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Hichem Lassoued

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DIMENSION PRESERVING RESOLUTIONS OF SINGULARITIES OF POISSON STRUCTURES

HICHEM LASSOUED

ABSTRACT. Some Poisson structures do admit resolutions by symplectic manifolds of the same dimension. We give examples and simple conditions under which such resolutions can not exist.

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INTRODUCTION

Poisson manifolds of dimension n are known to admit symplectic realizations of dimension $2n$ (see [5] or [3] for the holomorphic case). Recall that a *symplectic realization* of a Poisson manifold (M, π_M) is a triple of $(\Sigma, \Pi_\Sigma, \varphi)$ where (Σ, Π_Σ) is a symplectic¹ manifold and $\varphi : \Sigma \rightarrow M$ is a surjective submersion on (M, π_M) which is also a Poisson morphism.

It is obviously impossible to find a symplectic realization of a Poisson manifold of dimension $n = \dim M$, unless M is itself symplectic. But it is possible to modify the concept of realization by imposing only that φ is a surjective map, but not necessarily a submersion. More precisely, we define symplectic resolutions as follows:

Definition 1. *Let (M, π_M) be a real or complex Poisson manifold of dimension n . We call symplectic resolution a triple $(\Sigma, \pi_\Sigma, \varphi)$ where (Σ, Π_Σ) is a symplectic manifold of dimension n and $\varphi : \Sigma \rightarrow M$ is a surjective Poisson morphism.*

The term "symplectic resolution" has been used by Arnaud Beauville in the context of the algebraic geometry [1] and BaoHua Fu [6] for instance - see for instance [2] for lists of examples of those. The concept that we have introduced in above is consistent with this previously given meaning. It can not be really compared, because we work with in differential geometry, while they work in algebraic geometry, and singular points are for these authors points where the variety is singular, while for us singular points are those where the Poisson structure is singular. But the

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¹For us, a symplectic manifold is a Poisson manifold with an invertible bivector field.

Poisson structures they resolve are symplectic at regular points, and their resolutions are birational at these points. Something similar happens here: we are going to see that the necessary condition for the existence of a symplectic resolution is that the bivector π should be invertible (i.e. symplectic) on an open dense subset of M (see Proposition 1) and our resolutions are local diffeomorphisms at these points. As a consequence, it makes sense to use the same name.

Many examples of symplectic resolutions exist (see Section 1). But we give in this article a simple example of Poisson structure of dimension 2 which does not have reasonable symplectic resolutions. We use this result to exclude a large class of symplectic Poisson manifold which, although there are symplectic on a dense open subset, do not admit reasonable symplectic resolutions. In particular, in the holomorphic setting, we prove that such connected resolutions do not exist for non-symplectic Poisson manifolds.

Conventions: Throughout this article, we denote the Poisson structures by π or $\{\cdot, \cdot\}$ indifferently. We write π when we see it as a section of $\wedge^2 TM$ and by $\{\cdot, \cdot\}$ the associated skew-symmetric biderivation of the algebra of smooth or holomorphic functions on M . Also, we shall denote by π_X or $\{\cdot, \cdot\}_X$ a Poisson structure defined on the manifold X . As we shall in general not consider two Poisson structures on the same manifold, this notation is not ambiguous. When the Poisson structure is invertible, that is, a symplectic Poisson structure, we shall use a capital letter instead, e.g. Π_X .

1. EXAMPLES OF SYMPLECTIC RESOLUTIONS OF THE SAME DIMENSION IN DIMENSION 2

In this section, we give examples of manifolds of dimension 2 that do admit symplectic resolutions. Consider the Poisson structure on $M := \mathbb{R}^2$ (or an open subset of \mathbb{R}^2) given by:

$$(1.1) \quad \{x, y\}_M = f(x, y)$$

where x, y are the canonical coordinates to M , and $f(x, y)$ is a smooth function. We suppose that the Poisson manifold (M, π_M) described by (1.1) admits a symplectic resolution $(\Sigma, \Pi_\Sigma, \varphi)$. With Π_Σ the symplectic Poisson structure on Σ . Since $M = \mathbb{R}^2$, the map φ is of the form $\varphi = (u, v)$, where u and v are two smooth functions on Σ with values in \mathbb{R} . For any choice (p, q) of local coordinates on (Σ, Π_Σ) , the functions u and v are functions of the variables p and q .

Proposition 1. *Let Σ be a manifold, Π_Σ a symplectic Poisson structure on Σ , and $\varphi : \Sigma \rightarrow M = \mathbb{R}^2$ a surjective map, the pair $(\Sigma, \Pi_\Sigma, \varphi)$ is a symplectic resolution of the Poisson manifold (M, π_M) which is given by (1.1) if and only if*

$$(1.2) \quad \{p, q\}_\Sigma \left(\frac{\partial u}{\partial p} \frac{\partial v}{\partial q} - \frac{\partial u}{\partial q} \frac{\partial v}{\partial p} \right) = f(u, v)$$

where p, q are local coordinates on (Σ, Π_Σ) and $\varphi = (u, v)$.

