given in the coordinates (p, v) by $v = v_0$ and its image through φ is reduced to the point $(0, v_0)$. This completes the proof. \square

We now prove Theorem 2.12 for the holomorphic case. The real analytic case is similar.

We prove the lemma in the holomorphic case (the real analytic case is along the same lines).

Proof. Assume a semi-connected holomorphic symplectic resolution $(\Sigma, \Pi_\Sigma, \varphi)$ of (M, π_M) exists, with $\varphi = (u, v)$, where u, v are holomorphic functions on Σ with values in \mathbb{C} .

Let $\Gamma = u^{-1}(0)$, Γ_{sing} and Γ_{reg} be as defined above. As φ is surjective by assumption, $v(\Gamma) \subseteq \mathbb{C}$ in the intersection of M with the axis $\{x = 0\}$.

By Theorem 2.1 [Sard's theorem], applied to the differentiable function v , the critical values of v form a set of measure zero in \mathbb{C} . By construction, $v(\Gamma_{sing})$ is included in that subset and is therefore of measure 0 in \mathbb{C} .

Let us show that $v(\Gamma_{reg})$ is a countable or finite subset of \mathbb{C} . For any $\sigma \in \Gamma$, there exists by Lemma 2.13 a neighborhood U_σ of $\sigma \in \Sigma$ such that $v(\Gamma_{reg} \cap U_\sigma)$ is reduced to a point. The set Γ_{reg} is an open subset of the closed subset Γ . It is therefore a locally compact subset and we can extract out of any open cover of Γ_{reg} a finite or countable open cover. This implies that $v(\Gamma_{reg})$ is a finite or countable subset of \mathbb{C} . Hence $v(\Gamma_{reg})$ is of measure 0, since $v(\Gamma_{sing})$ is also of measure zero, then so is $v(\Gamma)$. This contradicts the surjectivity of φ , which imposes that $v(\Gamma)$ is an open subset of \mathbb{C} .

Therefore, no semi-connected symplectic resolution exist. \square

We now use Theorems 2.6 and 2.12 to give a class of Poisson manifolds that do not admit reasonable symplectic resolutions, although they are symplectic on a dense open subset.

We say that a submanifold N of a Poisson manifold (M, π_M) which can be smooth, real analytic or holomorphic, such that

$$(2.15) \quad T_n N \oplus \pi_M^\#|_n (T_n N^\perp) = T_n M, \quad \forall n \in N$$

is *Poisson-Dirac*. The usual definition is more general (see [12], Chapter 5) but we do need to use it here. Recall that any Poisson-Dirac submanifold admits a unique induced Poisson structure called *reduced Poisson structure* and denoted by $\{\cdot, \cdot\}_N$

which satisfies for all local functions F, G on N

$$\{F, G\}_N = \mathbf{i}_N^* \left\{ \tilde{F}, \tilde{G} \right\}_M,$$

where \tilde{F} and \tilde{G} are local extensions of F and G to M whose hamiltonian vector fields are tangent to N and $\mathbf{i}_N^* : N \hookrightarrow M$ is the inclusion map. Equation (2.15) implies that a such extensions always exist, see [12], chapter 5.

Remark 2.16. (see [18], corollary 2.11) Poisson-Dirac submanifolds of symplectic manifolds are exactly symplectic submanifolds, i.e, submanifolds on which the symplectic 2-form restricts to a non degenerate 2-form.

We now give a characterization of the Poisson-Dirac submanifolds, for more details see [12], page 163.

Proposition 2.17. *Let (M, π_M) be a smooth, real analytic or holomorphic Poisson manifold of dimension $2r + d$ and N a submanifold of M of dimension d . Then N is a Poisson-Dirac submanifold if and only if for all $n \in N$, N is locally defined as the zero set of $2r$ independent functions $p_1, \dots, p_r, q_1, \dots, q_r$ on M satisfying the following relations:*

$$(2.18) \quad \mathbf{i}_N^* \{p_i, q_j\}_M = \delta_i^j, \quad \mathbf{i}_N^* \{p_i, p_j\}_M = \mathbf{i}_N^* \{q_i, q_j\}_M = 0, \quad \forall i, j = 1, \dots, r.$$

Submanifolds N, S of M are said to be in direct sum at $n \in S \cap N$ if $T_n S \oplus T_n N = T_n M$.

