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ABSTRACT. For a class of $O(n+1,\mathbb{R})$ invariant measures on the Kepler manifold possessing finite moments of all orders, we describe the reproducing kernels of the associated Bergman spaces, discuss the corresponding asymptotic expansions of Tian-Yau-Zelditch, and study the relevant Hankel operators with conjugate holomorphic symbols. Related reproducing kernels on the minimal ball are also discussed. Finally, we observe that the Kepler manifold either does not admit balanced metrics, or such metrics are not unique.

1. INTRODUCTION

Let $n \geq 2$ and consider the Kepler manifold in $\mathbb{C}^{n+1}$ defined by

$$\mathbb{H} := \{z \in \mathbb{C}^{n+1} : z \bullet z = 0,\ z \neq 0\},$$

where $z \bullet w := z_1 w_1 + \cdots + z_{n+1} w_{n+1}$. This is the orbit of the vector $e = (1, i, 0, \ldots, 0)$ under the $O(n+1,\mathbb{C})$-action on $\mathbb{C}^{n+1}$; it is also well-known that $\mathbb{H}$ can be identified with the cotangent bundle of the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ minus its zero section. The unit ball of $\mathbb{H}$,

$$\mathbb{M} := \{z \in \mathbb{H} : |z|^2 = z \bullet \overline{z} < 1\}$$

as well as its boundary $\partial\mathbb{M} = \{z \in \mathbb{H} : |z| = 1\}$ are invariant under $O(n+1,\mathbb{C}) \cap U(n+1) = O(n+1,\mathbb{R})$, and in fact $\partial\mathbb{M}$ is the orbit of e under $O(n+1,\mathbb{R})$. In particular, there is a unique $O(n+1,\mathbb{R})$-invariant probability measure $d\mu$ on $\partial\mathbb{M}$, coming from the Haar measure on the (compact) group $O(n+1,\mathbb{R})$. Explicitly, denoting

$$\alpha := (n+1)\frac{(-1)^{j-1}}{z_j}\, dz_1 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_{n+1} \qquad \text{on } z_j \neq 0$$

(this is, up to constant factor, the unique $SO(n+1,\mathbb{C})$-invariant holomorphic n-form on $\mathbb{H}$, see [21]) and defining a $(2n-1)$-form ω on $\partial\mathbb{M}$ by

$$\omega(z)(V_1, \ldots, V_{2n-1}) := \alpha(z) \wedge \overline{\alpha(z)}(z, V_1, \ldots, V_{2n-1}), \qquad V_1, \ldots, V_{2n-1} \in T(\partial\mathbb{M}),$$

we then have $d\mu = \omega/\omega(\partial\mathbb{M})$ (where, abusing notation, we denote by ω also the measure induced by ω on $\partial\mathbb{M}$). It follows, in particular, that $d\mu$ is also invariant under complex rotations

$$z \mapsto \epsilon z, \qquad \epsilon \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}.$$

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For a finite (nonnegative Borel) measure $d\rho$ on $(0,\infty)$, we can therefore define a rotation-invariant measure $d(\rho\otimes\mu)$ on $\mathbb{H}$ by

$$\int_{\mathbb{H}} f\, d(\rho\otimes\mu) := \int_0^\infty \int_{\partial\mathbb{M}} f(t\zeta)\, d\mu(\zeta)\, d\rho(t),$$

and the (weighted) Bergman space

$$A^2_{\rho\otimes\mu} := \{f \in L^2(d(\rho\otimes\mu)) : f \text{ is holomorphic on } R\mathbb{M}\},$$

where $R := \sup\{t>0 : t\in \operatorname{supp}\rho\} = \sup\{|z| : z\in \operatorname{supp}\rho\otimes\mu\}$ (with the understanding that $R\mathbb{M}=\mathbb{H}$ if $R=+\infty$). It is standard that $A^2_{\rho\otimes\mu}$ is a reproducing kernel Hilbert space, that is, there exists a function $K_{\rho\otimes\mu}(x,y)\equiv K(x,y)$ on $R\mathbb{M}\times R\mathbb{M}$ for which $K(\cdot,y)\in A^2_{\rho\otimes\mu}$ for each y, $K(y,x)=\overline{K(x,y)}$, and

$$f(z) = \int_{R\mathbb{M}} f(w)K(z,w)\, d(\rho\otimes\mu)(w) \qquad \forall f\in A^2_{\rho\otimes\mu}.$$

Our goal in this paper is to give a description of these reproducing kernels, establish their asymptotics as ρ varies in a certain way (so-called Tian-Yau-Zelditch, or TYZ, expansion), and study the Hankel operators on $A^2_{\rho\otimes\mu}$. We also give an analogous description for the reproducing kernels on the minimal ball

$$\mathbb{B} := \{z\in\mathbb{C}^n : |z|^2 + |z\bullet z| < 1\},$$

which is the image of $\mathbb{M}$ under the 2-sheeted proper holomorphic mapping given by the projection onto the first n coordinates.

In more detail, let

$$q_k := \int_0^\infty t^k\, d\rho(t)$$

be the moments of the measure $d\rho$ (the values $q_k=+\infty$ being also allowed if the integral diverges). Our starting point is the following formula for the reproducing kernels (whose proof goes by arguments which are already quite standard).

Theorem 1. *For $z,w\in R\mathbb{M}$ with R as above,*

$$K_{\rho\otimes\mu}(z,w) = \sum_{l=0}^\infty \frac{(z\bullet\overline{w})^l}{d_l},$$

with

$$d_l := \frac{q_{2l}}{N(l)} \tag{1}$$

where

$$N(l) := \binom{l+n-1}{n-1} + \binom{l+n-2}{n-1} = \frac{(2l+n-1)(l+n-2)!}{l!(n-1)!}.$$

Next, recall that, quite generally, for an n-dimensional complex manifold M and a holomorphic line bundle L over M equipped with a Hermitian metric, the so-called Kempf distortion functions ϵ_l, $l=0,1,2,\dots$, are defined by

$$\epsilon_l(z) := \sum_j h(s_j(z), s_j(z)),$$

where $\{s_j\}_j$ is an orthonormal basis of the Hilbert space $L^2_{\rm hol}(L^{\otimes l},\omega^n)$ of holomorphic sections of the l-th tensor power $L^{\otimes l}$ of L square-integrable with respect to

the volume element ω^n on M, where $\omega = -\operatorname{curv} h$ (which is assumed to be positive); see Kempf [16], Rawnsley [24], Ji [15] and Zhang [29]. These functions are of importance in the study of projective embeddings and constant scalar curvature metrics (Donaldson [8]), where a prominent role is played, in particular, by their asymptotic behaviour as l tends to infinity: namely, one has

$$\epsilon_l(z) \approx l^n \sum_{j=0}^{\infty} a_j(z) l^{-j} \qquad \text{as } l \to +\infty$$

in the C^∞-sense, with some smooth coefficient functions $a_j(z)$, and $a_0(z) = 1$. This has been established in various contexts by Berezin [5] (for bounded symmetric domains), Tian [26] and Ruan [25] (answering a conjecture of Yau) and Catlin [7] and Zelditch [28] for M compact, Engliš [9] (for M a strictly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and h subject to some technical hypotheses), etc. If M is a domain in $\mathbb{C}^n$ and the line bundle is trivial (which certainly happens if M is simply connected; in this case one need not restrict to integer l, but may allow it to be any positive number), one can identify (holomorphic) sections of L with (holomorphic) functions on M, h with a positive smooth weight on M, $L^2_{\mathrm{hol}}(L^{\otimes l}, \omega^n)$ with $A^2_{h^l\omega^n} = A^2_{h^l \det[\partial\overline{\partial} \log \frac{1}{h}]}$, and

$$\epsilon_l(z) = h(z)^l K_{h^l \det[\partial\overline{\partial} \log \frac{1}{h}]}(z, z), \tag{2}$$

where we (momentarily) denote by K_w the weighted Bergman kernel with respect to a weight w on M (and similarly for A^2_w). In the context of our Kepler manifold, this has been studied by Gramchev and Loi [11] for $L = \mathbb{H} \times \mathbb{C}$ (i.e. the trivial bundle) and $h(z) = e^{-|z|}$ (so $\omega = \frac{i}{2}\partial\overline{\partial}|z|$; this turns out to be the symplectic form inherited from the isomorphism $\mathbb{H} \cong T^*\mathbb{S}^n \setminus \{\text{zero section}\}$ mentioned in the beginning of this paper [24]), who showed that

$$\epsilon_l(z) = l^n + \frac{(n-2)(n-1)}{2|z|} l^{n-1} + \sum_{k=2}^{n-2} \frac{b_k}{|z|^k} l^{n-k} + R_l(z), \tag{3}$$

with some constants b_k independent of z and remainder term $R_l(z) = O(e^{-cl|z|})$ for some $c > 0$ (i.e. exponentially small).

