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Collatz conjecture

The trivial cycle is unique (because a Collatz sequence that becomes periodic converges)

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Abstract

A Collatz sequence that is periodic after a certain rank reaches the value 1 (or converges). Therefore the trivial cycle is unique, and a Collatz sequence that has an upper bound becomes periodic, and then converges.

1 Introduction

The Collatz conjecture (or Syracuse conjecture, Ulam conjecture or $3x+1$ problem) claims that the following sequence of natural numbers reaches the value 1 after a certain rank (in this article, it will be specified that the sequence converges):

$s_0 \geq 1$, and for all natural number n :

$$s_{n+1} = \begin{cases} \frac{s_n}{2} & \text{if } s_n \text{ is even,} \\ 3s_n + 1 & \text{if } s_n \text{ is odd.} \end{cases}$$

$3p+1$ being even if the natural number p is odd, the compressed sequence (c_n) of the sequence (s_n) is defined as follows:

$c_0 \geq 1$, and for all natural number n :

$$c_{n+1} = \begin{cases} \frac{c_n}{2} & \text{if } c_n \text{ is even,} \\ \frac{3c_n+1}{2} & \text{if } c_n \text{ is odd.} \end{cases}$$

An uncompressed (or compressed) sequence that converges continues, after a certain rank, with the trivial cycle 1-4-2 (or 1-2) that is infinitely repeated.

2 General expression of an odd element of a Collatz sequence

Considering the extracted sequence (u_n) composed of the odd elements of the sequence (s_n) , two successive elements have the following relationship:

$$u_n = \frac{3u_{n-1} + 1}{2^{k'_n}} \tag{1}$$

Where k'_n is the number of divisions by 2 of the first even element of (s_n) (following u_{n-1}) before reaching the first successive odd element of (s_n) , i.e. u_n .

By developing the previous expression:

$$u_n = \frac{3^n u_0}{2^{\sum_{j=1}^n k'_j}} + \frac{3^{n-1}}{2^{\sum_{j=1}^n k'_j}} + \frac{3^{n-2}}{2^{\sum_{j=2}^n k'_j}} + \dots + \frac{3^0}{2^{k'_n}}$$

Then:

$$u_n = \frac{3^n u_0}{2^{\sum_{j=1}^n k'_j}} + \sum_{i=1}^n \frac{3^{n-i}}{2^{\sum_{j=i}^n k'_j}}$$

After factorization of the u_0 multiplier and simplification of the second term:

$$u_n = \frac{3^n}{2^{\sum_{j=1}^n k'_j}} \left[u_0 + \sum_{i=1}^n \left(3^{-i} * 2^{\sum_{j=1}^{i-1} k'_j} \right) \right]$$

Moreover:

$$\forall j \geq 1 : k'_j = 1 + k_j$$

Because the successor of an odd element of (s_n) is always an even element. k_j is the number of divisions by 2 of the first even element of (c_j) (following u_{j-1}) before reaching the first successive odd element of (c_j) , i.e. u_j .

Therefore:

$$\sum_{j=1}^n k'_j = n + \sum_{j=1}^n k_j \text{ and } \sum_{j=1}^{i-1} k'_j = (i-1) + \sum_{j=1}^{i-1} k_j$$

$\sum_{j=1}^n k_j$ is the number of the even elements, at the rank n , of the compressed sequence (c_n) that follow u_0 .

By introducing k_j , the expression of u_n becomes:

$$u_n = \left(\frac{3}{2}\right)^n \frac{1}{2^{\sum_{j=1}^n k_j}} \left[u_0 + \frac{1}{3} \sum_{i=1}^n \left(\left(\frac{2}{3}\right)^{i-1} * 2^{\sum_{j=1}^{i-1} k_j} \right) \right]$$

After a shift on the index i , the expression of u_n , depending on a given odd element u_0 , of n and of the n first elements of the sequence (k_n) , we have that:

$$u_n = \left(\frac{3}{2}\right)^n \frac{1}{2^{\sum_{j=1}^n k_j}} \left[u_0 + \frac{1}{3} \sum_{i=0}^{n-1} \left(\left(\frac{2}{3}\right)^i * 2^{\sum_{j=1}^i k_j} \right) \right]$$