Lemma 2.19. *Let (M, π) be a (smooth/real analytic/holomorphic) Poisson manifold of dimension $2r + d$, S a symplectic leaf of dimension $2r$ and $N \subset M$ a submanifold in direct sum with S at a point $n \in S \cap N$. For every symplectic resolution $(\Sigma, \Pi_\Sigma, \varphi)$ of (M, π) , there exists a neighborhood N' of n in N such that:*

- (1) N' is a Poisson-Dirac submanifold of (M, π) ,
- (2) $\varphi^{-1}(N')$ is a symplectic submanifold of Σ ,
- (3) the restriction $\varphi_{N'} : \varphi^{-1}(N') \rightarrow N'$ is a symplectic resolution for the reduced Poisson structures of $\varphi^{-1}(N')$ and N' respectively. Moreover, it is a proper (resp. semi-connected) symplectic resolution when $(\Sigma, \Pi_\Sigma, \varphi)$ is a proper (resp. semi-connected) symplectic resolution.

Proof. There exists, in a neighborhood U of n , Weinstein coordinates (p, q, z) such that the submanifold $N' := N \cap U$ is given by the equations: $p_1 = \dots = p_r = q_1 = \dots = q_r = 0$ and

$$\pi = \sum_{i=1}^r \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} + \sum_{i < j} \pi_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j},$$

with

$$\pi_{ij}(0) = 0 \quad \text{for } i, j = 1, \dots, d.$$

The functions $p_1, \dots, p_r, q_1, \dots, q_r$ having independent Hamiltonian vector fields at all points of U , their pull-back $\varphi^* p_1, \dots, \varphi^* p_r, \varphi^* q_1, \dots, \varphi^* q_r$ also have independent Hamiltonian vector fields at all point in $\varphi^{-1}(U)$. Therefore, they are independent functions. The zero locus that they define is $\varphi^{-1}(N')$ by construction, which is therefore a submanifold of Σ . Proposition 2.17 implies that N' and $\varphi^{-1}(N')$ are Poisson-Dirac submanifolds. This proves the first and second item.

For any function F on M such that $\{F, q_i\}|_{N'} = 0 = \{F, p_i\}|_{N'}$, for all $i = 1, \dots, r$, the relation $\{\varphi^* F, \varphi^* q_i\}|_{\varphi^{-1}(N')} = 0 = \{\varphi^* F, \varphi^* p_i\}|_{\varphi^{-1}(N')}$ holds for all $i = 1, \dots, r$. Hence, a function \tilde{F} on M such that the hamiltonian vector field $X_{\tilde{F}}$ is tangent to N' satisfies that the hamiltonian vector field $X_{\varphi^* \tilde{F}}$ is tangent to $\varphi^{-1}(N')$. Now, for all functions F, G on N' , we have by definition of the reduced structure

$$\{F, G\}_{N'} = \left\{ \tilde{F}, \tilde{G} \right\}_{|_{N'}},$$

where \tilde{F} and \tilde{G} are local extensions of F and G to M whose hamiltonian vector fields are tangents to N' . Hence

$$\varphi^* \{F, G\}_{N'} = \left(\varphi^* \circ \mathbf{i}_{N'}^* \left\{ \tilde{F}, \tilde{G} \right\} \right) = \mathbf{i}_{\varphi^{-1}(N')}^* \left\{ \varphi^* \tilde{F}, \varphi^* \tilde{G} \right\} = \{ \varphi^* F, \varphi^* G \}_{\varphi^{-1}(N')}.$$

This proves that φ restricts to a Poisson morphism from $\varphi^{-1}(N')$ to N' equipped with their reduced Poisson structures. This restriction is proper if φ is proper because compact subsets of N' are compact subsets of M , hence their inverse image are compact in Σ . Since this inverse image is included in $\varphi^{-1}(N')$, it is compact in $\varphi^{-1}(N')$.

Also the real analytic or holomorphic resolution $(\varphi^{-1}(N'), \pi_{\varphi^{-1}(N')}, \varphi_{\varphi^{-1}(N')})$ is semi-connected if $(\Sigma, \Pi_{\Sigma}, \varphi)$ is semi-connected. Let us prove this point by contradiction. Assume there exists a connected component C of $\varphi^{-1}(N')$ such that $\varphi(C)$ is included in the singular locus of $\pi_{N'}$. A singular point for $\pi_{N'}$ being also singular for π_M , it implies that $\varphi(C) \subseteq \mathbf{M}_{sing}$. Hence every point in Σ which is the image of a point in $C \subseteq \Sigma$ through the flow of the hamiltonien vector field $\mathcal{X}_{\varphi^* F}$, with F a local function on M , is mapped to \mathbf{M}_{sing} . Let Σ_0 be the set of such points. Since $\varphi^{-1}(N')$ is Poisson-Dirac in Σ , the submanifold C is Poisson-Dirac in Σ , there exists a neighborhood of C included in Σ_0 . This implies that an open subset of Σ containing C is mapped to \mathbf{M}_{sing} . Since Σ and φ are real analytic or holomorphic, the whole connected component of C has to be mapped to \mathbf{M}_{sing} , which contradicts the assumption. This proves the third item. \square

We recall that \mathbf{M}_{sing} is the set of points $n \in M$ where the Poisson structure degenerates.

