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Unstable manifold expansion for Vlasov-Fokker-Planck equation

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Abstract. We investigate the bifurcation of a homogeneous stationary state of Vlasov-Newton equation in one dimension, in presence of a small dissipation modeled by a Fokker-Planck operator. Depending on the relative size of the dissipation and the unstable eigenvalue, we find three different regimes: for a very small dissipation, the system behaves as a pure Vlasov equation; for a strong enough dissipation, the dynamics presents similarities with a standard dissipative bifurcation; in addition, we identify an intermediate regime interpolating between the two previous ones. This work relies on an unstable manifold expansion, performed using Bargman representation for the functions and operators analyzed. The resulting series are estimated with Mellin transform techniques.

Keywords: Vlasov-Fokker-Planck equation; unstable manifold expansion; Bargman representation; Mellin transform.

1. Introduction

Vlasov equation describes the behavior of a system of particles when the force felt by each particle is dominated by the mean-field created by all the others, while collisions are negligible. It plays of course a fondamental role in plasma physics and astrophysics, but also appears in many others fields, such as free electron lasers [1], non linear optics [2], sound propagation in bubbly fluids [3]... Vlasov equation does not possess any mechanism driving the dynamics towards thermal equilibrium, as it neglects collisional effects, as well as noise and friction. This induces a range of unusual behaviors: among those, we will be particularly interested in the peculiar bifurcations close to a weakly unstable stationary state, see [4], and [5] for a recent review.

While the collisionless hypothesis may be a very good approximation for the time scale considered, some kind of relaxation mechanism is usually present, even if small. For plasmas [6] and self gravitating systems [7], collisionnal effects provide this relaxation

mechanism; for cold atoms in a magneto-optical trap, there is a rather strong friction and velocity diffusion [8]; the dynamics of cold atoms in a cavity, although conservative in a first approximation, do contain friction and dissipation terms [9]. It is then natural to investigate the effect of a small relaxation mechanism on the specificities of the Vlasov dynamics. This is not a new endeavor: indeed only a few years after the prediction of Landau damping, Lenard and Bernstein have studied how a small velocity diffusion [10], modeled by a Fokker-Planck operator, would affect Landau's linear analysis. Their work has been since then complemented by many others: see [11, 12] in the context of plasma physics, or [13] for more general potentials. These studies all deal with the linearized Vlasov equation. In this paper, keeping the Fokker-Planck modeling for the relaxation mechanism, we address the question of the non linear dynamics close to a weakly unstable homogeneous stationary state.

In particular we will investigate how the peculiarities of Vlasov bifurcations survive a small Fokker-Planck dissipation. It was shown by J.D. Crawford that unstable manifold expansions for Vlasov equation are plagued by singularities [14, 15] when the real part of the unstable eigenvalue λ tends to 0. To be more specific, the dynamics on the unstable manifold reduces to the following equation, where A is the amplitude of the unstable mode:

$$\frac{dA}{dt} = \lambda A - c_3(\lambda)|A|^2 A + O(A^5). \tag{1}$$

It turns out that c_3, sometimes called the "Landau coefficient", is negative and diverges as λ^{-3} in the $\lambda \to 0^+$ limit, the divergences of the subsequent terms in the series being even more severe. These "Crawford singularities" should be regularized by the Fokker-Planck operator, and we will study what is their fate in the different regimes defined by the two small parameters, $\mathrm{Re}(\lambda)$ and the dissipation, which we will call γ. From now on, we assume λ is real, and thus replace $\mathrm{Re}(\lambda)$ by λ.

Our results include the identification of the following three regimes, characterized by different behaviors of the Landau coefficient:
i) When $\gamma \ll \lambda^3$, $c_3 \propto \lambda^{-3}$: the dissipation essentially has no effect.
ii) When $\lambda^3 \ll \gamma \ll \lambda^{3/4}$, $c_3 \propto \lambda\gamma^{-4/3}$: the dissipation induces a qualitative change in the dynamics; it provides a cut-off for the filamentation in velocity space. Nevertheless, the non linear terms are still dominated by highly oscillating modes in velocity, as in the first regime.
iii) When $\lambda^{3/4} \ll \gamma$, c_3 does not diverge. Nevertheless, we expect that the higher non linear orders may still show some weak singularities. A new qualitative change occurs: the nonlinear terms are now dominated by slowly oscillating modes in velocity.
The knowledge of c_3, combined with (1), allows us to guess the scaling of the saturation amplitude, ie the amplitude of the perturbation reached over timescales of order $1/\lambda$. These results are crucial to analyze a bifurcation of Vlasov equation in presence of a small dissipation, and are summarized on Fig. 1.

A similar interplay between a bifurcation in a continuous Hamiltonian system and a small dissipation has already been studied in the context of the weak instability of

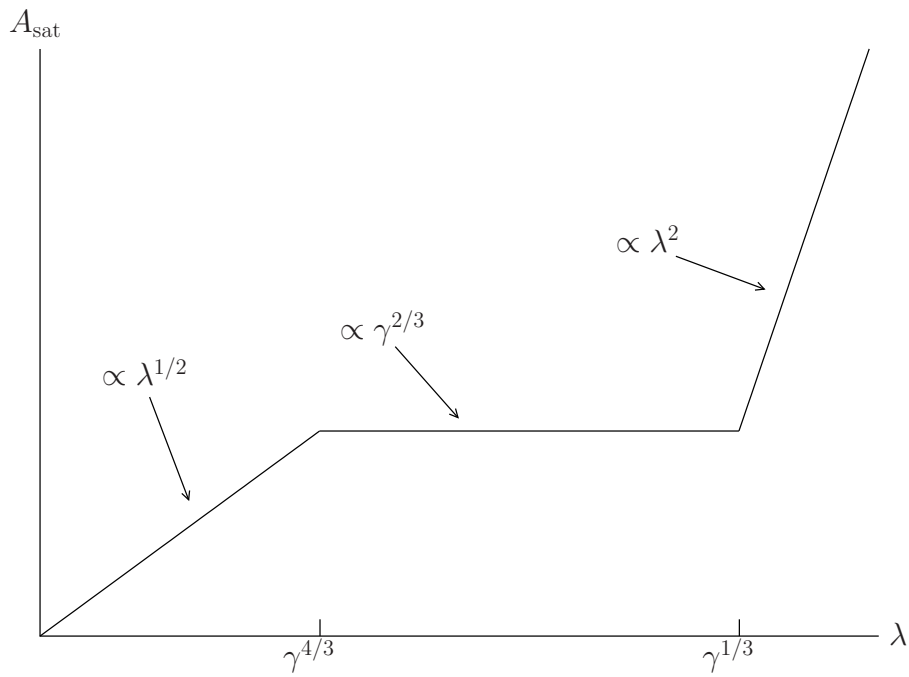

Figure 1. Schematic representation of the paper's main results. On the horizontal axis: the linear instability rate λ; on the vertical axis: the saturation amplitude (ie the amplitude reached by the perturbation over timescales of order $1/\lambda$). The dissipation coefficient γ is fixed. This picture assumes that both γ and λ are small. For $\lambda \gg \gamma^{1/3}$, the trapping scaling $A_{\text{sat}} \propto \lambda^2$, characteristic of Vlasov regime, appears. For $\lambda \ll \gamma^{4/3}$, the normal dissipative scaling $A_{\text{sat}} \propto \lambda^{1/2}$ is recovered. In between we predict a plateau with saturation amplitude $A_{\text{sat}} \propto \gamma^{2/3}$.

a 2D shear flow [16, 17], described by Euler equation plus a small viscosity. Regimes i) and ii) are found in this context [17]; regime iii), as well as the boundary between regimes ii) and iii), appear to be different, we will comment on this later. It is known (see for instance [18], as well as for in [19, 20] in a fluid dynamics context) that in the precise scaling regime $\gamma \propto \lambda^3$, the viscosity enters the equations at the same order as the "inviscid terms": this is compatible with [17] and our results.

In addition, while it is also known that the effective dynamics close to the bifurcation threshold in regime i) is infinite dimensional, we conjecture that it is possible to define a finite dimensional reduced dynamics in regimes ii) and iii). In other words, we expect that in regimes ii) and iii) (1) can be safely truncated at cubic order when λ tends to 0, despite the nonlinear singularities of higher order coefficients. A precise investigation of this conjecture is beyond the scope of this work.

Although we will limit ourselves to the simplest possible setting, in 1D and with periodic boundary conditions, the computations needed to answer these questions are fairly involved. To carry them out, we will make use of the Bargman representation of the Heisenberg algebra‡; this strategy appears to be new in this context. We obtain

‡ We are indebted to Gilles Lebeau for this idea. An alternative strategy is to use in a non linear

an intricate expression as a series for the Landau coefficient c_3; we then analyze this series in the different scaling regimes, sometimes with the help of the Mellin transform; however, one part of this expression, which we expect to be negligible, has resisted our analysis.

The article is organize as follows: In section 2 we introduce more precisely the Vlasov-Newton Fokker-Planck equation and set the problem. In section 3, we solve the linearized Vlasov-Newton Fokker-Planck equation in Bargman representation, providing the dispersion relation, eigenvectors and adjoint eigenvectors. This recovers already known results with a new method. We then turn to the case where the homogeneous stationary solution is weakly unstable, and provide a non linear unstable manifold expansion of the dynamics 3.3. This allows us to discuss the effect of the Fokker-Planck operator on the Crawford's singularities, our main result. We conclude with several remarks and open questions. Several technical parts are detailed in appendices.