Consequently:

$$u_n = \frac{1}{a_n} \left(u_0 + \frac{x_n}{3} \right) \quad (2)$$

x_n and a_n being defined by the following expressions:

$$\begin{aligned} x_n &= \sum_{i=0}^{n-1} \left(\left(\frac{2}{3}\right)^i * 2^{\sum_{j=1}^i k_j} \right) \\ a_n &= \left(\frac{2}{3}\right)^n * 2^{\sum_{j=1}^n k_j} \end{aligned} \quad (3)$$

For all n , x_n and a_n are strictly positive. Therefore the sequence (x_n) is strictly increasing. Moreover:

$$a_n = x_{n+1} - x_n \text{ and } x_n = \sum_{i=0}^{n-1} a_i \quad (4)$$

Notice that: $x_0 = 0$.

The Excel file used to verify the formula of the equation (2) tends to show that, for sufficient large n , $a_n \rightarrow +\infty$ and $\frac{x_n}{3a_n} \rightarrow 1$ (which corresponds to $u_n = 1$).

It is the case if the Collatz sequence becomes periodic (see the demonstration below).

3 Convergence of a periodic sequence after a certain rank

We hypothesize that the sequence (s_n) is periodic after a certain rank n_0 . Therefore it is the case of the sequence (u_n) .
If T is its period:

$$\forall n \geq n_0, u_{n+T} = u_n. \text{ In particular: } u_{n_0+T} = u_{n_0}$$

In the rest of this paragraph, the sequence (u_n) is considered after u_{n_0} : then the index n begins to 1 (for exemple, u_1 is noted u_{n_0+1} , successor of u_{n_0}).
Consequently, according to the equation (2), with u_{n_0} as the first reference element:

$$\frac{1}{a_T} \left(u_{n_0} + \frac{x_T}{3} \right) = u_{n_0}$$

Then:

$$(a_T - 1)u_{n_0} = \frac{x_T}{3}$$

Therefore, like u_{n_0} and x_T are strictly positive:

$$a_T > 1 \tag{5}$$

3.1 $a_n \rightarrow +\infty$ when $n \rightarrow +\infty$

For all n , the Euclidean division of n by T implies that: $n = q_n T + r_n$, with: $0 \leq r_n < T$.

According to the equation (3), by replacing n :

$$a_n = \left(\frac{2}{3} \right)^{q_n T + r_n} * 2^{\sum_{j=1}^{q_n T + r_n} k_j}$$

Then:

$$a_n = \left(\frac{2}{3} \right)^{q_n T} * 2^{\sum_{j=1}^{q_n T} k_j} * \left(\frac{2}{3} \right)^{r_n} * 2^{\sum_{j=q_n T + 1}^{q_n T + r_n} k_j}$$

The sequence (u_n) being periodic, the sequence (k'_n) is also periodic by reference to equation (1); therefore, it is also the case of the sequence (k_n) , which implies the following two equalities:

$$\sum_{j=1}^{q_n T} k_j = q_n * \sum_{j=1}^T k_j$$

$$\sum_{j=q_n T + 1}^{q_n T + r_n} k_j = \sum_{j=1}^{r_n} k_j$$

And consequently:

$$a_n = (a_T)^{q_n} * a_{r_n} \tag{6}$$

The expressions of a_T and a_{r_n} being as follows:

$$a_T = \left(\frac{2}{3}\right)^T * 2^{\sum_{j=1}^T k_j}$$

$$a_{r_n} = \left(\frac{2}{3}\right)^{r_n} * 2^{\sum_{j=1}^{r_n} k_j}$$

Moreover, for all n :

$$a_{r_n} \geq I, \text{ with: } I = \inf(a_i)_{i \in [0, T[}$$

r_n belonging to the interval $[0, T[$.

I is a strictly positive number because it is the case of a_i , for all i .

