



HAL
open science

On hyperbolic knots in S^3 with exceptional surgeries at maximal distance

Benjamin Audoux, Ana G Lecuona, Fionntan Roukema

► **To cite this version:**

Benjamin Audoux, Ana G Lecuona, Fionntan Roukema. On hyperbolic knots in S^3 with exceptional surgeries at maximal distance. Algebraic and Geometric Topology, 2018, 18 (4), pp.2371-2417. 10.2140/agt.2018.18.2371 . hal-01485358

HAL Id: hal-01485358

<https://hal.science/hal-01485358>

Submitted on 8 Mar 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON HYPERBOLIC KNOTS IN S^3 WITH EXCEPTIONAL SURGERIES AT MAXIMAL DISTANCE

BENJAMIN AUDOUX, ANA G. LECUONA, AND FIONNTAN ROUKEMA

ABSTRACT. Baker showed that 10 of the 12 classes of Berge knots are obtained by surgery on the minimally twisted 5-chain link. In this article we enumerate all hyperbolic knots in S^3 obtained by surgery on the minimally twisted 5-chain link that realise the maximal known distances between slopes corresponding to exceptional (lens, lens), (lens, toroidal), (lens, Seifert fibred spaces) pairs. In light of Baker's work, the classification in this paper conjecturally accounts for "most" hyperbolic knots in S^3 realising the maximal distance between these exceptional pairs. All examples obtained in our classification are realised by filling the magic manifold. The classification highlights additional examples not mentioned in Martelli and Petronio's survey of the exceptional fillings on the magic manifold. Of particular interest, is an example of a knot with two lens space surgeries that is not obtained by filling the Berge manifold.

1. INTRODUCTION

Let K be a knot in S^3 and consider its exterior $S^3 \setminus \nu(K)$ where $\nu(K)$ is a small open neighborhood of the knot. For a slope α (the isotopy class of an essential simple closed curve) on the boundary of the exterior of K , the closed manifold obtained from α -surgery (gluing a solid torus to the exterior of K by identifying the meridian to α) is denoted by $K(\alpha)$.

Suppose that K is hyperbolic, that is, its complement admits a Riemannian metric of constant sectional curvature -1 which is complete and of finite volume. Then Thurston's hyperbolic Dehn surgery theorem implies that all but finitely many slopes produce hyperbolic manifolds via surgery [Th]. The exceptional cases are called exceptional slopes and exceptional surgeries.

It is a consequence of the geometrization theorem that every exceptional surgery on a hyperbolic *link* is S^3 , a lens space, has an essential surface of non-negative Euler characteristic, or fibres over the sphere with three exceptional fibres. We now assign the following standard names to these classes of non-hyperbolic 3-manifolds following [G1]. We say that a manifold is of type D , A , S or T if it contains, respectively, an essential disc, annulus, sphere or torus, and of type S^H or T^H if it contains a Heegaard sphere or torus. Finally we denote by Z the type of small closed Seifert manifolds. Notice that $S^H = \{S^3\}$ and T^H is the set of lens spaces (including $S^1 \times S^2$).

The distance (minimal geometric intersection) between two slopes α, β on a torus is denoted $\Delta(\alpha, \beta)$. The *maximal distance between types of exceptional manifolds* \mathcal{C} and \mathcal{D} is defined as $\max\{\Delta(\alpha, \beta) \mid (X, \alpha, \beta) \in (\mathcal{C}, \mathcal{D})\}$ and denoted $\Delta(\mathcal{C}, \mathcal{D})$.

Quite some energy has been devoted in the literature to the understanding of exceptional slopes on hyperbolic manifolds. In the case of hyperbolic knot exteriors there are strong restrictions on their exceptional surgeries or fillings. The S^3 filling is unique [GL1] and (obviously) no knot exterior has a filling with an essential annulus or disc. Conjecturally no hyperbolic knot exterior has a reducible surgery [GAS]. So, there are nine possible exceptional pairs obtained by surgery on a hyperbolic knot in S^3 . Namely, the (S^3, T^H) , (S^3, T) , (S^3, Z) , (T^H, T^H) , (T^H, T) , (T^H, Z) , (T, T) , (T, Z) , (Z, Z) exceptional pairs.

The (S^3, T) pairs have been completely enumerated [GL2]. Examples of (S^3, Z) pairs have been constructed, see for example [Eud] and [Rou2]. The exceptional surgeries on the figure eight knot tell us that $\Delta(T^H, Z), \Delta(T, Z), \Delta(Z, Z) > 5$, and from [Ago] we know that there are only a finite number of examples realising these distances. The (S^3, T^H) pairs are conjecturally a subset of the Berge knots classified in [Be1]. It follows that, since the remaining three cases all involve a T^H surgery, an enumeration of the remaining three exceptional pairs is conjecturally an enumeration of a subset of Berge knots. Baker showed [Bak] that 10 of the 12 classes of Berge knots are obtained by surgery on the minimally twisted 5-chain link (5CL, see Figure 3). So, conjecturally, most of the hyperbolic knots realising (T^H, T^H) , (T^H, T) and (T^H, Z) exceptional pairs of slopes are obtained by surgery on 5CL.

In this article we enumerate all hyperbolic knots obtained from surgery on the 5CL that realise the maximum known distance between the exceptional filling types. We completely classify the knots arising in this manner and having two different lens space surgeries; having a lens space surgery and a toroidal surgery at distance 3; and having a lens space surgery and a small Seifert surgery at distance 2. In light of Baker's work, the classification in this article conjecturally accounts for most examples of hyperbolic knots with an exceptional pair of slopes at maximal distance. Our main result is the following:

Theorem 1.1. *Let K be a hyperbolic knot in S^3 obtained by surgery on the minimally twisted 5-chain link with two exceptional slopes α and β , and let $K(\alpha)$ be a lens space. The following statements are true:*

- *If $K(\beta)$ is a lens space then K is in found in Figure 1.*
- *If $K(\beta)$ is toroidal then the distance between α and β is at most three and if the distance equals three then K is found in Figure 2.*
- *If $K(\beta)$ fibres over the sphere with three exceptional fibres then the distance between α and β is at most two and if the distance equals two then K is found in Figure 2.*

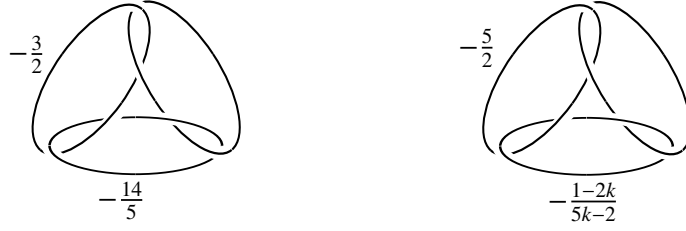


Figure 1: All hyperbolic knots with two lens space fillings obtained by surgery on the minimally twisted 5-chain link.

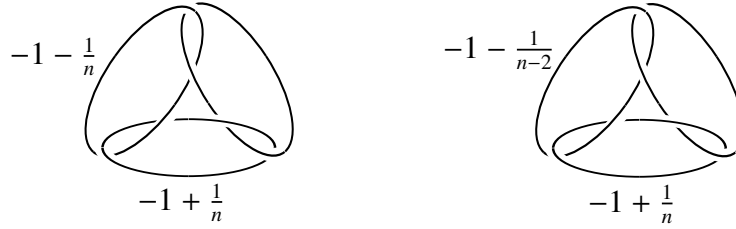


Figure 2: All hyperbolic knots with a lens space and a toroidal filling at distance 3, or a lens space and a Seifert filling at distance 2, obtained by surgery on 5CL.

Given M , an orientable cusped hyperbolic 3-manifold and τ a fixed topic component of the boundary of its compactification, it is a consequence of [LM] that 8 is a universal upper bound for $\Delta(\alpha_1, \alpha_2)$ for each exceptional pair $(M, \tau; \alpha_1, \alpha_2)$ (where with this notation we indicate that α_1 and α_2 are slopes on τ). The celebrated Gordon-Luecke theorem [GL1] can be formulated by saying that $\Delta(S^H, S^H) = 0$, the Cabling conjecture by saying that $\Delta(S, S^H) = -\infty$ [GAS], the Berge conjecture by saying that the Berge knots in [Be1] contain all exceptional pairs of type (S^H, T^H) , and the theorem of [GL2] by saying that the knots realizing $\Delta(\alpha_1, \alpha_2) = \Delta(S^H, T)$ are precisely the Eudave-Muñoz knots.

It is natural to generalise these types of questions by asking whether we can find $\Delta(C_1, C_2)$ for each pair of classes $C_1, C_2 \in \{S^H, S, T^H, T, D, A, Z\}$, and whether we can enumerate all $(M, \tau; \alpha_1, \alpha_2)$ of type (C_1, C_2) with $\Delta(\alpha_1, \alpha_2) = \Delta(C_1, C_2)$. A great deal is known, see [GL3] or [G2] for an overview.

If a knot in S^3 is not a torus knot or a satellite knot then its exterior is a hyperbolic 3-manifold. We can consider all $(M_K, \tau; \alpha_1, \alpha_2)$ when M_K is the exterior of a knot K in S^3 and ask what is $\Delta(C_1, C_2)$ and which $(M_K, \tau; \alpha_1, \alpha_2)$ of type (C_1, C_2) have $\Delta(\alpha_1, \alpha_2) = \Delta(C_1, C_2)$ for this subclass of hyperbolic manifolds. Of course, this is the same as asking what is the greatest value of $\Delta(\alpha_2, \alpha_3)$ among exceptional triples $(M, \tau; \alpha_1, \alpha_2, \alpha_3)$ of type (S^H, C, D) and which

$(M, \tau; \alpha_1, \alpha_2, \alpha_3)$ realise the maximum $\Delta(\alpha_2, \alpha_3)$. From this perspective, in this article we enumerate such (S^H, C, \mathcal{D}) triples obtained from the minimally twisted 5-chain link.

In order to state some of the noteworthy remarks coming from the analysis done to establish Theorem 1.1 we need to introduce some more notation. The chain links that are ubiquitous throughout this paper are depicted in Figure 3. We keep the notation conventions of [MPR] and denote the minimally twisted 5-chain link by 5CL and its exterior by M_5 ; the 4-chain link is denoted 4CL and its exterior is denoted M_4 . A -1 surgery on any component of 4CL gives a 3-chain link 3CL, whose exterior is denoted by M_3 . We closely reference the tables from [MP] which give a classification of the exceptional surgeries on the mirror $3CL^*$ shown in Figure 3. The exterior of this link is the “magic manifold” [GW] which we will denote by N . The manifold F , the exterior of the minimally twisted 4-chain link, will also appear extensively in the text.

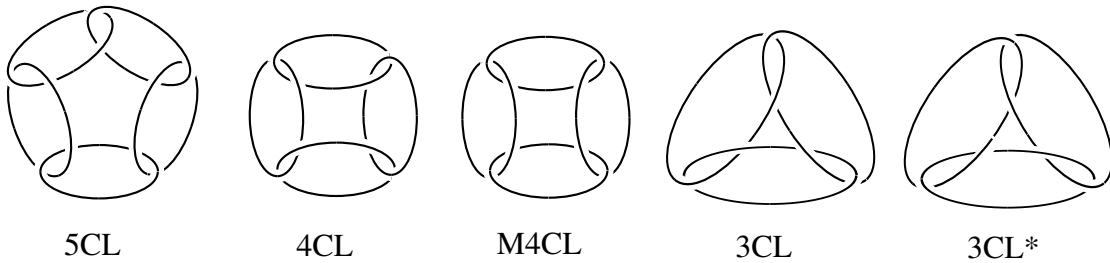


Figure 3: The minimally twisted 5-chain link 5CL, the 4-chain link 4CL, the minimally twisted 4-chain link M4CL and the 3-chain links 3CL and $3CL^*$. The exteriors of these links are respectively called M_5, M_4, F, M_3 and N .

The knots in Figures 1 and 2 are described by giving a filling instruction on two of the 3 boundary components of the magic manifold. The exceptional slopes on the knots $N(-\frac{3}{2}, -\frac{14}{5})$ and $N(-\frac{5}{2}, \frac{1-2k}{5k-2})$ from Figure 1 and corresponding fillings are found in Theorem 3.1. On the other hand, the exceptional slopes on $N(-1 + \frac{1}{n}, -1 - \frac{1}{n})$ and $N(-1 + \frac{1}{n}, -1 - \frac{1}{n-2})$ from Figure 2 and the corresponding fillings are found in Theorem 4.1. Furthermore, Theorems 3.1 and 4.1 go further and show that the three families of knots and the isolated example shown in Figures 1 and 2 are all distinct knots.

There is a unique hyperbolic knot in a torus with two non-trivial surgeries [Be2]; the exterior of this knot is called the Berge manifold, which will appear frequently in the text. It can be obtained by filling one of the 3 boundary components of the magic manifold N . Indeed, the Berge manifold is $N(-\frac{5}{2})$. Cutting, twisting and filling the boundary of the torus yields an infinite family of inequivalent knots in S^3 with two lens space fillings. This family is precisely the set of $N(-\frac{5}{2}, \frac{1-2k}{5k-2})$ from Theorem 1.1. We find particularly interesting the following fact.

Fact: the example $N(-\frac{3}{2}, -\frac{14}{5})$ is not obtained by surgery on the Berge manifold (Theorem 3.1).

The article [BDH] contains a complete description of all surgeries on the 5CL with three cyclic fillings. It is a more general question than our quest to find *knot exteriors* on the 5CL with two cyclic fillings and the techniques used in [BDH] are different from ours. A translation of our results into the language of [BDH] follows. The family $\{N(-\frac{5}{2}, \frac{1-2k}{5k-2})\}$ is the family $\{B_{(2k-1)/(5k-2)}\} \subset \{B_{p/q}\}$ from [BDH], and the isolated example $N(-\frac{3}{2}, -\frac{14}{5})$ is $A_{2,3}$ from [BDH].

A final remark, the family $N(-1 + \frac{1}{n}, -1 - \frac{1}{n})$ and its exceptional slopes and filling are highlighted in [MP, Table 17], but the distinct family $N(-1 + \frac{1}{n}, -1 - \frac{1}{n-2})$ is not highlighted in [MP].

1.1. Article structure. The results in this article are obtained by a careful analysis of the classifications of exceptional sets of slopes on surgeries of the minimally twisted 5-chain link given in [MP] and [Rou2]. The work done there, translates the enumeration of exceptional pairs realising maximal distances into finding the solutions to a (long) list of elementary diophantine equations. The translation necessitates a table by table analysis of the work given in [MP] and [Rou2]. A collection of easy (but technical) lemmas in the Appendix facilitates the translation and reduces the amount of work needed. The proofs of the main results are littered with references to results in the Appendix, [MP], and [Rou2]. Therefore, this article is best read with both articles and the Appendix in-hand.

Section 2 sets out the notation and conventions used throughout this article. Section 3 gives an enumeration of all exceptional (S^H, T^H, T^H) triples obtained by surgery on 5CL. Section 4 gives an enumeration of all exceptional (S^H, T^H, T) triples obtained by surgery on 5CL. Section 5 gives an enumeration of all exceptional (S^H, T^H, Z) triples obtained by surgery on 5CL. Sections 3-5 all proceed in the same way. The sections start with a precise statement about the enumeration of the exceptional triples. The results are established by first showing that all examples are obtained by surgery on 4CL, and then showing that all examples are obtained by surgery on 3CL. The final sections then enumerate all examples of exceptional triples obtained by surgery on 3CL.

1.2. Acknowledgements and remarks. We are thankful to Daniel Matignon and Luisa Paoluzzi for fruitful conversations. We also want to thank Ken Baker for interesting discussions. The second author is partially supported by the Spanish GEOR-MTM2014-55565. The third author was supported as a member of the Italian FIRB project ‘Geometry and topology of low-dimensional manifolds’ (RBFR10GHHH), internal funding from the University of Sheffield and the French ANR research project ‘‘VasKho’’ ANR-11-JS01-002-01.

2. NOTATION AND CONVENTIONS

In this section we set out notation and conventions used throughout the article. We will use the conventions on surgery instructions set out in [Rou2] which we briefly outline. For more

detailed descriptions, please refer to [Rou2]. Given an orientable compact 3-manifold X such that ∂X is a collection of tori, we use the term *slope* to indicate the isotopy class of a non-trivial unoriented loop on a component of ∂X . After fixing a choice of meridian and longitude on a boundary torus, a slope is naturally identified with an element in $\mathbb{Q} \cup \{\infty\}$. A *filling instruction* α for X is a set consisting of either a slope or the empty set for each component of ∂X . Throughout this article the components of the chain links are ordered cyclically, and the filling instructions are thus identified with tuples of elements in $\mathbb{Q} \cup \{\infty\}$. The *filling* $X(\alpha)$ is the manifold obtained by glueing one solid torus to ∂X for each non-empty slope in α . The meridian of the solid torus is glued to the slope.

A very related concept to that of a filling is a *surgery* on a link $L \subset S^3$. By definition, a surgery on L is a filling of the exterior of L , $S^3 \setminus \nu(L)$, where $\nu(L)$ is an open regular neighborhood of L . By a *surgery instruction* for L we mean a filling instruction on the exterior of L .

In the present article we will be concerned with *exceptional fillings*. If the interior of X is hyperbolic but the interior of $X(\alpha)$ is not, we say that α is an *exceptional filling instruction* for X and $X(\alpha)$ is an exceptional filling. We follow the notation used to describe the sets of exceptional slopes set out in [G2]. The set of exceptional slopes on a fixed toroidal boundary component τ of a hyperbolic 3-manifold X is denoted by $E_\tau(X)$, and the cardinality of $E_\tau(X)$ by $e_\tau(X)$. In our case τ will refer to the n^{th} component of the chain link with n components and is dropped throughout the article. A word of caution: when \mathcal{F} is a filling instruction on M_5 , we write the elements of $E(M_5(\mathcal{F}))$ with respect to the choice of bases on M_5 (and not $M_5(\mathcal{F})$!).

Denoting by C_i a type of manifolds (for example lens spaces, or reducible manifolds) one of the most recurrent notions in this article is that of an *exceptional* (C_1, \dots, C_n) n -tuple. By this we mean the following: if X is a hyperbolic 3-manifold and $\alpha_1, \dots, \alpha_n$ are exceptional slopes on a fixed toroidal boundary component of X , with $X(\alpha_i)$ a manifold of type C_i , then we say that $(X, \alpha_1, \dots, \alpha_n)$ is an exceptional (C_1, \dots, C_n) n -tuple and write $(X, \alpha_1, \dots, \alpha_n) \in (C_1, \dots, C_n)$. There is a notion of equivalence among exceptional tuples. We will say that two exceptional n -tuples $(X_1, \alpha_1, \dots, \alpha_n)$ and $(X_2, \beta_1, \dots, \beta_n)$ are *equivalent* if there exists a homeomorphism $h : X_1 \rightarrow X_2$ with $X_2(h(\alpha_i)) = X_2(\beta_i)$. When two n -tuples $(X_1, \alpha_1, \dots, \alpha_n)$, $(X_2, \beta_1, \dots, \beta_n)$ are equivalent we write $(X_1, \alpha_1, \dots, \alpha_n) \cong (X_2, \beta_1, \dots, \beta_n)$.

We now recall the following important notion introduced in [Rou2]: given α , a filling instruction on a manifold X , we say that α *factors through a manifold* Y if there exists some filling instruction $\alpha' \subset \alpha$ such that $Y = X(\alpha')$.

To describe the exceptional fillings on the minimally twisted 5-chain link, we follow the standard choice of notation used to describe graph manifolds set out in [Rou2]. Very briefly, if G is an orientable surface with k boundary components and Σ is G minus n discs, we can construct homology bases $\{(\mu_i, \lambda_i)\}$ on $\partial(\Sigma \times S^1)$. For coprime pairs $\{(p_i, q_i)\}_{i=1}^n$ with $|p_i| \geq 2$ we get a

Seifert manifold $(G, (p_1, q_1), \dots, (p_n, q_n))$ with fixed homology bases on its k boundary components. Given Seifert manifolds X and Y with boundary and orientable base surfaces as above and an element $B \in \text{GL}_2(\mathbb{Z})$, we define $X \cup_B Y$ unambiguously to be the quotient manifold $X \cup_f Y$, with $f : T \rightarrow T'$ where T and T' are arbitrary boundary tori of X and Y , and f acting on homology by B with respect to the fixed bases. Similarly one can define X/B when X has at least two boundary components.

As it happens in other papers on the subject, we employ a somehow more flexible notation for lens spaces than the usual one. We will write $L(2, q)$ for the real projective space, $L(1, q)$ for the 3-sphere, $L(0, q)$ for $S^2 \times S^1$ and $L(p, q)$ for $L(|p|, q')$ with $q \equiv q'$ modulo p and $0 < q' < |p|$, for any coprime p, q . Later in the paper, we will often be interested in understanding when $L(x, y) = S^3$ when x and y have some complicated expression. As $L(x, y) = S^3$ if and only if $x = 1$, to simplify matters, we will often replace “ y ” with “ \star ”.

Finally, throughout the text the symbols $\varepsilon, \varepsilon'$ etc. will all denote ± 1 .

3. (S^H, T^H, T^H) TRIPLES FROM 5CL

In this section we completely enumerate all exceptional (S^H, T^H, T^H) triples obtained by surgery on the 5CL. Each one of these triples can be thought of as a knot in S^3 with two different lens space surgeries. To state the main result of this section we need to first define the following family of 3-manifolds, where the parameter k is an integer:

$$A_k := \left(N\left(-\frac{5}{2}, \frac{1-2k}{5k-2}\right), \infty, -2, -1, \right).$$

Theorem 3.1. *The following statements are true:*

- If $(M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}), \alpha, \beta, \gamma) \in (S^H, T^H, T^H)$ then $(M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}), \alpha, \beta, \gamma)$ is equivalent to A_k for some k or to $(N(-\frac{3}{2}, -\frac{14}{5}), -2, -1, \infty)$.
- The sets of exceptional slopes and corresponding fillings of $N(-\frac{5}{2}, \frac{1-2k}{5k-2})$ for $k \neq 0$ and of $N(-\frac{3}{2}, -\frac{14}{5})$ are shown in Table 1.
- $(N(-\frac{3}{2}, -\frac{14}{5}), -2, -1, \infty) \not\cong A_k$ for all k , and all $N(-\frac{5}{2}, \frac{1-2k}{5k-2})$ are obtained by filling the Berge manifold.

