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### **Abstract**

The De Moivre's formula and the compound identities can be used to expand  $\sin n\theta$  and  $\cos n\theta$  into the powers of  $\cos \theta$  and  $\sin \theta$ , where  $n$  is an integer[1]. This paper shows the formulation of the formula to determine the co-efficients of  $\cos \theta$  and  $\sin \theta$ . The general formula of the expansion of  $\cos n\theta$  and  $\sin n\theta$  is obtained from the formulas of the co-efficients. It will be also shown that the co-efficients are made of binomial co-efficients[1].

# The expansion of $\sin n\theta$ and $\cos n\theta$ in terms of the powers of $\sin \theta$ and $\cos \theta$ respectively

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## Introduction

The De Moivre's formula and the compound identities can be used to expand  $\sin n\theta$  and  $\cos n\theta$  into the powers of  $\cos \theta$  and  $\sin \theta$ , where  $n$  is an integer. The expansion of  $\cos n\theta$  can be written in terms of powers of  $\cos \theta$ , for all positive values of  $n$ . e.g  $\cos 6\theta = 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1$ . [1]

However  $\sin n\theta$  can only be written in terms of  $\sin \theta$ , where  $n$  is an odd number. e.g  $\sin 5\theta = 5 \sin \theta - 20 \sin^3 \theta + 16 \sin^5 \theta$ . This is due to the fact that  $\sin 2\theta = 2 \cos \theta \sin \theta$ , which can cannot be simplified further. Therefore  $\sin n\theta$  must be expressed in terms of  $\sin 2\theta$  and  $\sin \theta$ , for even values of  $n$ . [1]

The De Moivre's formula can be written as follows [1]:

$$(\cos \theta + i \sin \theta)^n = \sum_{P=0}^n \left\{ \binom{n}{P} (\cos \theta)^n (i \sin \theta)^{n-P} \right\} \quad (1)$$

The expansion of  $\cos n\theta$  and  $\sin n\theta$  can be written as follows [1]:

$$\cos n\theta = \operatorname{Re} \left[ \sum_{P=0}^n \left\{ \binom{n}{P} (\cos \theta)^n (i \sin \theta)^{n-P} \right\} \right] \quad (2)$$

$$\sin n\theta = \operatorname{Im} \left[ \sum_{P=0}^n \left\{ \binom{n}{P} (\cos \theta)^n (i \sin \theta)^{n-P} \right\} \right] \quad (3)$$

The answers in (2) and (3) will be written in terms of  $\sin \theta$  and  $\cos \theta$ . In this paper, we want to find the general formula that express  $\sin n\theta$  and  $\cos n\theta$  into the powers of  $\cos \theta$  and  $\sin \theta$  respectively.

(2) and (3) suggest that we can expect the co-efficients of  $\cos \Theta$  and  $\sin \Theta$  to be made of the binomial co-efficients.

## 1 Development of the formula to determine the co-efficient of $\cos \Theta$ in the expansion of $\cos n\Theta$

### 1.1 Laws of determining co-efficient of $\cos \Theta$

The formulas have laws which govern them. It is essential to construct laws to simplify the proceedings of finding a formula.

*Note:*  $\cos n\Theta = 2^{(n-1)} \cos^n \Theta - A \cos^{n-2} \Theta + B \cos^{n-4} \Theta - C \cos^{n-6} \Theta + D \cos^{n-8} \Theta - \dots \pm n \cos \Theta / 1$ , where A, B, C, D... are co-efficients. The exponents of  $\cos \Theta$  corresponds to the value of n. i.e If n is an odd number then the exponent of  $\cos \Theta$  will be an odd number.

**Definition 1.1.1** The  $\cos \Theta$  that has an exponent(**E**) of n is called the **First Term**. i.e In the second term  $E=n-2$ , third term  $E=n-4$ ,...

**Definition 1.1.2** The first term has the co-efficient of  $2^{n-1}$ . The last term has a co-efficient of n, where n is an odd numbers. If n is an even number, the last term is always 1. Since the co-efficient of  $\cos \Theta$  and the value of 1 in the last terms are known, We don't consider them as terms. i.e The first term start at  $\cos(2 \times \Theta)$ . So any term start to exist at  $\cos(2 \times T_m \times \Theta)$ , where  $T_m$  is the term number. i.e In first term  $T_m=1$ , second term  $T_m=2$ ,...

**Remark 1.1.3 Note:** The first term and last term does exist and are ignored for simplification purposes.

**Definition 1.1.4** The number of times that the term existed since appeared for the first time is called the **subscript(S)**.

e.g. First term

$$\cos(2 \times \Theta) \Rightarrow S=1,$$

$$\cos(3 \times \Theta) \Rightarrow S=2,$$

$$\cos(10 \times \Theta) \Rightarrow S=9,$$

$$\cos(n \times \Theta) \Rightarrow S=E-1.$$

**Definition 1.1.5** The term number has a relationship with n and E, which can be written as follows:

$$T_m = \frac{1}{2} [n - E] + 1$$

### 1.2 Simplification of the co-efficients

#### 1.2.1 Expansions of $\cos n\Theta$

The expansion of  $\cos n\Theta$  in terms of the powers of  $\cos \Theta$ .

$$\cos \Theta = \cos \Theta$$

$$\begin{aligned}
\cos 2\Theta &= 2\cos^2\Theta - 1 \\
\cos 3\Theta &= 4\cos^3\Theta - 3\cos\Theta \\
\cos 4\Theta &= 8\cos^4\Theta - 8\cos^2\Theta + 1 \\
\cos 5\Theta &= 16\cos^5\Theta - 20\cos^3\Theta + 5\cos\Theta \\
\cos 6\Theta &= 32\cos^6\Theta - 48\cos^4\Theta + 18\cos^2\Theta - 1 \\
\cos 7\Theta &= 64\cos^7\Theta - 112\cos^5\Theta + 56\cos^3\Theta - 7\cos\Theta \\
\cos 8\Theta &= 128\cos^8\Theta - 256\cos^6\Theta + 160\cos^4\Theta - 32\cos^2\Theta + 1 \\
\cos 9\Theta &= 256\cos^9\Theta - 576\cos^7\Theta + 432\cos^5\Theta - 120\cos^3\Theta + 9\cos\Theta \\
\cos 10\Theta &= 512\cos^{10}\Theta - 1280\cos^8\Theta + 1120\cos^6\Theta - 400\cos^4\Theta + 50\cos^2\Theta - 1
\end{aligned}$$

As mentioned above, first term and the last term were only two terms that can be easily determined. So it essential to come with simplifications means. In order to simplify, there were adjustment were done on the co-efficients of the  $\cos \Theta$ .

$$\begin{aligned}
&\text{Let } n_1=1, n_2=2, n_3=3, \dots, n_n=n \\
&\text{and } 2^{(n_2-1)} = 2, 2^{(n_3-1)} = 4, \dots, 2^{(n_n-1)} = 2^{n-1}
\end{aligned}$$