2. Setting: the Vlasov-Newton Fokker-Planck equation

2.1. The equation

Our starting point is the Vlasov-Newton-Fokker-Planck equation, which describes, through their phase-space density $F(x, v, t)$, particles interacting through Newtonian interaction, and subjected to a friction and velocity diffusion. To keep the following computations as simple as possible, we stick to one dimension. For later convenience, we also normalize the length of the space interval to 2π. The equation reads:

$$\partial_t F + v\partial_x F - \partial_x \phi = \gamma \partial_v \left(vF + \partial_v F \right) \ , \ \Delta\phi = c \left(\int F dv - 1 \right) \ . \tag{2}$$

We take $c > 0$, which corresponds to a Newtonian (attractive) interaction, and $c < 0$ to a Coulombian (repulsive) one. We have chosen our units so that $k_B T = 1$, hence $f_0(v) = \frac{1}{(2\pi)^{3/2}} \mathrm{e}^{-v^2/2}$ is a stationary solution of this equation. It would be always stable for a repulsive interaction; since we are interested in the weakly unstable case, we assume $c > 0$. Our equation can be seen as a 1D self-gravitating model with periodic boundary conditions. Similar models have received attention as toy models for cosmology [21, 22], or to describe the dynamics of a cloud of trapped cold atoms [23].
We write $F(x, v, t) = f_0(v) + f(x, v, t)$ and we will study f, the perturbation. The equation for f reads:

$$\partial_t f = -v\partial_x f + \partial_x \phi[f] f_0'(v) + \partial_x \phi[f] \partial_v f + \gamma \partial_v \left(vf + \partial_v f \right) \ , \ \Delta\phi = c \int f dv \ . \tag{3}$$

context the velocity Fourier transform used in [10, 11, 12].

2.2. Linear and non linear parts

We split the right hand side of (3) in a linear and a non linear part:

$$\partial_t f = \mathcal{L} \cdot f + \mathcal{N}(f),$$

with

$$\begin{aligned} \mathcal{L} \cdot f &= -v\partial_x f + \partial_x \phi[f] f_0'(v) + \gamma \partial_v \left(v f + \partial_v f \right) \\ \mathcal{N}(f) &= \partial_x \phi[f] \partial_v f. \end{aligned}$$

We change the unknown function from f to $g = \mathrm{e}^{v^2/4} f$, in order to symmetrize the Fokker-Planck operator. Then

$$\partial_t g = L \cdot g + N(g), \tag{4}$$

with

$$L \cdot g = \mathrm{e}^{v^2/4} \mathcal{L} \mathrm{e}^{-v^2/4} \cdot g \ , \ N(g) = \mathrm{e}^{v^2/4} N(\mathrm{e}^{-v^2/4} g).$$

Fourier transforming (4) with respect to the space variable, we obtain:

$$\partial_t \hat{g}_k = L_k \cdot \hat{g}_k + \widehat{N(g)}_k,$$

with

$$L_k \cdot \hat{g}_k = \gamma \left(\left(\frac{1}{2} - \frac{v^2}{4} \right) \hat{g}_k + \partial_v^2 \hat{g}_k \right) - ikv\hat{g}_k + \frac{ic}{k(2\pi)^{3/2}} v \mathrm{e}^{-v^2/4} \int \hat{g}_k(w) \mathrm{e}^{-w^2/4} dw \ .$$

and

$$\widehat{N(g)}_k = \mathrm{e}^{v^2/4} \sum_l i(k-l) \phi[\widehat{\mathrm{e}^{-v^2/4} g}]_{k-l} \partial_v (\mathrm{e}^{-v^2/4} \hat{g}_l).$$

With $p = v/\sqrt{2}$, we obtain with a small abuse of notation, since we do not change the name of the functions):

$$\begin{aligned} L_k \cdot \hat{g}_k &= \frac{\gamma}{2} \left((1-p^2)\hat{g}_k + \partial_p^2 \hat{g}_k \right) - ik\sqrt{2} p \hat{g}_k + \frac{2ic}{k(2\pi)^{3/2}} p \mathrm{e}^{-p^2/2} \int \hat{g}_k(q) \mathrm{e}^{-q^2/2} dq \\ &= \gamma \left[-H_{\mathrm{OH}} - i \frac{k\sqrt{2}}{\gamma} p \right] \hat{g}_k + \frac{ic}{2\pi k} \langle E_0, \hat{g}_k \rangle_{L^2} E_1 \\ &= \gamma \left[-H_{\mathrm{OH}} - \frac{ik}{\gamma} \left(a + a^\dagger \right) \right] \hat{g}_k + \frac{ic}{2\pi k} \langle E_0, \hat{g}_k \rangle_{L^2} E_1 \ , \end{aligned} \tag{5}$$

where we have introduced the harmonic oscillator Hamiltonian on L^2

$$H_{\mathrm{OH}} = \frac{1}{2} \left(-\partial_p^2 + p^2 - 1 \right) \ ,$$

and the annihilation and creation operators on L^2

$$a = \frac{1}{\sqrt{2}} \left(\partial_p + p \right) \ , \ a^\dagger = \frac{1}{\sqrt{2}} \left(-\partial_p + p \right).$$

The $(E_n)_{n\in\mathbb{N}}$ are the normalized eigenstates of H_{OH} in L^2. In particular

$$E_0 = \frac{1}{\pi^{1/4}}\mathrm{e}^{-p^2/2}\ ,\ E_1 = \frac{\sqrt{2}}{\pi^{1/4}}p\mathrm{e}^{-p^2/2}.$$

The nonlinear operator reads:

$$\begin{aligned}
\widehat{N(g)}_k &= \mathrm{e}^{p^2/2}\sum_{l\neq k}\left[\frac{-i}{(k-l)}c\left(\int \mathrm{e}^{-p^2/2}\hat{g}_{k-l}(p)dp\right)\partial_p\left(\mathrm{e}^{-p^2/2}\hat{g}_l\right)\right]\\
&= \sum_{l\neq k}\left[\frac{-i}{(k-l)}c\left(\int \mathrm{e}^{-p^2/2}\hat{g}_{k-l}(p)dp\right)(\partial_p - p)\hat{g}_l\right]\\
&= \sum_{l\neq k}\left[\frac{-i}{(k-l)}c\left(\int \mathrm{e}^{-p^2/2}\hat{g}_{k-l}(p)dp\right)(-\sqrt{2}a^\dagger)\hat{g}_l\right]\\
&= \sum_{l\neq k}\left[\frac{ic\sqrt{2}\pi^{1/4}}{(k-l)}\langle E_0,\hat{g}_{k-l}\rangle a^\dagger\hat{g}_l\right] \qquad (6)
\end{aligned}$$

2.3. Bargman space

We see on (5) and (6) that the linear and nonlinear parts of the equation have a rather simple expression in terms of the Hermite functions, eigenfunctions of the harmonic oscillator. To exploit this remark, we shall use the Bargman representation which is particulary adapted to this problem, and which we quickly describe here. First we define the Bargman transform, which transforms an $L^2(\mathbb{R})$ function into an holomorphic one:

$$(B\varphi)(z) = \frac{1}{(\pi)^{3/4}}\int_{\mathbb{R}} \mathrm{e}^{-p^2/2+\sqrt{2}pz}\varphi(p)dp.$$

Let $\mathcal{H}_z$ be the space of holomorphic functions $u(z)$ such that

$$\iint |u(z)|^2 e^{-|z|^2}dzd\bar{z} < +\infty.$$

Equipped with the following scalar product:

$$\langle u,v\rangle_{\mathcal{H}_z} = \iint \bar{u}(z)v(z)\mathrm{e}^{-|z|^2}dzd\bar{z},$$

$\mathcal{H}_z$ is a Hilbert space. Furthermore the Bargman transform B is an isometry between $L^2(\mathbb{R})$, with the standard scalar product, and $\mathcal{H}_z$. We shall use the following orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\mathcal{H}_z$:

$$e_n(z) = \frac{1}{\sqrt{\pi}}\frac{z^n}{\sqrt{n!}}.$$

From now on, we shall only use scalar products on $\mathcal{H}_z$, and denote them simply by $\langle\cdot,\cdot\rangle$. In Bargman representation, the annihilation, creation and harmonic oscillator Hamiltonian operators are particularly simple:

$$a = \partial_z\ ,\ a^\dagger = z\ ,\ H_{\mathrm{OH}} = z\partial_z.$$

The spectrum of H_{OH} is $\mathbb{N}$, and we see that the (e_n) are eigenfunctions of H_{OH}. We can deduce that the Bargman transform maps the normalized Hermite functions $(E_n)_{n\in\mathbb{N}}$ into the $(e_n)_{n\in\mathbb{N}}$. In particular, the ground state $E_0 = \pi^{-1/4}\mathrm{e}^{-p^2/2}$ is mapped onto $e_0 = \pi^{-1/2}$.

3. Linear study

3.1. Dispersion relation and main eigenvector

The longest wavelength $k = 1$ mode is the most unstable, hence we study the operator L_1. From now on we forget the index 1, and we write $L = L_1$. Starting from (5), we write in Bargman representation

$$L \cdot g = \gamma \left[-H_{\mathrm{OH}} - \frac{i}{\gamma}\left(a + a^\dagger\right)\right] g + \frac{ic}{2\pi}\langle e_0, g\rangle e_1$$

Proposition 3.1 *Let the functions J_n be as defined in the Appendix A. The roots of the equation*

$$\Lambda(\gamma,\lambda) = 1 - \frac{c}{2\pi\gamma^2} J_1(1/\gamma, -\lambda/\gamma). \tag{7}$$

are eigenvalues of L. The eigenvector G associated to such an eigenvalue λ is $G = \sum_n G_n e_n$, with, for any $n \geq 1$

$$G_n = -\frac{c}{2\pi} G_0 \frac{1}{\sqrt{n!}} \left(\frac{-i}{\gamma}\right)^n (\lambda/\gamma) J_n(1/\gamma, -\lambda/\gamma),$$

and G_0 an arbitrary constant.