Then, for all n : $a_n \geq (a_T)^{q_n} * I$

If $n \rightarrow +\infty$, then $q_n \rightarrow +\infty$ and a_T is strictly greater than 1. We can conclude that:

$$\lim_{n \rightarrow +\infty} a_n = +\infty$$

And that:

$$\lim_{n \rightarrow +\infty} \frac{1}{a_n} = 0$$

3.2 $\frac{x_n}{a_n}$ converges when $n \rightarrow +\infty$

According to the equation (4) :

$$\frac{x_n}{a_n} = \frac{1}{a_n} \sum_{i=0}^{n-1} a_i = \sum_{i=0}^{n-1} \frac{a_i}{a_n}$$

Like for the index n previously, the Euclidian division of i by T is expressed by: $i = q_i T + r_i$, with : $0 \leq r_i < T$.

According to the equation (6):

$$\text{For all } i: a_i = (a_T)^{q_i} * a_{r_i} \tag{7}$$

The table below gives the values of q_i , r_i and a_i related to the values of the index i (from 0 to $n = q_n T + r_n$).

i	q_i	r_i	a_i
0	0	0	a_0
1	0	1	a_1
2	0	2	a_2
-	-	-	-
$T-1$	0	$T-1$	a_{T-1}
T	1	0	$a_T * a_0$
$T+1$	1	1	$a_T * a_1$
$T+2$	1	2	$a_T * a_2$
-	-	-	-
$2T-1$	1	$T-1$	$a_T * a_{T-1}$
$2T$	2	0	$a_T^2 * a_0$
$2T+1$	2	1	$a_T^2 * a_1$
$2T+2$	2	2	$a_T^2 * a_2$
-	-	-	-
$3T-1$	2	$T-1$	$a_T^2 * a_{T-1}$
-	-	-	-
$q_{n-1}T$	q_{n-1}	0	$a_T^{q_{n-1}} * a_0$
$q_{n-1}T+1$	q_{n-1}	1	$a_T^{q_{n-1}} * a_1$
$q_{n-1}T+2$	q_{n-1}	2	$a_T^{q_{n-1}} * a_2$
-	-	-	-
q_nT-1	q_{n-1}	$T-1$	$a_T^{q_{n-1}} * a_{T-1}$
q_nT	q_n	0	$a_T^{q_n} * a_0$
q_nT+1	q_n	1	$a_T^{q_n} * a_1$
q_nT+2	q_n	2	$a_T^{q_n} * a_2$
-	-	-	-
$q_nT + r_n - 1$	q_n	$r_n - 1$	$a_T^{q_n} * a_{r_n-1}$
$q_nT + r_n$	q_n	r_n	$a_T^{q_n} * a_{r_n}$

Therefore, by reference to this table and to the equation (4):

$$x_n = a_T^0(a_0+a_1+\dots+a_{T-1})+a_T^1(a_0+a_1+\dots+a_{T-1})+\dots+a_T^{q_{n-1}}(a_0+a_1+\dots+a_{T-1})+a_T^{q_n}(a_0+a_1+\dots+a_{r_n-1})$$

Then:

$$x_n = \left(\sum_{i=0}^{q_{n-1}} a_T^i \right) \left(\sum_{i=0}^{T-1} a_i \right) + a_T^{q_n} \left(\sum_{i=0}^{r_n-1} a_i \right)$$

We have:

$$x_T = \sum_{i=0}^{T-1} a_i \text{ and } \sum_{i=0}^{q_{n-1}} a_T^i = \frac{1 - a_T^{q_n}}{1 - a_T} \text{ (because } q_{n-1} = q_n - 1; \text{ see table) and } x_{r_n} = \sum_{i=0}^{r_n-1} a_i$$

Therefore:

$$x_n = \left(\frac{1 - a_T^{q_n}}{1 - a_T} \right) x_T + a_T^{q_n} x_{r_n}$$

Consequently, according to the previous formula and to the equation (6):

$$\frac{x_n}{a_n} = \frac{1}{(a_T)^{q_n} * a_{r_n}} \left[\left(\frac{1 - a_T^{q_n}}{1 - a_T} \right) x_T + a_T^{q_n} x_{r_n} \right]$$