### 1.2.2 Simplification of the expansions of $\cos n\Theta$

After simplification the expansion of  $\cos n\Theta$  above will look as follows:

$$\begin{aligned}
\cos \Theta &= 2^{(n_1-1)} \cos \Theta \\
\cos 2\Theta &= 2^{(n_2-1)} \cos^2 \Theta - 1 \\
\cos 3\Theta &= 2^{(n_3-1)} \cos^3 \Theta - n_3 \cos \Theta \\
\cos 4\Theta &= 2^{(n_4-1)} \cos^4 \Theta - (2^{(n_2-1)} + 2n_3) \cos^2 \Theta + 1 \\
\cos 5\Theta &= 2^{(n_5-1)} \cos^5 \Theta - (2^{(n_3-1)} + 2^{n_2} + 4n_3) \cos^3 \Theta + n_5 \cos \Theta \\
\cos 6\Theta &= 2^{(n_6-1)} \cos^6 \Theta - (2^{(n_4-1)} + 2^{n_3} + 2^{(n_2+1)} + 8n_3) \cos^4 \Theta + (2^{(n_2-1)} + 2n_3 \\
&+ 2n_5) \cos^2 \Theta + 1 \\
\cos 7\Theta &= 2^{(n_7-1)} \cos^7 \Theta - (2^{(n_5-1)} + 2^{n_4} + 2^{(n_3+1)} + 2^{(n_2+2)} + 16n_3) \cos^5 \Theta + \\
&(2^{(n_3-1)} + 2 \cdot 2^{n_2} + 8n_3 + 4n_5) \cos^3 \Theta - n_7 \cos \Theta \\
\cos 8\Theta &= 2^{(n_8-1)} \cos^8 \Theta - (2^{(n_6-1)} + 2^{n_5} + 2^{(n_4+1)} + 2^{(n_3+2)} + 2^{(n_2+3)} + 32n_3) \cos^6 \Theta \\
&+ (2^{(n_4-1)} + 2 \cdot 2^{n_3} + 6 \cdot 2^{n_2} + 24n_3 + 8n_5) \cos^4 \Theta - (2^{(n_2-1)} + 2n_3 + 2n_5 + 2n_7) \cos^2 \Theta \\
&+ 1 \\
\cos 9\Theta &= 2^{(n_9-1)} \cos^9 \Theta - (2^{(n_7-1)} + 2^{n_6} + 2^{(n_5+1)} + 2^{(n_4+2)} + 2^{(n_3+3)} + 2^{(n_2+4)} + \\
&64n_3) \cos^7 \Theta + (2^{(n_5-1)} + 2 \cdot 2^{n_4} + 6 \cdot 2^{n_3} + 16 \cdot 2^{n_2} + 64n_3 + 16n_5) \cos^5 \Theta - (2^{(n_3-1)} + \\
&3 \cdot 2^{n_2} + 12n_3 + 8n_5 + 4n_7) \cos^3 \Theta + n_9 \cos \Theta \\
\cos 10\Theta &= 2^{(n_{10}-1)} \cos^{10} \Theta - (2^{(n_8-1)} + 2^{n_7} + 2^{(n_6+1)} + 2^{(n_5+2)} + 2^{(n_4+3)} + \\
&2^{(n_3+4)} + 2^{(n_2+5)} + 128n_3) \cos^8 \Theta + (2^{(n_6-1)} + 2 \cdot 2^{n_5} + 6 \cdot 2^{n_4} + 16 \cdot 2^{n_3} + 40 \cdot 2^{n_2} + \\
&160n_3 + 32n_5) \cos^6 \Theta - (2^{(n_4-1)} + 3 \cdot 2^{n_3} + 12 \cdot 2^{n_2} + 48n_3 + 24n_5 + 8n_7) \cos^4 \Theta + \\
&(2^{(n_2-1)} + 2n_3 + 2n_5 + 2n_7 + 2n_9) \cos^2 \Theta - 1
\end{aligned}$$

We have established the co-efficients written in terms of  $2^{(n_n-1)}$  and  $n_n$ . Each term can be now written in a more simplified form called  $\{L : R\}$  form. The L side is occupied by elements which has  $2^{(n_n-1)}$  and the R side is occupied by elements which has structure of  $n_n$ .

### 1.3 The pattern of the co-efficients Of The Terms

Let us look at each term closely and study their pattern.

### 1.3.1 First Term

By the definition the first term start at  $\cos(2\Theta)$ , where  $S=1$ .

$$\begin{aligned}\cos 1\Theta &\longmapsto \{2^{(n_1-1)} : 0\} = 2^0 \{1 : 0\} \\ \cos 2\Theta &\longmapsto \{2^{(n_2-1)} : 0\} = 2^1 \{1 : 0\} \\ \cos 3\Theta &\longmapsto \{2^{(n_3-1)} : 0\} = 2^2 \{1 : 0\} \\ \cos 4\Theta &\longmapsto \{2^{(n_4-1)} : 0\} = 2^3 \{1 : 0\} \\ \cos 5\Theta &\longmapsto \{2^{(n_5-1)} : 0\} = 2^4 \{1 : 0\} \\ \cos 6\Theta &\longmapsto \{2^{(n_6-1)} : 0\} = 2^5 \{1 : 0\}\end{aligned}$$

### 1.3.2 Second Term

The Second term start at  $\cos(4\Theta)$ , where  $S=1$ .

$$\begin{aligned}\cos 4\Theta &\longmapsto \{2^{(n_2-1)} : 2n_3\} = 2^1 \{1 : 1n_3\} \\ \cos 5\Theta &\longmapsto \{2^{(n_3-1)}, 2^{n_2} : 4n_3\} = 2^2 \{1, 1 : 1n_3\} \\ \cos 6\Theta &\longmapsto \{2^{(n_4-1)}, 2^{n_3}, 2^{(n_2+1)} : 8n_3\} = 2^3 \{1, 1, 1 : 1n_3\} \\ \cos 7\Theta &\longmapsto \{2^{(n_5-1)}, 2^{n_4}, 2^{(n_3+1)}, 2^{(n_2+2)} : 16n_3\} = 2^4 \{1, 1, 1, 1 : 1n_3\} \\ \cos 8\Theta &\longmapsto \{2^{(n_6-1)}, 2^{n_5}, 2^{(n_4+1)}, 2^{(n_3+2)}, 2^{(n_2+3)} : 32n_3\} = 2^5 \{1, 1, 1, 1, 1 : 1n_3\} \\ \cos 9\Theta &\longmapsto \{2^{(n_7-1)}, 2^{n_6}, 2^{(n_5+1)}, 2^{(n_4+2)}, 2^{(n_3+3)}, 2^{(n_2+4)} : 64n_3\} = 2^6 \{1, 1, 1, 1, 1, 1 : 1n_3\}\end{aligned}$$

**Note:** There are  $S$ (subscript number) number of terms in the L side and there are  $(T_m - 1)$  terms in the R side. The common factor in the  $\{L : R\}$  form is  $2^S$ .

### 1.3.3 Third Term

The Third term start at  $\cos(6\Theta)$ , where  $S=1$ .

$$\begin{aligned}\cos 6\Theta &\longmapsto \{2^{(n_2-1)} : 2n_3, 2n_5\} = 2^1 \{1 : 1n_3, 1n_5\} \\ \cos 7\Theta &\longmapsto \{2^{(n_3-1)}, 2^{n_2} : 8n_3, 4n_5\} = 2^2 \{1, 2 : 2n_3, 1n_5\} \\ \cos 8\Theta &\longmapsto \{2^{(n_4-1)}, 2.2^{n_3}, 6.2^{(n_2+1)} : 24n_3, 8n_5\} = 2^3 \{1, 2, 3 : 3n_3, 1n_5\} \\ \cos 9\Theta &\longmapsto \{2^{(n_5-1)}, 2.2^{n_4}, 6.2^{n_3}, 16.2^{n_2} : 64n_3, 16n_5\} = 2^4 \{1, 2, 3, 4 : 4n_3, 1n_5\} \\ \cos 10\Theta &\longmapsto \{2^{(n_6-1)}, 2.2^{n_5}, 6.2^{n_4}, 16.2^{n_3}, 40.2^{n_2} : 160n_3, 32n_5\} = 2^5 \{1, 2, 3, 4, 5 : 5n_3, 1n_5\} \\ \cos 11\Theta &\longmapsto \{2^{(n_7-1)}, 2.2^{n_6}, 6.2^{n_5}, 16.2^{n_4}, 40.2^{n_3}, 96.2^{n_2} : 384n_3\} = \\ &2^6 \{1, 2, 3, 4, 5, 6 : 6n_3, 1n_5\}\end{aligned}$$

**Note:** There are  $S$ (subscript number) number of terms in the L side and there are  $(T_m - 1)$  terms in the R side. The common factor in the  $\{L : R\}$  form is  $2^S$ .