Remark If $k = 0$ then $N(-\frac{5}{2}, \frac{1-2k}{5k-2})$ is the exterior of the $(-2, 3, 7)$ pretzel knot which has 7 exceptional slopes, see [MP, Table A.4] for details.

We prove Theorem 3.1 by first considering, in Section 3.1, all $(M_5(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, T^H)$ with \mathcal{F} not factoring through M_4 . This set will turn out to be empty and we proceed in Section 3.2 to investigate the $(M_4(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, T^H)$ with \mathcal{F} not factoring through M_3 . Again, there will be no such examples and we will finally consider in Section 3.3 the case $(M_3(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, T^H)$. We will produce a complete list of examples, the family A_k and the isolated

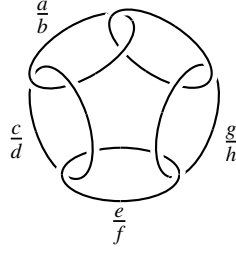
$k \in \mathbb{Z} \setminus \{0\},$	$E(N(-\frac{5}{2}, \frac{1-2k}{5k-2})) = \{-3, -2, -\frac{3}{2}, -1, 0, \infty\}$
$\beta \in E(N(-\frac{5}{2}, \frac{1-2k}{5k-2}))$	$N(-\frac{5}{2}, \frac{1-2k}{5k-2})(\beta)$
$\beta = \infty$	S^3
$\beta = -3$	$(D, (2, 1), (3, -2)) \cup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (2, 1), (3k-1, 5k-2))$
$\beta = -2$	$L(18-49k, 7-19k)$
$\beta = -\frac{3}{2}$	$(D, (2, 1), (3, 1)) \cup_{\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}} (D, (2, 1), (8k-3, 5k-2))$
$\beta = -1$	$L(49k-19, 31k-12)$
$\beta = 0$	$(D, (2, -1), (5k-2, 8k-3)) \cup_{\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}} (D, (2, 1), (3, 1))$
$E(N(-\frac{3}{2}, -\frac{14}{5})) = \{-3, -\frac{5}{2}, -2, -1, 0, \infty\}$	
$\beta \in E(N(-\frac{3}{2}, -\frac{14}{5}))$	$N(-\frac{3}{2}, -\frac{14}{5})(\beta)$
$\beta = \infty$	$L(32, -9)$
$\beta = -3$	$(S^2, (2, 1), (3, 2), (9, -5))$
$\beta = -\frac{5}{2}$	$(D, (2, 1), (3, 1)) \cup_{\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}} (D, (2, 1), (4, -5))$
$\beta = -2$	S^3
$\beta = -1$	$L(31, 17)$
$\beta = 0$	$(D, (2, 1), (5, -4)) \cup_{\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}} (D, (2, 1), (3, 1))$

Table 1: The exceptional slopes and corresponding fillings of hyperbolic knot exteriors in S^3 with two lens space fillings obtained by surgery on 5CL.

example in the statement of Theorem 3.1. The fact that the examples we find are all different is an easy consequence of the results in [MP] and is shown at the end of Section 3.3. Throughout the argument, easy (but technical) lemmas from the Appendix are referenced.

3.1. Hyperbolic knots with two lens surgeries arising from the 5-chain link. In this section we prove that if $M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})$ (below) is a hyperbolic knot exterior admitting two different lens space fillings then the instruction $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})$ factors through M_4 .

If $(M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}), \alpha, \beta, \gamma) \in (S^H, T^H, T^H)$ and $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})$ does not factor through M_4 then [Rou2, Theorem 4] tells us that there are two different scenarios to consider: there are 3 exceptional slopes, $E(M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})) = \{0, 1, \infty\}$; or there are 4 or 5 with $\{0, 1, \infty\} \subsetneq E(M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}))$,



and the manifold $M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})$ is equivalent to some $M_5(\mathcal{F})$ listed in [Rou2, Tables 6–11]. We study separately these two cases, starting with the latter one.

3.1.1. *The case $e(M_5(\mathcal{F})) > 3$.* A careful inspection of [Rou2, Tables 14–20] shows that if $e(M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})) > 3$, and a slope β yields a lens space or S^3 then it appears in Tables 17 or 18. Namely, $\mathcal{F} = (-2, \frac{p}{q}, 3, \frac{u}{v})$ [Rou2, Tables 17] or $\mathcal{F} = (-2, \frac{p}{q}, \frac{r}{s}, -2)$ [Rou2, Tables 18].

Moreover, [Rou2, Tables 17-18] tells us that if we want one of the exceptional slopes to not be in $\{0, 1, \infty\}$ (the case when $\{0, 1, \infty\}$ all correspond to cyclic fillings is considered in the next section) then we must assume that -1 is a T^H slope. If -1 corresponds to a T^H filling on $M_5(\mathcal{F})$ and $(M_5(\mathcal{F}), \alpha, -1, \gamma) \in (S^H, T^H, T^H)$, then we must necessarily assume that $\alpha, \gamma \neq 1$ because $\Delta(S^H, T^H) = \Delta(T^H, T^H) = 1$ [GL1]. Each of $-1, 0, \infty$ corresponding to a cyclic filling imposes conditions on the slopes of \mathcal{F} . We will now show that these conditions can not all be satisfied.

If $\mathcal{F} = (-2, \frac{p}{q}, \frac{r}{s}, -2)$ from [Rou2, Tables 18] and ∞ is a cyclic filling then $|s| = |q| = 1$, and so $\mathcal{F} = (-2, n, k, -2)$. From [Rou2, Tables 18] if $M_5(\mathcal{F})(0) \in S^H \cup T^H$ then $\frac{p}{q} = n = 1 + \frac{1}{m}$ or $\frac{r}{s} = k = 1 + \frac{1}{m}$. This implies that one of n or k is equal to 0, which makes $M_5(\mathcal{F})$ non-hyperbolic, or 2 which makes \mathcal{F} factor through M_4 (see Lemma A.4). This means that if \mathcal{F} is in [Rou2, Tables 18] and has three cyclic fillings then the three exceptional slopes corresponding to $S^H \cup T^H$ fillings on $M_5(\mathcal{F})$ are $\{0, 1, \infty\}$.

If $\mathcal{F} = (-2, \frac{p}{q}, 3, \frac{u}{v})$ from [Rou2, Tables 17] and -1 is a cyclic filling then $|p| = |u + v| = 1$, and so $\mathcal{F} = (-2, \frac{1}{n}, 3, 1 + \frac{1}{k})$. From [Rou2, Tables 17] if $M_5(\mathcal{F})(0) \in S^H \cup T^H$ then $\frac{u}{v} = 1 + \frac{1}{k} = 3$ which has no integer solutions in k , or $\frac{p}{q} = \frac{1}{n} = 1 + \frac{1}{m}$ which implies that $\frac{p}{q} = \frac{1}{2}$, or $\frac{u}{v} = 1 + \frac{1}{k} = 3 + \frac{1}{m}$ which implies that $\frac{u}{v} = 2$. Both $\frac{p}{q} = \frac{1}{2}$ and $\frac{u}{v} = 2$ are excluded by Lemma A.4. This means that if \mathcal{F} is in [Rou2, Tables 17] and has three cyclic fillings then the three exceptional slopes corresponding to $S^H \cup T^H$ fillings on $M_5(\mathcal{F})$ are $\{0, 1, \infty\}$.

We conclude then that, if $(M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}), \alpha, \beta, \gamma) \in (S^H, T^H, T^H)$ and \mathcal{F} does not factor through M_4 then $\{\alpha, \beta, \gamma\} = \{0, 1, \infty\}$.

3.1.2. *The case $E(M_5(\mathcal{F})) = \{0, 1, \infty\}$.* We now consider $(M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}), \alpha, \beta, \gamma) \in (S^H, T^H, T^H)$ with $\{\alpha, \beta, \gamma\} = \{0, 1, \infty\}$. We have

$$(1) \quad M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})(\infty) \stackrel{(35)}{=} F(-\frac{a}{b}, \frac{f}{e}, \frac{d}{c}, -\frac{g}{h})$$

Recall that if one of $a, b, c, d, e, f, g, h = 0$ then $M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})$ is non-hyperbolic (Lemma A.4). The enumeration of closed fillings of F is found in [Rou2, Table 4]. We will use (1) to translate instructions on M_5 to instructions on F and carefully consider the entries from [Rou2, Table 4]. In the analysis, T4. n will denote the n^{th} line of the $\frac{x}{y}$ column.

Considering the $T^H \cup S^H$ fillings of F listed in [Rou2, Table 4] we learn that, up to a D_4 permutation of slopes, T4.2–T4.5 is a complete list of necessary and sufficient conditions for $M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})(\infty) \in S^H \cup T^H$. The lines T4.2 and T4.3, which correspond to $\frac{p}{q} = 0$, can be ignored since by (1) and Lemma A.4 they yield a non hyperbolic filling.

The entry T4.4 tells us that if $M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})(\infty) = F(-\frac{a}{b}, \frac{f}{e}, \frac{d}{c}, -\frac{g}{h}) \in S^H \cup T^H$, then, taking into consideration the action of D_4 on F , one of the following conditions necessarily holds:

$$(i) \frac{a}{b} = \frac{1}{n} \ \& \ \frac{e}{f} = k \quad (ii) \frac{c}{d} = n \ \& \ \frac{g}{h} = \frac{1}{k} \quad (iii) \frac{a}{b} = \frac{1}{n} \ \& \ \frac{g}{h} = \frac{1}{k} \quad (iv) \frac{c}{d} = n \ \& \ \frac{e}{f} = k.$$

These conditions can all be identified using Lemma A.3. In fact, to identify for example case (i) with case (iii) it suffices to remark that:

$$\left(M_5\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}\right), \infty\right) \underset{(16)^3 \circ (18)}{\cong} \left(M_5\left(\frac{a}{a-b}, \frac{d-c}{d}, \frac{h}{g}, \frac{f}{e}\right), \infty\right).$$

In a similar way, case (i) can be identified with case (ii) using (16) and (17), and case (i) can be identified with case (iv) using (25).

The entry T4.5 tells us that if $M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})(\infty) = F(-\frac{a}{b}, \frac{f}{e}, \frac{d}{c}, -\frac{g}{h}) \in S^H \cup T^H$, then one of the following conditions necessarily holds:

$$(i) \frac{a}{b} = \frac{1}{n} \ \& \ \frac{c}{d} = \frac{\varepsilon-nk}{k} \quad (ii) \frac{a}{b} = \frac{\varepsilon-nk}{k} \ \& \ \frac{c}{d} = n \quad (iii) \frac{g}{h} = \frac{1}{n} \ \& \ \frac{e}{f} = \frac{\varepsilon-nk}{k} \quad (iv) \frac{g}{h} = \frac{\varepsilon-nk}{k} \ \& \ \frac{e}{f} = n$$

where $\varepsilon = \pm 1$. Case (i) is identified with Case (iii), and Case (ii) is identified with Case (iv) using (17). Case (i) is identified with Case (ii) using a composition of (16), (17) and (25).

Therefore, any $M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})$ with $(M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}), \alpha, \beta, \gamma) \in (S^H, T^H, T^H)$ and $\{\alpha, \beta, \gamma\} = \{0, 1, \infty\}$ is equivalent to one of:

$$M_5\left(\frac{1}{n}, \frac{c}{d}, k, \frac{g}{h}\right) \quad (\text{Family 1}) \quad \text{or} \quad M_5\left(\frac{1}{n}, \frac{\varepsilon-nk}{k}, \frac{e}{f}, \frac{g}{h}\right) \quad (\text{Family 2}).$$

We first consider the examples from Family 1. We have assumed that both 0 and 1 correspond to S^H or T^H slopes. Examining the 1 slope we obtain

$$M_5\left(\frac{1}{n}, \frac{c}{d}, k, \frac{g}{h}\right)(1) \underset{(36)}{=} F\left(\frac{1-n}{n}, \frac{c}{d}, k, \frac{g-h}{h}\right) \underset{\text{Lemma A.11}}{=} (D, (1-n, n), (k, 1)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (c, d), (g-h, h))$$

which has an essential torus unless $0, \pm 1 \in \{1-n, k, c, g-h\}$ (Lemma A.2). Since we are interested in hyperbolic manifolds and instructions not factoring through M_4 , we can use Lemma A.4 to rule out the possibilities $1-n, k \in \{0, \pm 1\}$ and $c, g-h = 0$. Without loss of generality (we are only interested in the quotients $\frac{c}{d}$ and $\frac{g}{h}$) we are then left with the cases $c = 1$ or $g = h + 1$.

Case $c = 1$: Turning now our attention to the slope 0 and writing $\frac{c}{d} = \frac{1}{m}$, it holds

$$M_5\left(\frac{1}{n}, \frac{1}{m}, k, \frac{g}{h}\right)(0) \stackrel{(37)}{=} F\left(\frac{n}{n-1}, 1-m, -\frac{h}{g}, k-1\right),$$

which, by Lemma A.2 and Lemma A.11, has an essential torus unless we are in the case $0, \pm 1 \in \{n, 1-m, -h, k-1\}$. This time Lemma A.4 leaves us with the necessary condition $h = \pm 1$, which translates to $\frac{g}{h} \in \mathbb{Z}$. Combining the two necessary conditions and writing $\frac{g}{h} = l$, we learn that

$$\begin{aligned} M_5\left(\frac{1}{n}, \frac{1}{m}, k, l\right)(1) &\stackrel{(36)}{=} F\left(\frac{1-n}{n}, \frac{1}{m}, k, l-1\right) \stackrel{(34)}{=} (D, (1-n, n), (k, 1)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (1, m), (l-1, 1)) \\ &\stackrel{(13)}{=} (S^2, (1+ml-m, 1-l), (1-n, n), (k, 1)) \in T^H \end{aligned}$$

only if $0, \pm 1 \in \{1+m(l-1), 1-n, k\}$ (Lemma A.2). Lemma A.4 is used to rule out any of these cases occurring.

Case $g - h = 1$: As before, turning now our attention to the slope 0 and writing $\frac{g}{h} = 1 + \frac{1}{m}$, it holds

$$M_5\left(\frac{1}{n}, \frac{c}{d}, k, \frac{m+1}{m}\right)(0) \stackrel{(37)}{=} F\left(\frac{n}{n-1}, \frac{c-d}{c}, -\frac{m}{m+1}, k-1\right),$$

which, unless $0, \pm 1 \in \{n, c-d, m, k-1\}$, will have an essential torus by Lemma A.2 and Lemma A.11. Just as in the preceding case, we can use Lemma A.4 to conclude that the only possibility is $c-d = \pm 1$, which is equivalent to $\frac{c}{d} = 1 + \frac{1}{l}$. Combining the necessary conditions obtained we learn that

$$\begin{aligned} M_5\left(\frac{1}{n}, \frac{l+1}{l}, k, \frac{m+1}{m}\right)(1) &\stackrel{(36)}{=} F\left(\frac{1-n}{n}, \frac{l+1}{l}, k, \frac{1}{m}\right) \stackrel{(34)}{=} (D, (1-n, n), (k, 1)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (l+1, l), (1, m)) \\ &\stackrel{(13)}{=} (S^2, (l+ml+m, -l-1), (1-n, n), (k, 1)) \in T^H \end{aligned}$$

only if $0, \pm 1 \in \{l+ml+m, 1-n, k\}$ (Lemma A.2). Lemma A.4 is used to rule out any of these cases occurring.

We proceed now to consider the examples from Family 2. The analysis follows verbatim the steps considered in the study of Family 1. We have assumed that both 0 and 1 correspond to S^H or T^H slopes. The manifold

$$M_5\left(\frac{1}{n}, \frac{\varepsilon-kn}{k}, \frac{e}{f}, \frac{g}{h}\right)(1) \stackrel{(36)}{=} F\left(\frac{1}{n}-1, \frac{\varepsilon-kn}{k}, \frac{e}{f}, \frac{g}{h}-1\right)$$

has an essential torus unless $0, \pm 1 \in \{1-n, \varepsilon-kn, e, g-h\}$ (Lemma A.2 and Lemma A.11). Lemma A.4 implies that $1-n, \varepsilon-kn \notin 0, \pm 1$ and $e, g-h \neq 0$. We are thus left with the possibilities $e = \pm 1$ and $g-h = \pm 1$.

Case $e = 1$: Writing $\frac{e}{f} = \frac{1}{m}$ we have

$$M_5\left(\frac{1}{n}, \frac{\varepsilon-kn}{k}, \frac{1}{m}, \frac{g}{h}\right)(0) \stackrel{(37)}{=} F\left(\frac{n}{n-1}, \frac{\varepsilon-(n+1)k}{k}, -\frac{h}{g}, \frac{1-m}{m}\right)$$

which has again an essential torus unless $0, \pm 1 \in \{n, \varepsilon - (n+1)k, h, 1-m\}$. Lemma A.4 leaves us with the necessary condition $h = \pm 1$ (note that every possible solution to $\varepsilon - (n+1)k = \pm 1, 0$ yields a non hyperbolic manifold or an instruction that factors through M_4). Now, calling $\frac{g}{h} = l$ we have

$$\begin{aligned} M_5\left(\frac{1}{n}, \frac{\varepsilon-kn}{k}, \frac{1}{m}, l\right)(1) &\stackrel{(36)}{=} F\left(\frac{1-n}{n}, \frac{\varepsilon-kn}{k}, \frac{1}{m}, l-1\right) \stackrel{(34)}{=} (D, (1-n, n), (1, m)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (\varepsilon-kn, k), (l-1, 1)) \\ &\stackrel{(13)}{=} (S^2, (n+m(1-n), n-1), (l-1, 1), (\varepsilon-nk, k)) \in T^H \end{aligned}$$

only when $0, \pm 1 \in \{n+m(1-n), l-1, \varepsilon-nk\}$ (Lemma A.2). These cases are all discounted using Lemma A.4.

Case $g - h = 1$: Turning now our attention to the slope 0 and writing $\frac{g}{h} = 1 + \frac{1}{m}$, it holds

$$M_5\left(\frac{1}{n}, \frac{\varepsilon-kn}{k}, \frac{e}{f}, \frac{m+1}{m}\right)(0) \stackrel{(37)}{=} F\left(\frac{n}{n-1}, \frac{\varepsilon-(n+1)k}{k}, -\frac{m}{m+1}, \frac{e-f}{f}\right)$$

which, unless $0, \pm 1 \in \{n, \varepsilon - (n+1)k, m, e-f\}$, will have an essential torus by Lemma A.2 and Lemma A.11. Once again we use Lemma A.4 to conclude that the only possibility is $e - f = \pm 1$, which is equivalent to $\frac{e}{f} = 1 + \frac{1}{l}$. Combining the necessary conditions obtained we learn that

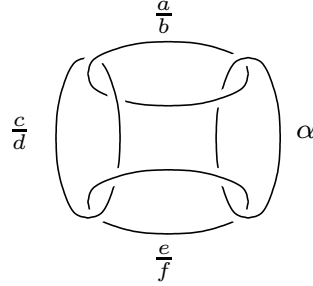
$$\begin{aligned} M_5\left(\frac{1}{n}, \frac{\varepsilon-kn}{k}, \frac{l+1}{l}, \frac{m+1}{m}\right)(1) &\stackrel{(36)}{=} F\left(\frac{1-n}{n}, \frac{\varepsilon-kn}{k}, \frac{l+1}{l}, \frac{1}{m}\right) \stackrel{(34)}{=} (D, (1-n, n), (l+1, l)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (\varepsilon-kn, k), (1, m)) \\ &\stackrel{(13)}{=} (S^2, (k+m(\varepsilon-kn), kn-\varepsilon), (1-n, n), (l+1, l)) \in T^H \end{aligned}$$

only if $0, \pm 1 \in \{k + m(\varepsilon - kn), 1 - n, l + 1\}$ (Lemma A.2). Lemma A.4 is used to rule out any of these cases occurring.

We conclude that if $(M_5(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, T^H)$ then \mathcal{F} factors through M_4 . This completes the argument and thus Section 3.1.

3.2. Hyperbolic knots with two lens surgeries arising from the 4-chain link. In this section we prove that if $M_4\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right)$ (below) is hyperbolic with three fillings in $S^H \cup T^H$ then the

instruction $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ factors through M_3 .



From [Rou2, Theorem 5] and a careful inspection of [Rou2, Tables 12, 21, 22] we deduce that if $(M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}), \alpha, \beta, \gamma) \in (S^H, T^H, T^H)$, then $e(M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})) = 4$ and $\{\alpha, \beta, \gamma\} \subset \{0, 1, 2, \infty\}$. Since $\Delta(S^H, T^H) = \Delta(T^H, T^H) = 1$ [GL1], it follows that either $\{\alpha, \beta, \gamma\} = \{1, 2, \infty\}$ or $\{\alpha, \beta, \gamma\} = \{0, 1, \infty\}$. For $\{\alpha, \beta, \gamma\} = \{1, 2, \infty\}$ as ordered sets, we have

$$(\dagger) \quad (M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}), \alpha, \beta, \gamma) \stackrel{\text{Lemma A.7 \& (29) \circ (16)^2}}{\cong} (M_5(-1, \frac{a-b}{b}, \frac{c}{d}, \frac{e}{f}), \alpha', \beta', \gamma'),$$

where $\{\alpha', \beta', \gamma'\} = \{0, 1, \infty\}$ as ordered sets. Now,

$$\begin{aligned} (\dagger) &\stackrel{(24)}{\cong} (M_5(\frac{f}{f-e}, -1, \frac{d}{d-c}, \frac{a-2b}{a-b}), 1 - (\alpha')^{-1}, 1 - (\beta')^{-1}, 1 - (\gamma')^{-1}) \\ &\stackrel{(29) \circ (16)}{\cong} (M_4(\frac{2f-e}{f-e}, \frac{2d-c}{d-c}, \frac{a-2b}{a-b}), 1 - (\alpha')^{-1}, 1 - (\beta')^{-1}, 1 - (\gamma')^{-1}) \end{aligned}$$

and $\{1 - (\alpha')^{-1}, 1 - (\beta')^{-1}, 1 - (\gamma')^{-1}\} = \{\infty, 0, 1\}$ as ordered sets. Hence, we may assume from here on that if $(M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}), \alpha, \beta, \gamma) \in (S^H, T^H, T^H)$ then $\{\alpha, \beta, \gamma\} = \{1, 2, \infty\}$.