Our second main result is the following.

Theorem 2. *Let $K_s = K_{\rho\otimes\mu}$ for*

$$d\rho(t) = 2cme^{-st^{2m}} t^{2mn-1}\, dt, \tag{4}$$

where c, m are fixed positive constants and $s > 0$. Then as $s \to +\infty$,

$$e^{-s|z|^{2m}} K_s(z, z) = \frac{2m^n}{(n-1)!c} s^n \sum_{j=0}^{n-1} \frac{b_j}{s^j |z|^{2mj}} + R_s(z), \tag{5}$$

where b_j are constants depending on m and n only,

$$b_0 = 1, \quad b_1 = \frac{(1-n)(mn-n+1)}{2m},$$

and $R_s(z) = O(e^{-\delta s|z|^{2m}})$ with some $\delta > 0$.

The result of Gramchev and Loi corresponds to $m = \frac{1}{2}$ and $c = \frac{2^{1-n}}{(n-1)!}$, so it is recovered as a special case. Our method of proof is a good deal simpler than in [11],

covers all $m > 0$, and is also extendible to other situations. We further note that $d(\rho \otimes \mu)$ with the ρ from (4) actually coincides (up to a constant factor) with ω^n for $\omega = \frac{i}{2}\partial\overline{\partial}(s|z|^{2m})$, in full accordance with (2) (taking $h(z) = e^{-|z|^{2m}}$, and with $l = 1, 2, \dots$ replaced by the continous parameter $s > 0$ as already remarked above).

There is also an analogous result for the Bergman-type weights on $\mathbb{M}$ corresponding to $d\rho(t) = \chi_{[0,1]}(t)(1-t^2)^s t^{2n-1}\, dt$, $s > -1$.

As for our last topic, recall that the Hankel operator $H_{\overline{g}}$, $g \in A^2_{\rho\otimes\mu}$, is the operator from $A^2_{\rho\otimes\mu}$ into $L^2(\rho \otimes \mu)$ defined by

$$H_{\overline{g}} f := (I - P)(\overline{g} f),$$

where $P : L^2(\rho \otimes \mu) \to A^2_{\rho\otimes\mu}$ is the orthogonal projection. This is a densely defined operator, which is (extends to be) bounded e.g. whenever g is bounded. For the (analogously defined) Hankel operators on the unit disc $\mathbb{D}$ in $\mathbb{C}$ or the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$, $n \geq 2$, criteria for the membership of $H_{\overline{g}}$ in the Schatten classes $\mathcal{S}^p$, $p > 0$, were given in the classical papers by Arazy, Fisher and Peetre [2] [1]: it turns out that for $p \leq 1$ there are no nonzero $H_{\overline{g}}$ in $\mathcal{S}^p$ on $\mathbb{D}$, while for $p > 1$, $H_{\overline{g}} \in \mathcal{S}^p$ if and only if $g \in B^p(\mathbb{D})$, the p-th order Besov space on $\mathbb{D}$; while on $\mathbb{B}^n$, $n \geq 2$, there are no nonzero $H_{\overline{g}}$ in $\mathcal{S}^p$ if $p \leq 2n$, while for $p > 2n$, again $H_{\overline{g}} \in \mathcal{S}^p$ if and only if $g \in B^p(\mathbb{B}^n)$. One says that there is a cut-off at $p = 1$ or $p = 2n$, respectively. The result remains in force also for $\mathbb{D}$ and $\mathbb{B}^n$ replaced any bounded strictly-pseudoconvex domain in $\mathbb{C}^n$, $n \geq 1$, with smooth boundary (Luecking [17]).

Our third main result shows that for the Bergman space $A^2(\mathbb{M}) := A^2_{\rho\otimes\mu}$ for $d\rho(t) = \chi_{[0,1]}(t)t^{2n-1}dt$ on $\mathbb{M}$, there is also a cut-off at $p = 2n$.

Theorem 3. *Let $p \geq 1$. Then the following are equivalent.*

(i) *There exists nonconstant $g \in A^2(\mathbb{M})$ with $H_{\overline{g}} \in \mathcal{S}^p$.*
(ii) *There exists a nonzero homogeneous polynomial g of degree $m \geq 1$ such that $H_{\overline{g}} \in \mathcal{S}^p$.*
(iii) *There exists $m \geq 1$ such that $H_{\overline{g}} \in \mathcal{S}^p$ for all homogeneous polynomials g of degree m.*
(iv) *$p > 2n$.*
(v) *$H_{\overline{g}} \in \mathcal{S}^p$ for any polynomial g.*

We remark that the proofs in [2] and [1] relied on the homogeneity of $\mathbb{B}^n$ under biholomorphic self-maps, and thus are not directly applicable for $\mathbb{M}$ with its much smaller automorphism group. The proof in [17] relied on $\overline{\partial}$-techniques, which probably could be adapted to our case of $\mathbb{M}$ i.e. of a smoothly bounded strictly pseudoconvex domain not in $\mathbb{C}^n$ but in a complex manifold, and furthermore having a singularity in the interior (cf. Ruppenthal [23]). Our method of proof of Theorem 3, which is close in spirit to those of [2] and [1], is, however, much more elementary.

The paper is organized as follows. The proof of Theorem 1, together with miscellaneous necessary prerequisites and the results for the minimal ball, occupies Section 2. Applications to the TYZ expansion appear in Section 3, and the results on Hankel operators in Section 4. The final section, Section 5, concludes by a small observation concerning the so-called balanced metrics on $\mathbb{H}$ in the sense of Donaldson [8].

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2. Reproducing kernels

Denote by $\mathcal{P}_k$, $k = 0, 1, 2, \dots$, the space of (restrictions to $\mathbb{H}$ of) polynomials on $\mathbb{C}^{n+1}$ homogeneous of degree k. Clearly, $\mathcal{P}_k$ is isomorphic to the quotient of the analogous space of k-homogeneous polynomials on all of $\mathbb{C}^n$ by the k-homogeneous component of the ideal generated by $z \bullet z$, and

$$\dim \mathcal{P}_k = \binom{k+n-1}{n-1} + \binom{k+n-2}{n-1} =: N(k).$$

For $z \in \mathbb{M}$ and $f \in L^2(\mathbb{S}^n, d\sigma)$, where σ stands for the normalized surface measure on $\mathbb{S}^n$, set

$$\hat{f}(z) := \int_{\mathbb{S}^n} f(\zeta) e^{\langle z,\zeta\rangle}\, d\sigma(\zeta), \tag{6}$$

and let $\mathcal{H}^k \equiv \mathcal{H}^k(\mathbb{S}^n)$ denote the subspace in $L^2(\mathbb{S}^n, d\sigma)$ consisting of spherical harmonics of degree k. It is then known [14] [27] that the functions $x \mapsto (x \bullet z)^k$, $z \in \mathbb{H}$, span $\mathcal{H}^k$, the functions $z \mapsto (z \bullet \zeta)^k$, $\zeta \in \mathbb{S}^n$, span $\mathcal{P}_k$, and the mapping $f \mapsto \hat{f}$ is an isomorphism of $\mathcal{H}^k$ onto $\mathcal{P}_k$, for each k. Using the Funke-Hecke theorem [19, p. 20], it then follows that [21] [20]

$$\int_{\partial\mathbb{M}} (z \bullet w)^k (\xi \bullet \overline{w})^l\, d\mu(w) = \delta_{kl} \frac{(z \bullet \xi)^k}{N(k)} \tag{7}$$

for all $z \in \mathbb{H}$, $\xi \in \mathbb{C}^{n+1}$; and, consequently, $\mathcal{P}_k$ and $\mathcal{P}_l$ are orthogonal in $L^2(d\mu)$ if $k \neq l$, while

$$\int_{\partial\mathbb{M}} f(w)(z \bullet \overline{w})^k\, d\mu(w) = \frac{f(z)}{N(k)} \tag{8}$$

for all $z \in \mathbb{H}$ and $f \in \mathcal{P}_k$.

If f is a function holomorphic on $R\mathbb{M}$ (for some $0 < R \leq \infty$), then it has a unique decomposition of the form

$$f = \sum_{k=0}^{\infty} f_k, \qquad f_k \in \mathcal{P}_k, \tag{9}$$

with the sum converging absolutely and uniformly on compact subsets of $R\mathbb{M}$ ([20, Lemma 3.1]). Let $\mathbf{s} = (s_0, s_1, \dots)$ be an arbitrary sequence of positive numbers. We denote by $A^2_{\mathbf{s}}$ the space of all functions f holomorphic in some $R\mathbb{M}$, $R > 0$, for which,

$$\|f\|^2_{\mathbf{s}} := \sum s_k \|f_k\|^2_{L^2(\partial\mathbb{M}, d\mu)} < \infty,$$

equipped with the natural inner product

$$\langle f, g\rangle_{\mathbf{s}} := \sum_{k=0}^{\infty} s_k \langle f_k, g_k\rangle_{L^2(\partial\mathbb{M}, d\mu)}$$

for $f = \sum_k f_k$, $g = \sum_k g_k$ as in (9). It is immediate that $A^2_{\mathbf{s}}$ is a Hilbert space which contains each $\mathcal{P}_k$, and the linear span of the latter (i.e. the space of all polynomials on $\mathbb{H}$) is dense in $A^2_{\mathbf{s}}$.