### 1.3.4 Fourth Term

The Fourth term start at  $\cos(8\Theta)$ , where  $S=1$ .

$$\begin{aligned}\cos 8\Theta &\longmapsto \{2^{(n_2-1)} : 2n_3, 2n_5, 2n_7\} = 2^1 \{1 : 1n_3, 1n_5, 1n_7\} \\ \cos 9\Theta &\longmapsto \{2^{(n_3-1)}, 3.2^{n_2} : 12n_3, 8n_5, 4n_7\} = 2^2 \{1, 3 : 3n_3, 2n_5, 1n_7\} \\ \cos 10\Theta &\longmapsto \{2^{(n_4-1)}, 3.2^{n_3}, 12.2^{n_2} : 48n_3, 24n_5, 8n_7\} = 2^3 \{1, 3, 6 : 6n_3, 3n_5, 1n_7\} \\ \cos 11\Theta &\longmapsto \{2^{(n_5-1)}, 3.2^{n_4}, 12.2^{n_3}, 40.2^{n_2} : 160n_3, 64n_5, 16n_7\} = 2^4 \{1, 3, 6, 10 : 10n_3, 4n_5, 1n_7\}\end{aligned}$$

$$\begin{aligned}
\cos 12\Theta &\longmapsto \{2^{(n_6-1)}, 3.2^{n_5}, 12.2^{n_4}, 40.2^{n_3}, 120.2^{n_2} : 480n_3, 160n_5, 32n_7, \} = \\
2^5 \{1, 3, 6, 10, 15 : 15n_3, 5n_5, 1n_7\} \\
\cos 13\Theta &\longmapsto \{2^{(n_7-1)}, 3.2^{n_6}, 12.2^{n_5}, 40.2^{n_4}, 120.2^{n_3}, 336.2^{n_2} : 1344n_3, 384n_5, 64n_7\} = \\
2^6 \{1, 3, 6, 10, 15, 21 : 21n_3, 6n_5, 1n_7\}
\end{aligned}$$

**Note:** There are S(subscript number) number of terms in the L side and there are  $(T_m - 1)$  terms in the R side. The common factor in the  $\{L : R\}$  form is  $2^S$ .

### 1.3.5 Fifth Term

The Fifth term start at  $\cos(10\Theta)$ , where  $S=1$ .

$$\begin{aligned}
\cos 10\Theta &\longmapsto \{2^{(n_2-1)} : 2n_3, 2n_5, 2n_7, 2n_9\} = 2^1 \{1 : 1n_3, 1n_5, 1n_7, 1n_9\} \\
\cos 11\Theta &\longmapsto \{2^{(n_3-1)}, 4.2^{n_2} : 16n_3, 12n_5, 8n_7, 4n_9\} = 2^2 \{1, 4 : 4n_3, 3n_5, 2n_7, 1n_9\} \\
\cos 12\Theta &\longmapsto \{2^{(n_4-1)}, 4.2^{n_3}, 20.2^{n_2} : 80n_3, 48n_5, 24n_7, 8n_9\} = 2^3 \{1, 4, 10 : 10n_3, 6n_5, 3n_7, 1n_9\} \\
\cos 13\Theta &\longmapsto \{2^{(n_5-1)}, 3.2^{n_4}, 12.2^{n_3}, 40.2^{n_2} : 320n_3, 160n_5, 64n_7, 16n_9\} = \\
2^4 \{1, 4, 10, 20 : 20n_3, 10n_5, 4n_7, 1n_9\} \\
\cos 14\Theta &\longmapsto \{2^{(n_6-1)}, 4.2^{n_5}, 20.2^{n_4}, 80.2^{n_3}, 280.2^{n_2} : 1120n_3, 480n_5, 160n_7, 32n_9\} = \\
2^5 \{1, 4, 10, 20, 35 : 35n_3, 15n_5, 5n_7, 1n_9\} \\
\cos 15\Theta &\longmapsto \{2^{(n_7-1)}, 4.2^{n_6}, 20.2^{n_5}, 80.2^{n_4}, 280.2^{n_3}, 896.2^{n_2} : 3584n_3, 1344n_5, 384n_7, 64n_9\} = \\
2^6 \{1, 4, 10, 20, 35, 56 : 56n_3, 21n_5, 6n_7, 1n_9\}
\end{aligned}$$

**Note:** There are S(subscript number) number of terms in the L side and there are  $(T_m - 1)$  terms in the R side. The common factor in the  $\{L : R\}$  form is  $2^S$ .

### 1.3.6 Sixth Term

The Sixth term start at  $\cos(10\Theta)$ , where  $S=1$ .

$$\begin{aligned}
\cos 12\Theta &\longmapsto \{2^{(n_2-1)} : 2n_3, 2n_5, 2n_7, 2n_9, 2n_{11}\} = 2^1 \{1 : 1n_3, 1n_5, 1n_7, 1n_9, 1n_{11}\} \\
\cos 13\Theta &\longmapsto \{2^{(n_3-1)}, 5.2^{n_2} : 20n_3, 16n_5, 12n_7, 8n_9, 4n_{11}\} = 2^2 \{1, 5 : 5n_3, 4n_5, 3n_7, 2n_9, 1n_{11}\} \\
\cos 14\Theta &\longmapsto \{2^{(n_4-1)}, 5.2^{n_3}, 30.2^{n_2} : 120n_3, 80n_5, 48n_7, 24n_9, 8n_{11}\} = \\
2^3 \{1, 5, 15 : 15n_3, 10n_5, 6n_7, 3n_9, 1n_{11}\} \\
\cos 15\Theta &\longmapsto \{2^{(n_5-1)}, 5.2^{n_4}, 30.2^{n_3}, 140.2^{n_2} : 560n_3, 320n_5, 160n_7, 64n_9, 16n_{11}\} = \\
2^4 \{1, 5, 15, 35 : 35n_3, 20n_5, 10n_7, 4n_9, 1n_{11}\} \\
\cos 16\Theta &\longmapsto \{2^{(n_6-1)}, 5.2^{n_5}, 30.2^{n_4}, 140.2^{n_3}, 560.2^{n_2} : 2240n_3, 1120n_5, 480n_7, 160n_9, 32n_{11}\} = \\
2^5 \{1, 5, 15, 35, 70 : 70n_3, 35n_5, 15n_7, 5n_9, 1n_{11}\} \\
\cos 17\Theta &\longmapsto \{2^{(n_7-1)}, 5.2^{n_6}, 30.2^{n_5}, 140.2^{n_4}, 560.2^{n_3}, 2016.2^{n_2} : 8064n_3, 3584n_5, 1344n_7, 384n_9, \\
64n_{11}\} = 2^6 \{1, 5, 15, 35, 70, 126 : 126n_3, 56n_5, 21n_7, 6n_9, 1n_{11}\}
\end{aligned}$$

**Note:** There are S(subscript number) number of terms in the L side and there are  $(T_m - 1)$  terms in the R side. The common factor in the  $\{L : R\}$  form is  $2^S$ .