We now proceed to examine the necessary conditions on the filling instruction $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ imposed from $\{\alpha, \beta, \gamma\} = \{1, 2, \infty\}$ corresponding to $S^H \cup T^H$ slopes.

3.2.1. *Necessary conditions from $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})(2)$.* We have

$$M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})(2) \stackrel{(33)}{=} (D, (a-b, b), (e-f, f)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (c, d), (2, -1)) \in S^H \cup T^H$$

only if $0, \pm 1 \in \{a-b, e-f, c\}$ by Lemma A.2. If $a-b=0, e-f=0, c=0$ then $\frac{a}{b}=1, \frac{e}{f}=1, \frac{c}{d}=0$ respectively, which are all excluded since we are only interested in the hyperbolic case (Lemma A.6). We continue with a case by case analysis:

The case $|a-b|=1$: We are working with $\frac{a}{b}$, so, up to a simultaneous change of signs of a and b , we may assume w.l.o.g. that $a-b=1$. This gives us, again by Lemma A.2,

$$\begin{aligned} M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})(2) &\stackrel{(33)}{=} (D, (1, b), (e-f, f)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (c, d), (2, -1)) \\ &\stackrel{(13)}{=} (S^2, (b(f-e)-f, e-f), (c, d), (2, -1)) \in S^H \cup T^H \end{aligned}$$

only if $0, \pm 1 \in \{c, b(f - e) - f\}$. The case $c = 0$ is directly excluded by Lemma A.6. If $b(f - e) - f = 0$ then $b = 1 + \frac{e}{f - e} = 1 + \frac{1}{\frac{f}{e} - 1}$. But b is an integer, so $\frac{f}{e} - 1 \in \{0, \pm 1\}$ and $\frac{e}{f} \in \{\frac{1}{2}, 1, \infty\}$ which is excluded by Lemma A.6.

So, up to changing the signs of c and d or of e and f , we may even assume w.l.o.g. that either $c = 1$ or $b(f - e) = 1 + f$;

The case $|e - f| = 1$: Lemma A.7 tells us that $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})(2) = M_4(\frac{e}{f}, \frac{c}{d}, \frac{a}{b})(2)$. So, any examples found in this case are contained in the case $|a - b| = 1$;

The case $|c| = 1$: Up to a simultaneous change of signs of c and d , we may assume w.l.o.g. that $c = 1$. We get

$$\begin{aligned} M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})(2) &\stackrel{(33)}{=} (D, (a-b, b), (e-f, f)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (1, d), (2, -1)) \\ &\stackrel{(13)}{=} (S^2, (e-f, f), (a-b, b), (1-2d, 2)) \in S^H \cup T^H \end{aligned}$$

only when $0, \pm 1 \in \{a - b, e - f, 1 - 2d\}$ by Lemma A.2. If $a - b$ or $e - f$ equals zero, then $\frac{a}{b}$, respectively $\frac{e}{f}$, equals 1 which is excluded by Lemma A.6. Since d is an integer, $1 - 2d \neq 0$ and if $1 - 2d = \pm 1$, then $d \in \{0, 1\} \Rightarrow \frac{c}{d} \in \{1, \infty\}$ which is again excluded by Lemma A.6.

So either $|a - b| = 1$ or $|e - f|$ equals 1. As previously remarked, Lemma A.7 allows us to assume w.l.o.g. that $|a - b| = 1$. Up to a simultaneous change of signs of a and b , we may assume that $a - b = 1$.

To summarise, if $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})(2) \in S^H \cup T^H$, then we may assume that one of the following sets of conditions holds:

$$\boxed{\begin{array}{ll} a - b = 1 & (C_2^0) \\ c = 1 & (C_2') \end{array}} (C_2^1) \quad \text{or} \quad \boxed{\begin{array}{ll} a - b = 1 & (C_2^0) \\ b(f - e) = 1 + f & (C_2'') \end{array}} (C_2^2).$$

3.2.2. *Necessary conditions from $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})(1)$.* We have

$$M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})(1) \stackrel{(32)}{=} (S^2, (a-2b, b), (c-d, c), (e-2f, f)) \in S^H \cup T^H$$

only if $0, \pm 1 \in \{a - 2b, c - d, e - 2f\}$ by Lemma A.2. If $0 \in \{a - 2b, c - d, e - 2f\}$, then one of $\frac{a}{b} = 2$, $\frac{c}{d} = 1$ or $\frac{e}{f} = 2$ which are all excluded by Lemma A.6. So, if $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})(1) \in S^H \cup T^H$ then one of the following conditions necessarily holds:

$$\boxed{a - 2b = \varepsilon} (C_1^1) \quad \text{or} \quad \boxed{c - d = \varepsilon} (C_1^2) \quad \text{or} \quad \boxed{e - 2f = \varepsilon} (C_1^3)$$

where $\varepsilon = \pm 1$.

3.2.3. *Arising from $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})(\infty)$.* We have

$$M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})(\infty) \underset{(30)}{=} (S^2, (a, b), (d, -c), (e, f)) \in S^H \cup T^H$$

only when $0, \pm 1 \in \{a, d, e\}$ by Lemma A.2. If one of a, d, e equals zero, then one of $\frac{a}{b} = 0, \frac{c}{d} = \infty, \frac{e}{f} = 0$ which are all excluded by Lemma A.6. So, if $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})(\infty) \in S^H \cup T^H$ then one of the following conditions necessarily holds:

$$\boxed{a = \eta} (C_\infty^1) \quad \text{or} \quad \boxed{d = \eta} (C_\infty^2) \quad \text{or} \quad \boxed{e = \eta} (C_\infty^3)$$

where $\eta = \pm 1$.

3.2.4. *Enumeration of $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ satisfying the necessary conditions.* We have shown that if $(M_4(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, T^H)$ then we may assume that \mathcal{F} is equivalent to a filling instruction $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ satisfying one of C_2^1 or C_2^2 , one of C_1^1, C_1^2 or C_1^3 and one of C_∞^1, C_∞^2 or C_∞^3 . We will now show that any such $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ must factor through M_3 :

$C_2^0 + C_1^1$: substituting $a - b = 1$ into $a - 2b = \varepsilon$ gives $b = 1 - \varepsilon \Rightarrow \frac{a}{b} = 1 + \frac{1}{1-\varepsilon} \in \{\frac{3}{2}, \infty\}$ which is excluded by Lemma A.6.

$C_2^0 + C_\infty^1$: substituting $a = \eta$ into $a - b = 1$ gives $\frac{a}{b} = \frac{\eta}{\eta-1} = 1 + \frac{1}{\eta-1} \in \{\frac{1}{2}, \infty\}$ which is excluded by Lemma A.6.

$C_2' + C_1^2$: substituting $c = 1$ into $c - d = \varepsilon$ gives $d = 1 - \varepsilon \Rightarrow \frac{c}{d} = \frac{1}{1-\varepsilon} \in \{\frac{1}{2}, \infty\}$ which is excluded by Lemma A.6.

$C_2' + C_\infty^2$: $\frac{c}{d} = \pm 1$ and which is excluded by Lemma A.6.

$C_2^2 + C_1^3$: C_1^3 implies $e - f = f + \varepsilon$ which we substitute into $b(f - e) = 1 + f$ to get $-b(f + \varepsilon) = 1 + f$. If $f = -\varepsilon$ then $\frac{e}{f} = -1$ which is excluded by Lemma A.6, and otherwise $-b = 1 + \frac{1-\varepsilon}{f+\varepsilon}$. If $\varepsilon = 1$, then $b = -1, a = 0$ and $\frac{a}{b} = 0$ which are excluded by Lemma A.6. If $\varepsilon = -1$ then $\frac{2}{f-1} = -b - 1$ is an integer, so $f - 1$ divides 2 and $f \in \{-1, 0, 1, 2, 3\}$. If $f = 3$, then $b = -2, a = -1$ and $\frac{a}{b} = \frac{1}{2}$ which is excluded by Lemma A.6. Otherwise, $\frac{e}{f} = 2 - \frac{1}{f} \in (1, \frac{3}{2}, 3, \infty)$ which is excluded by Lemma A.6;

$C_2^2 + C_\infty^3$: if $e = \eta$ then $f \neq \pm 1$ by Lemma A.6. Substituting $e = \eta$ into $b(f - e) = 1 + f$ gives $b(f - \eta) = 1 + f \Rightarrow b = 1 + \frac{1+\eta}{f-\eta}$. If $\eta = -1$, then $b = 1, a = 2$ and $\frac{a}{b} = 2$ which is excluded by Lemma A.6. If $\eta = 1$ then $\frac{2}{f-1} = b - 1$ is an integer, and $f - 1$ divides 2 which implies $f \in \{-1, 0, 1, 2, 3\}$. If $f = 3$, then $b = 2, a = 3$ and $\frac{a}{b} = \frac{3}{2}$ which is excluded by Lemma A.6. Otherwise, $\frac{e}{f} = \frac{1}{f} \in (-1, \frac{1}{2}, 1, \infty)$ which is excluded by Lemma A.6.

$C_1^2 + C_\infty^2$: we have $c = \eta + \varepsilon \Rightarrow \frac{c}{d} = \frac{\eta + \varepsilon}{\varepsilon \eta} = 1 + \varepsilon \eta \in \{0, 2\}$ which is excluded by Lemma A.6.

$C_1^3 + C_\infty^3$: we have $2f = \eta - \varepsilon \Rightarrow f \in \{0, \pm 1\}$, so $\frac{e}{f} \in \{\pm 1, \infty\}$ which is excluded by Lemma A.6.

We now observe that the above analysis is enough to conclude that no triple $C_2^n \cup C_1^m \cup C_\infty^k$ of conditions can hold:

- C_2^0 necessarily holds. The above analysis implies that C_1^1 or C_∞^1 do not hold.
- If C_2^2 holds then the above analysis shows that neither C_1^2 or C_∞^2 holds. Moreover, C_2^1 and C_∞^3 do not hold simultaneously. So, any (S^H, T^H, T^H) -triple of the form $(M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}), \alpha, \beta, \gamma)$ with $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ not factoring through M_3 necessarily satisfies C_2^2 .
- If C_2^2 holds then C_1^3 and C_∞^3 cannot hold. Moreover, C_1^2 and C_∞^2 cannot hold simultaneously.

We conclude that, as announced at the beginning of the section, if $(M_4(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, T^H)$ then \mathcal{F} factors through M_3 .

3.3. Hyperbolic knots with two lens surgeries arising from the 3–chain link. We now enumerate all the hyperbolic knots with two lens space surgeries obtained by surgery on the 3–chain link. We prove the following result:

Proposition 3.2. *If $(M_3(\frac{a}{b}, \frac{c}{d}), \alpha, \beta, \gamma) \in (S^H, T^H, T^H)$ then $(M_3(\frac{a}{b}, \frac{c}{d}), \alpha, \beta, \gamma)$ is equivalent to $(N(-\frac{3}{2}, -\frac{14}{5}), -2, -1, \infty)$ or some A_k .*

The enumeration of all (S^H, T^H, T^H) triples obtained by surgery on 3CL comes from a careful examination of the tables in [MP]. It should be noted that the classification of exceptional fillings on the exterior of the 3-chain link in [MP] is performed on the exterior of the mirror image 3CL*. The exterior of 3CL* is denoted N , and, of course, $M_3(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}) = N(-\frac{a}{b}, -\frac{c}{d}, -\frac{e}{f})$. For the sake of clarity when referencing tables, we adopt the convention in [MP].

All exceptional closed fillings of N are shown in [MP, Tables 2–4] and we consider each of the tables individually. First, we note that [MP, Table 4] involves no fillings of the form $L(\star, \star)$ so we restrict our attention to [MP, Tables 2–3]. In Table 3 there are some entries where $N(\frac{p}{q}, \frac{r}{s}, \frac{t}{u}) = L(\star, \star)$. However, in each case, applying Lemma A.10 we can conclude that if $\mathcal{F} \subset \{\frac{p}{q}, \frac{r}{s}, \frac{t}{u}\}$ with $|\mathcal{F}| = 2$ then $N(\mathcal{F})$ is non-hyperbolic. So, if $(M_3(\frac{a}{b}, \frac{c}{d}), \alpha, \beta, \gamma) \in (S^H, T^H, T^H)$ then $(M_3(\frac{a}{b}, \frac{c}{d}), \alpha, \beta, \gamma)$ is equivalent to some $(N(\frac{r}{s}, \frac{t}{u}), \alpha', \beta', \gamma')$ with $\alpha', \beta', \gamma' \in \{-3, -2, -1, 0, \infty\}$ and the $\frac{r}{s}, \frac{t}{u}$ can be found in [MP, Table 2] if all slopes are finite, or in [MP, Theorem 1.3] if some slope is ∞ . We now examine each case individually.

3.3.1. *Case 0 is an S^3 slope.* We see directly from [MP, Table 2] that if $N(\frac{r}{s}, \frac{t}{u})(0) = L(\star, \star)$ then $\frac{r}{s} = n$, $\frac{t}{u} = -4 - n + \frac{1}{m}$ and $N(\frac{r}{s}, \frac{t}{u})(0) = L(6m-1, 2m-1)$. So, if $N(\frac{r}{s}, \frac{t}{u})(0) = S^3$ then $m = 0$ and $\frac{t}{u} = \infty$ implies the manifold is non-hyperbolic (Lemma A.10).

3.3.2. *Case -1 is an S^3 slope.* We see directly from [MP, Table 2] that if $N(\frac{r}{s}, \frac{t}{u})(-1) = L(\star, \star)$ then $\frac{r}{s} = -3 + \frac{1}{n}$, and $N(\frac{r}{s}, \frac{t}{u})(-1) = L(2n(t+3u)-t-u, \star)$. If $L(2n(t+3u)-t-u, \star) = S^3$ then $2n(t+3u)-t-u = \pm 1$. By changing the signs of both t and u , we may assume w.l.o.g. that

$$(2) \quad 2n(t+3u) - t - u = 1.$$

Moreover, we know that $\Delta(S^H, T^H) = \Delta(T^H, T^H) = 1$ [GL1]. If $\Delta(\beta, -1) = 1$ then $\beta = \frac{1-k}{k}$ and we can quickly see that the only pairs of possibilities for the T^H slopes among the cases we are considering are $\{-2, \infty\}$ and $\{0, \infty\}$. From [MP, Theorem 1.3] we know that $N(\frac{r}{s}, \frac{t}{u})(\infty)$ is always a lens space.

We will now use (see [MP, Table 2]) to further refine the constraints, $\frac{r}{s} = -3 + \frac{1}{n}$ and (2), that we have found from imposing -1 to be a S^3 slope. This time we will analyse the restrictions we obtain by considering 0 and -2 to be lens space slopes and through this analysis we will enumerate all (S^3, T^H, T^H) triples. We will denote the new parameters with primes.

Case 0 is a T^H slope: Either $\frac{r}{s} = -3 + \frac{1}{n} = n'$ or $\frac{r}{s} = -3 + \frac{1}{n} = -4 - n' + \frac{1}{m'}$.

Case $\frac{r}{s} = -3 + \frac{1}{n} = n'$: In this case $n' = -2$ or -4 and $\frac{t}{u} = -4 - n' + \frac{1}{m'}$. The case $n' = -2$ is excluded by Lemma A.10. So, $n' = -4$ which implies $\frac{t}{u} = \frac{1}{m'}$ and $n = -1$. From (2), we have $-t(3 + 7m') = 1$ which cannot hold.

Case $\frac{r}{s} = -3 + \frac{1}{n} = -4 - n' + \frac{1}{m'}$: In this case $\frac{t}{u} = n'$ and from the former equality we deduce $n' + 1 = \frac{1}{m'} - \frac{1}{n} \in [-2, 2] \cup \{\infty\}$ so $n' \in [-3, 1] \cup \{\infty\}$. From Lemma A.10 we know that for $N(\frac{r}{s}, \frac{t}{u})$ to be hyperbolic the only possible value of $n' = \frac{t}{u}$ is 1 , which implies that $n = -1$. Substituting this information in (2) we obtain $-10u = 1$. A contradiction.

Case -2 is a T^H slope: Either $\frac{r}{s} = -3 + \frac{1}{n} = -2 + \frac{1}{n'}$ or $-2 + \frac{1}{n'} = \frac{t}{u}$.

Case $\frac{r}{s} = -2 + \frac{1}{n'} = -3 + \frac{1}{n}$: In this case $n = 2$ and, by (2), $(N(-\frac{5}{2}, \frac{t}{u}), -1, -2, \infty)$ is a (S^3, T^H, T^H) triple whenever $3t + 11u = 1$. Namely, for $t = 4 - 11k$ and $u = 3k - 1$ with any $k \in \mathbb{Z}$. That is $\boxed{(N(-\frac{5}{2}, \frac{4-11k}{3k-1}), -1, -2, \infty)}$ are (S^3, T^H, T^H) triples for every $k \in \mathbb{Z}$.

Case $-2 + \frac{1}{n'} = \frac{t}{u}$: In this case $\frac{t}{u} = \frac{1-2n'}{n'}$ so (2) becomes $2n(1+n')+n' = 2$ so $\frac{3}{2n+1} = n' + 1 \in \mathbb{Z}$. It follows that $n \in \{-2, -1, 0, 1\}$. For $n \in \{0, 1\}$, we have $\frac{r}{s} = -3 + \frac{1}{n} \in \{-2, \infty\}$ so the associated space is non-hyperbolic (Lemma A.10). For $n = -2$ and -1 we find that $\boxed{(N(-\frac{7}{2}, -\frac{5}{2}), -1, -2, \infty)}$ and $\boxed{(N(-4, -\frac{9}{4}), -1, -2, \infty)}$ are (S^3, T^H, T^H) triples.

3.3.3. *Case -2 is an S^3 slope.* We see directly from [MP, Table 2] that if $N(\frac{r}{s}, \frac{t}{u})(-2) = L(\star, \star)$ with $N(\frac{r}{s}, \frac{t}{u})$ hyperbolic then $\frac{r}{s} = -2 + \frac{1}{n}$, and $N(\frac{r}{s}, \frac{t}{u})(-2) = L((3n(t+2u)-2t-u), \star)$. So, up to simultaneously reversing the signs of t and u , we have $N(\frac{r}{s}, \frac{t}{u})(-2) = S^3$ if and only if w.l.o.g.

$$(3) \quad 3n(t+2u) - 2t - u = 1.$$

As in the previous section case $\Delta(S^3, T^H) = \Delta(T^H, T^H) = 1$, so the only possible pairs of T^H slopes are $\{-3, \infty\}$ and $\{-1, \infty\}$. We know that the ∞ -filling is always a lens space [MP, Theorem 1.3].

If -1 or -3 correspond to lens space slopes, then there are conditions on $\frac{r}{s}$ and $\frac{t}{u}$ (see [MP, Table 2]). We can use these conditions (the parameters are denoted with primes) in conjunction with $\frac{r}{s} = -2 + \frac{1}{n}$ (already established) to enumerate all (S^3, T^H, T^H) triples. The cases are now considered one at a time.

Case -1 is a T^H slope: From [MP, Table 2], either $-2 + \frac{1}{n} = -3 + \frac{1}{n'}$ or $\frac{t}{u} = -3 + \frac{1}{n'}$.

Case $-2 + \frac{1}{n} = -3 + \frac{1}{n'}$: In this case $n = -2$ and we find that $(N(-\frac{5}{2}, \frac{t}{u}), -1, -2, \infty)$ is a (S^3, T^H, T^H) triple whenever $8t + 13u + 1 = 0$. That is, for $t = 13k - 5$ and $u = 3 - 8k$ with any $k \in \mathbb{Z}$. So $(N(-\frac{5}{2}, \frac{13k-5}{3-8k}), -2, -1, \infty)$ is a (S^3, T^H, T^H) triple for every $k \in \mathbb{Z}$.

Case $-3 + \frac{1}{n} = \frac{t}{u}$: In this case $\frac{t}{u} = \frac{1-3n'}{n'}$ and (3) becomes $3n(1-n') + 5n' = 3$ so $\frac{2}{5-3n} = 1-n' \in \mathbb{Z}$. It follows that $n \in \{1, 2\}$. For $n = 1$, we have $\frac{r}{s} = -1$ which makes $N(\frac{r}{s}, \frac{t}{u})$ non-hyperbolic (Lemma A.10). For $n = 2$ we find that $(N(-\frac{3}{2}, -\frac{8}{3}), -2, -1, \infty)$ is a (S^3, T^H, T^H) triple.

Case -3 is a T^H slope: If $\frac{t}{u} = -2$ then $N(\frac{r}{s}, \frac{t}{u})$ is non-hyperbolic (Lemma A.10). So, from [MP, Table 2], $-2 + \frac{1}{n} = -1 + \frac{1}{n'}$ making $n = 2$ and $\frac{t}{u} = -1 + \frac{1}{m'} = \frac{1-m'}{m'}$. Using (3), we obtain $m' = -\frac{7}{3} \notin \mathbb{Z}$.