Then:

$$\frac{x_n}{a_n} = \frac{1}{a_{r_n}} \left[\left(\frac{1 - \left(\frac{1}{a_T}\right)^{q_n}}{a_T - 1} \right) x_T + x_{r_n} \right]$$

When $n \rightarrow +\infty$, then $q_n \rightarrow +\infty$ and $\frac{1}{a_T}$ is strictly lower than 1, according to (5). Consequently, $\left(\frac{1}{a_T}\right)^{q_n}$ tends to 0, which implies that the sequence $\left(\frac{x_n}{a_n}\right)$ is convergent because it tends to the strictly positive following limit:

$$\frac{1}{a_{r_n}} \left[\left(\frac{1}{a_T - 1} \right) x_T + x_{r_n} \right] = \frac{1}{a_{r_n}} (3u_{n_0} + x_{r_n}) = 3u_{r_n} \text{ (see equality before (5) and equation (2))}$$

Remark that the sequence $\left(\frac{x_n}{3a_n}\right)$ tends to the limit u_{r_n} , this natural number belonging to the following set of \mathbb{T} elements: $u_{n_0}, u_{n_0+1}, \dots, u_{n_0+T-1}$. As this limit is necessarily unique, that fact tends to show that there is only one element in this set, and then $T = 1$ and $u_{r_n} = u_{n_0}$. At this point of that demonstration, let simply precise that $\left(\frac{x_n}{3a_n}\right)$ converges.

3.3 $u_n \rightarrow 1$ if $n \rightarrow +\infty$

According to the equation (2) (replacing u_0 by u_{n_0}), u_n is the sum of two convergent sequences: $\frac{u_{n_0}}{a_n}$ and $\frac{1}{3} \left(\frac{x_n}{a_n}\right)$.

Therefore the sequence (u_n) converges when $n \rightarrow +\infty$. If l is its limit (l is greater or equal to 1 because it is the case of u_n , for all n). Note that we also have: $l = u_{r_n}$ (see end of section 3.2).

After a certain rank, u_n is equal to the number 1 (due to the fact this is a sequence of natural numbers).

According to the equation (1), and for sufficient large n :

$$u_{n+1} = \frac{3u_n + 1}{2^{1+k_{n+1}}}$$

Therefore:

$$l = \frac{3l + 1}{2^{1+k_{n+1}}}$$

Which implies the following equality:

$$l(2^{1+k_{n+1}} - 3) = 1$$

The product of these two natural numbers is equal to 1.

Consequently, each of these two numbers is equal to 1. Effectively, if $pq = 1$ (p and q being integers), q divides 1 (because $p = \frac{1}{q}$), which implies that $q = 1$ (1

is divisible only by itself), and then $p = 1$.

Therefore: $l = 1$.

Moreover $k_{n+1} = 1$, due to the fact that $2^{1+k_{n+1}} - 3 = 1$. The sequence (k_n) tends towards 1.

Therefore, $u_n = 1$ after a certain rank, and the Collatz sequence (s_n) converges.

Note that this last demonstration proves also that if two successive elements of the sequence (u_n) are equal, then this sequence converges and tends towards 1.

Moreover, we can remark that, at the end of section 3.2, we found that the limit of the sequence $(\frac{x_n}{3a_n})$ is equal to u_{r_n} ; as $T = 1$ and $l = 1$, we have $r_n = 0$, and then $u_{r_n} = u_{n_0} = 1$, which is consistent.

4 Conclusion

A Collatz sequence that is periodic after a certain rank converges, i.e. it reaches the value 1.

Therefore, the Collatz trivial cycle 1-4-2 (or 1-2 for a compressed sequence) is unique, which solves a half of the Collatz conjecture.

Note that a consequence of this demonstration is the following: a Collatz sequence that is upper bounded is convergent.

Effectively, such a sequence becomes periodic after a certain rank, with a maximum period that is equal to the number of odd elements of the sequence (u_n) lower than its maximum.

To prove the Collatz conjecture, it is for example sufficient to demonstrate that each Collatz sequence has an upper bound.