It can be also noted that the co-efficients are made of the binomial co-efficients of the Pascal triangle [2]. If the L side of the all the terms above is combined and give the following results:

$$1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1$$

1 2 3 4 5 6 7 8 9 10  
 1 3 6 10 15 21 28 36 45 55  
 1 4 10 20 35 56 84 120 165 220  
 1 5 15 35 70 126 210 330 495 715  
 1 6 21 56 126 252 462 792 1287 2002

#### 1.4 The construction of the formula

In the above array of numbers; We can notice that the terms are made of the binomial co-efficient, which is inline with the formula in (2) and (3). We can now be able to formulate the general formula for the co-efficients of  $\cos\Theta$ . Firstly, we have to write Table 1 using binomial notations [2] as follows:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \begin{pmatrix} 5 \\ 5 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \end{pmatrix} \begin{pmatrix} 7 \\ 7 \end{pmatrix} \begin{pmatrix} 8 \\ 8 \end{pmatrix} \begin{pmatrix} 9 \\ 9 \end{pmatrix} \\
 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix} \begin{pmatrix} 7 \\ 6 \end{pmatrix} \begin{pmatrix} 8 \\ 7 \end{pmatrix} \begin{pmatrix} 9 \\ 8 \end{pmatrix} \begin{pmatrix} 10 \\ 9 \end{pmatrix} \\
 \begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} \begin{pmatrix} 7 \\ 5 \end{pmatrix} \begin{pmatrix} 8 \\ 6 \end{pmatrix} \begin{pmatrix} 9 \\ 7 \end{pmatrix} \begin{pmatrix} 10 \\ 8 \end{pmatrix} \begin{pmatrix} 11 \\ 9 \end{pmatrix} \\
 \begin{pmatrix} 3 \\ 0 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} \begin{pmatrix} 7 \\ 4 \end{pmatrix} \begin{pmatrix} 8 \\ 5 \end{pmatrix} \begin{pmatrix} 9 \\ 6 \end{pmatrix} \begin{pmatrix} 10 \\ 7 \end{pmatrix} \begin{pmatrix} 11 \\ 8 \end{pmatrix} \begin{pmatrix} 12 \\ 9 \end{pmatrix} \\
 \begin{pmatrix} 4 \\ 0 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} \begin{pmatrix} 7 \\ 3 \end{pmatrix} \begin{pmatrix} 8 \\ 4 \end{pmatrix} \begin{pmatrix} 9 \\ 5 \end{pmatrix} \begin{pmatrix} 10 \\ 6 \end{pmatrix} \begin{pmatrix} 11 \\ 7 \end{pmatrix} \begin{pmatrix} 12 \\ 8 \end{pmatrix} \begin{pmatrix} 13 \\ 9 \end{pmatrix}$$

Where  $\begin{pmatrix} a \\ b \end{pmatrix} = \frac{a!}{b!(a-b)!}$ . [2]

Let's make the co-efficient of  $\cos^E \Theta$  in the expansion of  $\cos n\Theta$  be  $\mathbf{C}\cos\Theta_n^E$ .  
 e.g The co-efficient of  $\cos^{16}\Theta$  in the expansion of  $\cos 20\Theta$  is  $\mathbf{C}\cos\Theta_{20}^{16}$ .

$\mathbf{C}\cos\Theta_n^E$  can written in the binomial notation as follows:

$$\begin{aligned}
 \mathbf{C}\cos\Theta_8^6 &= 2^5 \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix} : \begin{pmatrix} 4 \\ 4 \end{pmatrix} n_3 \right\} \\
 \mathbf{C}\cos\Theta_{10}^6 &= 2^5 \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \end{pmatrix} : \begin{pmatrix} 5 \\ 4 \end{pmatrix} n_3, \begin{pmatrix} 4 \\ 4 \end{pmatrix} n_5 \right\} \\
 \mathbf{C}\cos\Theta_{12}^6 &= 2^5 \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \end{pmatrix} : \begin{pmatrix} 6 \\ 4 \end{pmatrix} n_3, \begin{pmatrix} 5 \\ 4 \end{pmatrix} n_5, \begin{pmatrix} 4 \\ 4 \end{pmatrix} n_7 \right\} \\
 \mathbf{C}\cos\Theta_{14}^6 &= 2^5 \left\{ \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 4 \end{pmatrix} : \begin{pmatrix} 7 \\ 4 \end{pmatrix} n_3, \begin{pmatrix} 6 \\ 4 \end{pmatrix} n_5, \begin{pmatrix} 5 \\ 4 \end{pmatrix} n_7, \begin{pmatrix} 4 \\ 4 \end{pmatrix} n_9 \right\} \\
 \mathbf{C}\cos\Theta_{16}^6 &= 2^5 \left\{ \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 3 \end{pmatrix}, \begin{pmatrix} 8 \\ 4 \end{pmatrix} : \begin{pmatrix} 8 \\ 4 \end{pmatrix} n_3, \begin{pmatrix} 7 \\ 4 \end{pmatrix} n_5, \begin{pmatrix} 6 \\ 4 \end{pmatrix} n_7, \begin{pmatrix} 5 \\ 4 \end{pmatrix} n_9, \begin{pmatrix} 4 \\ 4 \end{pmatrix} n_{11} \right\}
 \end{aligned}$$

We have seen that there are S(subscript number) number of terms in the L side and there are  $(T_m - 1)$  terms in the R side. The common factor in the  $\{L : R\}$  form is  $2^S$ .



Based on the findings above, we can write the general formula to determine the co-efficients of  $\cos\Theta$  in the expansion of  $\cos n\Theta$ . The General Formula is given by :

$$\mathbf{C}\cos\Theta_n^E = 2^S \left[ \sum_{P=0}^{S-1} \binom{R}{P} + \sum_{P=0}^{T_m-2} \binom{M}{S-1} (2P+3) \right]$$

where  $R = (P + T_m - 2)$  and  $M = (T_m + S - 3 - P)$ .

The sign of anyterm is given by the formula  $(-1)^{(T_m+1)}$ .

#### 1.4.1 An example of determining the co-efficients using the general formula

Example1: Determine  $\mathbf{C}\cos\Theta_{15}^7$  the co-efficient of  $\cos^7 \Theta$  in the expansion of  $\cos 15\Theta$  in terms of powers of  $\cos \Theta$ .

Given:  $n = 15$  and  $E = 7$ .

Step1: Determine the Term number( $T_m$ ).

$$T_m = \frac{1}{2} [15 - 7] + 1 = 5.$$

Step2: Determine the subscript number( $S$ )

$$S = E - 1 = 6.$$

Step3: Calculate the co-efficient.