Proposition 4. *Assume that*

$$R_{\mathbf{s}} := \liminf_{k\to\infty} \Big|\frac{N(k)}{s_k}\Big|^{-1/2k} > 0$$

(*the value* $R_{\mathbf{s}} = \infty$ *being also allowed*). *Then* $A^2_{\mathbf{s}}$ *is a reproducing kernel Hilbert space of holomorphic functions on* $R_{\mathbf{s}}\mathbb{M}$, *with reproducing kernel*

$$K_{\mathbf{s}}(x,y) = \sum_{k=0}^{\infty} \frac{N(k)}{s_k}(x\bullet\overline{y})^k. \tag{10}$$

Proof. By the definition of $R_{\mathbf{s}}$, the series (10) converges pointwise and locally uniformly for $x, y \in R_{\mathbf{s}}\mathbb{M}$. Moreover, for $g = K_{\mathbf{s}}(\cdot, y)$ we plainly have $g_k = \frac{N(k)}{s_k}(\cdot\bullet\overline{y})^k$ and by (7), $\|g_k\|^2_{L^2(\partial\mathbb{M},d\mu)} = \frac{N(k)}{s_k^2}(y\bullet\overline{y})^k$, so

$$\|g\|^2_{\mathbf{s}} = \sum_{k=0}^{\infty} \frac{N(k)}{s_k}(y\bullet\overline{y})^k = K_{\mathbf{s}}(y,y) < \infty.$$

Thus $K_{\mathbf{s}}(\cdot, y) \in A^2_{\mathbf{s}}$ for $y \in R_{\mathbf{s}}\mathbb{M}$. Furthermore, for $f \in A^2_{\mathbf{s}}$, by (8)

$$\begin{aligned}
\sum_k |f_k(y)| &\le \sum_k \Big|N(k)\int_{\partial\mathbb{M}} f_k(w)(y\bullet\overline{w})^k\,d\mu(w)\Big| \\
&= \sum_k s_k \Big|\Big\langle f_k, \frac{N(k)}{s_k}(\cdot\bullet\overline{y})^k\Big\rangle_{L^2(\partial\mathbb{M},d\mu)}\Big| \\
&\le \sum_k s_k\|f_k\|_{L^2(\partial\mathbb{M},d\mu)}\Big\|\frac{N(k)}{s_k}(\cdot\bullet\overline{y})^k\Big\|_{L^2(\partial\mathbb{M},d\mu)} \\
&\le \Big(\sum_k s_k\|f_k\|^2_{L^2(\partial\mathbb{M},d\mu)}\Big)^{1/2}\Big(\sum_k \Big\|\frac{N(k)}{s_k}(\cdot\bullet\overline{y})^k\Big\|^2_{L^2(\partial\mathbb{M},d\mu)}\Big)^{1/2} \\
&= \|f\|_{\mathbf{s}} K_{\mathbf{s}}(y,y)^{1/2},
\end{aligned}$$

implying that the series $\sum_k f_k$ converges locally uniformly on $R_{\mathbf{s}}\mathbb{M}$ and that the point evaluation $f \mapsto f(y)$ is continuous on $A^2_{\mathbf{s}}$ for all $y \in R_{\mathbf{s}}\mathbb{M}$. Finally, removing absolute values in the last computation shows that $f(y) = \langle f, K_{\mathbf{s}}(\cdot, y)\rangle_{\mathbf{s}}$, proving that $K_{\mathbf{s}}$ is the reproducing kernel for $A^2_{\mathbf{s}}$. □

Recall now that a sequence $\mathbf{s} = (s_k)_{k\in\mathbb{N}}$ is called a Stieltjes moment sequence if it has the form

$$s_k = s_k(\nu) := \int_0^\infty r^k\,d\nu(r)$$

for some nonnegative measure ν on $[0, +\infty)$, called a representing measure for $\mathbf{s}$; or, alternatively,

$$s_k = \int_0^\infty t^{2k}\,d\rho(t)$$

for the nonnegative measure $d\rho(t) = d\nu(t^2)$. These sequences have been characterized by Stieltjes in terms of a positive definiteness conditions. It follows from the above integral representation that each Stieltjes moment sequence is either nonvanishing, that is, $s_k > 0$ for all k, or else $s_k = c\delta_{0k}$ for all k for some $c \ge 0$. Fix a nonvanishing Stieltjes moment sequence $\mathbf{s} = (s_k)$. By the Cauchy-Schwarz

inequality we see that the sequence $\frac{s_{k+1}}{s_k}$ is nondecreasing and hence converges as $k \to +\infty$ to the radius of convergence $R_{\mathbf{s}}^2$ of the series

$$\sum_{k=0}^{\infty} \frac{N(k)}{s_k} z^k, \qquad z \in \mathbb{C}.$$

We can now prove our first theorem from the Introduction.

Proof of Theorem 1. Recall that we denoted $R = \sup\{t > 0 : t \in \operatorname{supp} \rho\}$. If $R = \infty$, then for any $r > 0$ we have $q_{2k} \geq \int_r^\infty t^{2k}\, d\rho(t) \geq r^{2k}\rho((r,\infty))$ with $\rho((r,\infty)) > 0$, so $\liminf_{k\to\infty} q_{2k}^{1/2k} \geq r$; thus $q_{2k}^{1/2k} \to +\infty$. If $R < \infty$, then the same argument shows that $\liminf_{k\to\infty} q_{2k}^{1/2k} \geq r$ for any $r < R$, while from

$$q_{2k} \leq R^{2k} \int_0^R d\rho = R^{2k}\rho([0,R])$$

with $\rho([0,R]) < \infty$ we get $\limsup_{k\to\infty} q_{2k}^{1/2k} \leq R$. Setting $s_k = q_{2k}$ we thus get in either case

$$R = R_{\mathbf{s}}.$$

Now for any $r < R$ and $f \in A^2_{\rho\otimes\mu}$, we have by the uniform convergence of (9)

$$\begin{aligned}\int_{r\mathbb{M}} |f|^2\, d(\rho\otimes\mu) &= \sum_{j,k} \int_{r\mathbb{M}} f_j\overline{f_k}\, d(\rho\otimes\mu)\\ &= \sum_{j,k} \int_0^r \int_{\partial\mathbb{M}} t^{j+k} f_j(\zeta)\overline{f_k(\zeta)}\, d\mu(\zeta)\, d\rho(t)\\ &= \sum_k \Big(\int_0^r t^{2k}\, d\rho(t)\Big)\|f_k\|^2_{L^2(\partial\mathbb{M},d\mu)}\end{aligned}$$

by the orthogonality of $\mathcal{P}_k$ and $\mathcal{P}_l$ for $j \neq k$. Letting $r \nearrow R$, we thus get

$$\|f\|^2_{\rho\otimes\mu} = \sum_k q_{2k}\|f_k\|^2_{L^2(\partial\mathbb{M},d\mu)} = \|f\|^2_{\mathbf{s}}.$$

Hence $A^2_{\rho\otimes\mu} \subset A^2_{\mathbf{s}}$, with equal norms. Since clearly $\mathcal{P}_l \in A^2_{\rho\otimes\mu}$ for each l and the span of $\mathcal{P}_l$ is dense in $A^2_{\mathbf{s}}$, it follows that actually $A^2_{\rho\otimes\mu} = A^2_{\mathbf{s}}$ and $\|f\|_{\rho\otimes\mu} = \|f\|_{\mathbf{s}}$ for any $f \in A^2_{\rho\otimes\mu}$. The claim now follows from the last proposition. □

As an example, let ϕ be a nonnegative integrable function on $(0,\infty)$, and consider the volume element

$$\alpha_\phi(z) := \phi(|z|^2)\frac{\alpha(z)\wedge\overline{\alpha(z)}}{(-1)^{n(n+1)/2}(2i)^n}.$$

Let A^2_ϕ and K_ϕ be the corresponding weighted Bergman space and its reproducing kernel, respectively.