The following lemma holds in smooth, real analytic and holomorphic contexts: It mainly uses Sard's theorem (Theorem 2.1). We prove it only in the smooth case, but the proof easily adapts.

Lemma 2.20. *Let (M, π_M) be a Poisson manifold of dimension $2n \geq 4$, such that \mathbf{M}_{sing} contains a submanifold $P \subset M$ of codimension one. If a symplectic resolution exists, then the bivector π_M can not be zero at all points in P .*

Proof. Consider a tubular neighborhood U of P in M with projection $\psi : U \rightarrow P$. Assume that $(\Sigma, \Pi_{\Sigma}, \varphi)$ is a symplectic resolution of (M, π_M) . The map $\psi \circ \varphi : \varphi^{-1}(U) \rightarrow P$ is surjective because both φ and ψ are surjective. By Sard's Theorem, there exists a regular value for $\psi \circ \varphi$, i.e. there exists a point $p \in P$ such that for every $\sigma \in (\psi \circ \varphi)^{-1}(U) \subset \Sigma$, the composition

$$d_{\varphi(\sigma)} \psi \circ d_{\sigma} \varphi : T_{\sigma} \Sigma \rightarrow T_p P$$

is a surjective linear map. Since φ is surjective, there exists $\sigma \in \Sigma$ such that $\varphi(\sigma) = p$. Such a point belongs to $(\psi \circ \varphi)^{-1}(U)$ by construction, and satisfies that

$d_{\varphi(\sigma)}\psi \circ d_{\sigma}\varphi$ is a surjective linear map. In turn, this implies that for any choice x_1, \dots, x_{2n-1} of local coordinates on P around M , the functions $(\psi \circ \varphi)^*x_1, \dots, (\psi \circ \varphi)^*x_{2n-1}$ are independent at the point σ .

Let us compute the Poisson brackets of these functions at the point σ . Since the map φ is a Poisson morphism, for all $i, j = 1, \dots, 2n-1$, then, if $\pi_M|_P = 0$

$$\begin{aligned} \{(\psi \circ \varphi)^*x_i, (\psi \circ \varphi)^*x_j\}_{\Sigma}(\sigma) &= \{\varphi^*\psi^*x_i, \varphi^*\psi^*x_j\}_{\Sigma}(\sigma) \\ &= \{\psi^*x_i, \psi^*x_j\}_M(\varphi(\sigma)) = 0. \end{aligned}$$

But it is impossible to have $2n-1$ independent functions on a symplectic manifold whose brackets are equal to zero at a given point. This completes the proof. \square

Theorem 2.21. *A smooth Poisson manifold (M, π_M) of dimension $2n$, which contains a submanifold of codimension one included in \mathbf{M}_{sing} , does not admit a proper symplectic resolution.*

Proof. We prove the theorem by induction on $2n = \dim(M)$. If $n = 1$. In a neighborhood of a submanifold $P \subset M$ of dimension 1, the Poisson structure reads $\{x, y\} = f(x, y)$ where (x, y) are local coordinates such that P is given, locally, by $x = 0$. This means that we are in the situation of Theorem 2.6 and initiates the recursion. We assume that the theorem is true for all $1, \dots, n$ and we show it for $n+1 = \frac{1}{2}\dim(M)$.

We assume that \mathbf{M}_{sing} contains a submanifold P of codimension 1. Assume, for the sake of contradiction that $(\Sigma, \Pi_{\Sigma}, \varphi)$ is a proper symplectic resolution. By Lemma 2.20, the rank of π_M on P can not be zero at every point of P . Let m be a point in P where $\pi_M|_m \neq 0$. If a symplectic resolution $(\Sigma, \Pi_{\Sigma}, \varphi)$ of (M, π_M) exists, then by Lemma 2.19, any submanifold N in direct sum with the symplectic leaf through m admits a neighborhood N' of m in N such that the map $\varphi_{N'} : \varphi^{-1}(N') \rightarrow N'$ is a symplectic resolution of dimension $2(n+1-r)$ with $2r$ the rank of π_M at m . Moreover $\varphi_{N'}$ is a proper map because φ is proper map. Now $P \cap N'$ is a submanifold of codimension one in N' , at least in a neighborhood of m in N' . We get a contradiction with the induction hypothesis. This shows the result. \square

We now give non-trivial example of a Poisson manifold that satisfies the condition of Theorem 2.21

Example 2.22. We consider the affine Lie algebra, that is to say the semi-direct product

$$\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R}) \ltimes \mathbb{R}^n,$$

equipped

$$[(A, u), (B, v)] = (AB - BA, Av - Bu).$$

We identify \mathfrak{g} and its dual vector space \mathfrak{g}^* with the help of the nondegenerate symmetric bilinear form defined for all $A, B \in \mathfrak{gl}_n(\mathbb{R})$ and $u, v \in \mathbb{R}^n$ by

$$(2.23) \quad \begin{aligned} \mathfrak{g} \otimes \mathfrak{g} &\rightarrow \mathbb{R} \\ \ll (A, u), (B, v) \gg &\mapsto \text{Tr}(AB) + \langle u, v \rangle. \end{aligned}$$