Proof. The map φ is Poisson if and only if $\{\varphi^*x, \varphi^*y\}_\Sigma = \varphi^*\{x, y\} = \varphi^*f(x, y)$, with (x, y) the coordinates on $M = \mathbb{R}^2$. As $u = \varphi^*x$ et $v = \varphi^*y$, this condition is equivalent to $\{u, v\}_\Sigma = f(u, v)$, the result is obtained by writing in the coordinates (p, q) the bracket $\{u, v\}_\Sigma$. \square

Example 1. This example has appeared in [7], Section 6. We construct a symplectic resolution of the Poisson structure on $M := \mathbb{R}^2$ given by :

$$(1.3) \quad \{x, y\}_M = x^2 + y^2,$$

where x, y denote the canonical coordinates of $M = \mathbb{R}^2$. Our candidate of symplectic resolution is given by $\Sigma := \mathbb{R}^2$ equipped with the canonical symplectic Poisson bracket $\{q, p\}_\Sigma = 1$, with (p, q) the coordinates on $\Sigma := \mathbb{R}^2$ (we use different letters for coordinates on M and Σ which by accident, in our example, happen to be isomorphic). We define a map φ from Σ to M by:

$$\begin{aligned} \varphi : \quad \Sigma &\rightarrow M \\ (p, q) &\mapsto (q \sin(pq), q \cos(pq)). \end{aligned}$$

It is easy to check that φ is surjective. We are left with the task of showing that this map φ is a Poisson morphism, for which it suffices to check that the condition given by Equation (1.2) is satisfied. This is done by direct computation: as $u(p, q) = q \sin(pq)$ and $v(p, q) = q \cos(pq)$, we have on the one hand $f(u, v) = (q \sin(pq))^2 + (q \cos(pq))^2 = q^2$, and on the other hand:

$$\begin{aligned} \frac{\partial u}{\partial p} \frac{\partial v}{\partial q} - \frac{\partial u}{\partial q} \frac{\partial v}{\partial p} &= \frac{\partial q \sin(pq)}{\partial p} \frac{\partial q \cos(pq)}{\partial q} - \frac{\partial q \sin(pq)}{\partial q} \frac{\partial q \cos(pq)}{\partial p} \\ &= q^2 (\sin^2(pq) + \cos^2(pq)) = q^2. \end{aligned}$$

Since $\{p, q\}_\Sigma = 1$ and $f(u, v) = q^2$. This proves the claim.

We can also construct a symplectic resolution of a Poisson structure more general than (1.3).

Example 2. We equip $\mathbb{R}^2 := M$ with the Poisson structure defined by

$$\{x, y\}_M = x^{2n} + y^{2m}, \text{ with } n \geq m > 1$$

where the couple (x, y) stands for the canonical coordinates of M , and n and m are integers with $n \geq m$. Our candidate of symplectic resolution is given by $\Sigma := \mathbb{R}^2$ with the Poisson bracket $\{p, q\}_\Sigma = q^{2n-2m} \sin^2(pq^{2m-1}) + \cos^2(pq^{2m-1})$, with (p, q) being the canonical coordinates of Σ . This Poisson bracket is symplectic, because $q^{2n-2m} \sin^2(pq^{2m-1}) + \cos^2(pq^{2m-1})$ is strictly positive and for all $p, q \in \mathbb{R}$. We define a map φ from Σ to M by:

$$\begin{aligned} \varphi : \quad \Sigma &\rightarrow M \\ (p, q) &\mapsto (q \sin(pq^{2m-1}), q \cos(pq^{2m-1})). \end{aligned}$$

To show that (Σ, φ) is a symplectic resolution of M , it is sufficient to check that the condition given by Equation (1.2) is satisfied. As $u(p, q) = q \sin(pq^{2m-1})$ and $v(p, q) = q \cos(pq^{2m-1})$, we have on the one hand, $f(u, v) = (q \sin(pq^{2m-1}))^{2n} + (q \cos(pq^{2m-1}))^{2m}$, and on the other hand, by direct computation:

$$\begin{aligned} \frac{\partial u}{\partial p} \frac{\partial v}{\partial q} - \frac{\partial u}{\partial q} \frac{\partial v}{\partial p} &= \frac{\partial q \sin(pq^{2m-1})}{\partial p} \frac{\partial q \cos(pq^{2m-1})}{\partial q} \\ &\quad - \frac{\partial q \sin(pq^{2m-1})}{\partial q} \frac{\partial q \cos(pq^{2m-1})}{\partial p} \\ &= q^{2m} (\sin^2(pq^{2m-1}) + \cos^2(pq^{2m-1})) = q^{2m}. \end{aligned}$$