Proof. The equation defining λ and G is:

$$-\gamma \left[H_{\mathrm{OH}} + \frac{i}{\gamma}(a + a^\dagger)\right] G + \frac{ic}{2\pi}\langle e_0, G\rangle e_1 = \lambda G$$

which can be rewritten, using the notation $G_n =< e_n, G >$:

$$\left[B(-i\sqrt{2}/\gamma) + \frac{\lambda}{\gamma}\right] G = \frac{ic}{2\pi\gamma} G_0 e_1,$$

where we have introduced the operator $B(i\xi)$ (following the notations in [24]):

$$B(i\xi) = H_{\mathrm{OH}} - \frac{i\xi}{\sqrt{2}}\left(a + a^\dagger\right).$$

We will now rely on the precise study of $B(i\xi)$ in [24] to proceed; the important definitions are given in appendix. Hence

$$G = \frac{ic}{2\pi\gamma} G_0 \left[B(-i\sqrt{2}/\gamma) + \frac{\lambda}{\gamma}\right]^{-1} \cdot e_1.$$

We now express G in Bargman representation, using the notations and results of [24] for $R(\xi,\lambda)$, the resolvant of $B(i\xi)$ (the ψ_α^β functions are defined in the appendix):

$$R(\xi,\lambda)\cdot z^\beta = [B(i\xi)-\lambda]^{-1}\cdot z^\beta = \sum_{\alpha\in\mathbb{N}} \psi_\alpha^\beta(\xi,\lambda) z^\alpha.$$

Then $G = ic/(2\pi\gamma)G_0 R(-\sqrt{2}/\gamma, -\lambda/\gamma)\cdot(z/\sqrt{\pi})$, that is, for all n

$$G_n = \frac{ic}{2\pi^{3/2}\gamma} G_0 \sqrt{n!}\sqrt{\pi}\psi_n^1(-\sqrt{2}/\gamma, -\lambda/\gamma).$$

For $n=0$, this yields the dispersion relation

$$\Lambda(\gamma,\lambda) = 1 - \frac{ic}{2\pi\gamma}\psi_0^1(-\sqrt{2}/\gamma, -\lambda/\gamma) = 0.$$

Now $\psi_0^1(\xi,\lambda) = (i\xi/\sqrt{2})J_1(|\xi|/\sqrt{2},\lambda)$ (the J_n functions are defined in the appendix). Hence

$$\Lambda(\gamma,\lambda) = 1 - \frac{c}{2\pi\gamma^2} J_1(1/\gamma, -\lambda/\gamma). \tag{8}$$

The roots of Λ are the eigenvalues of L. Furthermore, from [24] Eqs. (16.4.69) and (16.4.63), we have for $n>1$

$$\psi_n^1(\xi,\lambda) = \frac{1}{n!}\left(\frac{i\xi}{\sqrt{2}}\right)^{n-1}(-\lambda)J_n(|\xi|/\sqrt{2},\lambda).$$

Hence for $n>1$

$$G_n = \frac{ic}{2\pi\gamma} G_0 \frac{1}{\sqrt{n!}}\left(\frac{-i}{\gamma}\right)^{n-1}(\lambda/\gamma)J_n(1/\gamma,-\lambda/\gamma).$$

□

Remark: This computation of the dispersion relation (8) recovers the result of [10, 11, 12, 13], obtained by other means. In the limit $\gamma\to 0$, the dispersion relation (8) reduces to

$$\Lambda(0,\lambda) = 1 - \frac{c}{2\pi}\int_0^{+\infty} \mathrm{e}^{-s^2/2-\lambda s} s ds,$$

which can be shown to coincide with the classical direct computation of the analytically continued dispersion relation from the linearized Vlasov equation; a root of $\Lambda(0,\lambda)=0$ with $\mathrm{Re}(\lambda)>0$ is an eigenvalue of the linearized Vlasov operator, whereas a root with $\mathrm{Re}(\lambda)<0$ is a Landau pole, or a "resonance". Hence, the roots of $\Lambda(\gamma,\lambda)=0$, which are always true eigenvalues of the linearized Vlasov-Fokker-Planck operator when $\gamma>0$, approach the eigenvalues and Landau poles of the linearized Vlasov operator when $\gamma\to 0^+$. This can be seen as a kind of "stochastic stability" for the resonances of the linearized Vlasov operator, a phenomenon studied in other contexts: in fluid dynamics [20], for Pollicott-Ruelle resonances [25, 26], or for a Schrödinger operator [27].

Finally, we also see that G_n vanishes in the limit $\lambda\to 0$ for any $n>0$, which yields $G\propto e_0$ in this limit. This is consistent with the standard computation at $\gamma=0$.

Remark: We shall normalize the G eigenvector such that $\hat{\phi}_{k=1}[Ge^{ix}] = -c\sqrt{2}\int G(p)\mathrm{e}^{-p^2/2}dp = 1$. Hence from now on we take $G_0 = -1/(c\sqrt{2}\pi^{1/4})$.

3.2. Adjoint eigenvector

We shall use later the projection on the eigenvector G, provided by the corresponding adjoint eigenvector. The adjoint linear operator is

$$L^\dagger \cdot h = \gamma \left[-H_{\rm OH} + i\frac{\sqrt{2}}{\gamma} \right] h - \frac{ic}{2\pi} \langle e_1, h \rangle e_0.$$

Proposition 3.2 *Let $\lambda \in \mathbb{R}$ be a real eigenvalue of L. Then the eigenvector of $L^\dagger$ associated with the eigenvalue λ is $\tilde{G} = \sum_n \tilde{G}_n e_n$, with*

$$\tilde{G}_n = -\frac{c}{2\pi} \tilde{G}_1 \frac{1}{\sqrt{n!}} \left(\frac{i}{\gamma} \right)^{n+1} J_n(1/\gamma, -\lambda/\gamma),$$

with $\tilde{G}_1$ an arbitrary constant.

Proof. The eigenvalue equation reads (recall that we assume that the eigenvalue is real):

$$-\gamma[B(i\sqrt{2}/\gamma) + \lambda/\gamma]\tilde{G} = \frac{ic}{2\pi} \tilde{G}_1 e_0,$$

thus

$$\tilde{G} = -\frac{ic}{2\pi\gamma} \tilde{G}_1 R(\sqrt{2}/\gamma, -\lambda/\gamma) \cdot e_0;$$

this translates as

$$\tilde{G}_n = -\frac{ic}{2\pi\gamma} \sqrt{n!} \tilde{G}_1 \psi_n^0(\sqrt{2}/\gamma, -\lambda/\gamma) \ , \ \text{with } \psi_n^0(\xi, \lambda) = \frac{1}{n!} \left(\frac{i\xi}{\sqrt{2}} \right)^n J_n(|\xi|/\sqrt{2}, \lambda).$$

□

Remark: For $n = 1$, the computation above yields the dispersion relation again

$$1 + \frac{ic}{2\pi\gamma} \psi_1^0(\sqrt{2}/\gamma, -\lambda/\gamma) = 0.$$

Since $\psi_1^0(\xi, \lambda) = (i\xi/\sqrt{2}) J_1(|\xi|/\sqrt{2}, \lambda)$, this second expression for the dispersion coincides with the first one (8).

$\mathbb{P}$, the projection on Ge^{ix} is defined as $\mathbb{P} \cdot u = \frac{<\tilde{G}, \hat{u}_1>}{<\tilde{G}, G>} Ge^{ix}$. It will play a role in the nonlinear analysis; hence we need to control the scalar product $< \tilde{G}, G >$.

Proposition 3.3 *The scalar product $< \tilde{G}, G >$ has a finite non zero limit when $\gamma \to 0$, $\lambda \to 0$ (G_0 and $\tilde{G}_1$ are kept fixed).*

Proof.

$$\begin{aligned} < \tilde{G}, G > \ &= \sum_n \tilde{G}_n^* G_n \\ &= \tilde{G}_1^* G_0 \frac{ic}{2\pi\gamma} J_0(1/\gamma, -\lambda/\gamma) + \tilde{G}_1^* G_0 \frac{ic}{2\pi\gamma} (\lambda/\gamma) J_1(1/\gamma, \lambda/\gamma) \\ &\quad + \tilde{G}_1^* G_0 \sum_{n>1} \left(\left(\frac{c}{2\pi} \right)^2 \frac{(\lambda/\gamma)}{n!} \left(\frac{i}{\gamma} \right)^{2n+1} J_n^2(1/\gamma, -\lambda/\gamma) \right) \end{aligned} \tag{9}$$

By the remark after Lemma Appendix A.2, $yJ_0(y,-\lambda y)$ and $y^2J_1(y,-\lambda y)$ have a finite limit when $1/\gamma = y \to \infty$. Hence the first and second terms are not singular.

We now want to estimate the series, for $y \to \infty$ and $\lambda \to 0$. According to Lemma Appendix A.2, we introduce three characteristic values $N_2 = \lambda^{-2}$, $N_3 = y^{2/3}$ and $N_4 = y^2$. Using Lemma Appendix A.2 and Stirling formula, we can approximate for all $n \lesssim N_3$,

$$\frac{y^{2n+2}J_n^2(y,-\lambda y)}{n!} \quad \text{by} \quad \frac{\sqrt{\pi}\mathrm{e}^{-\lambda\sqrt{n}}}{\sqrt{n}}.$$

Indeed, if $n \gg \lambda^{-2}$ this is item ii), and if $n \ll \lambda^{-2}$ this is item i), since in this latter case $\lambda\sqrt{n} \ll 1$. Furthermore, for $n \gg N_3 = y^{2/3}$, and $n \ll y^2$, item iii) in Lemma Appendix A.2 tells us that

$$\frac{y^{2n+2}J_n^2(y,-\lambda y)}{n!} \ll \frac{\mathrm{e}^{-\lambda\sqrt{n}}}{\sqrt{n}}$$

Hence the term in the series is at most equivalent to $\sqrt{\pi}(-1)^n\mathrm{e}^{-\lambda\sqrt{n}}/\sqrt{n}$, which is a convergent series for any $\lambda > 0$. From Appendix B.1, we know that it has a finite limit when $\lambda \to 0^+$. Finally for $n \gg N_4 = y^2$, item iv) ensures that the term in the series is small, at least as an exponential; hence this large n part of the series is not singular either.

Remark: From this computation, we will be able to conclude in the next section that the normalization does not play any role in the divergences of the expansion; it does play a role of course to determine the precise value of the coefficients. Interestingly, the normalization factor is directly related to the derivative of the dispersion relation, as in the pure Vlasov case[15]and Kuramoto models [28, 36]; this suggests that this holds with some generality. More precisely, we have (see Appendix C):

$$< \tilde{G}, G >= G_0\tilde{G}_1^*\frac{ic}{2\pi}\partial_\lambda\Lambda(\gamma,\lambda).$$

From now on we choose $\tilde{G}_1 = -\dfrac{2\sqrt{2}\pi^{5/4}i}{\partial_\lambda\Lambda(\lambda)}$, so that $\langle\tilde{G}, G\rangle = 1$.

3.3. Non linear analysis

3.3.1. Preliminary remarks The Vlasov equation has an uncountable infinity of stable stationary states, and the asymptotic state reached by a growing perturbation is in general unknown; it is precisely one of the goals of expansions such as Crawford's to approximate this final state. However, as soon as a Fokker-Planck operator acts, no matter how small, the stable stationary states reduce to the stable and metastable thermodynamical equilibria. Then the possible final states of the dynamics are essentially known, and the question of their selection is much easier. The main question is now how the final state is reached, and this dynamics may still be non trivial. Indeed there are two dimensionless parameters λ, the linear growth rate, and γ, the relaxation rate related to the Fokker-Planck operator. We will see that the interplay between these two parameters defines different dynamical regimes.