Theorem 5. *We have*

$$K_\phi(z,w) = \frac{1}{(n-1)!c_{\mathbb{M}}}\Big[2tF^{(n-1)}(t) + (n-1)F^{(n-2)}(t)\Big]_{t=z\bullet\overline{w}},$$

where

$$c_{\mathbb{M}} = (n-1)\int_{\mathbb{M}} \frac{\alpha(z)\wedge\overline{\alpha(z)}}{(-1)^{n(n+1)/2}(2i)^n}$$

and

$$F(t) = \sum_{k=0}^{\infty} \frac{t^k}{c_k}, \qquad \text{where } c_k := \int_0^\infty t^k \phi(t)\, dt. \tag{11}$$

Proof. It was shown in [20, Lemma 2.1] that for any measurable function f on $\mathbb{H}$,

$$\int_{\mathbb{H}} f(z) \frac{\alpha(z) \wedge \overline{\alpha(z)}}{(-1)^{n(n+1)/2}(2i)^n} = 2c_{\mathbb{M}} \int_0^\infty \int_{\partial \mathbb{M}} f(t\zeta)\, t^{2n-3}\, d\mu(\zeta)\, dt,$$

with $c_{\mathbb{M}}$ as above. Thus $\alpha_\phi = \rho \otimes \mu$ for

$$d\rho(t) = 2c_{\mathbb{M}} t^{2n-3} \phi(t^2)\, dt,$$

and by the last theorem, K_ϕ is given by (10) with

$$s_k = \int_0^\infty t^{2k}\, d\rho(t) = c_{\mathbb{M}} \int_0^\infty t^{k+n-2} \phi(t)\, dt = c_{\mathbb{M}} c_{k+n-2}.$$

Now by an elementary manipulation,

$$\begin{aligned} \sum_k \binom{k+n-1}{n-1} \frac{t^k}{c_{k+n-2}} &= \frac{1}{(n-1)!} \sum_k \Big(\frac{d}{dt}\Big)^{n-1} \frac{t^{k+n-1}}{c_{k+n-2}} \\ &= \frac{1}{(n-1)!} \Big(\frac{d}{dt}\Big)^{n-1} (tF(t)) \\ &= \frac{1}{(n-1)!} \Big[tF^{(n-1)}(t) + (n-1)F^{(n-2)}(t) \Big], \end{aligned}$$

and similarly

$$\begin{aligned} \sum_k \binom{k+n-2}{n-1} \frac{t^k}{c_{k+n-2}} &= \frac{t}{(n-1)!} \sum_k \Big(\frac{d}{dt}\Big)^{n-1} \frac{t^{k+n-2}}{c_{k+n-2}} \\ &= \frac{t}{(n-1)!} F^{(n-1)}(t). \end{aligned}$$

Thus

$$\sum_k \frac{N(k) t^k}{c_{k+n-2}} = \frac{1}{(n-1)!} \Big[2tF^{(n-1)}(t) + (n-1)F^{(n-2)}(t) \Big]$$

and the assertion follows. □

Example. Take $\phi(r) = (1-r)^m$ for $r \in [0,1]$ and $\phi(r) = 0$ for $r > 1$, where $m > -1$. Then $c_k = \frac{k!\Gamma(m+1)}{\Gamma(m+k+2)}$ and we recover the formula from [20]

$$K_\phi(z,w) = \frac{\Gamma(n+m)}{(n-1)! c_{\mathbb{M}} \Gamma(m+1)} \frac{(n-1) + (n+1+2m) z \bullet \overline{w}}{(1 - z \bullet \overline{w})^{n+m+1}}$$

for the "standard" Bergman kernels on $\mathbb{M}$ (with respect to $\alpha \wedge \overline{\alpha}$).

Further applications of Theorem 1 will occur later in the sequel.

We conclude this section by considering the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ with respect to the "minimal norm" given by

$$N_*(z) = \sqrt{|z|^2 + |z \bullet z|}.$$

This norm was shown to be of interest in the study of several problems related to proper holomorphic mappings and the Bergman kernel, see [13], [22], [21], [20] and [18].

Suppose that ϕ is as before and consider the measure dV_ϕ on $\mathbb{C}^n$ with density $\phi(N_*^2)$ with respect to the Lebesgue measure. Namely,

$$dV_\phi(z) := \phi(|z|^2 + |z \bullet z|)\, dV(z)$$

where $dV(z)$ denotes the Lebesgue measure on $\mathbb{C}^n$ normalized so that the volume of the minimal ball is equal to one. We denote by $A^2_\phi(\mathbb{C}^n)$ the Bergman-Fock type space on $\mathbb{C}^n$ with respect to dV_ϕ, consisting of all measurable functions f on $\mathbb{C}^n$ which are holomorphic in the ball

$$\mathbb{B}_\phi := \{z \in \mathbb{C}^n : \sqrt{|z|^2 + |z \bullet z|} < R_\phi\}$$

and satisfy

$$\|f\|^2_\phi := \int_{\mathbb{C}^n} |f(z)|^2 dV_\phi(z) < +\infty. \tag{12}$$

Here R_ϕ is the square root of the radius of convergence of the series (11). We let $L^2_\phi(\mathbb{C}^n)$ denote the space of all measurable functions f in $\mathbb{C}^n$ verifying (12). Finally, we define the operators Δ_j, $j = 0, 1$, acting on power series in z by their actions on the monomials z^m as follows

$$(\Delta_0 z^m)(x, y) := 2 \sum_{k=0}^{[\frac{m-1}{2}]} \binom{m}{2k+1} x^{m-1-2k} y^k,$$

$$(\Delta_1 z^m)(x, y) := 2 \sum_{k=0}^{[\frac{m}{2}]} \binom{m}{2k} x^{m-2k} y^k, \qquad x, y \in \mathbb{C}.$$

If $f(z) = \sum_k c_k z^k$ is a power series of radius of convergence R, then the series

$$(\Delta_j f)(x, y) := \sum_k c_k (\Delta_j z^k)(x, y^2)$$

converges as long as $|x| + |y| < R$ and we have

$$(\Delta_0 f)(x, y) = \frac{f(x + y) - f(x - y)}{y}, \qquad y \neq 0,$$
$$(\Delta_1 f)(x, y) = f(x + y) + f(x - y).$$

Using these notations, we then have the following.

Theorem 6. *The space $A^2_\phi(\mathbb{C}^n)$ coincides with the closure of the holomorphic polynomials in $L^2_\phi(\mathbb{C}^n)$ and its reproducing kernel is given by*

$$K_{\phi,\mathbb{C}^n}(z, w) = \frac{(n+1)^2}{(n-1)! c_{\mathbb{M}}} \Big[2(z \bullet \overline{w}) \Delta_0(F^{(n-1)})(z \bullet \overline{w}, z \bullet z \cdot \overline{w \bullet w})$$
$$+ \Delta_1(F^{(n-1)})(z \bullet \overline{w}, z \bullet z \cdot \overline{w \bullet w})$$
$$+ (n-1) \Delta_0(F^{(n-2)})(z \bullet \overline{w}, z \bullet z \cdot \overline{w \bullet w}) \Big],$$

with F as in (11).

Proof. We will use the technique developed in [20]. Let $\mathrm{Pr} : \mathbb{C}^{n+1} \to \mathbb{C}^n$ be the projection onto the first n coordinates

$$\mathrm{Pr}(z_1, \ldots, z_n, z_{n+1}) = (z_1, \ldots, z_n)$$

and $\iota := \Pr|_{\mathbb{H}}$. Then $\iota : \mathbb{H} \to \mathbb{C}^n \setminus \{0\}$ is a proper holomorphic mapping of degree 2. The branching locus of ι consists of points with $z_{n+1} = 0$, and its image under ι consists of all $x \in \mathbb{C}^n \setminus \{0\}$ with $\sum_{j=1}^n x_j^2 = 0$. The local inverses Φ and Ψ of ι are given for $z \in \mathbb{C}^n \setminus \{0\}$ by

$$\Phi(z) = (z, i\sqrt{z \bullet z}),$$
$$\Psi(z) = (z, -i\sqrt{z \bullet z}).$$

In view of Lemma 3.1 of [21], we see that

$$\Phi^*(\alpha_\phi) = \frac{1+n}{i\sqrt{z \bullet z}} \phi(z)^{1/2} (-1)^n dz_1 \wedge \cdots \wedge dz_n,$$
$$\Psi^*(\alpha_\phi) = \frac{1+n}{-i\sqrt{z \bullet z}} \phi(z)^{1/2} (-1)^n dz_1 \wedge \cdots \wedge dz_n.$$

If $f : \mathbb{C}^n \to \mathbb{C}$ is a measurable function and $z \in \mathbb{H}$, we consider the operator $U = U_\phi$ by setting

$$(Uf)(z) := \frac{z_{n+1}}{\sqrt{2}(n+1)} (f \circ \iota)(z).$$

Using the same arguments as in the proof of Lemma 4.1 in [20], it can be shown that U is an isometry from $L^2_\phi(\mathbb{C}^n)$ into $L^2_\phi(\mathbb{H})$. More precisely, we have

$$\int_{\mathbb{H}} |Uf(z)|^2 \alpha_\phi(z) = \int_{\mathbb{C}^n} |f(z)|^2 \, dV_\phi(z).$$

In addition, the arguments used in the proof of part (2) of the latter lemma show that the image $\mathcal{E}_\phi(\mathbb{H})$ of $A^2_\phi(\mathbb{C}^n)$ under U is a closed proper subspace of $A^2_\phi(\mathbb{H})$, and U is unitary from $A^2_\phi(\mathbb{C}^n)$ onto $\mathcal{E}_\phi(\mathbb{H})$. From the technique used in the proof of Lemma 2.4 in [20], we the get the following lemma.