3.3.4. *Case -3 is an S^3 slope.* From [MP, Table 2], if $N(\frac{r}{s}, \frac{t}{u})(-3) = L(\star, \star)$ then either $\frac{t}{u} = -2$ (which is excluded by Lemma A.10) or $\frac{r}{s} = -1 + \frac{1}{n}$ and $\frac{t}{u} = -1 + \frac{1}{m}$. In the latter case we have $N(-1 + \frac{1}{n}, -1 + \frac{1}{m})(-3) = L((2n+1)(2m+1)-4, \star) = S^3$ if and only if $(2n+1)(2m+1) - 4 = \pm 1$; that is $(2n+1)(2m+1) = 3$ or 5 . Since both 3 and 5 are primes, it follows that either $2n+1$ or $2m+1$ is ± 1 . By symmetry, we may assume that $2n+1 = \pm 1$, making $n = -1$ or 0 which is excluded by Lemma A.10.

3.3.5. *Case ∞ is an S^3 slope.* From [MP, Theorem 1.3], $N(\frac{r}{s}, \frac{t}{u})(\infty) = L(tr-us, \star)$. So, ∞ is an S^3 slope if and only if

$$(4) \quad tr - us = \pm 1.$$

As before, we have $\Delta(S^3, T^H) = \Delta(T^H, T^H) = 1$ so the only possible pairs of T^H slopes are $\{-3, -2\}$, $\{-2, -1\}$ or $\{-1, 0\}$. Each T^H slope imposes conditions on $\frac{r}{s}, \frac{t}{u}$. We will use primes on the parameters to denote the conditions imposed from the smallest T^H -slope and double primes on the parameters coming from the conditions on the second T^H -slope.

Case 0 is a T^H -slope: From [MP, Table 2] we have $\frac{r}{s} = n'$ and $\frac{t}{u} = -4 - n' + \frac{1}{m'} = \frac{1-m'(n'+4)}{m'}$. Equation (4) becomes then $(1 - m'(n' + 4))n' = m' \pm 1$. According to Lemma A.15, we have $n' \in \{-5, -4, -3, -2, -1, 0, 1\}$. For $N(\frac{r}{s}, \frac{t}{u})$ to be hyperbolic, n' cannot be in $\{-3, -2, -1, 0\}$ (Lemma A.10). So we are left to consider the following cases $(n', m') \in \{(-5, -1), (-4, -5), (-4, -3), (1, 0)\}$. If $(n', m') = (-5, -1)$ then $\frac{t}{u} = 0$, and if $(n', m') = (1, 0)$ then $\frac{t}{u} = \infty$. So, these cases are both excluded by Lemma A.10. The other two cases $\left(N(-4, -\frac{1}{3}), \infty, -1, 0, \right)$ and $\left(N(-4, -\frac{1}{3}), \infty, -1, 0\right)$ are indeed (S^3, T^H, T^H) ([MP, Table 2]).

Case -2 and -1 are the T^H -slopes: In this case, either $-2 + \frac{1}{n'} = -3 + \frac{1}{n''}$ or, up to symmetry, $\left(\frac{r}{s}, \frac{t}{u}\right) = \left(\frac{1-2n'}{n'}, \frac{1-3n''}{n''}\right)$.

Case $-2 + \frac{1}{n'} = -3 + \frac{1}{n''}$: This means $n' = -2$. Up to symmetry, we may assume that $\frac{r}{s} = -\frac{5}{2}$. Up to a simultaneous change of sign of t and u , (4) becomes $5t + 2u = 1$.

We find $\left(N(-\frac{5}{2}, \frac{1-2k}{5k-2}), \infty, -2, -1\right)$ is a (S^3, T^H, T^H) triple for every $k \in \mathbb{Z}$.

Case $\left(\frac{r}{s}, \frac{t}{u}\right) = \left(\frac{1-2n'}{n'}, \frac{1-3n''}{n''}\right)$: We find that (4) becomes $2n' + 3n'' - 5n'n'' = 0$ or 2.

Case $2n' + 3n'' = 5n'n''$: In this case $n' = \frac{3n''}{5n''-2} \in \mathbb{Z}$. If $n'' \geq 0$ then $5n'' - 2 \leq 3n'' \Rightarrow n'' \leq 1$ and, if $n'' \leq 0$, $3n'' \leq 5n'' - 2$ then $n'' \geq 1$. It follows that $n'' = 0 \Rightarrow \frac{t}{u} = \infty$ or $n'' = 1 \Rightarrow \frac{t}{u} = -2$ which are both excluded by Lemma A.10.

Case $2n' + 3n'' = 2 + 5n'n''$: In this case $n' = \frac{3n''-2}{5n''-2} \in \mathbb{Z}$. If $n'' < 0$ then $0 < n' < 1$, and if $n'' \geq \frac{2}{3}$ then $0 < n' < 1$. So $n'' = 0$ and $\frac{t}{u} = \infty$ which is excluded by Lemma A.10.

Case -3 is a T^H -slope: From [MP, Table 2] we have $\frac{r}{s} = -2$ (which is excluded by Lemma A.10) or $\frac{r}{s} = \frac{1-n'}{n'}$ and $\frac{t}{u} = \frac{1-m'}{m'}$. In the latter case (4) becomes $n' + m' = 0$ or 2.

Using $\Delta(T^H, T^H) = 1$, if -3 is a T^H -slope then -2 is the only possible second T^H . Up to symmetry, we have $-1 + \frac{1}{n'} = -2 + \frac{1}{n''} \Rightarrow n' = -2$. Subbing this value into (4) with $\frac{t}{u} = \frac{1-m'}{m'}$ we find that either $m' = 2$ or 4. We find that $\left(N(-\frac{3}{2}, -\frac{1}{2}), \infty, -3, -2\right)$ and $\left(N(-\frac{3}{2}, -\frac{3}{4}), \infty, -3, -2\right)$ are (S^3, T^H, T^H) triples.

3.3.6. Identifying cases. In the above analysis we have proved above that the only (S^3, T^H, T^H) triples of the form $\left(N(\frac{r}{s}, \frac{t}{u}), \infty, \beta, \gamma\right)$ are:

- (i) $A_k := \left(N\left(-\frac{5}{2}, \frac{1-2k}{5k-2}\right), \infty, -2, -1\right)$ for $k \in \mathbb{Z}$;
- (ii) $B_k := \left(N\left(-\frac{5}{2}, \frac{4-11k}{3k-1}\right), -1, -2, \infty\right)$ for $k \in \mathbb{Z}$;
- (iii) $C_k := \left(N\left(-\frac{5}{2}, \frac{13k-5}{3-8k}\right), -2, -1, \infty\right)$ for $k \in \mathbb{Z}$;
- (iv) $\left(N\left(-\frac{7}{2}, -\frac{5}{2}\right), -1, -2, \infty\right)$;
- (v) $\left(N\left(-4, -\frac{9}{4}\right), -1, -2, \infty\right)$;
- (vi) $\left(N\left(-\frac{3}{2}, -\frac{8}{3}\right), -2, -1, \infty\right)$;
- (vii) $\left(N\left(-4, -\frac{1}{5}\right), \infty, -1, 0\right)$;
- (viii) $\left(N\left(-4, -\frac{1}{3}\right), \infty, -1, 0\right)$;
- (ix) $\left(N\left(-\frac{3}{2}, -\frac{1}{2}\right), \infty, -3, -2\right)$;
- (x) $\left(N\left(-\frac{3}{2}, -\frac{3}{4}\right), \infty, -3, -2\right)$.

In this list there are many repetitions. Indeed, using the first equality in [MP, Theorem 1.5], one can show that case (vii) is isomorphic to case (x), which in turn is isomorphic to case (v) and case (viii) is isomorphic to case (ix). Moreover, using the third equality in [MP, Theorem 1.5], we see that (ix) is isomorphic to case (vi), which is in turn isomorphic to (x). Summing up, all cases (v) to (x) are isomorphic and again via [MP, Theorem 1.5] we choose to fix the representative as $\left(N\left(-\frac{3}{2}, -\frac{14}{5}\right), -2, -1, \infty\right)$.

On the other hand we have that case (iv) is B_1 and using the second equality in [MP, Theorem 1.5], we see that $C_k \cong B_k \cong A_k$ for every $k \in \mathbb{Z}$.

3.3.7. Distinctness of examples. The Berge manifold is the unique hyperbolic knot exterior in a solid torus T with three distinct solid torus fillings [Gab]. The Berge manifold is equal to $N\left(-\frac{5}{2}\right)$ [MP]. By filling along a $\frac{1}{n}$ slope on ∂T we obtain a family of hyperbolic knot exteriors with two lens space fillings. As our enumeration of (S^3, T^H, T^H) triples obtained by surgery on 5CL is exhaustive, the family of (S^3, T^H, T^H) triples obtained by filling along a boundary component of the Berge manifold is $\left\{\left(N\left(-\frac{5}{2}, \frac{1-2k}{5k-2}\right), \infty, -2, -1\right)\right\}$. By considering the sets of exceptional fillings, we will now show that $N\left(-\frac{3}{2}, -\frac{14}{5}\right) \neq N\left(-\frac{5}{2}, \frac{1-2k}{5k-2}\right)$ for any k .

Using [MP, Tables 2–3] we can write down the set of exceptional slopes and fillings of $N\left(-\frac{5}{2}, \frac{1-2k}{5k-2}\right)$ and $N\left(-\frac{3}{2}, -\frac{14}{5}\right)$. The result is shown in Table 1. We immediately observe that $N\left(-\frac{5}{2}, \frac{1-2k}{5k-2}\right)$ has three distinct toroidal fillings for every k , and that $N\left(-\frac{3}{2}, -\frac{14}{5}\right)$ has only two toroidal filling. This shows $N\left(-\frac{3}{2}, -\frac{14}{5}\right) \neq N\left(-\frac{5}{2}, \frac{1-2k}{5k-2}\right)$ for any k .

4. (S^H, T^H, T) TRIPLES

In this section we are going to enumerate all (S^H, T^H, T) triples obtained by surgery on the five chain link and realizing the maximal distance. We know, from [Rou2, Theorem 1], that if $(M_5(\mathcal{F}), \beta, \gamma) \in (T^H, T)$, then $\Delta(\beta, \gamma) \leq 3$. In order to state the main theorem of this section we

need to first define the following families

$$B_n := \left(N\left(-1 + \frac{1}{n}, -1 - \frac{1}{n}\right), \infty, -3, 0\right) \quad \text{and} \quad C_n := \left(N\left(-1 + \frac{1}{n}, -1 - \frac{1}{n-2}\right), \infty, -3, 0\right).$$

We prove the following:

Theorem 4.1. *The following statements are true:*

- If $(M_5(\mathcal{F}), \alpha, \beta, \gamma) \in (S^3, T^H, T)$ with $\Delta(\beta, \gamma) = 3$, then $(M_5(\mathcal{F}), \alpha, \beta, \gamma) \in \{B_n\}$ for some $n \in \mathbb{Z} \setminus \{0, \pm 1\}$ or $(M_5(\mathcal{F}), \alpha, \beta, \gamma) \in \{C_n\}$ for some $n \in \mathbb{Z} \setminus \{0, \pm 1, 2, 3\}$.
- For all n and all k , B_n is not equivalent to C_k .
- Furthermore, if $n \in \mathbb{Z} \setminus \{\pm 1, 0, \pm 2\}$ then $E(B_n) = \{-3, -2, -1, 0, \infty\}$ and the exceptional fillings are shown in Table 2. If $n = \pm 2$ then B_n is the exterior of the pretzel knot $(-2, 3, 7)$ and $e(B_{\pm 2}) = 7$.
- Finally, if $n \in \mathbb{Z} \setminus \{\pm 1, 0, 2, 3\}$ then $E(C_n) = \{-3, -2, -1, 0, \infty\}$ and the exceptional fillings are shown in Table 2.

$n \in \mathbb{Z} \setminus \{\pm 1, 0, \pm 2\},$	$E(N(-1 + \frac{1}{n}, -1 - \frac{1}{n})) = \{-3, -2, -1, 0, \infty\}$
$\beta \in E(N(-1 + \frac{1}{n}, -1 - \frac{1}{n}))$	$N(-1 + \frac{1}{n}, -1 - \frac{1}{n})(\beta)$
$\beta = \infty$	S^3
$\beta = -3$	$L(4n^2+3, 2n^2+n+2)$
$\beta = -2$	$(S^2, (3, 2), (1+n, n), (1-n, n))$
$\beta = -1$	$(S^2, (2, 1), (1+2n, -n), (1-2n, n))$
$\beta = 0$	$(D, (n, 1+n), (n, n-1)) \cup \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} (D, (2, 1), (3, 1))$
$n \in \mathbb{Z} \setminus \{\pm 1, 0, 2, 3\},$	$E(N(-1 + \frac{1}{n}, -1 - \frac{1}{n-2})) = \{-3, -2, -1, 0, \infty\}$
$\beta \in E(N(-1 + \frac{1}{n}, -1 - \frac{1}{n-2}))$	$N(-1 + \frac{1}{n}, -1 - \frac{1}{n-2})(\beta)$
$\beta = \infty$	S^3
$\beta = -3$	$L(4n^2+8n-1, 2n^2-3n)$
$\beta = -2$	$(S^2, (1+n, n), (3-n, n-2), (3, 2))$
$\beta = -1$	$(S^2, (2, 1), (1+2n, -n), (5-2n, n-2))$
$\beta = 0$	$(D, (n, 1+n), (2-n, 2-n))3 - 3n \cup \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} (D, (2, 1), (3, 1))$

Table 2: The sets of exceptional slopes and fillings of all knot exteriors obtained by surgery on the minimally twisted 5-chain link realising $\Delta(T^H, T) = 3$ or $\Delta(T^H, Z) = 2$.

In Sections 4.1–4.3 we enumerate all $(M_5(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, T)$ with $\Delta(\beta, \gamma) = 3$ relying heavily on [MP] and [Rou2]. In Section 4.4, the various examples from the enumeration are studied to understand which among them are equivalent, and we show that $\{B_n\} \cap \{C_n\} = \emptyset$. Theorem 4.1 follows from this complete analysis carried out all through Section 4.

4.1. Triples from M_5 and M_4 . [Rou2, Theorem 4] gives a complete enumeration of $E(M_5(\mathcal{F}))$ for \mathcal{F} not factoring through M_4 . A careful inspection of [Rou2, Tables 6–11] allows us to conclude that if \mathcal{F} is a filling instruction on M_5 such that $E(M_5(\mathcal{F}))$ contains a pair of slopes at distance greater than 2 apart, then \mathcal{F} is equivalent to a filling instruction \mathcal{F}' found in [Rou2, Table 14]. In this table we can also find the class of $M_5(\mathcal{F}')(\alpha)$ for each $\alpha \in E(M_5(\mathcal{F}'))$; no $M_5(\mathcal{F}')(\alpha) = S^3$. Therefore, if $(M_5(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, T)$ with $\Delta(\beta, \gamma) = 3$, then \mathcal{F} factors through M_4 .

Similarly, [Rou2, Theorem 5] gives a complete enumeration of $E(M_4(\mathcal{F}))$ for \mathcal{F} not factoring through M_3 . If \mathcal{F} is a filling instruction on M_4 such that $E(M_4(\mathcal{F}))$ contains a pair of slopes at distance greater than 2 apart, then \mathcal{F} is equivalent to a filling instruction \mathcal{F}' found in [Rou2, Tables 21–22]. These tables provide also the class of $M_4(\mathcal{F}')(\alpha)$ for each $\alpha \in E(M_4(\mathcal{F}'))$; no $M_4(\mathcal{F}')(\alpha) = S^3$. Therefore, if $(M_4(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, T)$ with $\Delta(\beta, \gamma) = 3$ then \mathcal{F} factors through M_3 .

We conclude that if $(M_5(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, T)$ then \mathcal{F} factors through M_3 .

4.2. Exceptional triples from $M_3(\mathcal{F})$ with $e(M_3(\mathcal{F})) > 5$. As in the previous section, we recall that [MP] classifies the exceptional filling instructions and fillings on N , the exterior of the mirror image of 3CL. Of course $N = M_3$, but the sign of the slopes change sign; namely, $M_3(\alpha_1, \alpha_2, \alpha_3) = N(-\alpha_1, -\alpha_2, -\alpha_3)$. For the sake of clarity, as we work with the Tables in [MP], we work with the filling instructions on N which are identified with instructions on M_3 by changing signs at the end of the argument.

Any filling instruction \mathcal{F} on N consisting of two slopes and such that $e(N(\mathcal{F})) > 5$ can be found in [MP, Tables A.2–A.9]. The tables A.2, A.3, A.4 and A.9 each contain a finite list of $N(\mathcal{F})$. The remaining tables consist of four infinite families; Table A.5 considers $N(1, \frac{r}{s})$ with $\frac{p}{q}$ exceptional, Table A.6 considers $N(-\frac{3}{2}, \frac{r}{s})$ with $\frac{p}{q}$ exceptional, Table A.7 considers $N(-\frac{5}{2}, \frac{r}{s})$ with $\frac{p}{q}$ exceptional, and finally Table A.8 considers $N(-\frac{1}{2}, \frac{r}{s})$ with $\frac{p}{q}$ exceptional. We proceed to examine each of these tables in our quest for examples.

4.2.1. Examples arising from [MP, Tables A.2–A.4 and A.9]. The only hyperbolic knots (i.e. $N(\mathcal{F})$ with an S^3 filling) listed are $N(1, 2)$, also known as the Figure-8 knot, in Table A.2 and $N(-4, -\frac{1}{3})$, the $(-2, 3, 7)$ pretzel knot, in Table A.4. The former has no lens space filling while

the latter gives a unique (S^3, T^H, T) -triple with $\Delta(T^H, T) = 3$. So, from Tables A.2–A.4 and A.9 the only example we get is $\boxed{N(-4, -\frac{1}{3}), \infty, 0, -3) \in (S^3, T^H, T)}$.

4.2.2. *Examples arising from [MP, Table A.5].* In this table, the possible triples come from $N(\mathcal{F}) = N(1, \frac{r}{s})$ with $\frac{p}{q} \in E(N(\mathcal{F})) = \{-3, -2, -1, 0, 1, \infty\}$ exceptional. By inspection we conclude that in this table the only S^3 filling comes from $\frac{p}{q} = \infty$ when $r - s = \pm 1$. As we are interested in $\frac{r}{s}$, we may assume w.l.o.g. that $r = s + 1$. Moreover, if $N(1, \frac{r}{s})(\alpha)$ is toroidal then $\alpha = -3$ or $\alpha = 1$. We study both cases separately.

Case -3 is a toroidal slope on $N(1, \frac{r}{s})$: for $N(1, \frac{r}{s})(\frac{p}{q}) \in T^H$ with $\Delta(\frac{p}{q}, -3) = 3$ we need $\frac{p}{q} = 0$. Hence, Table A.5 combined with the fact that $r = s + 1$ tells us that either $\frac{r}{s} = 0$ or $\frac{r}{s} = 1 + \frac{1}{s} = -5 + \frac{1}{n}$. The former case gives rise to a non-hyperbolic space while the latter case cannot occur since $\frac{1}{n} - \frac{1}{s} \in (-2, 2) \cup \{\infty\} \not\cong 6$.

Case 1 is a toroidal slope on $N(1, \frac{r}{s})$: In this case, the T^H slope is necessarily $\frac{p}{q} = -2$ to realise $\Delta(1, \frac{p}{q}) = 3$. From Table A.5, the requirement that -2 is a lens space slope is $\frac{r}{s} = -2$ or $\frac{r}{s} = 1 + \frac{1}{s} = -2 + \frac{1}{n}$. The former case gives rise to a non-hyperbolic manifold while the latter case cannot occur since $\frac{1}{n} - \frac{1}{s} \in (-2, 2) \cup \{\infty\}$ and 3 is not in this set.

4.2.3. *Examples arising from [MP, Table A.6].* In this table, the possible triples come from $N(\mathcal{F}) = N(-\frac{3}{2}, \frac{r}{s})$ with $\frac{p}{q} \in E(N(\mathcal{F})) = \{-3, -\frac{5}{2}, -2, -1, 0, \infty\}$ exceptional. Table A.6 tells us that the possible S^3 slopes are $\frac{p}{q} = \infty, -3, -2, -1$. Examining each possible $\frac{p}{q}$ individually we find:

$\frac{p}{q} = \infty$: in this case, $\Delta(\frac{p}{q}, \alpha) \leq 2$ for all $\alpha \in E(N(\mathcal{F}))$.

$\frac{p}{q} = -3$: in this case we require $\frac{r}{s} = -1 + \frac{1}{n}$ and $6n + 7 = \pm 1$. It follows that $\frac{r}{s} = -2$ which implies that $N(\mathcal{F})$ is non-hyperbolic (Lemma A.10).

$\frac{p}{q} = -2$: in this case, Table A.6 requires $4r + 11s = \pm 1$ so that $\frac{r}{s} = \frac{1}{4k} - \frac{11}{4}$ where $k = \pm s$. The distance 3 pairs of slopes from $E(N(-\frac{3}{2}, \frac{r}{s}))$ are $(-3, 0)$ and $(-\frac{5}{2}, -1)$. In the first case, -3 must be the lens space surgery and $\frac{r}{s}$ is forced to be $-1 + \frac{1}{n}$ and so $-1 + \frac{1}{n} = \frac{1}{4k} - \frac{11}{4}$ from where we arrive to the contradiction $\frac{7}{4} = \frac{1}{4k} - \frac{1}{n} \leq \frac{5}{4}$.

In the second case -1 must be the lens space surgery and $\frac{r}{s}$ is forced to be $-3 + \frac{1}{n}$. So $-3 + \frac{1}{n} = \frac{r}{s} = \frac{1}{4k} - \frac{11}{4}$ or $4k - n = nk$. According to Lemma A.14 we have then $(n, k) \in \{(0, 0), (3, 3), (5, -5), (8, -2), (6, -3), (2, 1)\}$.

Case $(n, k) = (0, 0)$: then $\frac{r}{s} = \infty$ and the space is non-hyperbolic (Lemma A.10).

Case $(n, k) = (3, 3)$: then $\frac{r}{s} = -3 + \frac{1}{n} = -\frac{8}{3}$ is excluded from Table A.6.