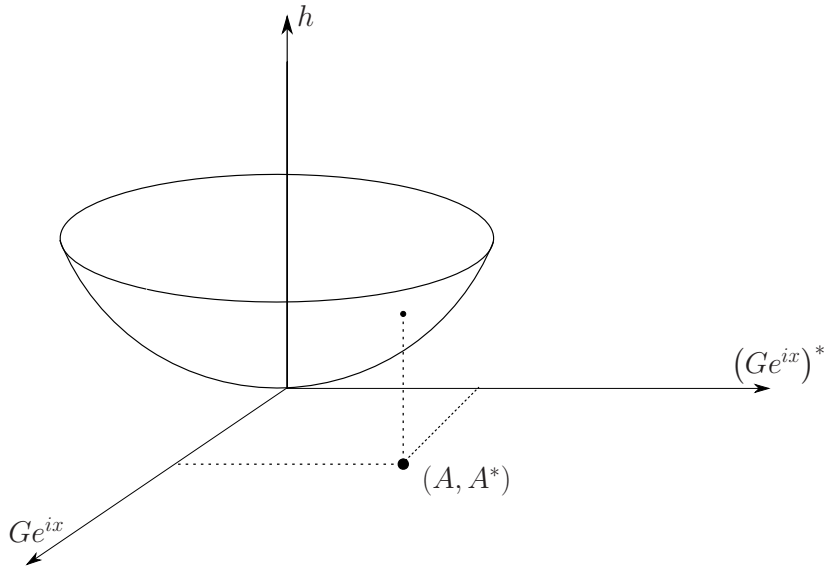

Figure 2. Schematic picture of a generic point of the unstable manifold h and its projection on the unstable eigenspace $\mathbb{P}h$; the coordinates of the projections are A, A^*.

3.3.2. The unstable manifold We follow here a standard route to perform the unstable manifold expansion. There are two unstable eigenvectors, associated with the same real eigenvalue $\lambda > 0$, that are complex conjugate of each other; we will keep for these eigenvectors the notations $G\mathrm{e}^{ix}$ and $G^*\mathrm{e}^{-ix}$. The unstable manifold is two dimensionnal, its tangent plane at $g = 0$ is spanned by the two unstable eigenvectors. We associate to each point h of the unstable manifold its projection onto the unstable eigenspace $\mathbb{P}h = AG\mathrm{e}^{ix} + A^*G^*\mathrm{e}^{-ix}$: this provides a parameterization of the manifold, at least locally. Fig.2 provides a schematic picture.

Assuming this schematic picture is correct, any function on the unstable manifold can be expanded in spatial Fourier series as follows:

$$h = AG\mathrm{e}^{ix} + A^*G^*\mathrm{e}^{-ix} + |A|^2H^{(0)}(p) + A^2H^{(2)}(p)\mathrm{e}^{2ix} + (A^*)^2H^{(-2)}(p)\mathrm{e}^{-2ix} + O((A, A^*)^3). \tag{10}$$

Indeed, the symmetries of the problem severely constrain the form of the expansion, see [28, 29] for details. Hence, at leading non linear order only the Fourier coefficients $-2, 0, 2$ play a role. They are computed in the following proposition.

Proposition 3.4 *At leading non linear order, the formal expansion of the unstable manifold is determined by the functions $H^{(0)} = U + U^*$, with $U = \sum_n U_n e_n$, $U_0 = 0$, $U_1 = i\frac{G_0}{\gamma+2\lambda}$, and, for $n \geq 2$*

$$U_n = -G_0\frac{c}{2\pi}\frac{n}{\gamma n + 2\lambda}\frac{1}{\sqrt{n!}}\left(\frac{-i}{\gamma}\right)^{n-2}\frac{\lambda}{\gamma^2}J_{n-1}(1/\gamma, -\lambda/\gamma) \tag{11}$$

and $H^{(2)} = \sum_n H_n^{(2)} e_n$ *with*

$$H_n^{(2)} = -(i/\gamma)\sum_k \left(\frac{\sqrt{m!}}{\sqrt{(k-1)!}} G_{k-1} \psi_n^{k-1}(2/\gamma, -2\lambda/\gamma) \right) + \frac{ic}{4\pi\gamma} H_0^{(2)} \sqrt{n!} \psi_n^1(2/\gamma, -2\lambda/\gamma)$$

$$H_0^{(2)} = \frac{1}{1 - \frac{ic}{4\pi\gamma}\psi_0^1(2/\gamma, -2\lambda/\gamma)} \sum_k G_{k-1} \frac{\psi_0^{k-1}(2/\gamma, -2\lambda/\gamma)}{\sqrt{(k-1)!}} \tag{12}$$

Proof. We assume the function g, which evolves under the full nonlinear dynamics, is on the unstable manifold. The non linear terms for the relevant Fourier modes $k = 0, 2$ are

$$\widehat{N(g)}_0 = i|A|^2 a^\dagger G - i|A|^2 a^\dagger G^*$$
$$\widehat{N(g)}_2 = -iA^2 a^\dagger G$$

The dynamical equation for g reads

$$\dot{A}G\mathrm{e}^{ix} + \dot{A}^* G^* \mathrm{e}^{-ix} + (\dot{A}A^* + \dot{A}^* A)H^{(0)} + 2\dot{A}AH^{(2)}\mathrm{e}^{2ix} + \ldots = \lambda AG\mathrm{e}^{ix} + \lambda A^* G^* \mathrm{e}^{-ix}$$
$$+|A|^2 L_0 H^{(0)} + A^2 L_2 H^{(2)} \mathrm{e}^{2ix} + cc + \widehat{N(g)}_1 \mathrm{e}^{ix} + cc + \widehat{N(g)}_0 + \widehat{N(g)}_2 \mathrm{e}^{2ix} + cc + \ldots \tag{13}$$

We first pick up the $k = 0$ Fourier component, to write an equation for $H^{(0)}$:

$$2\lambda H^{(0)} = L_0 H^{(0)} + (ia^\dagger G + cc);$$

the $k = 2$ Fourier component furnishes an equation for $H^{(2)}$:

$$2\lambda H^{(2)} = L_2 H^{(2)} - ia^\dagger G.$$

Recalling that $L_0 = -\gamma H_{\mathrm{OH}}$, we solve for $H^{(0)}$. We have $H^{(0)} = U + U^*$, with $U = \sum_{n\geq 0} U_n e_n$ solution of

$$(-\gamma H_{\mathrm{OH}} - 2\lambda)U = -i\sum_n G_n a^\dagger e_n.$$

This is particularly simple, as e_n is a basis of eigenvectors for the operator on the l.h.s. as well as for $a^\dagger$. Since $a^\dagger e_n = \sqrt{n+1}e_{n+1}$ we obtain $U_0 = 0$ and for $n \geq 1$

$$U_n = iG_{n-1}\frac{\sqrt{n}}{\gamma n + 2\lambda}$$
$$= -G_0 \frac{c}{2\pi\gamma}\frac{n}{\gamma n + 2\lambda}\frac{1}{\sqrt{n!}}\left(\frac{-i}{\gamma}\right)^{n-2}\frac{\lambda}{\gamma} J_{n-1}(1/\gamma, -\lambda/\gamma)$$

We now turn to $H^{(2)}$. We have, using the notation $B(i\xi) = H_{OH} - (i\xi/\sqrt{2})(a + a^\dagger)$:

$$[B(-2i\sqrt{2}/\gamma) + 2\lambda/\gamma]H^{(2)} = -(i/\gamma)a^\dagger G + \frac{ic}{4\pi\gamma}\mathbb{P}H^{(2)}.$$

Thus, with the notation $R(\xi, \lambda) = [B(i\xi) - \lambda]^{-1}$:

$$H^{(2)} = -(i/\gamma)R(-2\sqrt{2}/\gamma, -2\lambda/\gamma)a^\dagger G + \frac{ic}{4\pi\gamma}H_0^{(2)} R(-2\sqrt{2}/\gamma, -2\lambda/\gamma)e_1.$$

We now use

$$R(\xi,\lambda)e_n = \sum_m \frac{\sqrt{m!}}{\sqrt{n!}}\psi_m^n(\xi,\lambda)e_m \text{ and } a^\dagger G = \sum_{n\geq 1}\sqrt{n}G_{n-1}e_n$$

to compute $H_n^{(2)}$ for any n:

$$\begin{aligned} H_n^{(2)} &= -(i/\gamma)\sum_{k\geq 1}\left(\frac{\sqrt{n!}}{\sqrt{(k-1)!}}G_{k-1}\psi_n^k(2/\gamma,-2\lambda/\gamma)\right) + \frac{ic}{4\pi\gamma}H_0^{(2)}\sqrt{n!}\psi_n^1(2/\gamma,-2\lambda/\gamma) \\ H_0^{(2)} &= \frac{1}{1-\frac{ic}{4\pi\gamma}\psi_0^1(2/\gamma,-2\lambda/\gamma)}\sum_k G_{k-1}\frac{\psi_0^k(2/\gamma,-2\lambda/\gamma)}{\sqrt{(k-1)!}} \end{aligned}$$

This provides an explicit, but difficult to manipulate, expression for the $H_n^{(2)}$. □

3.3.3. The c_3 coefficient The leading non linear term for $k=1$ is at order A^3:

$$\widehat{N(g)}_1 = |A|^2 A\left(-ia^\dagger(U+U^*) + ia^\dagger H^{(2)} + ic\sqrt{2}\pi^{1/4}\frac{1}{2}\langle e_0, H^{(2)}\rangle a^\dagger G^*\right) \tag{14}$$

Projecting (13) on $G\mathrm{e}^{ix}$, we obtain the main equation

$$\dot{A} = \lambda A + \langle \tilde{G}, \widehat{N(g)}_1\rangle = \lambda A + (c_3^{(1)} + c_3^{(2)} + c_3^{(3)})|A|^2 A. \tag{15}$$

where the $c^{(i)}$ for $i=1,2,3$ correspond to the three terms on the r.h.s. of (14).