Lemma 7. *If Φ and Ψ are as before, then*

$$z_{n+1} K_{\phi,\mathbb{C}^n}(\iota z, w) = (n+1)^2 \left[\frac{K_\phi(z, \Phi(w))}{\overline{\Phi_{n+1}(w)}} + \frac{K_\phi(z, \Psi(w))}{\overline{\Psi_{n+1}(w)}} \right],$$

for all $z \in \mathbb{H}$, $w \in \mathbb{C}^n$.

The rest of the proof of the theorem now follows from the last lemma and the identities used in the proof of Theorem A in [20]. □

Example. Let $\phi(r) = e^{-cr}$, $c > 0$. Then $s_k = k!/c^{k+1}$, $F(t) = ce^{ct}$ and we obtain

$$K_{\phi,\mathbb{C}^n}(z, w) = \frac{2(n+1)^2 c^{n-2}}{(n-1)! c_{\mathbb{M}}} \times$$
$$e^{cz \bullet \overline{w}} \left[2c^2 (n - 1 + z \bullet \overline{w}) S(c^2 z \bullet z\overline{w \bullet w}) + cC(c^2 z \bullet z\overline{w \bullet w}) \right],$$

where we wrote for brevity $S(t) = \frac{\sinh\sqrt{t}}{\sqrt{t}}$, $C(t) = \cosh\sqrt{t}$.

3. TYZ expansions

Proof of Theorem 2. For the measure (4), we have by the change of variable $x = st^{2m}$,

$$q_{2k} = 2cm \int_0^\infty t^{2k} e^{-st^{2m}} \, t^{2mn-1} \, dt = \frac{c}{s} \int_0^\infty \left(\frac{x}{s}\right)^{\frac{2k+2mn}{2m}-1} e^{-x} \, dx = c \frac{\Gamma(\frac{k+mn}{m})}{s^{\frac{k+mn}{m}}}.$$

Applying Theorem 1, we get

$$K_s(x,y) = \sum_k \frac{s^{\frac{mn+k}{m}}(x\bullet\overline{y})^k N(k)}{c\Gamma(\frac{k+mn}{m})}$$

$$= \frac{s^n}{(n-1)!c}\Big[\Big(\frac{d}{dt}\Big)^{n-1}t^{n-1} + t\Big(\frac{d}{dt}\Big)^{n-1}t^{n-2}\Big]E_{\frac{1}{m},n}(t)\Big|_{t=s^{1/m}x\bullet\overline{y}}, \tag{13}$$

by a similar computation as in the proof of Theorem 5. Here $E_{\frac{1}{m},n}$ is the Mittag-Leffler function

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \qquad \alpha,\beta>0.$$

Recall now that as $z\to\infty$, $E_{\alpha,\beta}$ has the asymptotics

$$E_{\alpha,\beta}(z) = \begin{cases} \frac{1}{\alpha}\sum_{j=-N}^{N} z_j^{1-\beta}e^{z_j} + O\Big(\frac{1}{z}\Big), & |\arg z| \le \frac{\pi\alpha}{2}, \\ O\Big(\frac{1}{z}\Big), & |\arg(-z)| < \pi - \frac{\pi\alpha}{2}, \end{cases}$$

where N is the integer satisfying $N < \frac{\alpha}{2} \le N+1$ and $z_j = |z|^{1/\alpha}e^{(\arg z + 2\pi i j)/\alpha}$ with $-\pi < \arg z \le \pi$. See e.g. [4, §18.1, formulas (21)–(22)] (additional handy references are given in Section 7 of [6]). Furthermore, this asymptotic expansion can be differentiated termwise any number of times (see again Section 7 in [6] for details on this). In particular, for $t>0$ the term $j=0$ dominates all the others, and we therefore obtain

$$E_{\alpha,\beta}(t) = \frac{1}{\alpha}t^{(1-\beta)/\alpha}e^{t^{1/\alpha}} + O(e^{(1-\delta)t^{1/\alpha}}) \qquad \text{as } t\to+\infty \tag{14}$$

with some $\delta>0$. (One can take any $0<\delta<\frac{1}{\sqrt{2}}(1-\cos\frac{2\pi}{\alpha})$ for $\alpha>4$, and any $0<\delta<\frac{1}{\sqrt{2}}$ for $0<\alpha<4$.) Moreover, (14) remains in force when a derivative of any order is applied to the left-hand side and to the first term on the right-hand side.

Now by a simple induction argument,

$$\Big(\frac{d}{dt}\Big)^k t^{\gamma m}e^{t^m} = t^{\gamma m-k}e^{t^m}p_k(t^m), \tag{15}$$

where p_k are polynomials of degree k defined recursively by

$$p_0 = 1, \quad p_k(x) = (\gamma m - k + 1 + mx)p_{k-1}(x) + mxp'_{k-1}(x). \tag{16}$$

A short computation reveals that

$$p_k(x) = m^k x^k + [km^k\gamma + \tfrac{k(k-1)}{2}(m-1)m^{k-1}]x^{k-1} + \dots. \tag{17}$$

Taking $\alpha=\frac{1}{m}$, $\beta=n$, $\gamma=1-n$, an application of the Leibniz rule shows that

$$\Big[\Big(\frac{d}{dt}\Big)^{n-1}t^{n-1} + t\Big(\frac{d}{dt}\Big)^{n-1}t^{n-2}\Big]E_{\frac{1}{m},n}(t)$$

$$= \sum_{j=0}^{n-1}\binom{n-1}{j}E^{(j)}_{\frac{1}{m},n}(t)\Big[\frac{(n-1)!}{j!}t^j + \frac{(n-2)!}{(j-1)!}t^j\Big]$$

$$= \sum_{j=0}^{n-1}\binom{n-1}{j}\Big[\frac{(n-1)!}{j!} + \frac{(n-2)!}{(j-1)!}\Big]mt^{(1-n)m}e^{t^m}p_j(t^m) + O(e^{(1-\delta)t^m}), \tag{18}$$

so

$$\begin{aligned} e^{-t^m}\Big[\Big(\frac{d}{dt}\Big)^{n-1}t^{n-1}+t\Big(\frac{d}{dt}\Big)^{n-1}t^{n-2}\Big]E_{\frac{1}{m},n}(t) \\ &= \sum_{j=0}^{n-1}\binom{n-1}{j}\Big[\frac{(n-1)!}{j!}+\frac{(n-2)!}{(j-1)!}\Big]mt^{(1-n)m}p_j(t^m)+O(e^{-\delta t^m}) \\ &= \sum_{k=0}^{n-1}\frac{2m^n b_k}{t^{km}}+O(e^{-\delta t^m}) \end{aligned} \tag{19}$$

as $t\to+\infty$, where b_k are some constants depending only on m,n, with (after a small computation)

$$b_0=1,\qquad b_1=-\frac{(n-1)(mn-n+1)}{2m}$$

by (17). Setting $t=s^{1/m}|z|^2$ and substituting (19) into (13), we are done. □

As remarked in the Introduction, in the context of the TYZ expansions Theorem 2 corresponds to the situation of the trivial bundle $\mathbb{H}\times\mathbb{C}$ over $\mathbb{H}$, with Hermitian metric on the fiber given by $h(z)=e^{-s|z|^{2m}}$. The associated Kähler form $\omega=\frac{i}{2}\partial\overline{\partial}\log\frac{1}{h}=\frac{is}{2}\partial\overline{\partial}|z|^{2m}$ can be computed similarly as in [24] for $\frac{i}{2}\partial\overline{\partial}|z|$. Even without that, however, one can see what is the corresponding volume element ω^n: namely, since differentiation lowers the degree of homogeneity by 1, the density of ω^n with respect to the Euclidean surface measure on $\mathbb{H}$ must have homogeneity $n\cdot(2m-2)=2mn-2n$; and as the surface measure equals, up to a constant factor, to $t^{2n-1}dt\otimes d\mu$, we see that $\omega^n=d(\rho\otimes\mu)$ for $d\rho(t)=ct^{2mn-1}\,dt$, with some constant c. Thus

$$e^{-s|z|^{2m}}K_s(z,z)\equiv\epsilon_s(z)$$

is indeed precisely the Kempf distortion function for the above line bundle, and (5) is its asymptotic, or TYZ, expansion.