This proves the claim. Since $p, q_\Sigma = 1$ and $F(u, v) = q^{2m}$, this implies that any Poisson structure of the form

$$\{x, y\}_M = \kappa(x, y)(x^{2n} + y^{2m}),$$

admits a resolution, provided that the function $\kappa(x, y)$ is strictly positive on M : it suffices to consider the triple $(\Sigma, (\varphi^* \kappa) \Pi_\Sigma, \varphi)$.

Example 3. Examples 1 and 2 are Poisson structures with isolated singularities. We want to introduce an example with non-isolated singularities. We equip $M := \mathbb{R}^2$ with the bracket given in the canonical coordinates by $\{x, y\}_M = x$. We define $\Sigma := \mathbb{R}^2 \amalg \mathbb{R}^2 \amalg \mathbb{R}^2$ to be the disjoint union of three copies of \mathbb{R}^2 . There is a natural symplectic structure on Σ obtained by restricting each of the three copies canonical symplectic Poisson structure. Consider the map $\varphi : \Sigma \rightarrow M$ defined by :

$$\varphi := \begin{cases} (\exp(p), q) & \text{on the first copy} \\ (-\exp(p), q) & \text{on the second copy} \\ (0, q) & \text{on the third copy} \end{cases}$$

A direct computation using Condition (1.2) implies that the restriction on all three copies is a Poisson morphism, which is sufficient to make φ a Poisson morphism. The map φ is surjective, making the couple (Σ, φ) a symplectic resolution of (M, π) . Note that Σ is *not* connected.

Example 4. There is a general but non-satisfying construction for making a symplectic resolution of any Poisson manifold of even dimension. Let (M, π_M) be a Poisson manifold of dimension $2d$. For all symplectic leaf \mathcal{S} of π_M , consider the direct product $\Sigma_{\mathcal{S}} := \mathcal{S} \times \mathbb{R}^{2d-2s}$ where $2s$ is the dimension of \mathcal{S} . Equip $\Sigma_{\mathcal{S}}$ with the direct product Poisson symplectic structure $\Pi_{\Sigma_{\mathcal{S}}}$, with the understanding that \mathcal{S} , the symplectic leaf, comes equipped with its induced symplectic structure, and the vector space of even dimension \mathbb{R}^{2d-2s} comes equipped with any symplectic Poisson structure. The natural map $\varphi_{\mathcal{S}}$ obtained by projecting $\mathcal{S} \times \mathbb{R}^{2d-2s}$ onto \mathcal{S} , then by including \mathcal{S} into M is a Poisson map.

Now, let S be the set of all symplectic leaves. Let $\Sigma := \coprod_{\mathcal{S} \in S} \Sigma_{\mathcal{S}}$. All connected components of this manifold have dimension $2d$ and are symplectic manifolds, hence Σ is symplectic, with respect to a symplectic Poisson structure Π_Σ whose restriction to $\Sigma_{\mathcal{S}}$ is $\Pi_{\Sigma_{\mathcal{S}}}$. The map $\varphi : \Sigma \rightarrow M$ whose restriction to $\mathcal{S} \in S$ is $\varphi_{\mathcal{S}}$ is a surjective Poisson map. Hence, $(\Sigma, \Pi_\Sigma, \varphi)$ defines a Poisson resolution. In general, the symplectic leaves form a non-countable family hence Σ may not be separable (i.e. does not admit a dense countable subset).

2. OF POISSON STRUCTURES THAT DO NOT ADMIT SYMPLECTIC RESOLUTIONS OF THE SAME DIMENSION

We describe in this section broad classes of Poisson manifolds that can not admit reasonable symplectic resolutions. As shown in Example 4, it is very reasonable to assume that the symplectic resolutions are separable, since without such a condition they always exists, but they are not very interesting. The first result is the following.

Proposition 2. *A Poisson manifold (M, π) that admits a separable symplectic resolution is symplectic on an open dense subset.*

Proof. For all $\sigma \in \Sigma$ such that $d_\sigma\varphi$ is surjective, hence invertible, φ is a local diffeomorphism from a neighborhood of σ in Σ to a neighborhood of $\varphi(\sigma)$ in M . Hence, the bivector field π_M is symplectic at the point $\varphi(\sigma)$.