Proposition 3.5 *The Landau coefficent c_3 is given by the following expressions*

$$\begin{aligned} c_3^{(1)} &= -i\langle \tilde{G}, a^\dagger(U+U^*)\rangle, \\ c_3^{(2)} &= i\langle \tilde{G}, a^\dagger H^{(2)}\rangle, \\ c_3^{(3)} &= \frac{ic\pi^{1/4}\langle e_0, H^{(2)}\rangle}{\sqrt{2}}\langle \tilde{G}, a^\dagger G^*\rangle, \end{aligned}$$

and

$$\begin{aligned} \langle \tilde{G}, a^\dagger(U+U^*)\rangle &= \frac{-ic}{\pi\partial_\lambda\Lambda}\lambda\sum_{n\geq 3, n \text{ odd}}\frac{n(n-1)}{\gamma(n-1)+2\lambda}\frac{1}{\gamma^{2n}n!}J_{n-2}\left(\frac{1}{\gamma},-\frac{\lambda}{\gamma}\right)J_n\left(\frac{1}{\gamma},-\frac{\lambda}{\gamma}\right), \\ \langle \tilde{G}, a^\dagger G^*\rangle &= \tilde{G}_1^* G_0 - \frac{c^2\tilde{G}_1^* G_0^*}{4\pi^2}\lambda\sum_{n\geq 2}\frac{n}{\gamma^{2n+1}n!}J_{n-1}\left(\frac{1}{\gamma},-\frac{\lambda}{\gamma}\right)J_n\left(\frac{1}{\gamma},-\frac{\lambda}{\gamma}\right). \end{aligned}$$

Proof. These are simple computations using Props. 3.1, 3.2, 3.4, and $G_0\tilde{G}_1^* = 2\pi/(ic\partial_\lambda\Lambda)$. □

3.3.4. Asymptotic analysis of c_3 Our final task is to investigate the behavior of c_3 in the joint limit $\gamma \to 0^+, \lambda \to 0^+$. We first deal the series in $c_3^{(1)}$.

Proposition 3.6 *Assume $\lambda \to 0^+$ and $\gamma \to 0^+$:*

- *if $\lambda \gg \gamma^{1/3}$, then $c_3^{(1)}$ diverges as $1/\lambda^3$; more precisely, $c_3 \sim (-1/4)\lambda^{-3}$;*
- *if $\gamma^{4/3} \ll \lambda \ll \gamma^{1/3}$, then $c_3^{(1)} < 0$, and it diverges as $\lambda\gamma^{-4/3}$;*
- *if $\lambda \ll \gamma^{4/3}$, then $c_3^{(1)}$ does not diverge.*

Proof. First, a simple computation shows that

$$\partial_\lambda \Lambda(\lambda = 0) = \frac{c}{2\sqrt{2\pi}}.$$

Since the series is positive, the sign of $c_3^{(1)}$ is clear from Prop. 3.5.
The proof then relies on the remark that there are three characteristic values for n: $N_1 = \lambda/\gamma$, $N_2 = 1/\lambda^2$, and $N_3 = (1/\gamma)^{2/3}$. According to lemma Appendix A.2, the smallest between N_2 and N_3 provides an effective cut-off for the potentially diverging series. And the prefactor $n(n-1)/[\gamma(n-1) + 2\lambda]$ is equivalent to n/γ (resp. $n^2/(2\lambda)$) for $n \gg N_1$ (resp. $n \ll N_1$).

Regime $\lambda \gg \gamma^{1/3}$: the ordering is $N_2 \ll N_3 \ll N_1$, we have

$$\begin{aligned} c_3^{(1)} &\sim -\frac{c}{\pi\partial_\lambda\Lambda} 2\lambda \sum_{n \text{ odd}} \frac{n(n-1)}{\gamma(n-1)+2\lambda} \frac{1}{n!} \left(\frac{1}{\gamma}\right)^{n+1} J_n(1/\gamma, -\lambda/\gamma) \left(\frac{1}{\gamma}\right)^{n-1} J_{n-2}(1/\gamma, -\lambda/\gamma) \\ &\sim -\frac{c}{2\pi\partial_\lambda\Lambda} \sum_{n \text{ odd}} \frac{n^2}{e^{-n} n^n \sqrt{2\pi n}} \sqrt{\pi} \mathrm{e}^{-n/2+\frac{1}{2}n\ln n - \lambda\sqrt{n}} \sqrt{\pi} \mathrm{e}^{-(n-2)/2+\frac{1}{2}(n-2)\ln(n-2) - \lambda\sqrt{n-2}} \\ &\sim -\frac{c}{2\sqrt{2\pi}\partial_\lambda\Lambda} \sum_{n \text{ odd}} \sqrt{n} e^{-2\lambda\sqrt{n}}. \end{aligned} \tag{16}$$

From the first to the second line, we have neglected $\gamma(n-1)$ in front of 2λ (because $N_2 \ll N_1$), used Stirling formula, and the asymptotics of Appendix A for $y^{p+1}J_p$. From Appendix B.1 with $\alpha = 1/2$, we know the following asymptotics when $t \to 0^+$

$$\sum_{n\geq 1} t^{1/2} e^{-t\sqrt{n}} \sim \frac{4}{t^3} \text{ and } \sum_{n\geq 1} (-1)^n t^{1/2} e^{-t\sqrt{n}} = O(1).$$

Taking the difference, we obtain

$$\sum_{n\geq 1, n \text{ odd}} t^{1/2} e^{-t\sqrt{n}} \sim \frac{2}{t^3}$$

We conclude

$$c_3^{(1)} \sim -\frac{c}{2\sqrt{2\pi}\partial_\lambda\Lambda} \frac{2}{(2\lambda)^3} \sim -\frac{1}{4\lambda^3}$$

Regime $\lambda \ll \gamma^{1/3}$: the ordering is $N_1 \ll N_3 \ll N_2$. We have to compare the sum up to N_1, with prefactor $n^2/(2\lambda)$, and the sum between N_1 and N_3, with prefactor n/γ. The sum up to N_1 gives a contribution $N_1^{3/2} = (\lambda/\gamma)^{3/2}$ (if $\lambda \ll \gamma$, this contribution

disappears). The sum between N_1 and N_3 gives a contribution $\lambda N_3^{1/2}/\gamma = \lambda\gamma^{-4/3}$. Since $\lambda \ll \gamma^{1/3}$, the latter contribution always dominates, and the series is of order $\lambda\gamma^{-4/3}$ (it may be possible to compute the coefficient in front of the diverging factor, but since we will not use it, we do not pursue this route). If $\lambda \ll \gamma^{4/3}$, this diverging contribution disappears. □

The following proposition ensures that $c_3^{(3)}$ never provides the leading order.

Proposition 3.7 *Assume $\lambda \to 0^+$ and $\gamma \to 0^+$:*

- *if $\lambda \gg \gamma^{1/3}$, then the series part in $c_3^{(3)}$ diverges as $1/\lambda$;*
- *if $\lambda \ll \gamma^{1/3}$, then the series part in $c_3^{(3)}$ behaves as $\lambda\gamma^{-2/3}$. In particular, it diverges (slower than $1/\lambda$) if $\lambda \gg \gamma^{2/3}$, and tends to 0 for $\lambda \ll \gamma^{2/3}$.*

Proof: We set again $y = 1/\gamma$, a large parameter. We introduce again $N_2 = 1/\lambda^2$ and $N_3 = y^{2/3}$. Then, according to Appendix A, when $n \ll N_2$ and $n \ll N_3$

$$y^{n+1} J_n(y, -\lambda y) n^{-n/2} \mathrm{e}^{n/2} \underset{y,n,1/\lambda\to\infty,\ n\ll N_2, n\ll N_3}{\longrightarrow} \sqrt{\pi}.$$

Using Stirling formula and simplifying, we obtain, for large n, $n \ll N_2$ and $n \ll N_3$

$$\frac{n y^{2n+1}}{n!} J_{n-1}\left(y, -\lambda y\right) J_n\left(y, -\lambda y\right) \to \text{cste}$$

Furthermore, the smaller between N_2 and N_3 acts as a cut-off, since the term in the series becomes negligible for $n \gg N_2$ or $n \gg N_3$. Hence we have two cases:
i) $\lambda \gg \gamma^{1/3}$ corresponds to $N_2 \ll N_3$. Then the series is $\sim \lambda N_2 \sim 1/\lambda$.
ii) $\lambda \ll \gamma^{1/3}$ corresponds to $N_2 \gg N_3$. Then the series is $\sim \lambda N_3 \sim \lambda\gamma^{-2/3}$.
□

In view of the expression for $H^{(2)}$ given in 3.4, it is clear that the expression of $c_3^{(2)}$ is fairly complicated. As a consequence, we have not been able to provide an asymptotic analysis of $c_3^{(2)}$ when $\lambda \to 0$ and $\gamma \to 0$. Nevertheless, in all cases we are aware of where a similar unstable manifold expansion has been carried out, the contribution of $H^{(2)}$ is asymptotically negligible (see [15] for Vlasov equation without dissipation, [29, 36] for variants of the Kuramoto model, [17] for weakly viscous Euler equation). The following assumption seems then reasonable:
Assumption: The $c_3^{(2)}$ term is never dominant with respect to $c_3^{(1)}, c_3^{(3)}$ in the asymptotic regimes $(\lambda, \gamma) \to (0, 0)$.

Putting together this assumption, Props.3.6 and 3.7, we obtain our final result for the Landau coefficient c_3, announced in the introduction and that we repeat here. First, we see that at least in regime i) and ii) (ie $\gamma^{4/3} \ll \lambda$) c_3 is negative, which suggests a supercritical bifurcation.
Different regimes for the Landau coefficient c_3:
i) When $\gamma \ll \lambda^3$, $c_3 \sim (-1/4)\lambda^{-3}$;
ii) When $\lambda^3 \ll \gamma \ll \lambda^{3/4}$, $c_3 \propto \lambda\gamma^{-4/3}$;
iii) When $\lambda^{3/4} \ll \gamma$, c_3 does not diverge.

Based on these results and Eq.(15), we may now conjecture the scaling of the saturation amplitude A_{sat} for the instability:

- When $\gamma \ll \lambda^3$, $A_{\text{sat}} \propto \lambda^2$ (this is the standard "trapping scaling");
- When $\lambda^3 \ll \gamma \ll \lambda^{3/4}$, $A_{\text{sat}} \propto \gamma^{2/3}$;
- When $\lambda^{3/4} \ll \gamma$, $A_{\text{sat}} \propto \lambda^{1/2}$ (this is the standard scaling for a dissipative supercritical bifurcation).