Substitute Step1 and 2 into the general formula.

$$\begin{aligned} \mathbf{C}\cos\Theta_{15}^7 &= 2^6 \left[ \sum_{P=0}^5 \binom{P+3}{P} + \sum_{P=0}^3 \binom{8-P}{5} (2P+3) \right] \\ &= 2^6 \left\{ \binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \binom{6}{3} + \binom{7}{4} + \binom{8}{5} + \binom{8}{5} 3 + \binom{7}{5} 5 + \binom{6}{5} 7 + \binom{5}{5} 9 \right\} \\ &= 28\,800 \end{aligned}$$

## 2 Development of the formula to determine the co-efficient of $\sin\Theta$ in the expansion of $\sin n\Theta$

### 2.1 Laws of determining co-efficient of $\sin\Theta$

The expansion of  $\sin n\Theta$  have laws which are slightly different from that of  $\cos n\Theta$ .

**Remark 2.1.1** Recall that for the  $n$  of even numbers,  $\sin n\Theta$  must be expressed in terms of  $\sin \Theta$  and  $\sin 2\Theta$ . For  $n$  of even numbers;  $\sin n\Theta = \left[ \frac{n}{2} - A \right] \sin^2 \Theta + B \sin^4 \Theta - C \sin^6 \Theta + \dots \pm 2(n-2) \sin^{(n-2)} \Theta / \sin 2\Theta$ , where  $A, B, C, \dots$  are co-efficients.

If  $n$  is an odd number;  $\sin n\Theta = n\sin\Theta + A\sin^3\Theta - B\sin^5\Theta + C\sin^7\Theta - D\sin^9\Theta + \dots \pm 2^{(n-1)}\sin^n\Theta$ , where  $A, B, C, D, \dots$  are co-efficients.

**Definition 2.1.2** For  $n$  of odd numbers, the  $\sin\Theta$  that has an exponent( $E$ ) of  $n$  is called the **First Term**. i.e In the second term  $E=n-2$ , third term  $E=n-4, \dots$

For  $n$  of even numbers, the  $\sin\Theta$  that has an exponent( $E$ ) of  $(n-2)$  is called the **First Term**. i.e In the second term  $E=n-4$ , third term  $E=n-6, \dots$

**Definition 2.1.3** For  $n$  of odd numbers, The first term has the co-efficient of  $2^{n-1}$ . The last term has a co-efficients of  $n$ , where  $n$  is an odd numbers. Since the co-efficient of  $\sin\Theta$  is known, We don't consider it as terms. i.e The first term start at  $\sin(2 \times \Theta)$ . So any term start to exist at  $\sin(2 \times T_m \times \Theta)$ , where  $T_m$  is the term number. i.e In first term  $T_m=1$ , second term  $T_m=2, \dots$

If  $n$  is an even number; The first term has the co-efficient of  $2^{n-2}$ . The last term has a co-efficient of  $\frac{nn}{2}$ . Since the co-efficients of the first term and the last terms are known, We don't consider the first term and the last term as terms. i.e The first term start at  $\sin(4 \times \Theta)$ . So any term start to exist at  $\sin((2 \times T_m + 2) \times \Theta)$ , where  $T_m$  is the term number. i.e In first term  $T_m=1$ , second term  $T_m=2, \dots$

**Remark 2.1.4** The first term and last term does exist and are ignored for simplification purposes.

**Definition 2.1.5** For  $n$  of odd numbers; the number of times that the term existed since appeared for the first time is called the **subscript( $S$ )**, where  $S = (E - 1)$ .

If  $n$  is an even number; the number of times that the term existed since appeared for the first time is called the **subscript( $S$ )**, where  $S = \frac{n}{2} - 2 = \frac{E}{2}$ .

**Definition 2.1.6** For  $n$  of odd numbers; the term number has a relationship with  $n$  and  $E$ , which can be written as follows:

$$T_m = \frac{1}{2} [n - E] + 1$$

If  $n$  is an even number; the term number has a relationship with  $n$  and  $E$ , which can be written as follows:

$$T_m = \frac{1}{2} [n - E]$$

## 2.2 Simplification of the co-efficients

### 2.2.1 Expansions of $\sin n\Theta$

The expansion of  $\sin n\Theta$  in terms of the powers of  $\sin\Theta$ .

$$\sin\Theta = \sin\Theta$$

$$\sin 2\Theta = \sin 2\Theta$$

$$\sin 3\Theta = 3\sin\Theta - 4\sin^3\Theta$$

$$\sin 4\Theta = [2 - 4\sin^2\Theta]\sin 2\Theta$$

$$\sin 5\Theta = 5\sin\Theta - 20\sin^3\Theta + 16\sin^5\Theta$$

$$\sin 6\Theta = [3 - 16\sin^2\Theta + 16\sin^4\Theta]\sin 2\Theta$$

$$\begin{aligned}
\sin 7\Theta &= 7\sin\Theta - 56\sin^3\Theta + 112\sin^5\Theta - 64\sin^7\Theta \\
\sin 8\Theta &= [4 - 40\sin^2\Theta + 96\sin^4\Theta - 64\sin^6\Theta]\sin 2\Theta \\
\sin 9\Theta &= 9\sin\Theta - 120\sin^3\Theta + 432\sin^5\Theta - 576\sin^7\Theta + 256\sin^9\Theta \\
\sin 10\Theta &= [5 - 80\sin^2\Theta + 336\sin^4\Theta - 512\sin^6\Theta + 256\sin^8\Theta]\sin 2\Theta \\
\text{Let } n_1 &= 1, n_2 = 2, n_3 = 3, \dots, n_n = n \\
\text{and } 2^{(n_2-1)} &= 2, 2^{(n_3-1)} = 4, \dots, 2^{(n_n-1)} = 2^{n-1}
\end{aligned}$$

We have seen that the  $\{L : R\}$  form of the expansion of  $\cos n\Theta$  is simplification of the terms written in a form  $n_n = n$  and  $2^{(n-1)}$ .

### 2.2.2 Simplification of the expansions of $\sin n\Theta$

After simplification the expansion of  $\sin n\Theta$  above will look as follows:

$$\begin{aligned}
\sin 1\Theta &= 2^0 \{1 : 0\} \sin 1\Theta \\
\sin 2\Theta &= 2^0 \{1 : 0\} \sin 2\Theta \\
\sin 3\Theta &= n_3 \sin\Theta - 2^2 \{1 : 0\} \sin^3\Theta \\
\sin 4\Theta &= [n_2 - 2^2 \{1 : 0\} \sin^2\Theta] \sin 2\Theta \\
\sin 5\Theta &= n_5 \sin\Theta - 2^2 \{1, 1 : 1n_3\} \sin^3\Theta + 2^4 \{1 : 0\} \sin^5\Theta \\
\sin 6\Theta &= [n_3 - 2^2 \{0, 1 : 1n_3\} \sin^2\Theta + 2^4 \{1 : 0\} \sin^4\Theta] \sin 2\Theta \\
\sin 7\Theta &= n_7 \sin\Theta - 2^2 \{1, 1 : 2n_3, 1n_5\} \sin^3\Theta + 2^4 \{1, 1, 1, 1 : 1n_3\} \sin^5\Theta - 2^6 \{1 : 0\} \sin^7\Theta \\
\sin 8\Theta &= [n_4 - 2^2 \{1, 1 : 1n_3, 1n_5\} \sin^2\Theta + 2^4 \{0, 1, 2, 4 : 1n_3\} \sin^4\Theta - 2^6 \{1 : 0\} \sin^6\Theta] \sin 2\Theta \\
\sin 9\Theta &= n_9 \sin\Theta - 2^2 \{1, 3 : 3n_3, 2n_5, 1n_7\} \sin^3\Theta + 2^4 \{1, 2, 3, 4 : 4n_3, 1n_5\} \sin^5\Theta \\
&- 2^6 \{1, 1, 1, 1, 1, 1 : n_3\} \sin^7\Theta + 2^8 \{1 : 0\} \sin^9\Theta \\
\sin 10\Theta &= [n_5 - 2^2 \{0, 2 : 2n_3, 1n_5, 1n_7\} \sin^2\Theta + 2^4 \{1, 1, 2, 3 : 3n_3, 1n_5\} \sin^4\Theta \\
&- 2^6 \{0, 1, 1, 1, 1, 1 : n_3\} \sin^6\Theta + 2^8 \{1 : 0\} \sin^8\Theta] \sin 2\Theta \\
\sin 11\Theta &= n_{11} \sin\Theta - 2^2 \{1, 4 : 4n_3, 3n_5, 2n_7, 1n_9\} \sin^3\Theta + 2^4 \{1, 3, 6, 10 : 10n_3, 4n_5, 1n_7\} \sin^5\Theta \\
&- 2^6 \{1, 2, 3, 4, 5, 6 : 6n_3, 1n_5\} \sin^7\Theta + 2^8 \{1 : 0\} \sin^9\Theta \\
\sin 12\Theta &= [n_6 - 2^2 \{1, 2 : 2n_3, 2n_5, 1n_7, 1n_9\} \sin^2\Theta + 2^4 \{0, 2, 4, 7 : 7n_3, 3n_5, 1n_7\} \sin^4\Theta \\
&- 2^6 \{1, 1, 2, 3, 4, 5 : 5n_3, 1n_5\} \sin^6\Theta + 2^6 \{0, 1, 1, 1, 1, 1 : n_3\} \sin^8\Theta + 2^{10} \{1 : 0\} \sin^{10}\Theta] \sin 2\Theta
\end{aligned}$$