The bundles studied by Gramchev and Loi in [11] with $\omega=\frac{i}{2}\partial\overline{\partial}|z|$ correspond to $m=\frac{1}{2}$. Note that in that case, in agreement with [11], the lowest-order term in (5) actually vanishes, i.e. $b_{n-1}=0$. In fact, the constant term in p_k equals, by (16),

$$p_k(0)=(\gamma m-k+1)(\gamma m-k+2)\dots(\gamma m)=(-1)^k(-\gamma m)_k$$

(where $(\nu)_k:=\nu(\nu+1)\dots(\nu+k-1)$ is the usual Pochhammer symbol); thus for the lowest order term in the sum in (19) we get from (18)

$$\begin{aligned} 2m^n b_{n-1} &= m\sum_{j=0}^{n-1}\binom{n-1}{j}\Big[\frac{(n-1)!}{j!}+\frac{(n-2)!}{(j-1)!}\Big](-1)^j(-\gamma m)_j \\ &= (n-1)m(1-2m)\prod_{j=1}^{n-2}(j-(n-1)m) \end{aligned}$$

after some computation (using the Chu-Vandermonde identity), which vanishes for $m=\frac{1}{2}$. This explains why the summations stops at $k=n-2$ in (3).

We remark that in a completely analogous manner, one could also derive the asymptotics as $s\to+\infty$ of the reproducing kernels for the same weights $e^{-s|z|^{2m}}$ but

with respect to the density $\frac{\alpha(z)\wedge\overline{\alpha(z)}}{(-1)^{n(n+1)/2}(2i)^n}$ instead of $(\frac{i}{2}\partial\overline{\partial}|z|^{2m})^n$. By Theorem 5 the corresponding kernels are given by

$$K_s(z,w) = \frac{ms^{\frac{n-1}{m}}}{(n-1)!c_{\mathbb{M}}}\Big[2t\Big(\frac{d}{dt}\Big)^{n-1} + (n-1)\Big(\frac{d}{dt}\Big)^{n-2}\Big]E_{\frac{1}{m},\frac{1}{m}}(t)\Big|_{t=s^{\frac{1}{m}}z\bullet\overline{w}} \tag{20}$$

and the needed asymptotics of derivatives of $E_{\frac{1}{m},\frac{1}{m}}$ are worked out e.g. in Section 7 of [6] (or can be obtained from the formulas above in this section). We leave the details to the interested reader. For $m = 1$, (20) recovers the formula from Gonessa and Youssfi [12].

Finally, one can establish analogous "TYZ" asymptotics also for Bergman-type kernels on the unit ball $\mathbb{M}$ of $\mathbb{H}$; we limit ourselves to the following variant of Theorem 3.2 from [20].

Theorem 8. *For the weights corresponding to*

$$d\rho(t) = (1-t^2)^s t^{2n-1}\,dt, \qquad s > -1$$

(i.e. having the density $(1-|z|^2)^s$ with respect to the Euclidean surface measure) on $\mathbb{M}$, the corresponding reproducing kernels K_s of $A^2_{\rho\otimes\mu}$ are given by

$$K_s(z,w) = \frac{\Gamma(n+s+1)}{(n-1)!\Gamma(s+1)}\times \\ \Big[\frac{1}{(1-t)^{n+s+1}} + \frac{n+s+1}{t^{n-1}}\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j\frac{(1-t)^{j-n-s-1}-1}{n+s+1-j}\Big]_{t=z\bullet\overline{w}}.$$

Proof. Since

$$q_{2l} = \int_0^1 t^{2l}(1-t^2)^s t^{2n-1}\,dt = \frac{\Gamma(s+1)\Gamma(l+n)}{\Gamma(l+n+s+1)},$$

we get from Theorem 1

$$K_s(z,w) = \sum_{l=0}^{\infty}\frac{N(l)t^l\Gamma(l+n+s+1)}{(l+n-1)!\Gamma(s+1)}$$

where we have set for brevity $t = z\bullet\overline{w}$. Hence

$$\begin{aligned}\frac{(n-1)!\Gamma(s+1)}{\Gamma(n+s+1)}K_s(z,w) &= \sum_{l=0}^{\infty}t^l\Big[\frac{(n+s+1)_l}{l!} + \frac{(n+s+1)_l}{(l-1)!(l+n-1)}\Big] \\ &= \frac{1}{(1-t)^{n+s+1}} + \sum_{k=0}^{\infty}\frac{(n+s+1)_{k+1}t^{k+1}}{k!(n+k)} \\ &= \frac{1}{(1-t)^{n+s+1}} + (n+s+1)t\sum_{k=0}^{\infty}\frac{(n+s+2)_k t^k}{k!(n+k)}.\end{aligned}$$

The last sum can be written as

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(n+s+2)_k t^k}{k!(n+k)} &= \frac{1}{t^n}\int_0^t\sum_{k=0}^{\infty}\frac{(n+s+2)_k}{k!}x^{k+n-1}\,dx \\ &= \frac{1}{t^n}\int_0^t\frac{x^{n-1}}{(1-x)^{n+s+2}}\,dx\end{aligned}$$

$$= \frac{1}{t^n}\int_0^t \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{(x-1)^j}{(1-x)^{n+s+2}}\, dx$$

$$= \frac{1}{t^n}\sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j \frac{(1-t)^{j-n-s-1}-1}{s+n+1-j},$$

proving the theorem. □

Corollary 9. *In the setup from the last theorem,*

$$(1-|z|^2)^{s+n+1} K_s(z,z) \approx s^n \sum_{k=0}^{\infty} \frac{a_k(|z|^2)}{s^k} \qquad \text{as } s \to +\infty, \tag{21}$$

where $a_k(|z|^2)$ are some functions, with $a_0 = 2$.

Proof. With $t = |z|^2$, Theorem 8 shows that the left-hand side equals

$$\frac{(s+1)_n}{(n-1)!}\Big[1 + \frac{n+s+1}{t^{n-1}} \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j \frac{(1-t)^j - (1-t)^{s+n+1}}{n+s+1-j}\Big].$$

The term $(1-t)^{n+s+1}$ is exponentially small compared to the rest, and thus can be neglected. Noticing that $(s+1)_n$ is a polynomial in s of degree n while

$$\frac{n+s+1}{n+s+1-j} = 1 + \frac{j}{s(1+\frac{n+1-j}{s})} = 1 + \sum_{k=0}^{\infty} \frac{(j-n-1)^k j}{s^{k+1}},$$

the expansion (21) follows, as does the formula for a_0 upon a small computation. □

4. Hankel operators

Proof of Theorem 3. (i) $\Longrightarrow$ (ii) Let $g = \sum_k g_k$ be the homogeneous expansion (9) of g. For $\epsilon \in \mathbb{T}$, consider the rotation operator

$$U_\epsilon f(z) := f(\epsilon z), \qquad z \in \mathbb{M}.$$

Clearly U_ϵ is unitary on $A^2(\mathbb{M})$ as well as on $L^2(\mathbb{M})$, and

$$U_\epsilon H_{\overline{g}} U_\epsilon^* = H_{\overline{U_\epsilon g}}.$$

Thus also $H_{\overline{U_\epsilon g}} \in \mathcal{S}^p$ for all $\epsilon \in \mathbb{T}$. Furthermore, the action $\epsilon \mapsto U_\epsilon$ is continuous in the strong operator topology, i.e. $\epsilon \mapsto U_\epsilon f$ is norm continuous for each $f \in L^2(\mathbb{M})$. Now it was shown in Lemma on p. 997 in [2] that this implies that the map $\epsilon \mapsto H_{\overline{U_\epsilon g}}$ is even continuous from $\mathbb{T}$ into $\mathcal{S}^p$. Consequently, the Bochner integral

$$\int_0^{2\pi} e^{mi\theta} U_{e^{i\theta}} H_{\overline{g}} U_{e^{i\theta}}^* \frac{d\theta}{2\pi} = H_{\overline{g_m}}$$

also belongs to $\mathcal{S}^p$, for each m. As g is nonconstant, there exists $m \geq 1$ for which $g_m \neq 0$, and (ii) follows.