Now, since φ is surjective, it follows from Sard's Theorem (which is precisely valid for smooth maps between separable manifolds) that the image through φ of the set of points in Σ where the differential of φ is surjective (and therefore is a local diffeomorphism) is an open dense subset of M . This completes the proof. \square

For Poisson manifolds that are symplectic on an open dense subset, the singular points of the Poisson structures are exactly the points where the bivector field is not invertible. The following proposition will be used several times. A critical value of φ is a point $m \in M$ where $\sigma \in \Sigma$ with $\varphi(\sigma) = m$ and $d_\sigma\varphi$ is not surjective. A singular point of a Poisson structure is point m where $\pi_m^\# : T_m^*M \rightarrow T_mM$ is not of maximal rank.

Proposition 3. *Let $(\Sigma, \pi_\Sigma, \varphi)$ be a symplectic resolution of the Poisson manifold (M, π_M) .*

- (1) *Singular points of π_M coincides with the set of critical values of φ .*
- (2) *The differential $T_\sigma\varphi$ of φ at a point $\sigma \in \Sigma$ is invertible if and only if $\varphi(\sigma)$ is a regular point of π_M .*

Proofs. The map φ is a Poisson map if and only if the following diagram is commutative for all $m \in M$ and all $\sigma \in \Sigma$ with $\varphi(\sigma) = m$:

$$\begin{array}{ccc} T_\sigma^*\Sigma & \xrightarrow{\Pi_\sigma^\#} & T_\sigma\Sigma \\ T_\sigma^*\varphi \uparrow & & \downarrow T_\sigma\varphi \\ T_m^*M & \xrightarrow{\pi_m^\#} & T_mM. \end{array}$$

The commutativity of this diagram implies that $\pi_m^\#$ is invertible if and only if vertical arrows are invertible, i.e. if and only if $T_\sigma\varphi$ is invertible for all σ in the inverse image of m through φ . This proves the result. \square

Consider the Poisson structure given by:

$$(2.1) \quad \{x, y\}_M = f(x, y) = xg(x, y),$$

where x, y are local coordinates on $M := \mathbb{R}^2$ and $g(x, y)$ is a smooth function that does not vanish unless $x = 0$. We saw in Section 1 that, for $g = 1$, this Poisson structure admits a symplectic resolution for which Σ is a disjoint union of three copies of \mathbb{R}^2 . But we now show that (M, π_M) admits no symplectic resolution when we impose additional conditions. A resolution is said type proper when φ is a *proper* map, i.e. the inverse image though φ of a compact subset of M is a compact subset.

Theorem 2. *The Poisson manifold (M, π) described in (2.1) admits no proper symplectic resolution. It does not admit connected real analytic or holomorphic symplectic resolution.*

Of course, when we say "It does not admit real analytic or holomorphic symplectic resolution", we mean that we consider the case where $f(x, y)$ is real analytic or holomorphic, and in the holomorphic case, we place ourselves on an open subset of $M \subset \mathbb{C}^2$ a open subset that contains $(0, 0)$. We define resolutions in this

context as being resolutions in the previous sense, with $(\Sigma, \Pi_\Sigma, \varphi)$ real analytic or holomorphic.

Remark 1. *The following problem is still open: in the smooth case, does the Poisson manifold (M, π) described by (2.1) admit a connected symplectic resolution?*

Let us prove Theorem 2.

Proof. Let $(\Sigma, \Pi_\Sigma, \varphi)$ be a symplectic resolution of (M, π_M) . Since $M \simeq \mathbb{R}^2$, we write $\varphi = (u, v)$ with u, v smooth real valued functions on Σ . According to Sard's theorem [12], applied to the differentiable function $v : \Sigma \rightarrow \mathbb{R}$, function which is surjective since φ is surjective, the set of critical values of v admits a dense complement in \mathbb{R} .

Let $v_0 \in \mathbb{R}$ be outside the set of all critical values of v . Since v_0 is not a critical value, the inverse image by $v : \Sigma \rightarrow \mathbb{R}$ of v_0 is an union $(\mathcal{C}_i)_{i \in I}$ of curves. Since v is a submersion in a neighborhood of all point in $v^{-1}(v_0)$, the curves $(\mathcal{C}_i)_{i \in I}$ whose union form $v^{-1}(v_0)$ can be separated: that is to say there are open sets $(U_i)_{i \in I}$, with U_i containing \mathcal{C}_i for all indice $i \in I$, such that $U_i \cap U_j = \emptyset$ for all distinct $i, j \in I$.

Let us now call curves of *good type* curves in the previous set on there exists at least one point σ with $u(\sigma) \in [-1, 1]$. We claim that there are only finitely many curves of the good type. Let K be the inverse image through φ of $[-1, 1] \times \{v_0\} \subset M$. Curves of the good type are those that intersect K . Since φ is proper, this inverse image K is compact. But the open subsets $(U_i \cap K)_{i \in I}$ form a partition of K by open subsets. Now, in a partition of a compact set by open subset, only finitely many can be non-empty. Said otherwise, there is a finite subset $J \subset I$ such that $\cup_{j \in J} \mathcal{C}_j$ contains the inverse image of $[-1, 1] \times \{v_0\} \subset M$ through φ . Said otherwise, there are finitely many curves of the good type.