Under this identification, the coadjoint action becomes an action of \mathfrak{g} on \mathfrak{g} given at a point (B, v) by

$$(2.24) \quad (A, u) \mapsto \text{ad}_{(A, u)}^*(B, v) = (AB - BA + u^t \otimes v, A^t(v))$$

where A^t stands for the transpose of the matrix A , and

$$u^t \otimes v := \begin{pmatrix} u_1 v_1 & \cdots & u_i v_1 & \cdots & u_n v_1 \\ \vdots & & \vdots & & \vdots \\ u_1 v_j & \cdots & u_i v_j & \cdots & u_n v_j \\ \vdots & & \vdots & & \vdots \\ u_1 v_n & \cdots & u_i v_n & \cdots & u_n v_n \end{pmatrix}.$$

For B a diagonal matrix with pairwise different diagonal values and for v_1, \dots, v_n all different from zero, the linear map (2.24) is injective, and therefore bijective. Let us check this point. For all (A, u) in the kernel of (2.24), the diagonal terms of the matrix $AB - BA + u^t \otimes v$ are equal to $u_1 v_1, \dots, u_n v_n$. If they are zero, then $u = 0$. We then have $AB - BA = 0$, which implies that A is a diagonal matrix. But, then, the equality $A^t(v) = 0$ implies that all diagonal terms of the matrix A are equal to 0. This proves injectivity. Since there is at least one point (B, v) where the linear map (2.24) is bijective, and since the determinant of this map is a polynomial function in the coefficients of B and v , this linear map has to be bijective on a dense open subset.

Now, at every point α of the dual \mathfrak{g}^* of a Lie algebra \mathfrak{g} , the Lie-Poisson structure (or Kirillov-Kostant-Souriau structure) on \mathfrak{g}^* is given by the coadjoint action. In our case, the Lie-Poisson is transported on \mathfrak{g}^* through (2.23) and the map $\pi_{\mathfrak{g}}^{\#}$ is given as in (2.24). On the dense open subset of elements $(B, v) \in \mathfrak{g}$ such that (2.24) is injective, the Lie-Poisson structure $\pi_{\mathfrak{g}}$ is symplectic.

It is then natural to ask whether or not \mathfrak{g} admits proper symplectic resolutions. The answer is "no". Assume B is diagonalizable with pairwise distinct eigenvalues. Then so is B^t . Let e_1, \dots, e_n be a basis of eigenvectors of B^t . Assume $v = \sum_{i=1}^n \lambda_i e_i$ with $\lambda_1 = 0$. Then (2.24) is not injective because (A, u) with $u = 0$ and $A^t(e_1) = e_1$, $A^t(e_i) = 0$ for $i = 2, \dots, n$ is in the kernel of the coadjoint action (2.24).

In neighborhood of (B, v) with B diagonal with distinct eigenvalues and $v = (0, 1, 0, \dots, 0)$, the condition $\lambda_1 = 0$ defines a submanifold of codimension 1. The singular locus of $\pi_{\mathfrak{g}}$ contains therefore a submanifold of codimension 1, and Theorem 2.21 implies that no proper symplectic resolution exists.

Theorem 2.25. *A real analytic Poisson manifold (M, π_M) , which contains a submanifold of codimension one of singular points for π_M , does not admit a semi-connected symplectic resolution.*

The proof of this result is identical to that of Theorem 2.21. We simply use Theorem 2.12 instead of Theorem 2.6 in the case where $\dim(M) = 2$ for the recursion hypothesis in order to say that M , in this case, does not admit a semi-connected symplectic resolution.

In the holomorphic case, the result is much stronger because the submanifold of codimension one that appears in both Theorems 2.21 and 2.25 always exist.

Theorem 2.26. *A holomorphic Poisson manifold, does not admit a semi-connected symplectic resolution, unless it is symplectic.*

Proof. A Poisson manifold (M, π_M) that admits a symplectic resolution must be symplectic on an open dense subset by Proposition 2.3. For such a manifold, the multivector field $\pi_M^n = \pi_M \wedge \cdots \wedge \pi_M$ (n times, where $2n$ is the dimension of M) is a section of a vector bundle of rank 1, namely $\wedge^{2n} TM$ which vanishes precisely at

singular points. If it vanishes at least in one point, i.e, if (M, π_M) is not symplectic, there is also a submanifold of codimension one where it vanishes (by Weierstrass preparation theorem). Theorem 2.21 allows to conclude that no semi-connected symplectic resolution exists. □

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