Case $(n, k) = (5, -5)$: we obtain $\boxed{N(-\frac{3}{2}, -\frac{14}{5}), -2, -1, -\frac{5}{2})}$ which is a (S^3, T^H, T) triple with $\Delta(T^H, T) = 3$.

Case $(n, k) = (8, -2)$: then $r = \frac{1 \pm 11k}{4} \notin \mathbb{Z}$.

Case $(n, k) = (6, -3)$: then $r = \frac{1 \pm 11k}{4} \notin \mathbb{Z}$.

Case $(n, k) = (2, 1)$: then $r = \frac{1 \pm 11k}{4} = -\frac{10}{2} \notin \mathbb{Z}$.

$\frac{p}{q} = -1$: In this case Table A.6 requires that $\frac{r}{s} = -3 + \frac{1}{n}$ and $6n + 1 = \pm 1$ which has no solutions.

4.2.4. *Examples arising from [MP, Table A.7].* In this table, the possible triples come from $N(\mathcal{F}) = N(-\frac{5}{2}, \frac{r}{s})$ with $\frac{p}{q} \in E(N(\mathcal{F})) = \{-3, -2, -\frac{3}{2}, -1, 0, \infty\}$ exceptional. Table A.7 tells us that the possible lens space slopes are $\frac{p}{q} = \infty, -2, -1$. However, none of these slopes are at distance equal to 3 from any other slopes in $E(N(\mathcal{F}))$. So no examples are found in Table A.7.

4.2.5. *Examples arising from [MP, Table A.8].* In this table, the possible triples come from $N(\mathcal{F}) = N(-\frac{1}{2}, \frac{r}{s})$ with $\frac{p}{q} \in E(N(\mathcal{F})) = \{-4, -3, -2, -1, 0, \infty\}$ exceptional. Table A.8 tells us that the possible S^3 slopes are $\frac{p}{q} = \infty, -3, -2, -1$. If $\frac{p}{q} = -3, -2, -1$ corresponds to an S^3 filling, then we would have either $\frac{r}{s} = \infty$, which makes $N(\mathcal{F})$ non-hyperbolic (Lemma A.10), or $n \notin \mathbb{Z}$. So, if a triple exists in Table A.8, then the S^3 slope is necessarily ∞ and $\frac{r}{s} = -2 + \frac{1}{s}$.

The only pairs of slopes at distance 3 apart in $E(N(\mathcal{F}))$ are $(-4, -1)$ and $(-3, 0)$. Table A.8 tells us that -4 and 0 can not correspond to lens space fillings.

If -1 is a \mathbf{T}^H slope: then Table A.8 requires $\frac{r}{s} = -3 + \frac{1}{k}$, but $\frac{r}{s} = -2 + \frac{1}{s}$ from before, and so $\frac{r}{s} = -\frac{5}{2}$, which is excluded in Table A.8.

If -3 is a \mathbf{T}^H slope: then Table A.8 requires $\frac{r}{s} = -1 + \frac{1}{k}$, but $\frac{r}{s} = -2 + \frac{1}{s}$ from before, and so $\frac{r}{s} = -\frac{3}{2}$, which is excluded in Table A.8.

4.3. **Exceptional triples arising from $N(\mathcal{F})$ with $e(N(\mathcal{F})) = 5$.** The same arguments presented at the beginning of Section 3.3 reduce the study of the cases coming from [MP, Theorem 1.3 and Tables 2–4] to just Table 2 and Theorem 1.3; namely the hyperbolic $N(\frac{r}{s}, \frac{t}{u})$ with $E(N(\frac{r}{s}, \frac{t}{u})) = \{-3, -2, -1, 0, \infty\}$. Notice that such $N(\frac{r}{s}, \frac{t}{u})(\frac{p}{q})$ are toroidal only when $\frac{p}{q} = -3$ or 0 .

4.3.1. *Case $\frac{p}{q} = -3$ is the T -filling.* In this case, [MP, Table 2] gives us the conditions that $\frac{r}{s}, \frac{t}{u} \neq -1 - \frac{1}{n}$. We also require the lens space slope to be at distance 3 from the toroidal slope so we require $\frac{p}{q} = 0$ to be a lens space slope.

For $\frac{p}{q} = 0$ to be the lens space slope we need $\{\frac{r}{s}, \frac{t}{u}\} = \{n, -4 - n + \frac{1}{m}\}$, and we may assume that $\frac{r}{s} = n$ and $\frac{t}{u} = -4 - n + \frac{1}{m}$. The possible S^3 -slopes are $-1, -2$ and ∞ .

Case $\frac{p}{q} = -1$ is the S^3 -slope: From [MP, Table 2] we know that either $\frac{r}{s} = -3 + \frac{1}{n'} = n$ or $\frac{t}{u} = -3 + \frac{1}{n'} = -4 - n + \frac{1}{m}$.

Case $\frac{r}{s} = -3 + \frac{1}{n'} = n$: then $\frac{1}{n'} = 3 + n \in \mathbb{Z}$ so $3 + n = \pm 1$ and $n = -4$ or $n = -2$. For $N(\mathcal{F})$ to be hyperbolic, n is necessarily -4 (Lemma A.10), so $\frac{t}{u} = \frac{1}{m}$. However, from [MP, Table 2] we get $N(\mathcal{F})(-1) = N(-4, \frac{1}{m})(-1) = L(-3 - 7m, \star) \neq S^3$.

Case $\frac{t}{u} = -3 + \frac{1}{n'} = -4 - n + \frac{1}{m}$: then $1+n = \frac{1}{m} - \frac{1}{n'} \in [-2, 2]$, so $n \in \{-3, -2, -1, 0, 1\}$. $N(\mathcal{F})$ is hyperbolic only when $n = 1$ (Lemma A.10). In this case $n' = -1$ and $\frac{t}{u} = -4$. However, again from [MP, Table 2], $N(\mathcal{F})(-1) = N(1, -4)(-1) = L(-10, \star) \neq S^3$.

Case $\frac{p}{q} = -2$ is the S^3 -slope: From [MP, Table 2] we know that either $\frac{r}{s} = -2 + \frac{1}{n'} = n$ or $\frac{t}{u} = -2 + \frac{1}{n'} = -4 - n + \frac{1}{m}$.

Case $\frac{r}{s} = -2 + \frac{1}{n'} = n$: in this case $\frac{1}{n'} = 2 + n \in \mathbb{Z}$ so $n = -3$ or $n = -1$. In both cases $N(\mathcal{F})$ is non-hyperbolic (Lemma A.10).

Case $\frac{t}{u} = -2 + \frac{1}{n'} = -4 - n + \frac{1}{m}$: then $2+n = \frac{1}{m} - \frac{1}{n'} \in [-2, 2]$, so $n \in \{-4, -3, -2, -1, 0\}$. For $N(\mathcal{F})$ to be hyperbolic, $\frac{r}{s} = n$ is necessarily -4 , but then $n' = 1$ and $\frac{t}{u} = -2 + \frac{1}{n'} = -1$, which makes $N(\mathcal{F})$ non-hyperbolic by Lemma A.10.

Case $\frac{p}{q} = \infty$ is the S^3 -slope: From [MP, Theorem 1.3] we obtain the identities $N(\mathcal{F})(\infty) = N(n, \frac{1-m(n+4)}{m})(\infty) = L((1-m(n+4))n - m, \star)$. So $N(\mathcal{F})(\infty) = S^3$ only when $(1-m(n+4))n = m \pm 1$. Lemma A.15 tells us that we have then $n \in \{-5, -4, -3, -2, -1, 0, 1\}$. For $N(\mathcal{F}) = N(n, \frac{1-m(n+4)}{m})$ to be hyperbolic, n cannot be in $\{-3, -2, -1, 0\}$ (Lemma A.10), so we are left with the cases $(n, m) \in \{(-5, -1), (-4, -5), (-4, -3), (1, 0)\}$. The cases $(n, m) = (-5, -1), (1, 0)$ yield again a non-hyperbolic $N(\mathcal{F})$, while $(n, m) = (-4, -5), (-4, -3)$ give the (S^H, T^H, T) triples $\left(N(-4, -\frac{1}{5}), \infty, 0, -3\right)$ and $\left(N(-4, -\frac{1}{3}), \infty, 0, -3\right)$ with $\Delta(T^H, T) = 3$.

4.3.2. *Case $\frac{p}{q} = 0$ is a T -filling.* To have a (T^H, T) pair of slopes at distance 3 apart, we require $N(\frac{r}{s}, \frac{t}{u})(-3) \in T^H$. For this to be the case, [MP, Table 2] tells us that either $\frac{r}{s} = -2$, in which case $N(\mathcal{F})$ is non-hyperbolic, or that we may assume $\frac{r}{s} = -1 + \frac{1}{n}$ and $\frac{t}{u} = -1 + \frac{1}{m}$. The S^3 -slope is now one of $-1, -2$ or ∞ .

Case $\frac{p}{q} = -2$ is the S^3 -slope: From [MP, Table 2] we know that, without loss of generality, $\frac{r}{s} = -2 + \frac{1}{n'} = -1 + \frac{1}{n}$ so $n' = 2$ and $\frac{t}{u} = \frac{1-m}{m}$. Table 2 then tells us that $N(\mathcal{F})(-2) = L(4 + 7m, \star) \neq S^3$.

Case $\frac{p}{q} = -1$ is the S^3 -slope: [MP, Table 2] tells us that, with out loss of generality, $\frac{r}{s} = -3 + \frac{1}{n'} = -1 + \frac{1}{n}$ so $n' = 1$ and $\frac{r'}{s'} = -2$ which makes $N(\mathcal{F})$ non-hyperbolic by Lemma A.10.

Case $\frac{p}{q} = \infty$ is the S^3 -slope: By [MP, Theorem 1.3], we have the identities: $N(\mathcal{F})(\infty) = N(\frac{1-n}{n}, \frac{1-m}{m})(\infty) = L(1 - n - m, \star)$. So, if $N(\mathcal{F})(\infty) = S^3$ then $1 - n - m = \pm 1 \Leftrightarrow n + m \in \{0, 2\}$. The cases $n + m = 0$ and $n + m = 2$ give (S^3, T^H, T) triples with $\Delta(T^H, T) = 3$. These families are, respectively, $B_n := \left(N(-1 + \frac{1}{n}, -1 - \frac{1}{n}), \infty, -3, 0\right)$ for $n \in \mathbb{Z} \setminus \{0, \pm 1\}$ and $C_n := \left(N(-1 + \frac{1}{n}, -1 - \frac{1}{n-2}), \infty, -3, 0\right)$ for $n \in \mathbb{Z} \setminus \{0, \pm 1, 2, 3\}$.

4.4. **Conclusion.** All the examples found in Sections 4.1 – 4.3 are:

- $B_n = N\left(-1 + \frac{1}{n}, -1 - \frac{1}{n}\right)(\infty, -3, 0)$ for $n \in \mathbb{Z} \setminus \{0, \pm 1\}$;
- $C_n = N\left(-1 + \frac{1}{n}, -1 - \frac{1}{n-2}\right)(\infty, -3, 0)$ for $n \in \mathbb{Z} \setminus \{0, \pm 1, 2, 3\}$;
- $N\left(-\frac{3}{2}, -\frac{14}{5}\right)\left(-2, -1, -\frac{5}{2}\right)$;
- $N\left(-4, -\frac{1}{3}\right)(\infty, 0, -3)$;
- $N\left(-4, -\frac{1}{5}\right)(\infty, 0, -3)$.

Using [MP, Theorem 1.5], it is easy to see that:

$$\begin{aligned} \left(N\left(-\frac{3}{2}, -\frac{14}{5}\right), -2, -1, -\frac{5}{2}\right) &\cong \left(N\left(-4, -\frac{1}{5}\right), \infty, 0, -3\right) \cong \left(N\left(-\frac{3}{2}, -\frac{3}{4}\right), \infty, -3, 0\right) \cong C_{-2}. \\ \left(N\left(-4, -\frac{1}{3}\right), \infty, 0, -3\right) &\cong \left(N\left(-\frac{3}{2}, -\frac{1}{2}\right), \infty, -3, 0\right) \cong B_{-2}. \end{aligned}$$

This completes the proof that every (S^H, T^H, T) triple with $\Delta(T^H, T) = 3$ is equivalent to some B_n or C_n . We now show that these families are distinct by showing that the set of exceptional fillings of B_n is different from the set of exceptional fillings of C_k .

Using [MP, Theorem 1.3] and [MP, Table 2] we can write down the exceptional slopes and fillings of $N\left(-1 + \frac{1}{n}, -1 - \frac{1}{n}\right)$. The result is shown in Table 2. We note that both B_n and C_k have unique lens space fillings. We have $B_n(-3) = L(4n^2+3, 2n^2+n+2)$ and $C_k(-3) = L(4k^2+8k-1, 2k^2-3k)$. If $B_n(-3) = C_k(-3)$ then the order of their fundamental groups are equal. It is well known that $\pi_1(L(p, q))$ is the cyclic group of order p (see for example [Rol, Exercise 9.B.5]). So, if $B_n(-3) = C_k(-3)$ then $3 + 4n^2 = 4k^2 + 8k - 1 \Leftrightarrow 4(n - k)(k + n) = 2(4k + 1)$ which implies that $2 \mid 4k + 1$, a contradiction. Hence, $\{B_n\} \cap \{C_k\} = \emptyset$ and the proof of Theorem 4.1 is complete.

Remark 4.2. $B_{-2} = B_2$ is the exterior of the $(-2, 3, 7)$ pretzel knot. In this case, $e(B_{\pm 2}) = 7$ and the exceptional slopes and fillings can be found in [MP, Table A.2]. In the second family, $E(C_{-2}) = \{-3, -\frac{5}{2}, -2, -1, 0, \infty\}$, $C_{-2}(\alpha)$ is found in Table 2 for $\alpha \in E(C_{-2}) \setminus \{-\frac{5}{2}\}$, and $C_{-2}(-\frac{5}{2}) = (D, (2, 1), (3, 1)) \cup \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} (D, (2, 1), (5, 4))$.

5. (S^H, T^H, Z) TRIPLES

In this section we are going to enumerate all (S^H, T^H, Z) triples obtained by surgery on the five chain link and realizing the maximal distance. It turns out that all such triples are obtain by surgery on the 3CL. The main result of this section is the following.

Theorem 5.1. *If $(M_5(\frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{x}{y}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ then $\Delta(\beta, \gamma) \leq 2$. If $(M_5(\frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{x}{y}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ and $\Delta(\beta, \gamma) = 2$ then either*

$$\left(M_5\left(\frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{x}{y}\right), \alpha, \beta, \gamma\right) \cong \left(N\left(-1 + \frac{1}{n}, -1 - \frac{1}{n}\right), \infty, -3, -1\right), \text{ or}$$

$$\left(M_5\left(\frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{x}{y}\right), \alpha, \beta, \gamma\right) \cong \left(N\left(-1 + \frac{1}{n}, -1 + \frac{1}{2-n}\right), \infty, -3, -1\right).$$

We will prove this theorem in three steps. In Section 5.1, we will show that if $(M_5(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ and $\Delta(\beta, \gamma) \geq 2$ then \mathcal{F} factors through M_4 . Then, in Section 5.2, we will show that if $(M_4(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ and $\Delta(\beta, \gamma) \geq 2$ then \mathcal{F} factors through M_3 . Finally, in Section 5.3 we show that if $(N(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ then $\Delta(\beta, \gamma) \leq 2$. All triples with $\Delta(\beta, \gamma) = 2$ are enumerated to complete the proof of Theorem 5.1.

Remark Note that the knot exteriors in Theorem 5.1 are the same as the knot exteriors in Theorem 4.1. Therefore, we know that these examples are distinct and that the exceptional slopes and fillings of $N\left(-1 + \frac{1}{n}, -1 - \frac{1}{n}\right)$ and $N\left(-1 + \frac{1}{n}, -1 + \frac{1}{2-n}\right)$ are found in Table 2.

5.1. (S^H, T^H, Z) triples from M_5 . If $(M_5(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ and \mathcal{F} does not factor through M_4 then, from [Rou2, Theorem 4], we have $E(M_5(\mathcal{F})) = \{0, 1, \infty\}$, or \mathcal{F} is equivalent to one of the surgery instructions in [Rou2, Tables 14 – 20]. Clearly, if $E(M_5(\mathcal{F})) = \{0, 1, \infty\}$ then no two exceptional slopes are at distance greater than 2 apart. Hence, any such (S^H, T^H, Z) triple realising $\Delta(T^H, Z) \geq 2$ is to be found in [Rou2, Tables 14 – 20]. In these tables, each filling instruction \mathcal{F} is shown together with $E(M_5(\mathcal{F}))$ and the class of each $M_5(\mathcal{F})(\alpha)$ for $\alpha \in E(M_5(\mathcal{F}))$ (which may depend on parameters). It is immediately clear that if $(M_5(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ then it is found in Table 17 with $\mathcal{F} = (-2, \frac{p}{q}, 3, \frac{u}{v})$ or it is found in Table 18 with $\mathcal{F} = (-2, \frac{p}{q}, \frac{r}{s}, -2)$. In both cases, it is also clear that $\Delta(\beta, \gamma) \leq 2$, and that the (Z, T^H) slopes realising $\Delta(\beta, \gamma) = 2$ must be $(-1, 1)$ or $(1, -1)$ and that the S^H slope must be 0 or ∞ . We proceed with a case by case analysis.

Case $\mathcal{F} = (-2, \frac{p}{q}, 3, \frac{u}{v})$: By (23), we may assume that 0 corresponds to the S^H slope. [Rou2, Table 17] tells us that if 0 is an S^H slope then either $\frac{p}{q} = 1 + \frac{1}{n}$ and $|(3+2n)u - (7+6n)v| = 1$, or $\frac{u}{v} = 3 + \frac{1}{k}$ and $|(3+2k)p - (1+2k)q| = 1$. We also obtain the condition from [Rou2, Table 17] that $|p| = 1$ or $|u + v| = 1$ if -1 corresponds to a type Z or T^H surgery.

We will now show that these conditions are incompatible.

Case $\frac{p}{q} = 1 + \frac{1}{n}$ and $|(3+2n)u - (7+6n)v| = 1$: We have $\frac{p}{q} = \frac{n+1}{n}$. [Rou2, Table 17] tells us that if -1 corresponds to a type Z or T^H slope then $|p| = 1$ or $|u + v| = 1$. If $|p| = 1$ and $\frac{p}{q} = \frac{n+1}{n}$ then $|p| = |n+1| = 1$. This occurs only when $n = -2$ or $n = 0$ and both cases are excluded from Table 17 in [Rou2].

If $|u + v| = 1$ then $\frac{u}{v} = \frac{1-k}{k}$. Subbing $\frac{u}{v} = \frac{1-k}{k}$ into $|(3+2n)u - (7+6n)v| = 1$ and solving for k in terms of n gives

$$k = \frac{4+2n}{10+8n} \quad \text{or} \quad k = \frac{1+n}{5+4n}.$$

In the first case, k is an integer if and only if $n = -1, -2 \Rightarrow \frac{p}{q} = 0, \frac{1}{2}$ which, by Lemma A.4, make \mathcal{F} exceptional or factor through M_4 respectively. In the second case, k is an integer if and only if $n = -1$ which makes $M_5(\mathcal{F})$ non-hyperbolic.

Case $\frac{u}{v} = 3 + \frac{1}{k}$ and $|(3 + 2k)p - (1 + 2k)q| = 1$: We have $\frac{u}{v} = \frac{1+3k}{k}$. [Rou2, Table 17] tells us that if -1 corresponds to a type Z or T^H slope then $|p| = 1$ or $|u + v| = 1$. In the case $|u + v| = 1$ we get $|u + v| = |1 + 4k| = 1$ only when $k = 0$ which makes $M_5(\mathcal{F})$ non-hyperbolic by Lemma A.4. If $|p| = 1$ then $\frac{p}{q} = \frac{1}{n}$ and $|(3 + 2k)p - (1 + 2k)q| = |(3 + 2k) - (1 + 2k)n| = 1$. Solving for n in terms of k gives

$$n = 1 + \frac{3}{1 + 2k} \quad \text{or} \quad n = 1 + \frac{1}{1 + 2k}.$$

In the former case n is an integer if and only if $k \in \{-2, -1, 0, 1\}$. If $k = -1, 0$ then, by Lemma A.4, \mathcal{F} factors through M_4 or is exceptional respectively. If $k = -2, 1$ then $n = -1, 1$ respectively, and, by Lemma A.4, \mathcal{F} factors through M_4 or is exceptional respectively.

In the latter case n is an integer if and only if $k = -1$ or 0 . If $k = 0$ then $\frac{u}{v} = \infty$ making \mathcal{F} exceptional by Lemma A.4. If $k = -1$ then $\frac{u}{v} = 2$ and \mathcal{F} factors through M_4 by Lemma A.4.

Case $\mathcal{F} = (-2, \frac{p}{q}, \frac{r}{s}, -2)$: From [Rou2, Table 18] we see that for -1 to be a type Z or T^H slope we need $|q| = 1$ or $|s| = 1$. By (17), we may assume that $|q| = 1$. The same table tells us that if 1 corresponds to a type Z or T^H slope then $|p| = 1$ or $|r| = 1$. Since we are assuming $|q| = 1$, the case $|p| = 1$ is excluded from Table 18, so we require $|r| = 1$. Examining the table we see that the only possible S^H slope is 0 , and that 0 as an S^H necessitates either $\frac{p}{q} = 1 + \frac{1}{n}$ or $\frac{r}{s} = 1 + \frac{1}{n}$. However, when $|q| = 1$ the only solutions are $\frac{p}{q} = 0, 2$, and when $|r| = 1$ the only solution is $\frac{r}{s} = \frac{1}{2}$. These three solutions are excluded from Table 18.

5.2. (S^H, T^H, Z) triples from M_4 . If $(M_4(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ with \mathcal{F} not factoring through M_3 , then $E(M_4(\mathcal{F})) = \{0, 1, 2, \infty\}$ or \mathcal{F} is equivalent to a filling instruction listed in [Rou2, Tables 21 – 22]. In these tables no $M_4(\mathcal{F})$ has an S^3 filling. Hence $\Delta(\beta, \gamma) \leq 2$ and if $\Delta(\beta, \gamma) = 2$ then $\{\beta, \gamma\} = \{0, 2\}$ and $\alpha \in \{\infty, 1\}$.