Final remarks:

(i) In regime i), we recover not only the trapping scaling, but also the universal $-1/4$ prefactor, obtained without dissipation in [15].

(ii) Notice that in regimes i) and ii), the dominant contribution to c_3 is a diverging series; this means that high order Hermite coefficients (ie large n), corresponding to highly oscillating velocity profiles, provide the dominant contribution. In regime ii), the dissipation γ plays a role in the cut-off of the diverging series, contrary to regime i). In regime iii), high order Hermite coefficients have a negligible contribution.

(iii) It is interesting to compare more precisely with the literature on weakly unstable 2D shear flows. In [17], the regimes i) $c_3 \propto \lambda^{-3}$ and ii) $c_3 \propto \lambda\gamma^{-4/3}$ also appear. However, the regime iii) $c_3 = O(1)$ is different, and the boundary between regimes ii) and iii) is different too. A possible explanation is that when the dissipative time scale is shorter than the linear instability time scale (ie $\lambda \ll \gamma$), it is necessary to add an external force to maintain the background shear flow. By contrast, maintaining the gaussian velocity distribution in the present Vlasov-Fokker-Planck setting does not require any extra force, since it is stationary for the dissipation operator.

(iv) The $\lambda \sim \gamma^{1/3}$ boundary already appeared in the literature on Vlasov or 2D Euler equations: in the derivation of the Single Wave Model, taking $\gamma \propto \lambda^3$ is the right scaling to ensure that dissipation enters in the equation at the same order as the "Vlasov terms" [19, 18, 20]. This is consistent with our finding that for $\gamma \ll \lambda^3$, the dissipation has no effect at leading order, while for $\gamma \gg \lambda^3$ it qualitatively modifies the problem.

(v) For $\lambda \gg \gamma^{1/3}$, c_3 behaves as in the pure Vlasov case; it seems safe to conjecture that this conclusion remains true at higher orders as well, and that c_{2p+1} diverges as $1/\lambda^{4p-1}$.

(vi) In the pure Vlasov case, it is known that rescaling time and amplitude as $A(t) = \lambda^2\alpha(\lambda t)$, all terms in the expansion in powers of A contribute at the same order to the equation for α [15]; it is thus impossible to safely truncate the series to obtain a simple ordinary differential equation, which is usually understood as a manifestation of the fact that the effective dynamics close to the bifurcation is actually infinite dimensional [5]. Here, we may conjecture that as soon as $\gamma \gg \lambda^3$ under a rescaling $A(t) = \gamma^{2/3}\alpha(\lambda t)$, the series can be safely truncated, yielding an effective ordinary differential equation for the reduced dynamics. While a full investigation of this

conjecture is beyond the scope of this work, we present in Appendix D a partial computation for the 5th order coefficient, which supports it.

(vii) It is worth noting that the bifurcation of the standard Kuramoto model [30], which shares some similarities with Vlasov equation, do not present the same kind of divergences [28, 29], and has been tackled at a rigorous mathematical level [31, 32, 33]. One may then wonder if the regimes ii) and iii) of Vlasov-Fokker-Planck equation may be also amenable to a mathematical treatment. All these conjectures go well beyond the scope of this work.

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Appendix A. The J_n and ψ_α^β functions

We summarize here some results of [24]. Define the functions

$$J_n(y,\lambda) = \int_0^1 t^{y^2-\lambda} \mathrm{e}^{(1-t)y^2} (1-t)^n \frac{dt}{t}.$$

The functions ψ_α^β define the resolvent $(B(i\xi)-\lambda)^{-1}$ in Bargman representation:

$$(B(i\xi)-\lambda)^{-1} z^\beta = \sum_{\alpha\in\mathbb{N}} \psi_\alpha^\beta z^\alpha.$$

Prop. 16.4.4 in [24] provide the following expression:

Proposition Appendix A.1 *For any $\beta \in \mathbb{N}$*

$$\sum_{\alpha\in\mathbb{N}} \psi_\alpha^\beta(\xi,\lambda) z^\alpha = \sum_{k,\beta_1+\beta_2=\beta} J_{k+\beta_1}\left(|\xi|/\sqrt{2},\lambda\right) \frac{(\beta_1+\beta_2)!}{k!\beta_1!\beta_2!} \left(iz\xi/\sqrt{2}\right)^k \left(-z+i\xi/\sqrt{2}\right)^{\beta_1} z^{\beta_2} \tag{A.1}$$

We will need to study, for large n, y (y will be taken to be $1/\gamma$) and $1/\lambda$

$$\begin{aligned} a_n(y,\lambda) &= y^{n+1} J_n(y,-y\lambda) \\ &= y^{n+1} \int_0^1 \mathrm{e}^{y^2(1-t+\ln t)+\lambda y \ln t} \frac{(1-t)^n}{t} dt \\ &= y^{n+1} \int_0^1 \mathrm{e}^{y^2(x+\ln(1-x))+\lambda y \ln(1-x)} \frac{x^n}{1-x} dx \\ &= y^{n+1} \int_0^1 \mathrm{e}^{\varphi(x)} dx \end{aligned}$$

$$\text{with } \varphi(x) = y^2(x+\ln(1-x)) + \lambda y \ln(1-x) + n\ln x - \ln(1-x) \tag{A.2}$$

Lemma Appendix A.2 *Depending on how n, y and $1/\lambda$ tend to infinity, there are several regimes:*
Case i) If $n^{3/2}y^{-1} \ll 1$ and $\lambda\sqrt{n} \ll 1$:

$$\lim_{\substack{y,n,\lambda^{-1}\to\infty \\ n^{3/2}y^{-1}\ll 1, \lambda\sqrt{n}\ll 1}} a_n(y,\lambda)\mathrm{e}^{n/2-\frac{1}{2}n\ln n} = \sqrt{\pi}.$$

Case ii) If $n^{3/2}y^{-1} \ll 1$ and $\lambda\sqrt{n} \gg 1$:

$$\lim_{\substack{y,n,\lambda^{-1}\to\infty \\ n^{3/2}y^{-1}\ll 1, \lambda\sqrt{n}\gg 1}} a_n(y,\lambda)\mathrm{e}^{n/2-\frac{1}{2}n\ln n}\mathrm{e}^{\lambda\sqrt{n}} = \sqrt{\pi}.$$

Case iii) If $n^{3/2}y^{-1} \gg 1$, $ny^{-2} \ll 1$: then for any quantity $C(n,y)$ such that $C(n,y) \ll n^{3/2}y^{-1}$ (note in particular that $C(n,y)$ can tend to infinity almost as fast as $n^{3/2}y^{-1}$)

$$\limsup_{\substack{y,n,1/\lambda\to\infty \\ n^{3/2}y^{-1}\gg 1}} a_n(y,\lambda)\mathrm{e}^{n/2-\frac{1}{2}n\ln n}\mathrm{e}^{\lambda\sqrt{n}}e^{C(n,y)} \leq 1$$

Case iv) If $ny^{-2} \gg 1$: then there is $\alpha > 0$ such that

$$\limsup_{\substack{y,n,1/\lambda\to\infty \\ ny^{-2}\geq 1}} a_n(y,\lambda)\mathrm{e}^{n/2-\frac{1}{2}n\ln n}\mathrm{e}^{\alpha n} \leq 1$$

Remark: For cases iii) and iv) we do not seek to be as precise as for cases i and ii); we will only need the fact that for $n^{3/2y^{-1}} \gg 1$, $a_n(y,\lambda)\mathrm{e}^{n/2-\frac{1}{2}n\ln n}$ is small enough.
Proof: Our starting point is (A.2). Let us first assume that the integral is concentrated close to $x = 0$, which will be checked self consistently below. Then it is legitimate to Taylor expand around $x = 0$; we have

$$\varphi(x) = y^2\left(-\frac{x^2}{2} - \frac{x^3}{3}\right) - \lambda xy + n\ln x - \lambda y\frac{x^2}{2} + \dots$$

Higher order terms will not contribute to the final result. We differentiate in order to find the maximum:

$$\varphi'(x) = y^2\left(-x - x^2\right) - \lambda y + \frac{n}{x} - \lambda yx + \dots$$

At leading order, we obtain $x^* = x_0 = \sqrt{n}/y$. This is compatible with the above hypotheses as soon as $\mathbf{n \ll y^2}$, that is for cases i), ii) and iii). At following order, we write $x^* = x_0 + x_1$, and get

$$x_1 = -\frac{n}{2y^2} \text{ if } n \gg \lambda y \ , \ x_1 = -\frac{\lambda}{2y} \text{ if } n \ll \lambda y.$$

Introducing into the expansion for φ, we obtain

$$\varphi(x^*) = -\frac{1}{2}n + \frac{1}{2}n\ln n - n\ln y - \lambda\sqrt{n} - \frac{1}{3}\frac{n^{3/2}}{y^3} + \text{smaller terms}.$$

Furthermore, the second derivative is

$$\varphi"(x^*) = -2y^2 + o\left(y^2\right).$$

We approximate now the computation of a_n as a gaussian integral

$$\begin{aligned} a_n(y,\lambda) &\simeq y^{n+1}\mathrm{e}^{\varphi(x^*)}\int_0^1 \mathrm{e}^{-\frac{1}{2}\varphi"(x^*)(x-x^*)^2}dx \\ &\simeq y^{n+1}\mathrm{e}^{\varphi(x^*)}\int_{-yx^*}^{y(1-x^*)} \mathrm{e}^{-u^2}du \\ &\simeq \sqrt{\pi}\mathrm{e}^{-n/2+\frac{1}{2}n\ln n}\mathrm{e}^{-\lambda\sqrt{n}}\mathrm{e}^{-\frac{1}{3}\frac{n^{3/2}}{y}}\mathrm{e}^{\text{smaller terms}}. \end{aligned} \tag{A.3}$$