(ii) $\Longrightarrow$ (iii) Assume that $H_{\overline{g}} \in \mathcal{S}^p$ for some $0 \neq g \in \mathcal{P}_m$. For any transform $\kappa \in O(n+1, \mathbb{R})$, the corresponding composition

$$U_\kappa f(z) := f(\kappa z), \qquad z \in \mathbb{M},$$

again acts unitarily on $A^2(\mathbb{M})$ as well as on $L^2(\mathbb{M})$, and

$$U_\kappa H_{\overline{g}} U_\kappa^* = H_{\overline{U_\kappa g}},$$

so that also $H_{\overline{U_\kappa g}} \in \mathcal{S}^p$ for all $\kappa \in O(n+1,\mathbb{R})$. Likewise, the composition $V_\kappa : f \mapsto f \circ \kappa$ sends the space $\mathcal{H}^m$ of spherical harmonics into itself, and the isomorphism (6) satisfies $\widehat{V_\kappa f} = U_\kappa \hat{f}$. Now it is known [19] that the representation $\kappa \mapsto V_\kappa^{-1}$ of $O(n+1,\mathbb{R})$ on $\mathcal{H}^m$ is irreducible, that is, for any nonzero $f \in \mathcal{H}^m$ the span of the translates $V_\kappa f$, $\kappa \in O(n+1,\mathbb{R})$, is dense in $\mathcal{H}^m$. Consequently, for any nonzero $g \in \mathcal{P}_m$, the span of the translates $U_\kappa g$, $\kappa \in O(n+1,\mathbb{R})$, of g by κ is dense in $\mathcal{P}_m$. As $\mathcal{P}_m$ has finite dimension $N(m)$, this actually means that $\mathcal{P}_m$ consists just of all linear combinations of $U_{\kappa_j} g$ for some tuple $\kappa_1, \dots, \kappa_{N(m)}$ in $O(n+1,\mathbb{R})$. Since $H_{\overline{U_{\kappa_j} g}} \in \mathcal{S}^p$ for each j, it follows by linearity that $H_{\overline{h}} \in \mathcal{S}^p$ for all $h \in \mathcal{P}_m$, showing that (iii) holds (for the same m as in (ii)).

(iii) $\Longrightarrow$ (iv) By hypothesis, we have in particular $H_{\overline{z^\nu}} \in \mathcal{S}^p$ for all multiindices ν with $|\nu| = m$; thus the operator

$$H := \sum_{|\alpha|=m} \binom{m}{\alpha} H^*_{\overline{z^\alpha}} H_{\overline{z^\alpha}} \tag{22}$$

belongs to $\mathcal{S}^p \cdot \mathcal{S}^p = \mathcal{S}^{p/2}$.

Recall that the Toeplitz operator T_ϕ, $\phi \in L^\infty(\mathbb{M})$, is the operator on $A^2(\mathbb{M})$ defined by

$$T_\phi f = P(\phi f)$$

(P being, as before, the orthogonal projection in L^2 onto A^2). One thus has

$$\|H_\phi f\|^2 = \|\phi f\|^2 - \|T_\phi f\|^2, \tag{23}$$

and, by the reproducing property of the Bergman kernels,

$$T_\phi f(x) = \int_{\mathbb{M}} f(y)\phi(y)K(x,y)\,dy,$$

where we started to write for brevity (for the duration of this proof) just dy instead of $d(\rho \otimes \mu)(y)$. With the notation from Theorem 1, we thus have, for any $\xi \in \mathbb{M}$,

$$\begin{aligned}
T_{\overline{z^\alpha}}(z \bullet \xi)^l(x) &= \int_{\mathbb{M}} (y \bullet \xi)^l \overline{y}^\alpha \sum_k \frac{(x \bullet \overline{y})^k}{d_k}\,dy \\
&= \sum_k \int_{\mathbb{M}} (y \bullet \xi)^l \partial_x^\alpha \frac{k!}{(k+|\alpha|)!} \frac{(x \bullet \overline{y})^{k+|\alpha|}}{d_k}\,dy \\
&= \sum_k \frac{k!}{(k+|\alpha|)! d_k} \partial_x^\alpha \int_{\mathbb{M}} (y \bullet \xi)^l (x \bullet \overline{y})^{k+|\alpha|}\,dy \\
&= \sum_k \frac{k!}{(k+|\alpha|)! d_k} \partial_x^\alpha \int_0^1 t^{l+k+|\alpha|}\,d\rho(t) \int_{\partial\mathbb{M}} (y \bullet \xi)^l (x \bullet \overline{y})^{k+|\alpha|}\,d\mu(y) \\
&= \sum_k \frac{k!}{(k+|\alpha|)! d_k} \partial_x^\alpha \delta_{l,k+|\alpha|} d_l (x \bullet \xi)^l
\end{aligned}$$

by (7) and (1). This vanishes for $l < |\alpha|$, while for $l \geq |\alpha|$ it equals

$$\frac{(l-|\alpha|)!}{l!} \frac{d_l}{d_{l-|\alpha|}} \partial_x^\alpha (x \bullet \xi)^l = \frac{d_l}{d_{l-|\alpha|}} \xi^\alpha (x \bullet \xi)^{l-|\alpha|}.$$

Declaring d_l to be ∞ for negative l, we thus obtain for all l

$$T_{\overline{z^\alpha}}(z \bullet \xi)^l = \frac{d_l}{d_{l-|\alpha|}} \xi^\alpha (z \bullet \xi)^{l-|\alpha|}.$$

For any $\xi, \eta \in \mathbb{M}$, we then get by (23)

$$\begin{aligned}\langle H(\cdot \bullet \xi)^l, (\cdot \bullet \eta)^k\rangle &= \int_{\mathbb{M}} \sum_{|\alpha|=m} \binom{m}{\alpha} |z^\alpha|^2 (z \bullet \xi)^l (\overline{z} \bullet \overline{\eta})^k \, dz \\ &\qquad - \frac{d_l^2}{d_{l-m}^2} \int_{\mathbb{M}} \sum_{|\alpha|=m} \binom{m}{\alpha} \xi^\alpha \overline{\eta^\alpha} (z \bullet \xi)^{l-m} (\overline{z} \bullet \overline{\eta})^{k-m} \, dz \\ &= \int_{\mathbb{M}} |z|^{2m} (z \bullet \xi)^l (\overline{z} \bullet \overline{\eta})^k \, dz - \frac{d_l^2}{d_{l-m}^2} (\xi \bullet \overline{\eta})^m \int_{\mathbb{M}} (z \bullet \xi)^{l-m} (\overline{z} \bullet \overline{\eta})^{k-m} \, dz \\ &= \int_0^1 t^{2m+k+l} \, d\rho(t) \int_{\partial\mathbb{M}} (z \bullet \xi)^l (\overline{z} \bullet \overline{\eta})^k \, d\mu(z) \\ &\qquad - \frac{d_l^2}{d_{l-m}^2} (\xi \bullet \overline{\eta})^m \int_0^1 t^{l+k-2m} \, d\rho(t) \int_{\partial\mathbb{M}} (z \bullet \xi)^{l-m} (\overline{z} \bullet \overline{\eta})^{k-m} \, d\mu(z) \\ &= \delta_{kl} \frac{q_{2l+2m}}{N(l)} (\xi \bullet \overline{\eta})^l - \frac{d_l^2}{d_{l-m}^2} (\xi \bullet \overline{\eta})^m \delta_{kl} q_{2l-2m} \frac{(\xi \bullet \overline{\eta})^{l-m}}{N(l-m)}\end{aligned}$$

by (7) and (1) again. Comparing this with

$$\begin{aligned}\langle (\cdot \bullet \xi)^l, (\cdot \bullet \eta)^k\rangle &= \int_0^1 t^{l+k} \, d\rho(t) \int_{\partial\mathbb{M}} (z \bullet \xi)^l (\overline{z} \bullet \overline{\eta})^k \, d\mu(z) \\ &= \delta_{kl} \frac{q_{2l}}{N(l)} (\xi \bullet \overline{\eta})^l = \delta_{kl} d_l (\xi \bullet \overline{\eta})^l,\end{aligned}$$

we thus see that, for all l,

$$\langle H(\cdot \bullet \xi)^l, (\cdot \bullet \eta)^k\rangle = \Big(\frac{q_{2l+2m}}{q_{2l}} - \frac{d_l}{d_{l-m}}\Big) \langle (\cdot \bullet \xi)^l, (\cdot \bullet \eta)^k\rangle.$$

Since $(\cdot \bullet \xi)^l$, $\xi \in \mathbb{M}$, span $\mathcal{P}_l$, it follows that

$$\langle H f_l, g_k\rangle = \Big(\frac{q_{2l+2m}}{q_{2l}} - \frac{d_l}{d_{l-m}}\Big) \langle f_l, g_k\rangle$$

for any $f_l \in \mathcal{P}_l$ and $g_k \in \mathcal{P}_k$. In other words, the operator H is diagonalized by the orthogonal decomposition $A^2(\mathbb{M}) = \bigoplus_k \mathcal{P}_k$, and

$$H = \bigoplus_l \Big(\frac{q_{2l+2m}}{q_{2l}} - \frac{d_l}{d_{l-m}}\Big) \, I|_{\mathcal{P}_l}.$$

Recalling that $\dim \mathcal{P}_k = N(k)$, this means that

$$H \in \mathcal{S}^{p/2} \iff \sum_l \Big(\frac{q_{2l+2m}}{q_{2l}} - \frac{d_l}{d_{l-m}}\Big)^{p/2} N(l) < \infty. \tag{24}$$

Now for our measure $d\rho(t) = \chi_{[0,1]}(t) t^{2n-1} \, dt$, one has $q_{2k} = \frac{1}{2k+2n}$, so

$$\begin{aligned}d_k = \frac{q_{2k}}{N(k)} &= \frac{(n-1)!k!}{2(k+n)(2k+n-1)(k+n-2)!} \\ &= \frac{(n-1)!}{4k^n} \Big[1 - \frac{n + \frac{n-1}{2} + \frac{(n-1)(n-2)}{2}}{k} + O\Big(\frac{1}{k^2}\Big)\Big]\end{aligned}$$

and

$$\frac{d_k}{d_{k-m}} = 1 - \frac{mn}{k} + O\Big(\frac{1}{k^2}\Big)$$

as $k \to +\infty$, while

$$\frac{q_{2k+2m}}{q_{2k}} = \frac{k+n}{k+n+m} = 1 - \frac{m}{k} + O\Big(\frac{1}{k^2}\Big).$$

Thus

$$\Big(\frac{q_{2k+2m}}{q_{2k}} - \frac{d_k}{d_{k-m}}\Big) \sim \frac{(n-1)m}{k}$$

and the sum (24) is finite if and only if (recall that $n \geq 2$ and $m > 0$)

$$+\infty > \sum_k k^{-p/2} N(k) \sim \sum_k k^{n-1-p/2}$$

i.e. if and only if $p > 2n$. Thus we have proved (iv).