### 2.3 The formula of the co-efficients of $\sin\Theta$ , where $n$ is an odd number

In this subsection, we will study the pattern of the terms where  $n$  is an odd number. Then later, we will write the general formula of these expansion.

#### 2.3.1 First Term

$$\begin{aligned}
\sin 1\Theta &\mapsto 2^0 \{1 : 0\} \\
\sin 3\Theta &\mapsto 2^2 \{1 : 0\} \\
\sin 5\Theta &\mapsto 2^4 \{1 : 0\}
\end{aligned}$$

#### 2.3.2 Second Term

$$\begin{aligned}
\sin 5\Theta &\mapsto 2^2 \{1, 1 : 1n_3\} \\
\sin 7\Theta &\mapsto 2^4 \{1, 1, 1, 1 : 1n_3\} \\
\sin 9\Theta &\mapsto 2^6 \{1, 1, 1, 1, 1, 1 : 1n_3\}
\end{aligned}$$

**Note:** There are  $S$ (subscript number) number of terms in the L side and there are  $(T_m - 1)$  terms in the R side. The common factor in the  $\{L : R\}$  form is  $2^S$ .

### 2.3.3 Third Term

$$\begin{aligned}\sin 7\Theta &\mapsto 2^2 \{1, 2 : 2n_3, 1n_5\} \\ \sin 9\Theta &\mapsto 2^4 \{1, 2, 3, 4 : 4n_3, 1n_5\} \\ \sin 11\Theta &\mapsto 2^6 \{1, 2, 3, 4, 5, 6 : 6n_3, 1n_5\}\end{aligned}$$

**Note:** There are  $S$ (subscript number) number of terms in the L side and there are  $(T_m - 1)$  terms in the R side. The common factor in the  $\{L : R\}$  form is  $2^S$ .

### 2.3.4 The construction of the General Formula

Let's make the co-efficient of  $\sin^E \Theta$  in the expansion of  $\sin n\Theta$  be  $\mathbf{Csin}\Theta_n^E$ . e.g The co-efficient of  $\sin^{16}\Theta$  in the expansion of  $\sin 20\Theta$  is  $\mathbf{Csin}\Theta_{20}^{16}$ .

Based on the three terms above; It can said that terms in the of the expansion of  $\sin n\Theta$ , where  $n$  is an odd number are the same as that of  $\cos n\Theta$  of the same kind. i.e They have the same formula, which is written as follows:

$$\mathbf{Csin}\Theta_{n_{odd}}^E = 2^S \left[ \sum_{P=0}^{S-1} \binom{R}{P} + \sum_{P=0}^{T_m-2} \binom{M}{S-1} (2P+3) \right]$$

where  $R = (P + T_m - 2)$  and  $M = (T_m + S - 3 - P)$ .

## 2.4 The formula of the co-efficients of $\sin\Theta$ , where $n$ is an even number

In this subsection, we will study the pattern of the terms where  $n$  is an even number. Then later, we will write the general formula of the expansion.

### 2.4.1 First Term

By definition, the second term start to exist at  $\sin 4\Theta$ .

$$\begin{aligned}\sin 2\Theta &\mapsto 2^0 \{1 : 0\} \\ \sin 4\Theta &\mapsto 2^2 \{1 : 0\} \\ \sin 6\Theta &\mapsto 2^4 \{1 : 0\} \\ \sin 8\Theta &\mapsto 2^6 \{1 : 0\}\end{aligned}$$

### 2.4.2 Second Term

The second term start to exist at  $\sin 6\Theta$ .

$$\begin{aligned}\sin 6\Theta &\mapsto 2^2 \{0, 1 : 1n_3\} \\ \sin 8\Theta &\mapsto 2^4 \{0, 1, 1, 1 : 1n_3\} \\ \sin 10\Theta &\mapsto 2^6 \{0, 1, 1, 1, 1, 1 : 1n_3\}\end{aligned}$$

$$\sin 12\Theta \mapsto 2^8 \{0, 1, 1, 1, 1, 1, 1, 1 : 1n_3\}$$

**Note:** There are E(exponent) number of terms in the L side and there are  $(T_m - 1)$  terms in the R side. The common factor in the  $\{L : R\}$  form is  $2^E$ .

### 2.4.3 Third Term

The Third term start to exist at  $\sin 8\Theta$ .

$$\sin 8\Theta \mapsto 2^2 \{1, 1 : 1n_3, 1n_5\}$$

$$\sin 10\Theta \mapsto 2^4 \{1, 1, 2, 3 : 3n_3, 1n_5\}$$

$$\sin 12\Theta \mapsto 2^6 \{1, 1, 2, 3, 4, 5 : 5n_3, 1n_5\}$$

$$\sin 14\Theta \mapsto 2^8 \{1, 1, 2, 3, 4, 5, 6, 7 : 7n_3, 1n_5\}$$

**Note:** There are E(exponent) number of terms in the L side and there are  $(T_m - 1)$  terms in the R side. The common factor in the  $\{L : R\}$  form is  $2^E$ .

### 2.4.4 Fourth Term

The Fourth term start to exist at  $\sin 10\Theta$ .

$$\sin 10\Theta \mapsto 2^2 \{0, 2 : 2n_3, 1n_5, 1n_7\}$$

$$\sin 12\Theta \mapsto 2^4 \{0, 2, 4, 7 : 7n_3, 3n_5, 1n_7\}$$

$$\sin 14\Theta \mapsto 2^6 \{0, 2, 4, 7, 11, 16 : 16n_3, 5n_5, 1n_7\}$$

$$\sin 16\Theta \mapsto 2^8 \{0, 2, 4, 7, 11, 16, 22, 29 : 29n_3, 7n_5, 1n_7\}$$

**Note:** There are E(exponent) number of terms in the L side and there are  $(T_m - 1)$  terms in the R side. The common factor in the  $\{L : R\}$  form is  $2^E$ .