The "smaller terms" are at most of order n^2/y^2, which may be a large or small quantity.
Case i): $\lambda\sqrt{n} \ll 1$ and $\frac{n^{3/2}}{y} \ll 1$. Hence the two corresponding exponentials can be replaced by one, and the same thing is valid for the "smaller terms".
Case ii): $\lambda\sqrt{n} \gg 1$ and $\frac{n^{3/2}}{y} \ll 1$. Hence the "smaller terms" exponential can be replaced by one, and we have to keep the $\mathrm{e}^{-\lambda\sqrt{n}}$ term.
Case iii): $\frac{n^{3/2}}{y} \gg 1$.The "smaller terms" may be much larger than 1, but are necessarily much smaller than $n^{3/2}/y$; hence we can remove them, at the expense of replacing $n^{3/2}/y$ by any slightly smaller function $C(n,y)$; we keep $\lambda\sqrt{n}$, which may be large or small.
Case iv): When $n \gg y^2$, φ reaches its maximum at x^* close to 1. At leading order $x^* \sim 1 - y^2/n$, $\varphi''(x^*) \sim -n^2/y^2$, and $\varphi(x^*) \sim y^2\ln(y^2/n)$. A gaussian approximation yields

$$a_n(y,\lambda) \sim y^n e^{y^2\ln(y^2/n)+o(y^2\ln(y^2/n))};$$

now writing $y^n = n^{n/2}(y^2/n)^{n/2}$, we have

$$a_n(y,\lambda)e^{-\frac{1}{2}n\ln n+n/2} \sim e^{n/2+(y^2+n/2)\ln(y^2/n)} \ll Ce^{-\alpha n},$$

where the last inequality is because $\ln(y^2/n) \to -\infty$. $\square$

Remark: We will also use the fact that for any n, $a_n(y,\lambda)$ has a finite limit when $y \to \infty, \lambda \to 0$ and n fixed. It is an easy extension of case i) above.

Appendix B. Analytic continuation of Dirichlet series and Mellin transform

For $\alpha > -1$ a real number, we want to study the behavior as $\lambda \to 0^+$ of the functions

$$\varphi_\alpha^+(\lambda) = \sum_{n\geq 1} n^\alpha \mathrm{e}^{-\lambda\sqrt{n}} \quad \text{and} \quad \varphi_\alpha^-(\lambda) = \sum_{n\geq 1}(-1)^n n^\alpha \mathrm{e}^{-\lambda\sqrt{n}}.$$

They fall in the category of Dirichlet series

$$f(\lambda) = \sum_{n\geq 0} c_n g(\mu_n\lambda), \tag{B.1}$$

with $\mu_n = \sqrt{n}$, $c_n = n^\alpha$ or $c_n = (-1)^n n^\alpha$, and $g(y) = \mathrm{e}^{-y}$. We have the following:

Proposition Appendix B.1 *Let $\alpha > -1$.*

$$\varphi_\alpha^+(\lambda) \underset{\lambda\to 0^+}{\sim} \frac{\Gamma\big(2(\alpha+1)\big)}{\lambda^{2(\alpha+1)}} \ , \ \varphi_\alpha^-(\lambda) \underset{\lambda\to 0^+}{\to} C_\alpha.$$

Proof: We use Mellin transforms:

Definition Appendix B.2 *Let f be a locally integrable function on $\mathbb{R}_+$. Its Mellin transform Mf is defined as*

$$Mf(s) = \int_0^\infty f(x)x^{s-1}dx.$$

Under appropriate conditions on f, this integral can sometimes be guaranteed to converge on a strip in $\mathbb{C}$, $\alpha < \mathrm{Re}(s) < \beta$, called "the fondamental strip". On this strip, Mf is analytic, and it may be meromorphically continuable in a larger strip, or in $\mathbb{C}$. The important point is that the poles of this meromorphic continuation are in direct correspondence with the asympototic behavior of $f(x)$: a real simple pole σ on the left of the fundamental strip contributes in the asymptotic expansion a term $R_\sigma x^{-\sigma}$, where R_σ is the residue of (the continued) $Mf(s)$ at the pole σ (see [35]).

A straightforward computation shows that for a Dirichlet series (B.1)

$$Mf(s) = F(s)Mg(s),$$

with

$$F(s) = \sum_n \frac{c_n}{\mu_n^s}.$$

We now specialize this to our case. First we note that the Mellin transform of the exponential is defined for $\mathrm{Re}(s) > 0$, and is the Γ function. Then, for $\mathrm{Re}(s) > 2(\alpha+1)$

$$\begin{aligned} \sum_{n\geq 1} \frac{n^\alpha}{n^{s/2}} &= \zeta(s/2-\alpha) \\ \sum_{n\geq 1} \frac{(-1)^n n^\alpha}{n^{s/2}} &= -\eta(s/2-\alpha), \end{aligned} \tag{B.2}$$

where ζ is the Riemann ζ function and η is the Dirichlet η function. We conclude that for $\mathrm{Re}(s) > \max[0, 2(\alpha+1)] > 2(\alpha+1)$

$$\begin{aligned} M\varphi_\alpha^+(s) &= \zeta(s/2-\alpha)\Gamma(s), \\ M\varphi_\alpha^-(s) &= -\eta(s/2-\alpha)\Gamma(s). \end{aligned}$$

From these expressions, it is clear that $M\varphi_\alpha^+$ and $M\varphi_\alpha^-$ can be meromorphically continued to the whole complex plane. It is known that $\Gamma(s)$ has simple poles at $s = 0$ and the negative integers. Since the Riemann $\zeta(z)$ function has its rightmost pole at $z = 1$, which is simple and with residue 1, the continued $M\varphi_\alpha^+$ has its rightmost pole at $s = 2(\alpha+1)$ (remember $\alpha > -1$), with residue $2\Gamma\big(2(\alpha+1)\big)$. Exploiting the

correspondence between these poles and the asymptotic behavior of the functions $\varphi_\alpha^+(\lambda)$ and $\varphi_\alpha^-(\lambda)$ when $\lambda \to 0^+$, we obtain:

$$\varphi_\alpha^+(\lambda) \sim 2\Gamma(2(\alpha+1))\lambda^{-2(\alpha+1)}.$$

Similarly, since the Dirichlet η function is holomorphic (see for instance [34]), the continued $M\varphi_\alpha^-$ has simple poles at 0 and the negative integers. For φ_α^-, the dominant pole (ie the one with the largest real part) is then 0, it is simple, hence we conclude that the dominant term in the asymptotic expansion of φ_α^- is a constant. In other words, φ_α^- has a finite limit when $\lambda \to 0^+$. $\square$

Appendix C. Computation of the normalization factor $\langle \tilde{G}, G \rangle$

The dispersion relation reads, with $y = 1/\gamma$:

$$\Lambda(y,\lambda) = 1 - \frac{y^2 c}{2\pi} J_1(y, -\lambda y) = 0. \tag{C.1}$$

Introducing the definition of the function J_1:

$$\begin{aligned}
\partial_\lambda \Lambda(y,\lambda) &= -\frac{y^3 c}{2\pi} \int_0^1 t^{y^2+\lambda y} \mathrm{e}^{(1-t)y^2} (1-t) \ln t \frac{dt}{t} \\
&= \frac{y^3 c}{2\pi} \int_0^1 t^{y^2+\lambda y} \mathrm{e}^{(1-t)y^2} (1-t) \sum_{n\geq 1} \frac{(1-t)^n}{n} \frac{dt}{t} \\
&= \frac{yc}{2\pi} \sum_{n\geq 1} \frac{y^2 J_{n+1}(y, -\lambda y)}{n}.
\end{aligned} \tag{C.2}$$

We now make use of the recurrence relation (16.4.63) in [24]:

$$\text{For } n > 0: \ n(J_n - J_{n-1}) + y^2 J_{n+1} + \lambda y J_n = 0 \ ,$$

and

$$\text{for } n = 0 : y^2 J_1 + \lambda y J_0 = 1. \tag{C.3}$$

We obtain

$$\begin{aligned}
\partial_\lambda \Lambda(y,\lambda) &= \frac{yc}{2\pi} \sum_{n\geq 1} \frac{y^2 J_{n+1}(y,-\lambda y)}{n} = -\frac{yc}{2\pi} \sum_{n\geq 1} \left(J_n - J_{n-1} + \frac{\lambda y}{n} J_n \right) \\
&= \frac{yc}{2\pi} \left(J_0(y,-\lambda y) - \lambda y \sum_{n\geq 1} \frac{J_n(y,-\lambda y)}{n} \right),
\end{aligned} \tag{C.4}$$

where we have used $\lim_{n\to\infty} J_n(y,-y\lambda)=0$ (see Appendix A). Coming back to $\langle\tilde{G},G\rangle$:

$$\begin{aligned}\langle\tilde{G},G\rangle &= \sum_n \tilde{G}_n^* G_n = \sum_n \frac{icy}{2\pi}\tilde{G}_1^*\frac{1}{\sqrt{n!}}(-iy)^n J_n(y,-\lambda y)G_n \\ &= G_0\tilde{G}_1^*\left(\frac{icy}{2\pi}J_0(y,-\lambda y)+\left(\frac{icy}{2\pi}\right)^2(\lambda y)\sum_{n\geq 1}\frac{1}{n!}(-iy)^{2n-1}J_n^2(y,-\lambda y)\right) \\ &= G_0\tilde{G}_1^*\frac{icy}{2\pi}\left(J_0(y,-\lambda y)-\frac{c}{2\pi}(\lambda y)\sum_{n\geq 1}\frac{(-y^2)^n}{n!}J_n^2(y,-\lambda y)\right)\end{aligned} \tag{C.5}$$