We will now show that we may assume that ∞ corresponds to the S^3 slope. We have

$$\begin{aligned} M_4\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}\right) &\stackrel{(29)}{=} M_5\left(\frac{a}{b}, \frac{c-d}{d}, -1, \frac{e-f}{f}, \frac{g}{h}\right) \stackrel{(20)}{=} M_5\left(\frac{g}{g-h}, \frac{b-a}{b}, -1, \frac{d}{c-d}, \frac{2f-e}{f}\right) \\ &\stackrel{(29)}{=} M_4\left(\frac{g}{g-h}, \frac{2b-a}{b}, \frac{c}{c-d}, \frac{2f-e}{f}\right). \stackrel{\text{Lemma A.7}}{=} M_4\left(\frac{2b-a}{b}, \frac{c}{c-d}, \frac{2f-e}{f}, \frac{g}{g-h}\right) \end{aligned}$$

Hence

$$\left(M_4\left(\frac{p}{q}, \frac{r}{s}, \frac{u}{v}\right), 1, 0, 2\right) \cong \left(M_4\left(\frac{2q-p}{q}, \frac{r}{r-s}, \frac{2v-uu}{v}\right), \infty, 0, 2\right)$$

and we are allowed to assume that ∞ corresponds to the S^3 slope. We will now show that we may also assume without loss of generality that the (S^H, T^H, Z) triple of slopes is $(\infty, 0, 2)$. We have

$$\begin{aligned} M_4\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}\right) & \stackrel{(29)}{=} M_5\left(\frac{a}{b}, \frac{c-d}{d}, -1, \frac{e-f}{f}, \frac{g}{h}\right) \stackrel{(18)}{=} M_5\left(-1, \frac{h-g}{h}, \frac{a}{a-b}, \frac{2d-c}{d}, \frac{f}{e-f}\right) \\ & \stackrel{(16) \circ (16)}{=} M_5\left(\frac{2d-c}{d}, \frac{f}{e-f}, -1, \frac{h-g}{h}, \frac{a}{a-b}\right) \stackrel{\text{Lemma A.7 \& (29)}}{=} M_4\left(\frac{a}{a-b}, \frac{2d-c}{d}, \frac{e}{e-f}, \frac{2h-g}{h}\right), \end{aligned}$$

which gives us

$$\left(M_4\left(\frac{p}{q}, \frac{r}{s}, \frac{u}{v}\right), \infty, 0, 2\right) \cong \left(M_4\left(\frac{p}{p-q}, \frac{2s-r}{s}, \frac{u}{u-v}\right), \infty, 2, 0\right).$$

The result is that if $(M_4(\mathcal{F}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ with \mathcal{F} not factoring through M_3 and $\Delta(\beta, \gamma) \geq 2$, then we may assume that $\alpha = \infty$, $\beta = 0$, and $\gamma = 2$.

We set the filling instruction on M_4 to be $\mathcal{F} = (\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$. By (30), we know that $M_4(\mathcal{F})(\infty) \in Z$ unless one of $a, b, c, d, e, f \in \{0, \pm 1\}$. If one of $a, b, c, d, e, f = 0$, then one of $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \{0, \infty\}$ and $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ is non-hyperbolic (Lemma A.6). We conclude that if

$$\left(M_4\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right), \infty, 0, 2\right) \in (S^H, T^H, Z), \text{ then one of } a, d, e = \pm 1.$$

We see from Identity (31) and Lemma A.2 that the slope 0 is toroidal unless $0 \in \{f, b, c-2d\}$ or $\pm 1 \in \{f, b, c-2d\}$. If $f, b = 0$ or $c-2d = 0$ then one of $\frac{a}{b}, \frac{e}{f} = \infty$ or $\frac{c}{d} = 2$ and $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ is non-hyperbolic (Lemma A.6). So if 0 corresponds to a T^H slope then necessarily $\pm 1 \in \{f, b, c-2d\}$ meaning $\frac{e}{f}, \frac{a}{b} = n$, or $\frac{c}{d} = 2 + \frac{1}{k}$. Using Lemma A.7 we may assume that one of the following conditions hold:

$$(5) \quad \frac{a}{b} = n,$$

$$(6) \quad \frac{c}{d} = 2 + \frac{1}{k}.$$

If $\frac{a}{b} = n$ then (31) implies

$$M_4\left(n, \frac{c}{d}, \frac{e}{f}\right)(0) \stackrel{(31)}{=} (D, (f, -e), (1, 2-n)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (2, 1), (c-2d, d)) \stackrel{(13)}{=} (S^2, (2, 1), (c-2d, d), (e+fn-2f, f)).$$

Since we have fixed 0 to be the T^H slope, we require $0, \pm 1 \in \{c-2d, e+f(n-2)\}$ (otherwise $M_4(\mathcal{F})(0) \in Z$). We only need to consider the cases $|e+f(n-2)| = 1$: if $|c-2d| = 1$ then Identity (6) is satisfied (so we are in a subset of this case), $c-2d = 0$ implies $\frac{c}{d} = 2$ (which is excluded by Lemma A.6), while $e+f(n-2) = 0$ implies

$$M_4\left(n, \frac{c}{d}, \frac{e}{f}\right)(0) \stackrel{(14)}{=} (S^2, (2, 1), (c-2d, d), (0, f)) = L(2, 1) \# L(c-2d, d)$$

which is a lens space if and only if $c - 2d = \pm 1$ (which means again that Identity (6) is satisfied). So for $(M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}), \infty, 0, 2) \in (S^H, T^H, Z)$ we necessarily require one of

$$\frac{a}{b} = n \quad \text{and} \quad \frac{e}{f} = 2 - n + \frac{1}{k}$$

$$\frac{c}{d} = 2 + \frac{1}{k}.$$

We see from Identity (33) that the slope 2 is toroidal unless one of $0 \in \{a - b, e - f, c\}$ or $\pm 1 \in \{a - b, e - f, c\}$. If $a - b = 0$, $e - f = 0$, $c = 0$ then $\frac{a}{b} = 1$, $\frac{e}{f} = 1$, $\frac{c}{d} = 0$ respectively and $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ is not hyperbolic (Lemma A.6). So if 2 corresponds to a type Z slope then $\pm 1 \in \{a - b, e - f, c\}$ meaning $\frac{e}{f}, \frac{a}{b} = 1 + \frac{1}{p}$, or $\frac{c}{d} = \frac{1}{m}$.

Collecting the necessary conditions for $(\infty, 0, 2)$ to be a (S^H, T^H, Z) triple found above, we see that we require at least one condition from each column of Table 3.

$\infty \rightsquigarrow S^H$	$0 \rightsquigarrow T^H$	$2 \rightsquigarrow Z$
$a = \pm 1$	$\frac{c}{d} = 2 + \frac{1}{k}$	$\frac{a}{b} = 1 + \frac{1}{p}$
$d = \pm 1$	$\frac{a}{b} = n$ and $\frac{e}{f} = 2 - n + \frac{1}{k}$	$\frac{e}{f} = 1 + \frac{1}{q}$
$e = \pm 1$		$\frac{c}{d} = \frac{1}{m}$

Table 3: Necessary conditions for $(\infty, 0, 2)$ to be a (S^3, T^H, Z) triple.

We will now discount the possibility $\frac{a}{b} = n$ and $\frac{e}{f} = 2 - n + \frac{1}{k}$. The condition $\frac{a}{b} = 1 + \frac{1}{p}$ from column 3 of Table 3 is discounted because taken together with $\frac{a}{b} = n$ we get $\frac{a}{b} = 0, 2$ which implies M is non-hyperbolic by Lemma A.6. While the condition that $\frac{e}{f} = 1 + \frac{1}{q}$ from column 3 in Table 3 is discounted because $\frac{e}{f} = 2 - n + \frac{1}{k} = 1 + \frac{1}{q}$ implies $n \in \{0, 1, 2\}$ meaning $\frac{a}{b} \in \{0, 1, 2\}$ and so M is non-hyperbolic by Lemma A.6. So we necessarily require

$$(7) \quad \frac{c}{d} = \frac{1}{m}.$$

The condition $a = \pm 1$ from column 1 of Table 3 is discounted because $\frac{a}{b} = \frac{\pm 1}{b} = n \Rightarrow n = \pm 1$ meaning $M_4(\mathcal{F})$ is non-hyperbolic or $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ factors through M_3 (Lemma A.6). The condition $e = \pm 1$ from column 1 of Table 3 is discounted because $\frac{e}{f} = 2 - n + \frac{1}{k} \Rightarrow n \in \{1, 2, 3\}$ and these cases are again excluded by Lemma A.6. So we necessarily require $d = \pm 1$. However, $d = \pm 1$ together with (7) implies that $\frac{c}{d} = \pm 1$ meaning M is non-hyperbolic or that $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ factors through N by Lemma A.6.

We will now discount the possibility that $\frac{c}{d} = 2 + \frac{1}{k}$. If 0 is a T^H slope and $\frac{c}{d} = 2 + \frac{1}{k}$ then the condition that $d = \pm 1$ in the first column of Table 3 implies $\frac{c}{d} \in \{1, 3\}$ which means $M_4(\frac{a}{b}, \frac{2k+1}{k}, \frac{e}{f})$

is non-hyperbolic (Lemma A.6). Namely

$$(8) \quad \frac{c}{d} = 2 + \frac{1}{k} \quad \text{implies} \quad d \neq \pm 1.$$

When $\frac{c}{d} = 2 + \frac{1}{k}$, Identity (31) implies

$$\begin{aligned} M_4\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right)(0) &\stackrel{(31)}{=} (D, (f, -e), (b, 2b-a)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (2, 1), (1, k)) \\ &\stackrel{(13)}{=} (S^2, (2k+1, -1), (f, -e), (b, 2b-a)). \end{aligned}$$

So for $M = M_4\left(\frac{a}{b}, \frac{2k+1}{k}, \frac{e}{f}\right)(0)$ to be a lens space we require $k = 0$ or $k = -1$ (meaning M is non-hyperbolic by Lemma A.6) or $b = \pm 1$ or $f = \pm 1$ ($b, f = 0$ makes $M_4(\mathcal{F})$ non-hyperbolic). Using Lemma A.7 we may assume that $b = \pm 1$. If $b = \pm 1$ then the only possible condition from column 1 of Table 3 is $e = \pm 1$ ($d = \pm 1$ is excluded by (8) and $a = \pm 1$ implies $\frac{a}{b} = \pm 1$ meaning M factors through N or is non-hyperbolic by Lemma A.6). From column 3 of Table 3 we see that either $\frac{c}{d} = \frac{1}{m}$ or $\frac{e}{f} = 1 + \frac{1}{p}$ or $\frac{a}{b} = 1 + \frac{1}{p}$. If $\frac{c}{d} = \frac{1}{m} = 2 + \frac{1}{k}$ then $\frac{c}{d} = 1$ and $M_4(\mathcal{F})$ is non-hyperbolic by Lemma A.6. If $\frac{e}{f} = 1 + \frac{1}{p} = \frac{1}{k}$ then $\frac{e}{f} = \frac{1}{2}$ and \mathcal{F} factors through M_3 by Lemma A.6. This implies that the only possible condition from column 3 of Table 3 is $\frac{a}{b} = 1 + \frac{1}{p}$. Summing up, we obtain the following conditions

$$\frac{a}{b} = 1 + \frac{1}{p}, \quad \frac{c}{d} = 2 + \frac{1}{k}, \quad \frac{e}{f} = \frac{1}{m}.$$

If $M_4\left(\frac{p+1}{p}, \frac{2k+1}{k}, \frac{1}{m}\right)$ is non-hyperbolic and does not factor through M_3 then $p \neq 0, \pm 1, \pm 2$ and $m \neq 0, \pm 1, 2$ by Lemma A.6. We will now use the assumption that $M_4(\mathcal{F})(\infty) = S^3$ to show that if $(M_4\left(\frac{p+1}{p}, \frac{2k+1}{k}, \frac{1}{m}\right), \infty, 0, 2) \in (S^H, T^H, Z)$ then $p = 0, \pm 1, \pm 2$ or $m = 0, \pm 1, \pm 2$. From (30) we see that

$$M_4\left(\frac{p+1}{p}, \frac{2k+1}{k}, \frac{1}{m}\right)(\infty) \stackrel{(30)}{=} (S^2, (p+1, p), (k, -2k-1), (1, m)) \stackrel{(15)}{=} L_{(-p+1)(2k+1)+kp+kmp+km, \star}.$$

So, ∞ is an S^3 slope on $M_4\left(\frac{p+1}{p}, \frac{2k+1}{k}, \frac{1}{m}\right)$ if and only if $((m-1)p + m - 2)k = 1 \pm 1 + p$. If $(m-1)p + m - 2 = 0$ then $1 \pm 1 + p = 0$ which makes $p = 0$ or $p = -2$. If $p = 0$ then $M_4\left(\frac{p+1}{p}, \frac{2k+1}{k}, \frac{1}{m}\right)$ is non-hyperbolic by Lemma A.6, and if $p = -2$ then $\left(\frac{p+1}{p}, \frac{2k+1}{k}, \frac{1}{m}\right)$ factors through M_3 by Lemma A.6. This means that ∞ is an S^3 slope on $M_4\left(\frac{p+1}{p}, \frac{2k+1}{k}, \frac{1}{m}\right)$ if and only if one of the following holds

$$(9) \quad k = \frac{p}{(m-1)p + m - 2},$$

$$(10) \quad k = \frac{2 + p}{(m-1)p + m - 2}.$$

In (9) we see that $(m-1)p + m - 2$ divides p . We already know that $p \neq 0, \pm 1, \pm 2$, so we are only left to study the cases $p > 2$ and $p < 2$. If $p > 2$ then

$$\begin{aligned} -p \leq (m-1)p + m - 2 \leq p &\Rightarrow 2 \leq m(p+1) \text{ and } (m-2)(p+1) \leq 0 \\ &\Rightarrow m \geq 1 \text{ and } m \leq 2 \Rightarrow m = 1, 2. \end{aligned}$$

As we argued before $m = 1, 2$ yields a $M_4(\frac{p+1}{p}, \frac{2k+1}{k}, \frac{1}{m})$ which factors through M_3 .

If $p < -2$ then

$$\begin{aligned} p \leq (m-1)p + m - 2 \leq -p &\Rightarrow 0 \leq (m-2)(p+1) \text{ and } m(p+1) \leq 2 \\ &\Rightarrow m \leq 2 \text{ and } m \geq -1 \Rightarrow m = -1, 0, 1, 2. \end{aligned}$$

Once more, Lemma A.6 rules out these possibilities for m .

In (10) we see that $(m-1)p + m - 2$ divides $p+2$. As before we study two cases: if $p > 2$ then

$$\begin{aligned} -2 - p \leq (m-1)p + m - 2 \leq 2 + p &\Rightarrow 0 \leq m(p+1) \text{ and } (m-2)(p+1) \leq 2 \\ &\Rightarrow m \geq 0 \text{ and } m \leq 2 \Rightarrow m = 1, 2. \end{aligned}$$

On the other hand, if $p < -2$ then

$$\begin{aligned} 2 + p \leq (m-1)p + m - 2 \leq -2 - p &\Rightarrow 2 \leq (m-2)(p+1) \text{ and } m(p+1) \leq 0 \\ &\Rightarrow m < 2 \text{ and } m \geq 0 \Rightarrow m = 0, 1. \end{aligned}$$

Once again, all these possible values for m are ruled out by Lemma A.6.

We have now completed the proof that if $(M_5(\frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{x}{y}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ and $\Delta(\beta, \gamma) \geq 2$ then $(\frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{x}{y})$ factors through N .

5.3. (S^H, T^H, Z) triples from N .

Proposition 5.2. *If $(N(\frac{a}{b}, \frac{c}{d}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ then $\Delta(\beta, \gamma) \leq 2$.*

Proof. [MP, Corollary A.6] tells us that if $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ and $\Delta(\beta, \gamma) > 2$ then either $e(N(\frac{r}{s}, \frac{t}{u})) = 5$ or $N(\frac{r}{s}, \frac{t}{u})$ is found in [MP, Tables A.2–A.9]. If $e(N(\frac{r}{s}, \frac{t}{u})) = 5$ then, it is a consequence of [MP, Theorem 1.1] that $E(N(\frac{r}{s}, \frac{t}{u})) = \{\infty, -3, -2, -1, 0\}$. Since we are interested in slopes at distance greater than two then we conclude $\{\beta, \gamma\} = \{0, -3\}$.

We first suppose that $\{\beta, \gamma\} = \{0, -3\}$. In this case, we can see in [MP, Table 2] that if $N(\frac{r}{s}, \frac{t}{u})$ is hyperbolic with $N(\frac{r}{s}, \frac{t}{u})(-3) \in T^H$ then $\frac{r}{s} = -1 + \frac{1}{n}$ and $\frac{t}{u} = -1 + \frac{1}{m}$. Moreover, if $N(\frac{r}{s}, \frac{t}{u})(0) \in Z$, [MP, Table 2] tells us that one of $\frac{r}{s} = -1 + \frac{1}{n}$ or $\frac{t}{u} = -1 + \frac{1}{m}$ is an integer, which makes one of $\frac{r}{s}, \frac{t}{u} \in \{-3, 0\}$, implying $N(\frac{r}{s}, \frac{t}{u})$ non-hyperbolic by Lemma A.10.

If $\{\beta, \gamma\} = \{0, -3\}$ with $N(\frac{r}{s}, \frac{t}{u})$ hyperbolic and $N(\frac{r}{s}, \frac{t}{u})(-3) \in Z$, the same table tells us that $-1 + \frac{1}{n} \in \{\frac{r}{s}, \frac{t}{u}\}^1$. Lemma A.8 allows us to assume that $\frac{r}{s} = -1 + \frac{1}{n}$. If $N(\frac{r}{s}, \frac{t}{u})(0) \in T^H$, then [MP, Table 2] tells us that $\{\frac{r}{s}, \frac{t}{u}\} = \{k, -4 - k + \frac{1}{m}\}$. The case when $\frac{r}{s}$ is an integer was discounted in the previous paragraph, so we assume that $\frac{r}{s} = -1 + \frac{1}{n} = -4 - k + \frac{1}{m}$ and $\frac{t}{u} = k$. As $n = \pm 1$ makes $N(\frac{r}{s}, \frac{t}{u})$ non-hyperbolic (Lemma A.10), we have $\frac{r}{s} = -1 + \frac{1}{n} = -4 - k + \frac{1}{m} \Rightarrow -1 \leq 3 + k = \frac{1}{m} - \frac{1}{n} \leq 1 \Rightarrow -4 \leq k \leq -2$. If $-3 \leq \frac{t}{u} = k \leq -2$ then $N(\frac{r}{s}, \frac{t}{u})$ is non-hyperbolic (Lemma A.10), and if $k = -4$ then $\frac{t}{u} = -4$ and $\frac{r}{s} = -1 + \frac{1}{n} = -4 - (-4) + \frac{1}{n} \Rightarrow \frac{r}{s} = -\frac{1}{2}$. However, $N(\frac{r}{s}, \frac{t}{u}) = N(-4, -\frac{1}{2})$ is non-hyperbolic (see [MP, Table 1]).

We now consider the case when $N(\frac{r}{s}, \frac{t}{u})$ is found in [MP, Tables A.2 – A.9]. It is immediately clear that the only $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ in Tables A.2 – A.4 and Table A.9 is the triple obtained from the $(-2, 3, 7)$ pretzel knot, and in this case $\Delta(\beta, \gamma) = 2$.

If $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ with $\Delta(\beta, \gamma) > 2$ is found in [MP, Table A.5] then $E(N(\frac{r}{s}, \frac{t}{u})) = \{-3, -2, -1, 0, 1, \infty\}$ and one of $\beta, \gamma = 1$. However, this table tells us $N(\frac{r}{s}, \frac{t}{u})(1) \notin T^H \cup Z$. So, [MP, Table A.5] produces no $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ with $\Delta(\beta, \gamma) > 2$.

If $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ with $\Delta(\beta, \gamma) > 2$ is found in [MP, Table A.6] then $E(N(\frac{r}{s}, \frac{t}{u})) = \{-3, -\frac{5}{2}, -2, -1, 0, \infty\}$ and one of $\beta, \gamma = -\frac{5}{2}$. This table tells us $N(\frac{r}{s}, \frac{t}{u})(-\frac{5}{2}) \notin T^H$ and that $N(\frac{r}{s}, \frac{t}{u})(-\frac{5}{2}) \in Z$ only when $\frac{r}{s} = -2 + \frac{1}{n}$. If $\Delta(\beta, -\frac{5}{2}) > 2$ then $\beta \in \{-1, 0\}$. Note that β is not a T^H slope. If $\beta = -1$ is a T^H slope then [MP, Table A.6] tells us that $\frac{r}{s} = -2 + \frac{1}{n} = -3 + \frac{1}{k} \Rightarrow \frac{r}{s} = -\frac{5}{2}$ which is actually excluded from this table. We conclude that [MP, Table A.6] produces no $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ with $\Delta(\beta, \gamma) > 2$.

If $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ with $\Delta(\beta, \gamma) > 2$ is found in [MP, Table A.7] then $E(N(\frac{r}{s}, \frac{t}{u})) = \{-3, -2, -\frac{3}{2}, -1, 0, \infty\}$ and one of $\beta, \gamma = -\frac{3}{2}$. This table tells us $N(\frac{r}{s}, \frac{t}{u})(-\frac{3}{2}) \notin T^H$ and that $N(\frac{r}{s}, \frac{t}{u})(-\frac{3}{2}) \in Z$ only when $\frac{r}{s} = -2 + \frac{1}{n}$. If $\Delta(\beta, -\frac{3}{2}) > 2$ then $\beta \in \{-3, 0\}$. However we can directly see in the table that $N(\frac{r}{s}, \frac{t}{u})(\beta) \notin T^H$ for $\beta \in \{-3, 0\}$. Summing up, [MP, Table A.7] produces no $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ with $\Delta(\beta, \gamma) > 2$.