Consider the point $(0, v_0) \in M := \mathbb{R}^2$. This point is a singular point of π_M by the definition thereof. Consider a point $\sigma \in \Sigma$ in its inverse image. This point σ belongs to a curve \mathcal{C}_{j_0} for a certain $j_0 \in I$. We are going to show that the image of \mathcal{C}_{j_0} by φ is reduced to the point $(0, v_0)$, or, equivalently, that u identically vanishes on this curve \mathcal{C}_{j_0} .

Since the function v has a differential that does not vanish at the point σ , there exists another local function p , defined on a neighborhood of σ , such that the pair (p, v) form local Darboux coordinates on an open subset $U_\sigma \subset \Sigma$. On $U_\sigma \subset \Sigma$ the map φ reads $\varphi : (p, q) \rightarrow (u(p, v), v)$ while the restriction $U_\sigma \subset \Sigma$ of the curve \mathcal{C}_{j_0} is given by $v = v_0$. By Proposition 1, since the map φ is a Poisson morphism, the following differential equation in the variable p holds for all value of v in a neighborhood of v_0 :

$$(2.3) \quad \frac{\partial u}{\partial p}(p, v) = f(u(p, v), v).$$

In particular, for $v = v_0$, we obtain the differential equation:

$$(2.4) \quad \begin{cases} \frac{\partial u}{\partial p}(p, v_0) = f(u(p, v_0), v_0) \\ \text{and } u(p_0, v_0) = 0. \end{cases}$$

where (p_0, v_0) are coordinates of the point σ . Since $f(0, v) = 0$ for all value of v and in particular $f(0, v_0) = 0$, the Cauchy-Lipschitz theorem imply that the differential equation (2.3) admits for solution the zero function, i.e. $u(p, v_0) = 0$ for all p close to zero. In particular, the function u vanishes on the restriction of the curve \mathcal{C}_{j_0} to

a neighborhood of $\sigma \in \Sigma$. One can repeat the above deduction in a neighborhood of every point in \mathcal{C}_{j_0} , which proves that the function u vanishes identically on the whole curve \mathcal{C}_{j_0} .

As previously stated, the inverse image by $v : \Sigma \rightarrow \mathbb{R}$ of v_0 is a union of curves. The conclusion of the previous lines is that there are two types of such curves, those on which the restriction of the function u is never equal to 0 (curves that we call curves of the *first type*) and those where u vanishes identically, (curves that we call curves of the *second type*). Since the map φ is a surjection, there is necessarily at least one curve of the first type is of the good type. Also all curves of the second type are of the good type.

Since there are only finitely many curves of the good type, those which are both of the good type and of the first type and those the second type can be separated by two open subsets V and W of Σ . We now consider a sequence $(y_n)_{n \in \mathbb{N}} \in \Sigma$ such that $\varphi(y_n) = (\frac{1}{n}, v_0)$. For all $n \geq 1$ Since the map φ is proper, out of the sequence y_n , we can extract a convergent subsequence. Let $\tilde{y} \in \Sigma$ be its limit. By construction, $\varphi(\tilde{y}) = (0, v_0)$. For any $n \in \mathbb{N}$, the element y_n belongs to a curve of the first type and of the good type, and therefore is in V , but its limit has to belong to a curve of the good type and of the second type, and is therefore in W . This contradicts the assumption that $V \cap W = \emptyset$, this completes the proof. \square

We now look at the real analytic or holomorphic case (i.e, the case where the function f in (2.1) is a real analytic or holomorphic function, and we impose that so are Σ, Π_Σ and φ . We will start with a lemma:

Lemma 1. *Let (Σ, Π, φ) be a connected symplectic resolution of the Poisson structure in (2.1). For any point $\sigma \in \Sigma$ such as $d_\sigma v \neq 0$ et $\varphi(\sigma) \in \mathbf{M}_{sing}$, there exists a neighborhood U_σ of σ such as $\varphi(U_\sigma) \cap \mathbf{M}_{sing} = \{\varphi(\sigma)\}$ where $\mathbf{M}_{sing} \subset M$ is the set $\{x = 0\}$, i.e. is the singular set of π_M .*

Proofs. For any point $\sigma \in \Sigma$ such that $d_\sigma v \neq 0$, there exists a neighborhood U_σ of σ in Σ and a function p defined on Σ such that the pair (p, v) are local Darboux coordinates on Σ . the map φ reads in these coordinates as :

$$\begin{aligned} \varphi : U_\sigma \subset \Sigma &\rightarrow M \\ (p, q) &\mapsto (u(p, q), q) \end{aligned}$$