Now we re-express the series, with $a=y^2+\lambda y$:

$$\begin{aligned}\sum_{n\geq 1}\frac{(-y^2)^n}{n!}J_n^2(y,-\lambda y) &= \int_0^1\int_0^1 (ut)^{a-1}\mathrm{e}^{y^2(1-t+1-u)}\sum_{n\geq 1}\frac{((-y^2)(1-t)(1-u))^n}{n!}\,du\,dt \\ &= \int_0^1\int_0^1 (ut)^{a-1}\mathrm{e}^{y^2(1-t+1-u)}\left(e^{-y^2(1-t)(1-u)}-1\right)du\,dt \\ &= \int_0^1 t^{a-1}\mathrm{e}^{y^2(1-t)}\int_0^1 u^{a-1}\mathrm{e}^{y^2(1-u)t}\,du\,dt - J_0^2(y,-\lambda y) \\ &= \int_0^1 t^{a-1}\mathrm{e}^{y^2(1-t)}\left(e^{ty^2}\left(ty^2\right)^{-a}\gamma(a,ty^2)\right)dt - J_0^2(y,-\lambda y) \\ &= y^2e^{y^2}(y^2)^{-a}\int_0^1\frac{\gamma(a,y^2t)}{t}\,dt - J_0^2(y,-\lambda y) \\ &= -y^2e^{y^2}(y^2)^{-a}\int_0^1 \mathrm{e}^{-y^2t}(y^2t)^{a-1}\ln t\,dt - J_0^2(y,-\lambda y) \\ &= \sum_{n\geq 1}\frac{J_n(y,-\lambda y)}{n} - J_0^2(y,-\lambda y);\end{aligned} \tag{C.6}$$

we have used the incomplete Gamma function [37] $\gamma(a,z)=\int_0^z t^{a-1}e^{-t}dt$ (not to be confused with the friction parameter γ), and an integration by part to get the sixth equality. Replacing in (C.5) we get

$$\langle\tilde{G},G\rangle = G_0\tilde{G}_1^*\frac{icy}{2\pi}\left(J_0(y,-\lambda y)\left(1+\lambda y\frac{c}{2\pi}J_0(y,-\lambda y)\right)-\lambda y\frac{c}{2\pi}\sum_{n\geq 1}\frac{J_n(y,-\lambda y)}{n}\right) \tag{C.7}$$

Using (C.3) with (C.1) gives

$$\begin{aligned}\langle\tilde{G},G\rangle &= G_0\tilde{G}_1^*\frac{ic^2y}{(2\pi)^2}\left(J_0(y,-\lambda y)-\lambda y\sum_{n\geq 1}\frac{J_n(y,-\lambda y)}{n}\right) \\ &= G_0\tilde{G}_1^*\frac{ic}{2\pi}\partial_\lambda\Lambda(\lambda).\end{aligned} \tag{C.8}$$

We had set $G_0=-1/(c\sqrt{2}\pi^{1/4})$. Hence we choose $\tilde{G}_1=-\dfrac{2\sqrt{2}\pi^{5/4}i}{\partial_\lambda\Lambda(\lambda)}$, so that $\langle\tilde{G},G\rangle=1$.

Appendix D. A first computation at 5th order

The dominant term for c_3 comes for $H^{(0)}$, the zeroth Fourier coefficient of h. Actually, $H^{(0)}$ can be expanded as

$$H^{(0)}(p) = H^{(0,0)} + |A|^2 H^{(0,1)} + O(|A|^4),$$

and only the leading order $H^{(0,0)}$ contributes to c_3. $H^{(0,1)}$ can be computed by pushing further the non linear expansion, and identifying the terms with the same powers in A and A^*. One obtains the following equation

$$(4\lambda - L_0) \cdot H^{(0,1)} = u,$$

where u is a source term depending on things we have already computed. In particular, u contains a term equal to $-2c_3 H^{(0,0)}$. We call $H_0^{(0,1)}$ the solution of

$$(4\lambda - L_0) \cdot H_0^{(0,1)} = -2c_3 H^{(0,0)},$$

and we make the assumption that the leading contribution to c_5 is given by

$$\langle \tilde{G}, a^\dagger H_0^{(0,1)} \rangle.$$

There are several other terms contributing to c_5; we expect that they are never dominant, but have no proof.

Since $H^{(0,0)} = U + U^*$, we now solve the equation in X

$$(4\lambda - L_0) \cdot X = U,$$

where we recall that $U = \sum_{n \geq 0} U_n e_n$. Writing $X = \sum_n X_n e_n$, we obtain

$$X_n = \frac{U_n}{\gamma n + 4\lambda} = -G_0 \frac{c\sqrt{2}}{2\pi\gamma} \frac{n}{(\gamma n + 2\lambda)(\gamma n + 4\lambda)} \frac{1}{\sqrt{n!}} \left(\frac{-i}{\gamma}\right)^{n-2} \frac{\lambda}{\gamma} J_{n-1}(1/\gamma, -\lambda/\gamma).$$

We now have to compute $\langle \tilde{G}, a^\dagger X \rangle = \sum_n \tilde{G}_n^* \sqrt{n} X_{n-1}$. Using Prop.3.2, we have to estimate a series, whose term A_n is (with C a constant whose precise value may vary, and which is of no consequence regarding the asymptotic behavior of the series):

$$A_n = C \frac{\lambda y^2 n(n-1)}{[(n-1) + 4\lambda y][(n-1) + 2\lambda y]} \frac{1}{n!} y^{n+1} J_n(y, -\lambda y) y^{n-1} J_{n-2}(y, -\lambda y);$$

we have used here the notation $y = 1/\gamma$. For n large, we use Stirling formula for the $1/n!$, and Appendix A for the $y^{l+1} J_l$ terms. According to Appendix A, we have to distinguish different cases, depending on the size of n with respect to the characteristic values $N_2 = 1/\lambda^2$, and $N_3 = y^{2/3}$; furthermore, another characteristic value for n appear: $N_1 = \lambda y$.

Case 1: $\lambda \gg \gamma^{1/3}$, $1 \ll N_2 \ll N_3 \ll N_1$
In this case, for $1 \ll n \ll N_2$, we also have $n \ll N_3$, and

$$\frac{1}{n!} y^{n+1} J_n(y, -\lambda y) y^{n-1} J_{n-2}(y, -\lambda y) \sim C n^{-3/2},$$

hence for $1 \ll n \ll N_2$

$$A_n \sim C n^{1/2} \lambda^{-1}.$$

The sum of these terms up to N_2 gives a contribution $C\lambda^{-1} N_2^{3/2} \sim C\lambda^{-4}$. For $n \gg N_2$, A_n is very small, and this gives a non diverging contribution to the series. Including the overall c_3 factor which behaves as λ^{-3}, we find finally

$$c_5 \sim C\lambda^{-7}.$$

This is consistent with the results of [15] for $\gamma = 0$.

Case 2: $\gamma \ll \lambda \ll \gamma^{1/3}$, $1 \ll N_1 \ll N_3 \ll N_2$
In this case, for $1 \ll n \ll N_1$, we also have $n \ll N_2$ and $n \ll N_3$, thus, thanks to Appendix A

$$\frac{1}{n!} y^{n+1} J_n(y, -\lambda y) y^{n-1} J_{n-2}(y, -\lambda y) \sim C n^{-3/2};$$

hence for $1 \ll n \ll N_1$

$$A_n \sim C n^{1/2} \lambda^{-1}.$$

The sum of these terms up to N_1 gives a contribution $C\lambda^{-1} N_1^{3/2} \sim C(\lambda y^3)^{1/2}$.
For $N_1 \ll n \ll N_3$, we have

$$A_n \sim C\lambda y^2 n^{-3/2}.$$

The sum of these terms from N_1 to N_2 gives a contribution (notice this is a convergent series, we estimate it by its first term) $C\lambda y^2 N_1^{-3/2} = C(y/\lambda)^{1/2}$. Since $\lambda \gg 1/y$, this is smaller than the contribution from the sum up to N_1, and we neglect it. For $n \gg N_3$, A_n is very small and does not contribute a diverging term. Overall, including the factor $c_3 \sim C\lambda y^{4/3}$, we find

$$c_5 \sim C\lambda^{3/2} y^{17/6}.$$

For $\lambda \propto y^{-1/3}$, this coincides with the previous case.

Case 3: $\lambda \ll \gamma$, $N_1 \ll 1 \ll N_3 \ll N_2$
As above, for $n \gg N_3$, A_n is very small and these values do not contribute. For $1 \leq n \ll N_3$, we are always in the $n \gg N_1$ regime, thus we have, using Appendix A

$$A_n \sim C\lambda y^2 n^{-3/2}.$$

This is a convergent series; hence we estimate its sum from $n = 1$ to $n \simeq N_3$ by its first term λy^2. To include the factor c_3, we have to distinguish $\gamma^{4/3} \ll \lambda \ll \gamma$, where $c_3 \sim C\lambda y^{4/3}$, and $\lambda \ll \gamma^{4/3}$, where $c_3 = O(1)$.

Case 3a: $\gamma^{4/3} \ll \lambda \ll \gamma$

$$c_5 \sim C\lambda^2 y^{10/3}.$$

Case 3b: $\lambda \ll \gamma^{4/3}$

$$c_5 \sim C\lambda y^2.$$

In the standard situation where only one, or two, eigenvalues cross the imaginary axis at the bifurcation point, while the others keep a finite, negative, real part, the coefficients $c_3, c_5, \ldots$ do not diverge. In this case, if $c_3 < 0$ (ie supercritical bifurcation), the reduced dynamics suggests that the amplitude A first grows then saturates at a value $A_{\text{sat}} \propto \lambda^{1/2}$, when the linear growth and the non linear term $-c_3|A|^2A$ balance. For this amplitude, it is clear that the next nonlinear term $c_5|A|^4A$ is of order $\lambda^{5/2}$, which is much smaller than λA and $c_3|A|^2A$. This suggests that the series for the reduced dynamics can be truncated at order A^3, still providing an exact description for the evolution of A in the limit $\lambda \to 0$.

For $\lambda \gg \gamma^{1/3}$, the situation is exactly as in the pure Vlasov case [15], and is very different. The $O(A^3)$ non linearity suggests a saturation amplitude $A_{\text{sat}} \sim \lambda^2$; hence $c_5|A|^4A \sim \lambda^{-3}$ is of the same order of magnitude as the two first terms. The series then cannot be safely truncated, no matter how small λ is. It is natural to conjecture that this is related to the fact that one should look for an infinite dimensional reduced dynamics in this case [5].

For $\gamma \ll \lambda \ll \gamma^{1/3}$, the $O(A^3)$ non linearity suggests a saturation amplitude $A_{\text{sat}} \sim y^{2/3}$. This yields with **Case 2** above $c_5|A|^4A \sim (\lambda^3 y)^{1/2}$, while $\lambda A, c_3|A|^2A = O(\lambda y^{2/3})$. Hence $c_5|A|^4A$ is negligible at saturation for any $\gamma \ll \lambda \ll \gamma^{1/3}$, which suggests that the series can be truncated, and that the dynamics can be reduced to finite dimension.

For $\lambda \ll \gamma$ the c_5 divergence is even weaker; thus, as in **Case 2** we expect that the series can be truncated and the dynamics reduced to finite dimension.

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