Finally, if $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ with $\Delta(\beta, \gamma) > 2$ is found in [MP, Table A.8] then $E(N(\frac{r}{s}, \frac{t}{u})) = \{-4, -3, -2, -1, 0, \infty\}$ and one of $\beta, \gamma = -4$. This table tells us $N(\frac{r}{s}, \frac{t}{u})(-4) \notin T^H$ and that $N(\frac{r}{s}, \frac{t}{u})(-4) \in Z$ only when $\frac{r}{s} \in \mathbb{Z}$. If $\Delta(\beta, -4) > 2$ then $\beta \in \{-1, 0\}$. We can see in this table that $N(\frac{r}{s}, \frac{t}{u})(0) \notin T^H$, and that $N(\frac{r}{s}, \frac{t}{u})(-1) \in T^H$ only if the integer $\frac{r}{s} = -3 + \frac{1}{n} \Rightarrow \frac{r}{s} = -2, -4$. These cases are excluded from the table. So, [MP, Table A.8] produces no $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ with $\Delta(\beta, \gamma) > 2$. \square

¹A word of caution: in [MP, Table 2], as one can read in the arXiv preprint of this article, the relation between the entries on the $\frac{r}{s}$ column and the $\frac{t}{u}$ column is more intricate than what a reader might appreciate in the published version.

Proposition 5.3. *If $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ and $\Delta(\beta, \gamma) = 2$ then either*

$$\begin{aligned} (N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) &\cong (N(-1 + \frac{1}{n}, -1 - \frac{1}{n}), \infty, -3, -1), \text{ or} \\ (N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) &\cong (N(-1 + \frac{1}{n}, -1 + \frac{1}{2-n}), \infty, -3, -1). \end{aligned}$$

Proof. We know from [MP, Corollary A.6] and [MP, Theorem 1.1] that either $E(N(\frac{r}{s}, \frac{t}{u})) = \{0, -1, -2, -3, \infty\}$ or the set $\{0, -1, -2, -3, \infty\} \subset E(N(\frac{r}{s}, \frac{t}{u}))$ and $N(\frac{r}{s}, \frac{t}{u}), E(N(\frac{r}{s}, \frac{t}{u}))$ are found in [MP, Tables A.2–A.9].

For $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ we separate our enumeration into two cases:

Case 1: $\{\alpha, \beta, \gamma\} \not\subset \{0, -1, -2, -3, \infty\}$, in which no examples will be found and

Case 2: $\{\alpha, \beta, \gamma\} \subset \{0, -1, -2, -3, \infty\}$.

Case 1: In this case, one of $\alpha, \beta, \gamma \notin \{0, -1, -2, -3, \infty\}$ and $N(\frac{r}{s}, \frac{t}{u}), E(N(\frac{r}{s}, \frac{t}{u}))$ are found in [MP, Tables A.2–A.9]. It is immediately clear that the only $(N(\frac{a}{b}, \frac{c}{d}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ in [MP, Tables A.2–A.4 and Table A.9] is the $(-2, 3, 7)$ pretzel knot exterior $(N(-\frac{3}{2}, -\frac{1}{2}), \infty, 0, -2)$.

If an (S^H, T^H, Z) triple $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ with one of $\alpha, \beta, \gamma \notin \{0, -1, -2, -3, \infty\}$ comes from [MP, Table A.5] then $\frac{t}{u} = 1$ and $E(N(\frac{r}{s}, \frac{t}{u})) = \{-3, -2, -1, -0, 1, \infty\}$. So, one of $\alpha, \beta, \gamma = 1$. However, in this table $N(1, \frac{r}{s})(1) \notin T^H \cup S^H \cup Z$.

If an (S^H, T^H, Z) triple $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ with one of $\alpha, \beta, \gamma \notin \{0, -1, -2, -3, \infty\}$ comes from [MP, Table A.6] then $\frac{t}{u} = -\frac{3}{2}$ and $E(N(\frac{r}{s}, \frac{t}{u})) = \{-3, -\frac{5}{2}, -2, -1, -0, \infty\}$. So, one of $\alpha, \beta, \gamma = -\frac{5}{2}$. In this table $N(\frac{r}{s}, -\frac{3}{2})(-\frac{5}{2})$ is only in $S^H \cup T^H \cup Z$ if $\frac{r}{s} = -2 + \frac{1}{n}$, in which case $N(-2 + \frac{1}{n}, -\frac{3}{2})(-\frac{5}{2}) \in Z$. We have $\Delta(\beta, -\frac{5}{2}) = 2$ for $\beta \in E(N(\frac{r}{s}, \frac{t}{u}))$ only when $\beta = \infty$.

We can see from Table A.6 that the only possible S^3 slopes on hyperbolic $N(\frac{r}{s}, -\frac{3}{2})$ are ∞ (which is a T^H slope in our case) and -2 . In the table we can read that $N(\frac{r}{s}, -\frac{3}{2})(-2) = S^3$ if and only if $|4r+11s| = 1$. In our case, $\frac{r}{s} = \frac{1-2n}{n}$, so $|4r+11s| = 1$ if and only if $n = -1 \Rightarrow \frac{r}{s} = -3$ which is excluded by Lemma A.10. So if [MP, Table A.6] contains an $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ with $\Delta(\beta, \gamma) = 2$, then $\alpha, \beta, \gamma \in \{0, -1, -2, -3, \infty\}$.

If an (S^H, T^H, Z) triple $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ with one of $\alpha, \beta, \gamma \notin \{0, -1, -2, -3, \infty\}$ comes from [MP, Table A.7] then $\frac{t}{u} = -\frac{5}{2}$ and $E(N(\frac{r}{s}, \frac{t}{u})) = \{-3, -2, -\frac{3}{2}, -1, 0, \infty\}$. This table tells us that $N(\frac{r}{s}, -\frac{5}{2})(-\frac{3}{2})$ is in $S^H \cup T^H \cup Z$ only if $\frac{r}{s} = -2 + \frac{1}{n}$, in which case $N(-2 + \frac{1}{n}, -\frac{5}{2})(-\frac{3}{2}) \in Z$. The only possible S^H slopes found in this table are $\infty, -2$ or -1 . [MP, Table A.7] tells us that

$$N\left(-2 + \frac{1}{n}, -\frac{3}{2}\right)(\infty) = N\left(\frac{1-2n}{n}, -\frac{3}{2}\right)(\infty) = L(5(1-2n)+2n, \star)$$

which is S^3 if and only if $|5 - 8n| = 1$ which has no integer solutions. The possibility for the -2 slope is

$$N\left(-2 + \frac{1}{n}, -\frac{3}{2}\right)(-2) = N\left(\frac{1-2n}{n}, -\frac{3}{2}\right)(-2) = L(8(1-2n)+13n, \star)$$

which is S^3 if and only if $|8 - 3n| = 1 \Rightarrow n = 3$ which is excluded from [MP, Table A.7]. Finally, the -1 slope yields

$$N\left(-2 + \frac{1}{n}, -\frac{3}{2}\right)(-1) = N\left(\frac{1-2n}{n}, -\frac{3}{2}\right)(-1) = L(3(1-2n)+11n, \star)$$

which is S^3 if and only if $|3 + 5n| = 1$ which has no integer solutions.

We conclude that if [MP, Table A.7] contains an $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ with $\Delta(\beta, \gamma) = 2$, then $\alpha, \beta, \gamma \in \{0, -1, -2, -3, \infty\}$.

If an (S^H, T^H, Z) triple $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ with one of $\alpha, \beta, \gamma \notin \{0, -1, -2, -3, \infty\}$ comes from [MP, Table A.8] then $\frac{t}{u} = -\frac{1}{2}$ and $E(N(\frac{r}{s}, \frac{t}{u})) = \{-4, -3, -2, -1, 0, \infty\}$. This table tells us that $N(\frac{r}{s}, -\frac{1}{2})(-4)$ is in $S^H \cup T^H \cup Z$ only if $\frac{r}{s} = n$, in which case $N(n, -\frac{1}{2})(-4) \in Z$. Moreover, we see in this table that the only possible S^H slope with $N(\frac{r}{s}, -\frac{1}{2})$ hyperbolic is ∞ . Since

$$N\left(n, -\frac{1}{2}\right)(\infty) = L(n+2, \star)$$

is S^3 if and only if $n = -3$, which makes $N(n, -\frac{1}{2})$ non-hyperbolic by Lemma A.10, we conclude once again that if [MP, Table A.8] contains an $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ with $\Delta(\beta, \gamma) = 2$, then $\alpha, \beta, \gamma \in \{0, -1, -2, -3, \infty\}$.

Case 2: We now consider the case $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ with $\Delta(\beta, \gamma) = 2$ and $\{\alpha, \beta, \gamma\} \subset \{-3, -2, -1, 0, \infty\}$. All examples can be constructed from [MP, Table 2]. We will write T2. n to refer to line n in the $\frac{t}{u}$ column of [MP, Table 2] **from the arXiv preprint**.

We will first show that we may assume that ∞ corresponds to the S^H slope. The distance between the S^H and the T^H slope is 1 [GL1]. So, when ∞ is not an S^H slope, and the distance between the T^H and Z slopes is 2 we have six choices of (S^H, T^H, Z) slopes:

- (1) The (S^H, T^H, Z) triple of slopes correspond to $(-3, -2, 0)$
- (2) The (S^H, T^H, Z) triple of slopes correspond to $(-2, -3, -1)$
- (3) The (S^H, T^H, Z) triple of slopes correspond to $(-2, -1, -3)$
- (4) The (S^H, T^H, Z) triple of slopes correspond to $(-1, -2, 0)$
- (5) The (S^H, T^H, Z) triple of slopes correspond to $(-1, 0, -2)$
- (6) The (S^H, T^H, Z) triple of slopes correspond to $(0, -1, -3)$

If we are in Case (1) then $N(\frac{r}{s}, \frac{t}{u})$ is hyperbolic and $N(\frac{r}{s}, \frac{t}{u})(-3) = L(x, y)$. T2.4 tells us that $\frac{r}{s} = -1 + \frac{1}{n}$ and $\frac{t}{u} = -1 + \frac{1}{m}$ (all other lines of Table 2 with $\frac{t}{u} = -3$ make $N(\frac{r}{s}, \frac{t}{u})$ non-hyperbolic or $N(\frac{r}{s}, \frac{t}{u})(-3) \neq L(x, y)$). Since 0 is a type Z slope, we have to look for $\frac{r}{s}, \frac{t}{u}$ in T2.16. This implies that $\frac{r}{s} = n' = -1 + \frac{1}{n}$ which yields $n' \in \{-2, 0\}$ making $N(\frac{r}{s}, \frac{t}{u})$ non-hyperbolic.

If we are in Case (2) then T2.4 tells us that $\frac{r}{s} = -1 + \frac{1}{n}$ and $\frac{t}{u} = -1 + \frac{1}{m}$, and T2.9 tells us that one of $\frac{r}{s}, \frac{t}{u}$ equals $-2 + \frac{1}{k}$. By Lemma A.8 we may assume that $\frac{r}{s} = -1 + \frac{1}{n} = -2 + \frac{1}{m} \Rightarrow \frac{r}{s} = -\frac{3}{2}$.

However, Identity (1.3) in [MP, Proposition 1.5] gives us:

$$(11) \quad \left(N\left(-\frac{3}{2}, \frac{t}{u}\right), -2, \beta, \gamma\right) \cong \left(N\left(-\frac{3}{2}, \frac{2t+5u}{t+2u}\right), \infty, -\frac{2\beta+5}{\beta+2}, -\frac{2\gamma+5}{\gamma+2}\right).$$

So all examples found in Case (2) are contained in the case when ∞ is the S^3 slope (considered below).

If we are in Case (3) then T2.5 tells us that one of $\frac{r}{s}, \frac{t}{u} \in \{-1 + \frac{1}{n}\}$. Lemma A.8 tells us that we may assume $\frac{r}{s} = -1 + \frac{1}{n}$. T2.12 tells us that one of $\frac{r}{s}, \frac{t}{u} = -3 + \frac{1}{k}$. As we have $\frac{r}{s} = -1 + \frac{1}{n}$ and $\frac{r}{s} = -2$ is excluded by Lemma A.10, we require $\frac{t}{u} = -3 + \frac{1}{k}$. T2.9 requires one of $\frac{r}{s}, \frac{t}{u} = -2 + \frac{1}{n}$. So, the possible (S^H, T^H, Z) triples in this case are $\left(N\left(-\frac{3}{2}, -3 + \frac{1}{k}\right), -2, -1, -3\right)$ and $\left(N\left(-1 + \frac{1}{n}, -\frac{5}{2}\right), -2, -1, -3\right)$. Equation (11) tells us that any examples of the form $\left(N\left(-\frac{3}{2}, -3 + \frac{1}{k}\right), -2, -1, -3\right)$ are contained in the analysis of the case when ∞ is the S^3 slope (considered below). T2.9 tells us that filling along our S^3 slope, -2 , gives us

$$N\left(-1 + \frac{1}{n}, -\frac{5}{2}\right)(-2) = N\left(-\frac{5}{2}, \frac{1-n}{n}\right)(-2) = L(3(-2)(1-n+2n)-2(1-n)-n, \star) = L(-8-6n, \star) \neq S^3.$$

If we are in Case (4) then T2.9 tells us that one of $\frac{r}{s}, \frac{t}{u} = -2 + \frac{1}{n}$. Lemma A.8 tells us that we may assume $\frac{r}{s} = -2 + \frac{1}{n}$. T2.12 tells us that one of $\frac{r}{s}, \frac{t}{u} = -3 + \frac{1}{k}$. So, either $\frac{r}{s} = -2 + \frac{1}{n} = -3 + \frac{1}{k}$ (i.e. $n = -2 = -k$), or $\frac{r}{s} = -2 + \frac{1}{n}$ and $\frac{t}{u} = -3 + \frac{1}{k}$. In the former case, T2.16 requires $\frac{t}{u} = k$ and filling along the S^3 slope, -1 , gives

$$N\left(-\frac{5}{2}, k\right)(-1) = L(2(2)(k+3)-k-1, \star) = L(3k+11, \star) = S^3 \text{ only when } k = -4.$$

So, $\left(N\left(-\frac{5}{2}, -4\right), -1, -2, 0\right) \in (S^H, T^H, Z)$. In the latter case, T2.16 requires one of $\frac{r}{s} = -2 + \frac{1}{n}$, $-3 + \frac{1}{k}$ to be an integer. The slopes $-1, -2, -3$ are excluded by Lemma A.10, and the remaining case is $\frac{t}{u} = -4$. In this case, by T2.12, filling along the S^3 slope, -1 , gives

$$N\left(-4, \frac{1-2n}{n}\right)(-1) = L(2(-1)(1-2n+3n)-(1-2n)-n, \star) = L(-3-n, \star) = S^3 \text{ only when } n = -1 \text{ or } n = -4.$$

This gives the example $\left(N\left(-\frac{5}{2}, -4\right), -1, -2, 0\right) \in (S^H, T^H, Z)$ (already listed) and the example

$$\left(N\left(-\frac{9}{4}, -4\right), -1, -2, 0\right) \in (S^H, T^H, Z).$$

If we are in Case (5) then T2.12 tells us that one of $\frac{r}{s}, \frac{t}{u} = -3 + \frac{1}{n}$. Lemma A.8 tells us that we may assume $\frac{r}{s} = -3 + \frac{1}{n}$. T2.12 tells us that $\left\{\frac{r}{s}, \frac{t}{u}\right\} = \left\{k, -4 - k + \frac{1}{m}\right\}$. If $\frac{r}{s} = -3 + \frac{1}{n} \in \mathbb{Z}$ then, by Lemma A.10, we necessarily have $\frac{r}{s} = k = -4$ and $\frac{t}{u} = \frac{1}{m}$. T2.12 tells us that our S^3 slope, -1 , gives us

$$N\left(-4, \frac{1}{m}\right) = L(2(-1)(1+3m)-1-m, \star) = L(-3-7m, \star) \neq S^3.$$

On the other hand, if $\frac{r}{s} = -3 + \frac{1}{n} = -4 - k + \frac{1}{m}$ and $\frac{t}{u} = k$ then $k = -1 + \frac{1}{m} - \frac{1}{n} \Rightarrow k \in \{-3, -2, 0, 1\}$. The cases $k \in \{-3, -2, 0\}$ are excluded by Lemma A.10. If $\frac{t}{u} = k = 1$ then $n = -1 \Rightarrow \frac{r}{s} = -4$.

T2.12 tells us that our S^3 slope, -1 , gives us

$$N(-4, 1) = L(2(-1)(1+3)-1-1, \star) \neq S^3.$$

If we are in Case (6) then [MP, Table 2] tells us that $N(\frac{r}{s}, \frac{t}{u})(0) = S^3$ only if $\frac{t}{u} = \infty$ which makes $N(\frac{r}{s}, \frac{t}{u})$ non-hyperbolic.

The arguments so far allow us to assume that $(N(\frac{r}{s}, \frac{t}{u}), \infty, \beta, \gamma) \in (S^H, T^H, Z)$, with the slopes $\{\beta, \gamma\} \subset \{0, -1, -2, -3\}$ and $\Delta(\beta, \gamma) = 2$. We immediately see that either $\{\beta, \gamma\} = \{0, -2\}$ or $\{\beta, \gamma\} = \{-1, -3\}$ giving us four cases to examine:

Case 2.1: $(N(\frac{r}{s}, \frac{t}{u}), \infty, -2, 0) \in (S^H, T^H, Z)$

Case 2.2: $(N(\frac{r}{s}, \frac{t}{u}), \infty, 0, -2) \in (S^H, T^H, Z)$

Case 2.3: $(N(\frac{r}{s}, \frac{t}{u}), \infty, -1, -3) \in (S^H, T^H, Z)$

Case 2.4: $(N(\frac{r}{s}, \frac{t}{u}), \infty, -3, -1) \in (S^H, T^H, Z)$

In all cases for ∞ to be an S^H slope we require

$$(12) \quad |rt - su| = 1.$$

In **Case 2.1** we see from [MP, Table 2] that $N(\frac{r}{s}, \frac{t}{u})(-2) \in T^H$ implies that $-2 + \frac{1}{n} \in \{\frac{r}{s}, \frac{t}{u}\}$ and $N(\frac{r}{s}, \frac{t}{u})(0) \in Z$ implies that one of $\frac{r}{s}, \frac{t}{u}$ is an integer k . If $-2 + \frac{1}{n} = k$ then $k = -3, -1$ (which are excluded by Lemma A.10). So we are interested in $N(k, \frac{1-2n}{n})$ with ∞ an S^H slope. By condition (12) we find

$$k = \frac{n+1}{1-2n} \quad \text{or} \quad k = \frac{n-1}{1-2n}.$$

For k to be an integer in the first equality we require $n = -1, 0, 1$ or $k = -1$ which are all excluded by Lemma A.10. For k to be an integer in the second equality we require $n = 0, 1$ which are all excluded by Lemma A.10. So, Case 2.1 produces no examples.

In **Case 2.2** we see in [MP, Table 2] that $N(\frac{r}{s}, \frac{t}{u})(0) \in T^H$ implies that $\{\frac{r}{s}, \frac{t}{u}\} = \{n, -4 - n + \frac{1}{m}\}$. By Lemma A.8 we may assume that $(\frac{r}{s}, \frac{t}{u}) = (n, \frac{1-4m-nm}{m})$. Identity (12) implies that

$$n(1 - 4m - nm) - m = \pm 1 \Rightarrow m = \frac{n+1}{n^2 + 4n + 1} \quad \text{or} \quad m = \frac{n-1}{n^2 + 4n + 1}.$$

This implies $n \in \{-3, -1, 0\}$ in the first case and $n \in \{-5, -3, -2, -1, 0, 1\}$ in the second case. We know that $n \in \{-3, -2, -1, 0\}$ are excluded by Lemma A.10. In the second case, if $n = -5$ then $m = -1$ which implies $\frac{t}{u} = 0$ which is excluded by Lemma A.10. If $n = 1$ in the second case then $m = 0$ which implies $\frac{t}{u} = \infty$ which is excluded by Lemma A.10. So, the Case 2.2 produces no examples.