### 2.4.5 Fifth Term

The Fifth term start to exist at  $\sin 12\Theta$ .

$$\sin 12\Theta \mapsto 2^2 \{1, 2 : 2n_3, 2n_5, 1n_7, 1n_9\}$$

$$\sin 14\Theta \mapsto 2^4 \{1, 2, 6, 13 : 13n_3, 7n_5, 3n_7, 1n_9\}$$

$$\sin 16\Theta \mapsto 2^6 \{1, 2, 6, 13, 24, 40 : 40n_3, 16n_5, 5n_7, 1n_9\}$$

$$\sin 18\Theta \mapsto 2^8 \{1, 2, 6, 13, 24, 40, 62, 91 : 91n_3, 29n_5, 7n_7, 1n_9\}$$

**Note:** There are E(exponent) number of terms in the L side and there are  $(T_m - 1)$  terms in the R side. The common factor in the  $\{L : R\}$  form is  $2^E$ .

### 2.4.6 Sixth Term

The Sixth term start to exist at  $\sin 14\Theta$ .

$$\sin 14\Theta \mapsto 2^2 \{0, 3 : 3n_3, 2n_5, 2n_7, 1n_9, 1n_{11}\}$$

$$\sin 16\Theta \mapsto 2^4 \{0, 3, 9, 22 : 22n_3, 13n_5, 7n_7, 3n_9, 1n_{11}\}$$

$$\sin 18\Theta \mapsto 2^6 \{0, 3, 9, 22, 46, 86 : 86n_3, 40n_5, 16n_7, 5n_9, 1n_{11}\}$$

$$\sin 20\Theta \mapsto 2^8 \{0, 3, 9, 22, 46, 86, 148, 239 : 239n_3, 91n_5, 29n_7, 7n_9, 1n_{11}\}$$

**Note:** There are E(exponent) number of terms in the L side and there are  $(T_m - 1)$  terms in the R side. The common factor in the  $\{L : R\}$  form is  $2^E$ .

### 2.4.7 The construction of the General Formula

Let us look at the L side of the expansion of the terms above. We will write their pattern from the second term up to sixth term. 0 1 1 1 1 1 1 1 1 1

1 1 2 3 4 5 6 7 8 9 10

0 2 4 7 11 16 22 29

1 2 6 13 24 40 62 91

0 3 9 22 46 86 148 239

We can notice that the terms are made of a strange array of numbers, which is not binomial notation. It is essential to study this array in details. Let us examine the pattern of this array in terms of the binomial co-efficients.

$$1 = 1 - 0; 1 = 1 - 0; 1 = 1 - 0; 1 = 1 - 0; 1 = 1 - 0; 1 = 1 - 0$$

$$0 = 1 - 1; 1 = 2 - 1; 2 = 3 - 1; 3 = 4 - 1; 4 = 5 - 1; 5 = 6 - 1; 6 = 7 - 1$$

$$1 = 1 - 1 + 1; 2 = 3 - 2 + 1; 4 = 6 - 3 + 1; 5 = 10 - 4 + 1$$

$$0 = 1 - 1 + 1 - 1; 2 = 4 - 3 + 2 - 1; 6 = 10 - 6 + 3 - 1; 13 = 20 - 10 + 4 - 1$$

$$1 = 1 - 1 + 1 - 1 + 1; 3 = 5 - 4 + 3 - 2 + 1; 9 = 15 - 10 + 6 - 3 + 1; 22 = 35 - 20 + 10 - 4 + 1$$

We have seen from the second array(pattern of the first array) above that the co-efficients are made of binomial co-efficients in a strange form. However the first elements of the second array contradict with the first elements of the first array. In order to illustrate this contradiction, let's represent the second array in the format of the first array.

1 1 1 1 1 1 1 1 1 1

0 1 2 3 4 5 6 7 8 9 10

1 2 4 7 11 16 22 29

0 2 6 13 24 40 62 91

1 3 9 22 46 86 148 239

The difference between the two arrays is the replacement of 1 and 0 of the first array by 0 and 1. However we should recall that the first array is the correct array for finding the co-efficients in the expansion of  $\sin n\Theta$ . i.e We have to find the notation of replacing the value 1 and 0 of the second array by 0 and 1.

**Remark 2.4.1** The Boolean notation is made for inputs of 1's and 0's in the logic circuits, where 1 represent switch on and 0 represent switch off. The Boolean algebra uses the symbol  $\bar{A}$  for inverting the value of A. e.g. If the input A is 1 then output will be  $\bar{1} = 0$ . [3]

The second array can now be written in the correct form, which is same as the first array using the Boolean equation.

$$0 = \bar{1} = 1 - 0; 1 = 1 - 0; 1 = 1 - 0; 1 = 1 - 0; 1 = 1 - 0; 1 = 1 - 0$$

$$1 = \bar{0} = 1 - 1; 1 = 2 - 1; 2 = 3 - 1; 3 = 4 - 1; 4 = 5 - 1; 5 = 6 - 1; 6 = 7 - 1$$

$$0 = \bar{1} = 1 - 1 + 1; 2 = 3 - 2 + 1; 4 = 6 - 3 + 1; 5 = 10 - 4 + 1$$

$$1 = \bar{0} = 1 - 1 + 1 - 1; 2 = 4 - 3 + 2 - 1; 6 = 10 - 6 + 3 - 1; 13 = 20 - 10 + 4 - 1 \\ 0 = \bar{1} = 1 - 1 + 1 - 1 + 1; 3 = 5 - 4 + 3 - 2 + 1; 9 = 15 - 10 + 6 - 3 + 1; 22 = \\ 35 - 20 + 10 - 4 + 1$$

Let's represent this table in a binomial notation.[2]

$$0 = \bar{1} = \binom{0}{0} - 0; 1 = \binom{1}{1} - 0; 1 = \binom{2}{2} - 0.$$

$$1 = \bar{0} = \binom{1}{0} - \binom{0}{0}; 1 = \binom{2}{1} - \binom{1}{1}; 2 = \binom{3}{2} - \binom{2}{2}.$$

$$0 = \bar{1} = \binom{2}{0} - \binom{1}{0} + \binom{0}{0}; 2 = \binom{3}{1} - \binom{2}{1} + \binom{1}{1}; 4 = \binom{4}{2} - \binom{3}{2} + \binom{2}{2}.$$

$$1 = \bar{0} = \binom{3}{0} - \binom{2}{0} + \binom{1}{0} - \binom{0}{0}; 2 = \binom{4}{1} - \binom{3}{1} + \binom{2}{1} - \binom{1}{1}.$$

$$0 = \bar{1} = \binom{4}{0} - \binom{3}{0} + \binom{2}{0} - \binom{1}{0} + \binom{0}{0}; 3 = \binom{5}{1} - \binom{4}{1} + \binom{3}{1} - \binom{2}{1} + \binom{1}{1}.$$

We can now write the general formula for each element in an array. The general formula for the elements which made up the co-efficients is shown below:

$$\binom{a}{b} = \binom{a}{b} - \binom{a-1}{b} + \binom{a-2}{b} - \binom{a-3}{b} + \binom{a-4}{b} - \binom{a-5}{b} + \dots \pm \binom{b}{b} \quad (4)$$

where a and b are integers greater or equal to 0.

The first element will be written as  $\binom{a}{0}$ . i.e The final answer of the first element must be converted.