Let (p_0, v_0) be the coordinates of a point $\sigma \in \Sigma$ such that $\varphi(\sigma) = (0, v_0) \in \mathbf{M}_{sing}$. Proposition 1 implies that Relations (2.3) are satisfied. Consider the function $h : v \rightarrow u(p_0, v)$. Since the function h is real analytic or holomorphic, and h has a zero at $v = v_0$, there are two possibilities : either this function is identically equal to zero or it has an isolated zero at v_0 . Equation (2.3) and Cauchy-Lipschitz theorem give that if h is identically equal to 0 on a neighborhood of v_0 , the function u vanishes at all point of U_σ . Since the function u is real analytic or holomorphic, this implies that $u = 0$ on the whole connected manifold Σ . But this is impossible because this contradicts the surjectivity of φ . Hence v_0 is necessarily an isolated zero of h and Equation (2.3) and Cauchy-Lipschitz theorem imply that $u(p, v)$ can not be zero for $v \neq v_0$ while $u(p, v_0) = 0$ for all p close to p_0 . More precisely, there exists $\eta', \eta'' \in \mathbb{R}_+$ such that the inequalities $|v - v_0| < \eta'$ and $|p - p_0| < \eta''$ give a neighborhood U_σ of σ such as $\varphi(U_\sigma) \cap \mathbf{M}_{sing}$ is reduced to a point, namely the point $(0, v_0) = \varphi(\sigma)$. \square

We now show Theorem 2 in the real analytic or holomorphic cases.

Proofs. Let $(\Sigma, \Pi_\Sigma, \varphi)$ be a connected real analytic or holomorphic symplectic resolution. By construction $\varphi = (u, v)$ with u, v are two real analytic or holomorphic functions on Σ with values in \mathbb{R} or \mathbb{C} . We recall that \mathbf{M}_{sing} is the set $\{x = 0\} \subset \mathbb{C}^2$.

Let Γ be the set of the points $\sigma \in \varphi^{-1}(\mathbf{M}_{sing})$ where v is regular (i.e. $d_\sigma v \neq 0$). As φ is supposed to be surjective, the union of the set of critical values of v with $v(\Gamma)$ is $\mathbf{M}_{sing} \cong \mathbb{R}$ or \mathbb{C} . By Sard's Theorem [12], applied to the differentiable function v , which is surjective because φ is surjective, the critical values of v form a set of measure zero in $\mathbf{M}_{sing} \cong \mathbb{R}$ or \mathbb{C} .

Moreover, for any $\sigma \in \Gamma$, there exists by Lemma 1 a neighborhood V_σ of $\sigma \in \Sigma$ such that $\varphi(V_\sigma) \cap \mathbf{M}_{sing}$ is reduced to a point $(0, v_0) = \varphi(\sigma)$. The set Γ being a closed subset of the open subset of all regular points of v , it is a locally compact set and we can extract out of any open cover of Γ a finite or countable open cover. In particular, one can extract from the open sets cover $(V_\sigma)_{\sigma \in \Gamma}$ a finite or countable family $\{V_i\}_{i \in \mathbb{N}}$. This implies that $\varphi(\Gamma)$, just as $v(\Gamma)$, is a finite or countable set.

Now, \mathbf{M}_{sing} is the union of the set of critical values of v with $v(\Gamma)$. But the first set is a set of measure zero (by Sard's Theorem [12]) while the second is finite or a countable set. This is impossible. This completes the proof. \square

We now enlarge the class of Poisson manifolds that do not admit symplectic resolutions, although they are symplectic on a dense open subset. We refer to [8] Chapter 5 for the definition of Poisson-Dirac submanifolds of Poisson manifolds and recall that any submanifold N of a Poisson manifold (M, π) whose tangent space $T_n N$ is in direct sum with the tangent space of the symplectic leaf \mathcal{S}_n through n , namely such that $T_n N \oplus T_n \mathcal{S}_n = T_n M$, is Poisson-Dirac in a neighborhood of n . we also recall that any submanifold N of a Poisson manifold M such that $T_n N \oplus \pi^\#(T_n N^\perp) = T_n M$ is Poisson-Dirac. Recall also that any Poisson-Dirac submanifold admits an unique induce Poisson structure called *reduced Poisson structure*.