In **Case 2.3** [MP, Table 2] tells us that $N(\frac{r}{s}, \frac{t}{u})(-1) \in T^H$ implies that $-3 + \frac{1}{n} \in \{\frac{r}{s}, \frac{t}{u}\}$ and $N(\frac{r}{s}, \frac{t}{u})(-3) \in Z$ implies that $-1 + \frac{1}{m} \in \{\frac{r}{s}, \frac{t}{u}\}^2$. If the slope $-3 + \frac{1}{n} = -1 + \frac{1}{m}$ then $n = 1$ and

²see footnote in Section 5.3.

our slope is -2 which is forbidden by Lemma A.10. So, by Lemma A.8, we may assume that $(\frac{r}{s}, \frac{t}{u}) = (-3 + \frac{1}{n}, -1 + \frac{1}{m})$. Identity (12) implies that

$$(1 - 3n)(1 - m) - nm = \pm 1 \Rightarrow m = \frac{3n}{2n - 1} \text{ or } m = \frac{3n - 2}{2n - 1}.$$

This implies $n \in \{-1, 0, 1, 2\}$ in the first case and $n \in \{0, 1\}$ in the second case. The cases $n = 0, 1$ make $\frac{r}{s} = \infty, -2$ respectively, which are excluded by Lemma A.10. If $n = 1$ in the first case then $m = 1 \Rightarrow \frac{t}{u} = 0$ which is excluded by Lemma A.10. If $n = 2$ in the first case then $m = 2$ which implies $\frac{r}{s} = -\frac{r}{s}$ and $\frac{t}{u} = -\frac{1}{2}$. This gives us the example

$$\left(N\left(-\frac{5}{2}, -\frac{1}{2}\right), \infty, -1, -3 \right) \in (S^H, T^H, Z).$$

In **Case 2.4** we see in [MP, Table 2] that $N(\frac{r}{s}, \frac{t}{u})(-3) \in T^H$ implies that $\{\frac{r}{s}, \frac{t}{u}\} = \{-1 + \frac{1}{n}, -1 + \frac{1}{m}\}$ and $N(\frac{r}{s}, \frac{t}{u})(-1) \in Z$ implies that $-3, -3 + \frac{1}{k} \notin \{\frac{a}{b}, \frac{c}{d}\}$. The condition $(\frac{r}{s}, \frac{t}{u}) = (-1 + \frac{1}{n}, -1 + \frac{1}{m})$ and (12) imply that

$$(1 - n)(1 - m) - nm = \pm 1 \Rightarrow m = 1 \pm 1 - n.$$

This gives us

$$\left(N\left(-1 + \frac{1}{n}, -1 - \frac{1}{n}\right), \infty, -3, -1 \right) \in (S^H, T^H, Z)$$

and

$$\left(N\left(-1 + \frac{1}{n}, -1 + \frac{1}{2-n}\right), \infty, -3, -1 \right) \in (S^H, T^H, Z).$$

We have thus proved the following: if $(N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) \in (S^H, T^H, Z)$ and $\Delta(\beta, \gamma) = 2$ then

$$\begin{aligned} (N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) &\cong (N(-\frac{5}{2}, -4), -1, -2, 0), \text{ or} \\ (N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) &\cong (N(-\frac{9}{4}, -4), -1, -2, 0), \text{ or} \\ (N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) &\cong (N(-\frac{5}{2}, -\frac{1}{2}), \infty, -1, -3), \text{ or} \\ (N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) &\cong (N(-1 + \frac{1}{n}, -1 - \frac{1}{n}), \infty, -3, -1), \text{ or} \\ (N(\frac{r}{s}, \frac{t}{u}), \alpha, \beta, \gamma) &\cong (N(-1 + \frac{1}{n}, -1 + \frac{1}{2-n}), \infty, -3, -1). \end{aligned}$$

From [MP, Table A.4] we know that $N(-\frac{5}{2}, -4)$ and $N(-\frac{5}{2}, -\frac{1}{2})$ are both equal to the $(-2, 3, 7)$ pretzel knot exterior $N(-\frac{3}{2}, -\frac{1}{2})$, which is also $N(-1 - \frac{1}{n}, -1 + \frac{1}{n})$ for $n = 2$. We have

$$\begin{aligned} N\left(-\frac{9}{4}, -4\right) &\stackrel{\text{Lemma A.9}}{=} M_3\left(\frac{9}{4}, 4\right) \stackrel{\text{Lemma A.6}}{=} M_4\left(\frac{5}{4}, -1, 3\right) \stackrel{(16) \circ (29) \circ (16)}{=} M_5\left(\frac{1}{4}, -1, -2, 3\right) \\ &\stackrel{(17) \circ (20)}{=} M_5\left(-2, -1, -\frac{1}{2}, \frac{3}{4}\right) \stackrel{\text{Lemma A.6} \circ (29)}{=} M_3\left(\frac{3}{2}, \frac{3}{4}\right) \stackrel{\text{Lemma A.9}}{=} \left(-\frac{3}{2}, -\frac{3}{4}\right) \\ &= N\left(-1 + \frac{1}{n}, -1 + \frac{1}{2-n}\right) \text{ with } n = -2. \end{aligned}$$

We observed in Section 4.4 that the two families are distinct. This completes the proof. \square

APPENDIX A. FACTS USED LIBERALLY THROUGHOUT THIS ARTICLE

The classification in this article comes from a careful consideration of the tables found in [MP] and [Rou2]. Often, cases considered in the enumeration are identified and/or discounted using technical results, most of which are found in [MP] and [Rou2]. To keep this article as self-contained as possible we list the technical lemmas that are used in this article.

A.1. Identities between graph manifolds. The following lemma consists of a list of identities between graph manifolds which are found in both [Rou2] and [MP]. Details can be found in [FM].

Lemma A.1. *The following identities on graph manifolds hold:*

$$(13) \quad (D, (1, b), (c, d)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (e, f), (g, h)) = (D, (e, f), (g, h)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (1, b), (c, d)) \\ = (S^2, (e, f), (g, h), (d+bc, -c))$$

$$(14) \quad (S^2, (a, b), (c, d), (0, 1)) = L(a, b) \# L(c, d)$$

$$(15) \quad (S^2, (a, b), (c, d), (1, e)) = L(a(d+ce)+bc, \star)$$

The following obvious lemma is used throughout the article.

Lemma A.2. *If $(D, (a, b), (c, d)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (e, f), (g, h))$ or $(S^2, (a, b), (c, d), (e, f))$ is a lens space or S^3 , then one of $|a|, |c|, |e|$ or $|g|$ is less than or equal to 1.*

A.2. Concerning surgery instruction on 5CL.

Lemma A.3 ([Rou2, Lemma 2.2]). *The action of $\text{Aut}(M_5)$ on surgery instructions on 5CL is generated by (16)–(28). Moreover, for $18 \leq n \leq 28$ each (n) corresponds to the action of a distinct element of $\text{Aut}(M_5)/G$ where G is the subgroup generated by the elements (16)–(17) corresponding to the generators of the link symmetry group of 5CL.*

$$(16) \quad (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \mapsto (\alpha_5, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

$$(17) \quad (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \mapsto (\alpha_5, \alpha_4, \alpha_3, \alpha_2, \alpha_1)$$

$$(18) \quad \left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}, \frac{i}{j}\right) \mapsto \left(\frac{f}{e}, \frac{j-i}{j}, \frac{a}{a-b}, \frac{d-c}{d}, \frac{h}{g}\right)$$

$$(19) \quad \left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}, \frac{i}{j}\right) \mapsto \left(\frac{b}{b-a}, \frac{i-j}{i}, \frac{e-f}{e}, \frac{d}{d-c}, \frac{g}{h}\right)$$

$$(20) \quad \left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}, \frac{i}{j}\right) \mapsto \left(\frac{i}{i-j}, \frac{b-a}{b}, \frac{f}{e}, \frac{d}{c}, \frac{h-g}{h}\right)$$

$$(21) \quad \left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}, \frac{i}{j}\right) \mapsto \left(\frac{j}{j-i}, \frac{e}{f}, \frac{b}{b-a}, \frac{c-d}{c}, \frac{g-h}{g}\right)$$

$$(22) \quad \left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}, \frac{i}{j}\right) \mapsto \left(\frac{a}{a-b}, \frac{e}{e-f}, \frac{i}{i-j}, \frac{c}{c-d}, \frac{g}{g-h}\right)$$

$$(23) \quad \left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}, \frac{i}{j}\right) \mapsto \left(\frac{h}{g}, \frac{j}{i}, \frac{f-e}{f}, \frac{c}{c-d}, \frac{b-a}{b}\right)$$

$$(24) \quad \left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}, \frac{i}{j}\right) \mapsto \left(\frac{h}{h-g}, \frac{a}{b}, \frac{f}{f-e}, \frac{c-d}{c}, \frac{i-j}{i}\right)$$

$$(25) \quad \left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}, \frac{i}{j}\right) \mapsto \left(\frac{g}{g-h}, \frac{f-e}{f}, \frac{b}{a}, \frac{d}{c}, \frac{j-i}{j}\right)$$

$$(26) \quad \left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}, \frac{i}{j}\right) \mapsto \left(\frac{g-h}{g}, \frac{f}{f-e}, \frac{i}{j}, \frac{d}{d-c}, \frac{a-b}{a}\right)$$

$$(27) \quad \left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}, \frac{i}{j}\right) \mapsto \left(\frac{h-g}{h}, \frac{b}{a}, \frac{j}{i}, \frac{d-c}{d}, \frac{e}{e-f}\right)$$

$$(28) \quad \left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}, \frac{i}{j}\right) \mapsto \left(\frac{a-b}{a}, \frac{e-f}{e}, \frac{h}{h-g}, \frac{c}{d}, \frac{j}{j-i}\right).$$

Lemma A.4 ([Rou2, Theorem 4 and Eq. (70)]). *The following statements hold:*

- If $0, 1, \infty \in \{\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}, \frac{i}{j}\}$ then $M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}, \frac{i}{j})$ is non-hyperbolic.
- If $-1, \frac{1}{2}, 2 \in \{\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}, \frac{i}{j}\}$ then $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}, \frac{i}{j})$ factors through M_4 .

As highlighted in [MPR]:

Lemma A.5. *The following identity holds:*

$$(29) \quad M_5\left(\frac{a}{b}, \frac{c}{d}, -1, \frac{e}{f}, \frac{g}{h}\right) = M_4\left(\frac{a}{b}, \frac{c+d}{d}, \frac{e+f}{f}, \frac{g}{h}\right).$$

A.3. Concerning surgery instructions on 4CL. From [Rou2] we have the following identities:

$$(30) \quad M_4\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right)(\infty) = (S^2, (a, b), (d, -c), (e, f)),$$

$$(31) \quad M_4\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right)(0) = (D, (f, -e), (b, 2b-a)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (2, 1), (c-2d, d)),$$

$$(32) \quad M_4\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right)(1) = (S^2, (a-2b, b), (c-d, c), (e-2f, f)),$$

$$(33) \quad M_4\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right)(2) = (D, (a-b, b), (e-f, f)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (c, d), (2, -1)).$$

Lemma A.6 ([Rou2, Theorem 5 and Eq. (69)]). *The following statements hold:*

- If $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})$ is an instruction on 4CL and one of the slopes is in $\{0, 1, 2, \infty\}$ then $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})$ is non-hyperbolic.
- If $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})$ is an instruction on 4CL and one of the slopes is in $\{-1, \frac{1}{2}, \frac{3}{2}, 3\}$ then $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})$ factors through M_3 . In particular, $M_4(\frac{a}{b}, -1, \frac{c}{d}, \frac{e}{f}) = M_3(\frac{a}{b} + 1, \frac{c}{d} + 1, \frac{e}{f})$

The action induced from the link symmetry group of 4CL tells us:

Lemma A.7. For a filling instruction $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})$ on M_4 we have

$$M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}) = M_4(\frac{e}{f}, \frac{c}{d}, \frac{a}{d}, \frac{g}{h}) \quad \text{and} \quad M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}) = M_4(\frac{g}{h}, \frac{a}{b}, \frac{c}{d}, \frac{e}{f}).$$

A.4. Concerning surgery instructions on 3CL.

Lemma A.8. If $\sigma \in S_3$ and $(\alpha_1, \alpha_2, \alpha_3)$ is a filling instruction on N then

$$N(\alpha_1, \alpha_2, \alpha_3) = N(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)}).$$

Lemma A.9. For all filling instructions it holds $M_3(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}) = N(-\frac{a}{b}, -\frac{c}{d}, -\frac{e}{f})$.

The following Lemma is contained in the statement of [MP, Corollary A.6].

Lemma A.10. The following statements hold:

- If $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ is an instruction on 3CL and one of the slopes is $\{0, -1, -2, -3, \infty\}$ then $N(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ is non-hyperbolic.
- If $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ is an instruction on 3CL and one of the slopes is $\{-4, -\frac{5}{2}, -\frac{3}{2}, -1, -\frac{1}{2}, 1\}$ then $N(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ is found in Tables 8–15 of [MP].

A.5. Concerning surgery instructions on M4CL. [Rou2, Proposition 2.1] gives us a complete enumeration of the Dehn fillings on F , the exterior of the minimally twisted 4 chain link. We have:

Lemma A.11. For slopes $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}$ on M4CL the following identity holds:

$$(34) \quad F(\frac{a}{b}, \frac{e}{f}, \frac{c}{d}, \frac{g}{h}) = (D, (a, b), (c, d)) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} (D, (e, f), (g, h))$$

In fact, “most” exceptional fillings of M_5 are obtained by filling F (c.f. [Rou2, Proposition 3.1]).

Lemma A.12. The following identities hold:

$$(35) \quad M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})(\infty) = F(-\frac{a}{b}, \frac{f}{e}, \frac{d}{c}, -\frac{g}{h})$$

$$(36) \quad M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})(1) = F(\frac{a-b}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g-h}{h})$$

$$(37) \quad M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})(0) = F(\frac{b}{b-a}, \frac{c-d}{c}, -\frac{h}{g}, \frac{e-f}{f})$$

Consequently,

Lemma A.13. *The following identities hold:*

$$(38) \quad F\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}\right) = M_5\left(-\frac{a}{b}, \frac{f}{e}, \frac{d}{c}, -\frac{g}{h}\right)(\infty)$$

$$(39) \quad F\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}\right) = M_5\left(\frac{a+b}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g+h}{h}\right)(1)$$

$$(40) \quad F\left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h}\right) = M_5\left(\frac{a-b}{a}, \frac{d}{d-c}, -\frac{h}{g}, \frac{f+e}{f}\right)(0)$$

A.6. Some elementary diophantine equations.

Lemma A.14. *For $(n, s) \in \mathbb{Z}^2$, we have*

- $s - n = ns \implies (n, s) \in \{(0, 0), (2, -2)\}$;
- $2s - n = ns \implies ((n, s) \in \{(0, 0), (1, 1), (3, -3), (4, -2)\}$;
- $4s - n = ns \implies ((n, s) \in \{(0, 0), (3, 3), (5, -5), (8, -2), (6, -3), (2, 1)\}$;
- $s - n = 3ns \implies (n, s) \in \{(0, 0)\}$;
- $2s - n = 3ns \implies ((n, s) \in \{(0, 0), (1, -1)\}$;
- $4s - n = 3ns \implies ((n, s) \in \{(0, 0), (1, 1), (2, -1)\}$;
- $8s - n = 3ns \implies (n, s) \in \{(0, 0), (3, -3), (2, 1), (4, -1)\}$;
- $5s - n = 3ns \implies ((n, s) \in \{(0, 0), (2, -2)\}$;
- $s - n = -5ns \implies (n, s) \in \{(0, 0)\}$;
- $2s - n = -5ns \implies (n, s) \in \{(0, 0)\}$;
- $4s - n = -5ns \implies (n, s) \in \{(0, 0), (-1, 1)\}$;
- $8s - n = -5ns \implies (n, s) \in \{(0, 0), (-2, 1)\}$;
- $3s - n = -5ns \implies (n, s) \in \{(0, 0)\}$.

Proof. Here, we consider equations of the form $\alpha s - n = \beta ns$ for some $\alpha, \beta \in \mathbb{Z}$. They are solved by induction on the number of prime factor of α .

Indeed, we first note that $s \mid n$ and $n \mid \alpha s$.

- If actually $n \mid s$, then $s = \pm n$ and n satisfies either $(\alpha - 1)n = \beta n^2$ or $(\alpha + 1)n = \beta n^2$. It follows that $(n, s) = (0, 0)$, or $\left(\frac{\alpha-1}{\beta}, \frac{\alpha-1}{\beta}\right)$ if $\frac{\alpha-1}{\beta} \in \mathbb{Z}$, or $\left(\frac{\alpha+1}{\beta}, -\frac{\alpha+1}{\beta}\right)$ if $\frac{\alpha+1}{\beta} \in \mathbb{Z}$.
- If $n \nmid s$ then $n = kn'$ with some prime divisor of α , but then $\frac{\alpha}{k}n' - s = \beta n's$ and by induction, we know all such (n'_0, s_0) and each of them leads to a solution (kn'_0, s_0) .

□

Lemma A.15. *If m, n are integers such that $(1 - m(n + 4))n = m \pm 1$ then*

$$(m, n) \in \{(-5, -1), (-4, -3), (-4, -5), (-3, 1), (-3, 2), (-2, 1), (-1, 0), (-1, 1), (0, -1), (0, 1), (1, 0)\}.$$

Proof. Then $m(1 + n(n + 4)) = n \pm 1$. So either $m = 0$ or $(1 + n(n + 4)) \mid n \pm 1$.

Case $m = 0$: then $n = \pm 1$.

Case $m(1 + n(n + 4)) \mid n + 1$: then $1 + n(n + 4) \leq |n + 1|$.

If $n + 1 \geq 0$: then $n(n + 3) \leq 0$ so $n \in \{-3, -2, -1, 0\}$. Only -1 and 0 satisfy $n + 1 \geq 0$, leading to solutions $(n, m) \in \{(-1, 0), (0, 1)\}$.

If $n + 1 \leq 0$: then $n^2 + 5n + 2 \leq 0$ so $n \in \left[-\frac{5-\sqrt{17}}{2}, \frac{-5+\sqrt{17}}{2}\right] \cap \mathbb{Z} = \{-4, -3, -2, -1\}$. If $n = -2$, then $m = \frac{1}{3} \notin \mathbb{Z}$. Other cases lead to solutions $(n, m) \in \{(-4, -3), (-3, 1), (-1, 0)\}$.

Case $m(1 + n(n + 4)) \mid n - 1$: $1 + n(n + 4) \leq |n - 1|$.

If $n - 1 \geq 0$: then $(n + 1)(n + 2) \leq 0$ so $n \in \{-2, -1\}$ and doesn't satisfy $n - 1 \geq 0$.

If $n - 1 \leq 0$: then $n(n + 5) \leq 0$ so $n \in \{-5, -4, -3, -2, -1, 0\}$ leading to solutions $(n, m) \in \{(-5, -1), (-4, -5), (-3, 2), (-2, 1), (-1, 1), (0, -1)\}$.

□

REFERENCES

- [Ago] AGOL, I., *Bounds on exceptional Dehn filling II*, *Geom. Topol.* **14** (2010) 1921–1940.
- [Bak] BAKER, K. L., *Surgery descriptions and volumes of Berge knots II: Descriptions of the Minimally Twisted Five Chain Link*, *J. Knot Theory Ramifications*, **17** (2008).
- [BDH] BAKER, K. L., DOLESHAL, B. G., HOFFMAN, N., *On manifolds with multiple lens space fillings*, *Bol. Soc. Mat. Mex.* **20** (2014) 405–447.
- [Be1] BERGE, J., *Some knots with surgeries yielding lens spaces*, preprint.
- [Be2] BERGE, J., *The knots in $D^2 \times S^1$ which have nontrivial Dehn surgeries that yield $D^2 \times S^1$* , *Topology and its Applications*, **39** (1991) 1-19.
- [CGLS] CULLER, M., GORDON, C. MCA., LUECKE, J., SHALEN, P.B., *Dehn surgery on knots*, *Ann. of Math.* **125** (1987), 237-300.
- [Eud] EUDAVE-MUÑOZ, M., *On hyperbolic knots with Seifert fibered Dehn surgeries*, *Topology and its Applications* **121** (2002), 119-141.
- [FM] Fomenko, A. T., Matveev, S. V., “Algorithmic and Computer Methods for Three-Manifolds”, Kluwer Academic Publishers, (1997).
- [Gab] GABAI, D., *Surgery on knots in solid tori*, *Topology* **28** (1989), 1-6.
- [GAS] GONZÁLEZ-ACUÑA, F., SHORT, H., *Knot surgery and primeness*, *Math. Proc. Cambridge Philos. Soc.* **99** (1986), 89-102.
- [G1] GORDON, C., MCA. *Small surfaces and Dehn fillings*, *Geometry and Topology Monographs* **42**, Proceedings of the Kirbyfest, 1999, 177-199.
- [G2] GORDON, C., MCA. *Dehn surgery and 3-manifolds*, MROWKA, T.S., OZSVÁTH (Eds.) IAS/Park City Mathematics Series, **16** (2006).
- [GL1] GORDON, C. MCA., LUECKE, J., *Knots are determined by their complements* *J. Amer. Math. Soc.* **2** (1989), 371–415.
- [GL2] GORDON, C. MCA., LUECKE, J., *Non-integral toroidal Dehn surgeries*, *Comm. Anal. Geom.* **12** (2004), 417-485.

- [GL3] GORDON, C. MCA., LUECKE, J., *Dehn surgeries on knots creating essential tori, I*, *Comm. Anal. Geom.* **3** (1995), 371-644.
- [GW] GORDON, CAMERON MCA.; WU, YING-QING, *Toroidal and annular Dehn fillings*. *Proc. London Math. Soc.* **3** 78 (1999), 662–700.
- [LM] Lackenby, M., Meyerhoff, R., *The Maximal Number of Exceptional Dehn Surgeries*, *Invent. math.* 191 (2) (2013) 341-382.
- [MP] MARTELLI, B., PETRONIO, C., *Dehn filling of the “magic” 3-manifold*, *Comm. Anal. Geom.* **14** (2006), 969-1026.
- [MPR] MARTELLI, B., PETRONIO, C., ROUKEMA, F.W.M., *Exceptional Dehn surgery on the minimally twisted 5-chain link*, available at http://arxiv.org/PS_cache/arxiv/pdf/1109/1109.0903v1.pdf
- [Rol] ROLFSEN, D., “Knots and Links”, AMS Chelsea Publishing, (1976).
- [Rou1] ROUKEMA, F.W.M., *Dehn surgery on the minimally twisted five-chain link*, Ph.D. thesis, supervised by C. Petronio and B. Martelli 2011, available at <http://etd.adm.unipi.it/theses/available/etd-09212011-102615/>
- [Rou2] ROUKEMA, F.W.M., *Exceptional Slopes on Manifolds of Small Complexity*, available at <https://arxiv.org/abs/1210.1824>
- [Th] THURSTON, W.P., *The geometry and topology of three-manifolds*, Princeton Univ. Math. Dept. (1978) available at <http://library.msri.org/books/gt3m/>

AIX MARSEILLE UNIV, CNRS, CENTRALE MARSEILLE, I2M, MARSEILLE, FRANCE
E-mail address: Benjamin.Audoux@univ-amu.fr

AIX MARSEILLE UNIV, CNRS, CENTRALE MARSEILLE, I2M, MARSEILLE, FRANCE
E-mail address: ana.lecuona@univ-amu.fr

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SHEFFIELD, UK
E-mail address: f.roukema@sheffield.ac.uk