(iv) $\Longrightarrow$ (v) We have seen in the last computation that for $p > 2n$, the operator (22) belongs to $\mathcal{S}^{p/2}$ for any $m \geq 1$. Since all the summands $H^*_{\overline{z}^\alpha} H_{\overline{z}^\alpha}$ in (22) are nonnegative operators, it follows that even $H^*_{\overline{z}^\alpha} H_{\overline{z}^\alpha} \in \mathcal{S}^{p/2}$ for all α, i.e. $H_{\overline{z}^\alpha} \in \mathcal{S}^p$ for all α. By linearity, $H_{\overline{g}} \in \mathcal{S}^p$ for any polynomial g, proving (v).

The last implication (v) $\Longrightarrow$ (i) is trivial, and thus the proof of Theorem 3 is complete. □

Note that the only place where $d\rho$ entered was in the computation in the part (iii) $\Longrightarrow$ (iv). Thus Theorem 3 remains in force also for any measure $d\rho$ for which

$$q_{2k} = ak^r[1 + \tfrac{b}{k} + O(k^{-2})] \quad \text{as } k \to +\infty$$

for some $a \neq 0$ and $b, r \in \mathbb{R}$; because one then has $\frac{q_{2k+2m}}{q_{2k}} = 1 + \frac{mr}{k} + O(k^{-2})$, $\frac{d_k}{d_{k-m}} = 1 + \frac{(r-n+1)m}{k} + O(k^{-2})$ and again $(\frac{q_{2k+2m}}{q_{2k}} - \frac{d_k}{d_{k-m}}) \sim \frac{(n-1)m}{k}$. In particular, Theorem 3 thus also holds for the "standard" weighted Bergman spaces on $\mathbb{M}$ corresponding to $d\rho(t) = \chi_{[0,1]}(t)(1-t^2)^\alpha t^{2n-1}\, dt$, $\alpha > -1$.

5. Balanced metrics

Returning to the formalism reviewed in connection with the TYZ expansions, recall that a Kähler form $\omega = \frac{i}{2}\partial\overline{\partial}\log\frac{1}{h}$ on a domain in $\mathbb{C}^n$ (or the Hermitian metric associated to ω) is called *balanced* if the corresponding weighted Bergman kernel satisfies

$$h(z) K_{h\det[\partial\overline{\partial}\log\frac{1}{h}]}(z,z) \equiv \text{const.} \quad (\neq 0); \tag{25}$$

that is, if and only if the corresponding Kempf distortion function ϵ_1 is a nonzero constant. More generally, for $\alpha > n$ one calls ω *α-balanced* if it is balanced and h is commensurable to $\operatorname{dist}(\cdot, \partial\Omega)^\alpha$ at the boundary. (For $\alpha \leq n$, the corresponding Bergman space degenerates just to the zero function, thus $K_{h\det[\partial\overline{\partial}\log\frac{1}{h}]} \equiv 0$ and the left-hand side in (25) is constant zero.) This definition turns out to indeed depend only on $\omega = \frac{i}{2}\partial\overline{\partial}\log\frac{1}{h}$ and not on the particular choice of h for a given ω, and also can be extended from domains to manifolds with line bundles as before; see Donaldson [8], Arezzo and Loi [3] and [10] for further details.

The simplest example of α-balanced metric is $h(z) = (1-|z|^2)^\alpha$, $\alpha > 1$, on the unit disc $\mathbb{D}$ in $\mathbb{C}$; one then gets $\det[\partial\overline{\partial}\log\frac{1}{h}] = \frac{\alpha}{(1-|z|^2)^2}$ and $K_{h\det[\partial\overline{\partial}\log\frac{1}{h}]}(z,z) = \frac{\alpha-1}{\pi\alpha}(1-|z|^2)^{-\alpha}$, so that (25) holds with the constant $\frac{\alpha-1}{\pi\alpha}$. Similarly for $h(z) = (1-|z|^2)^\alpha$, $\alpha > n$, on the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$, where the constant turns out to be $\frac{\Gamma(\alpha)}{\Gamma(\alpha-n)\alpha^n\pi^n}$. The only known examples of bounded domains with balanced metrics are invariant metrics on bounded homogeneous domains (in particular, on bounded

symmetric domains). In the unbounded setting, every dilation of the Euclidean metric on $\mathbb{C}^n$ is balanced (with $h(z) = e^{-\alpha|z|^2}$, $\alpha > 0$, so that $A^2(\mathbb{C}^n, h\det[\partial\overline{\partial}\log\frac{1}{h}])$ is just the familiar Fock space). Balanced metrics are known to exist in abundance on compact manifolds [8]; the existence of balanced metrics e.g. on bounded strictly pseudoconvex domains with smooth boundary is still an open problem, and their uniqueness for a given α is an open problem even on the unit disc.

We conclude this paper by the following simple observation concerning balanced metrics on our Kepler manifold $\mathbb{H}$. Note that for $n \geq 3$, $\mathbb{H}$ is known to be simply connected.

Theorem 10. *Let $n = \dim_{\mathbb{C}} \mathbb{H} \geq 3$ and $\alpha > n$. Then either there does not exist any α-balanced metric on $\mathbb{H}$, or it is not unique.*

Proof. Suppose there exists a unique α-balanced Kähler form $\omega = \frac{i}{2}\partial\overline{\partial}\log\frac{1}{h}$ on $\mathbb{H}$. It was shown in [10] that the image of an α-balanced metric on a simply connected domain under a biholomorphic map is again α-balanced (with the same α). (Simple connectivity is needed for some powers of Jacobians to be single-valued when $\frac{\alpha}{n+1}$ is not an integer.) Consequently, the Kähler form ω, or, equivalently, the associated Hermitian metric $g_{j\overline{k}}$, at the point $e = (1, i, 0, \ldots, 0) \in \mathbb{H}$ has to be invariant under any biholomorphic self-map of $\mathbb{H}$ fixing e, in particular, under all elements of the isotropy subgroup $L := \{\kappa \in O(n+1, \mathbb{C}) : \kappa e = e\}$ of e in $O(n+1, \mathbb{C})$. The unit sphere with respect to $g_{j\overline{k}}$ in the tangent space $T_e\mathbb{H}$ of $\mathbb{H}$ at e thus has to be invariant under L. Now $g_{j\overline{k}}$ being a Riemannian metric, the unit sphere is diffeomorphic to $\partial\mathbb{B}^n \cong \mathbb{S}^{2n-1}$, in particular, it is a compact manifold. However, L is easily seen to have orbits that extend to infinity. For instance, the matrix

$$A_z := \left(\begin{array}{ccc|c} 1+\frac{z^2}{2} & \frac{iz^2}{2} & iz & \\ \frac{iz^2}{2} & 1-\frac{z^2}{2} & -z & 0 \\ -iz & z & 1 & \\ \hline & 0 & & I \end{array}\right)$$

is easily checked to belong to L for any $z \in \mathbb{C}$, while the orbit of $\overline{e} = (1, -i, 0, \ldots, 0) \in T_e\mathbb{H}$ under A_z,

$$A_z\overline{e} = (z^2 + 1, i(z^2 - 1), -2iz)$$

evidently does not stay in any compact subset as $z \to \infty$. We have reached a contradiction. □

The same argument applies, in principle, to any manifold whose isotropy subgroup L of biholomorphic self-maps that stabilize some basepoint e contains a complex one-parameter subgroup (such as the A_z, $z \in \mathbb{C}$, above): by Liouville's theorem, the orbit of any tangent vector under L cannot stay bounded without being constant, and it follows that each element of L acts trivially (i.e. reduces to the identity) on the tangent space at e. Consequently, if there exists $\kappa \in L$ with $d\kappa_e \neq I$, then for each α, the manifold either does not admit any α-balanced metric, or such metric is not unique.

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