Lemma 2. *Let (M, π) be a Poisson manifold, S be a symplectic leaf and $N \subset M$ a submanifold transversal at a point $n \in S \cap N$. For every symplectic resolution $(\Sigma, \Pi_\Sigma, \varphi)$ of (M, π) , there exists a neighborhood N' of n in N such that:*

- (1) $\varphi^{-1}(N')$ is a submanifold of Σ ,
- (2) this submanifold is Poisson-Dirac,
- (3) the restriction $\varphi_{N'} : \varphi^{-1}(N') \rightarrow N'$ is a symplectic resolution for the reduced Poisson structures of $\varphi^{-1}(N')$ and N' respectively.

Proofs. There exists, in a neighborhood U of n , Weinstein coordinates (p, q, z) such that the submanifold $N' := N \cap U$ is given by the equations: $p_1 = \dots = p_r = q_1 = \dots = q_r = 0$ (see [8] Chapter 1 for the definition of Weinstein's coordinate). The functions $p_1, \dots, p_r, q_1, \dots, q_r$ having independent Hamiltonian vector fields at all points of U , their pull-back $\varphi^* p_1, \dots, \varphi^* p_r, \varphi^* q_1, \dots, \varphi^* q_r$ also have independent Hamiltonians vector fields at all point in $\varphi^{-1}(U)$. Therefore, their pull-backs through φ are independent functions. The zero locus that they define is $\varphi^{-1}(N')$ by construction, which is therefore a submanifold that we call Σ_N . This proves the first item.

A function f on M such that the hamiltonian vector field X_f tangent to N satisfies that the hamiltonian vector field $X_{\varphi^* f}$ is tangent to Σ_N . This is because if $\{F, q_i\}|_N = 0 = \{F, p_i\}|_N$ then $\{\varphi^* F, \varphi^* q_i\}|_{\Sigma_N} = 0 = \{\varphi^* F, \varphi^* p_i\}|_{\Sigma_N}$, since φ is a Poisson morphism. Yet, N and Σ_N are sub-manifolds of Poisson-Dirac, because

they are defined by the functions $(p_1, \dots, p_r, q_1, \dots, q_r)$ and $(\varphi^*p_1, \dots, \varphi^*p_r, \varphi^*q_1, \dots, \varphi^*q_r)$ respectively, whose Poisson matrix is invertible. This proves the second item.

Now, for all $F, G \in C^\infty(N)$, we have $\{\varphi^*F, \varphi^*G\}_{\Sigma_N} = \left\{ \varphi^*\tilde{F}, \varphi^*\tilde{G} \right\}_{\Sigma_N} = \varphi^*\left(\left\{ \tilde{F}, \tilde{G} \right\}_{\Sigma_N} \right)$ where \tilde{F} and \tilde{G} are local extensions of F and G . This proves that φ is a Poisson morphism from Σ_N to N . A Poisson-Direc sub-manifold of a symplectic manifold being itself symplectic, this proves third item. \square

Corollary 1. *A Poisson manifold (M, π) of dimension $2r+2$ symplectic on a dense open subset*

- (1) *admits a submanifold P of codimension 1 which included in to the singular locus of π_M ,*
- (2) *has at least one point $n \in P$ where the rank of π is $2r$,*

does not admit a proper symplectic resolution. In the real analytic or holomorphic cases, under the same assumptions it does not admit a symplectic connected resolution.

Proofs. Let F be a local function in a neighborhood of n . Let us prove that the Hamiltonian vector field X_F of F is tangent to P . Since P has codimension 1, if there exists a point $n' \in P$ where X_F is not tangent to P , every point in a neighborhood of n' in M is obtained from a point of P by following the flow of X_F . But since every point in P is a singular point for π_M , and since the flow of X_F is made of Poisson diffeomorphisms, this would imply that every point in a neighborhood of n' in M is a singular point, which contradicts the assumption on π_M .

Let N be any submanifold of dimension 2 transversal to the symplectic leaf \mathcal{S}_n through n where $n \in P$ is a point where the rank of π_M is $2r$. By the previous point S is a submanifold of P . Since N is transversal to \mathcal{S}_n , $P \cap N$ is a curve in N , at least in a neighborhood of $n \in N$. We denote by P_N this curve.

Suppose that (M, π_M) admits a proper symplectic resolution. Then, by Lemma 2, there is a neighborhood N' of n in N that does admit a proper symplectic resolution.

But upon shrinking N' if necessary, we can assume that N' is diffeomorphic to an open subset of \mathbb{R}^2 , isomorphism under which the curve P_N is given by the equation $x = 0$, with x, y being the canonical coordinates on \mathbb{R}^2 . The Poisson structure on N' being now of the form (2.1), it does not admit proper symplectic resolutions by Theorem 2, which contradicts the previous statement. Hence (M, π_M) does not admit proper symplectic resolutions. The real analytic or holomorphic cases are similar. \square

Lemma 3. *Let (M, π) be a Poisson manifold of dimension $n \geq 4$, such that the singular locus of π contains a submanifold $P \subset M$ of codimension 1. If a symplectic resolution exists, then the bivector π can not be zero at all points in P .*

Proofs. Consider a projection $\psi : U \rightarrow P$ where U is a neighborhood of P in M . Let (Σ, Π, φ) be a symplectic resolution of (M, π) . The map $\psi \circ \varphi : \varphi^{-1}(U) \rightarrow P$ is surjective because both φ and ψ are surjective. By Sard's Theorem, there exists regular value for $\psi \circ \varphi$, i.e. there exists point $p \in P$ such that for every $\sigma \in \varphi^{-1}(U) \subset \Sigma$, the composition $T_{\varphi(\sigma)}\psi \circ T_\sigma\varphi$ is a surjective linear map. Since φ is

surjective, there exists $\sigma \in \Sigma$ such that $\varphi(\sigma) = p$. Such a point belongs to $\varphi^{-1}(U)$ by construction, and satisfies that $T_{\varphi(\sigma)}\psi \circ T_{\sigma}\varphi$ is a surjective linear map. In turn, this implies that for any choice x_1, \dots, x_{n-1} of local coordinates on P around M , the functions $(\psi \circ \varphi)^*x_1, \dots, (\psi \circ \varphi)^*x_{n-1}$ are linearly independent at the point $\sigma \in \varphi^{-1}(U)$.

Let us compute the Poisson brackets of these functions at the point σ . Since the map φ is a Poisson morphism, for all $i, j = 1, \dots, n-1$:

$$\begin{aligned} \{(\psi \circ \varphi)^*x_i, (\psi \circ \varphi)^*x_j\}_{\Sigma}(\sigma) &= \{\varphi^*\psi^*x_i, \varphi^*\psi^*x_j\}_{\Sigma}(\sigma) \\ &= \{\psi^*x_i, \psi^*x_j\}_P(\varphi(\sigma)) = 0 \end{aligned}$$

But it is impossible to have $n-1$ independent functions on a symplectic manifold whose brackets are equal to zero at a given point. This completes the proof. \square

Theorem 5. *A Poisson manifold (M, π) symplectic on an open dense subset which contains a submanifold of codimension 1 of singular points for π_M , does not admit a proper symplectic resolution. When (M, π_M) is a holomorphic Poisson manifold no connected symplectic resolution exists.*

Proofs. We prove the theorem by induction on $\frac{1}{2} \dim(M)$. If $\frac{1}{2} \dim(M) = 1$, the theorem reduces to the statement of Theorem 2 in local coordinates. We assume that the theorem is true until $n = \frac{1}{2} \dim(M)$ and we show it for $n+1 = \frac{1}{2} \dim(M)$. We assume that the singular locus of π contains a submanifold P of codimension 1. By Lemma 3, the rank of π_M on P can not be zero. Let m be a point where in P where $\pi_m \neq 0$. If a symplectic resolution $(\Sigma, \pi_{\Sigma}, \varphi)$ of (M, π_M) exists, by Lemma 2, the inverse image of any submanifold N transverse to the symplectic leaf through m by φ is a symplectic submanifold of Σ and the restriction $\varphi_N : \varphi^{-1}(N) \rightarrow N$ is a symplectic resolution of dimension $2(n+1-r)$ with $2r$ the rank of π_M at m . Since the singular locus of π contains a submanifold of codimension 1, we get a contradiction with the induction hypothesis. This shows the result. \square

Corollary 2. *A holomorphic Poisson manifold does not admit connected symplectic resolutions unless it is symplectic.*

Proofs. A Poisson manifold (M, π_M) that admits a symplectic realization must be symplectic on an open dense subset by Proposition 2. For such a manifold, the multivector field $\pi^n = \pi \wedge \dots \wedge \pi$ (n times, where $2n$ is the dimension of M) is a section of a vector bundle of rank 1, namely $\wedge^{2n}TM$ which vanishes precisely at critical points. If it vanishes in at least one point, i.e, if (M, π_M) is not symplectic, there is also a submanifold of codimension 1 where it vanishes (by Weierstrass preparation theorem - which implies that the zero locus of any holomorphic function contains regular points, around which it is simply a submanifold of codimension 1). Theorem 5 allows to conclude that no symplectic realization exist. \square

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LABORATOIRE D'ANALYSE NUMERIQUE ET D'OPTIMISATION ET DE STATISTIQUE L.A.N.O.S, UNIVERSITÉ BADJI MOKHTAR -ANNABA- B.P.12 ANNABA, 23000 ALGÉRIE.

INSTITUT ELIE CARTAN DE LORRAINE I.E.C.L., UMR 7502 , UNIVERSITÉ DE LORRAINE, ÎLE DU SAULCY 57010 METZ, FRANCE