The correct array can be represented in a new notation as follows:

$$\begin{bmatrix} \bar{0} \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \end{bmatrix} \begin{bmatrix} 6 \\ 6 \end{bmatrix} \begin{bmatrix} 7 \\ 7 \end{bmatrix} \begin{bmatrix} 8 \\ 8 \end{bmatrix} \begin{bmatrix} 9 \\ 9 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} \begin{bmatrix} 11 \\ 11 \end{bmatrix} \begin{bmatrix} 12 \\ 12 \end{bmatrix} \begin{bmatrix} 13 \\ 13 \end{bmatrix} \\ \begin{bmatrix} \bar{1} \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} \begin{bmatrix} 6 \\ 5 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \end{bmatrix} \begin{bmatrix} 8 \\ 7 \end{bmatrix} \begin{bmatrix} 9 \\ 8 \end{bmatrix} \begin{bmatrix} 10 \\ 9 \end{bmatrix} \begin{bmatrix} 11 \\ 10 \end{bmatrix} \begin{bmatrix} 12 \\ 11 \end{bmatrix} \begin{bmatrix} 13 \\ 12 \end{bmatrix} \begin{bmatrix} 14 \\ 13 \end{bmatrix} \\ \begin{bmatrix} \bar{2} \\ 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \end{bmatrix} \begin{bmatrix} 9 \\ 7 \end{bmatrix} \begin{bmatrix} 10 \\ 8 \end{bmatrix} \begin{bmatrix} 11 \\ 9 \end{bmatrix} \begin{bmatrix} 12 \\ 10 \end{bmatrix} \begin{bmatrix} 13 \\ 11 \end{bmatrix} \begin{bmatrix} 14 \\ 12 \end{bmatrix} \begin{bmatrix} 15 \\ 13 \end{bmatrix} \\ \begin{bmatrix} \bar{3} \\ 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \end{bmatrix} \begin{bmatrix} 8 \\ 5 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \end{bmatrix} \begin{bmatrix} 10 \\ 7 \end{bmatrix} \begin{bmatrix} 11 \\ 8 \end{bmatrix} \begin{bmatrix} 12 \\ 9 \end{bmatrix} \begin{bmatrix} 13 \\ 10 \end{bmatrix} \begin{bmatrix} 14 \\ 11 \end{bmatrix} \begin{bmatrix} 15 \\ 12 \end{bmatrix} \begin{bmatrix} 16 \\ 13 \end{bmatrix} \\ \begin{bmatrix} \bar{4} \\ 0 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \end{bmatrix} \begin{bmatrix} 9 \\ 5 \end{bmatrix} \begin{bmatrix} 10 \\ 6 \end{bmatrix} \begin{bmatrix} 11 \\ 7 \end{bmatrix} \begin{bmatrix} 12 \\ 8 \end{bmatrix} \begin{bmatrix} 13 \\ 9 \end{bmatrix} \begin{bmatrix} 14 \\ 10 \end{bmatrix} \begin{bmatrix} 15 \\ 11 \end{bmatrix} \begin{bmatrix} 16 \\ 12 \end{bmatrix} \begin{bmatrix} 17 \\ 13 \end{bmatrix} \\ \begin{bmatrix} \bar{5} \\ 0 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \end{bmatrix} \begin{bmatrix} 8 \\ 3 \end{bmatrix} \begin{bmatrix} 9 \\ 4 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \end{bmatrix} \begin{bmatrix} 11 \\ 6 \end{bmatrix} \begin{bmatrix} 12 \\ 7 \end{bmatrix} \begin{bmatrix} 13 \\ 8 \end{bmatrix} \begin{bmatrix} 14 \\ 9 \end{bmatrix} \begin{bmatrix} 15 \\ 10 \end{bmatrix} \begin{bmatrix} 16 \\ 11 \end{bmatrix} \begin{bmatrix} 17 \\ 12 \end{bmatrix} \begin{bmatrix} 18 \\ 13 \end{bmatrix} \\ \begin{bmatrix} \bar{6} \\ 0 \end{bmatrix} \begin{bmatrix} 7 \\ 1 \end{bmatrix} \begin{bmatrix} 8 \\ 2 \end{bmatrix} \begin{bmatrix} 9 \\ 3 \end{bmatrix} \begin{bmatrix} 10 \\ 4 \end{bmatrix} \begin{bmatrix} 11 \\ 5 \end{bmatrix} \begin{bmatrix} 12 \\ 6 \end{bmatrix} \begin{bmatrix} 13 \\ 7 \end{bmatrix} \begin{bmatrix} 14 \\ 8 \end{bmatrix} \begin{bmatrix} 15 \\ 9 \end{bmatrix} \begin{bmatrix} 16 \\ 10 \end{bmatrix} \begin{bmatrix} 17 \\ 11 \end{bmatrix} \begin{bmatrix} 18 \\ 12 \end{bmatrix} \begin{bmatrix} 19 \\ 13 \end{bmatrix}$$

At this point, we are in a position of writing the general formula for the expansion of  $\sin n\Theta$ , where  $n$  is an even number. Let's examine some couple of terms then write the general formula.

$$\begin{aligned}\text{Csin}\Theta_{12}^8 &= 2^8 \left\{ \binom{0}{0}, \binom{1}{1}, \binom{2}{2}, \binom{3}{3}, \binom{4}{4}, \binom{5}{5}, \binom{6}{6}, \binom{7}{7} : \binom{7}{7} n_3 \right\} \\ \text{Csin}\Theta_{14}^8 &= 2^8 \left\{ \binom{1}{0}, \binom{2}{1}, \binom{3}{2}, \binom{4}{3}, \binom{5}{4}, \binom{6}{5}, \binom{7}{6}, \binom{8}{7} : \binom{8}{7} n_3, \binom{7}{7} n_5 \right\} \\ \text{Csin}\Theta_{16}^8 &= 2^8 \left\{ \binom{2}{0}, \binom{3}{1}, \binom{4}{2}, \binom{5}{3}, \binom{6}{4}, \binom{7}{5}, \binom{8}{6}, \binom{9}{7} : \binom{9}{7} n_3, \binom{8}{7} n_5, \binom{7}{7} n_7 \right\} \\ \text{Csin}\Theta_{18}^8 &= 2^8 \left\{ \binom{3}{0}, \binom{4}{1}, \binom{5}{2}, \binom{6}{3}, \binom{7}{4}, \binom{8}{5}, \binom{9}{6}, \binom{10}{7} : \binom{10}{7} n_3, \binom{9}{7} n_5, \binom{8}{7} n_7, \binom{7}{7} n_9 \right\}\end{aligned}$$

We have seen that there are  $E$ (Exponent value) number of terms in the L side and there are  $(T_m - 1)$  terms in the R side. The common factor in the  $\{L : R\}$  form is  $2^S$ .

Based on the findings above, we can write the general formula to determine the co-efficient of  $\sin\Theta$  in the expansion of  $\sin n\Theta$ . We will use the cap on the general formula to remind us to invert the first entry on the co-efficient. The General Formula is given by :

$$\begin{aligned}\text{Csin}\Theta_{n_{even}}^E &= 2^E \left[ \sum_{P=0}^{E-1} \binom{R}{P} + \sum_{P=0}^{T_m-2} \binom{M}{E-1} (2P+3) \right] \\ &= 2^E \binom{T_m+E}{E+1}\end{aligned}$$

where  $R = (P + T_m - 2)$  and  $M = (T_m + E - 3 - P)$ .

#### 2.4.8 An example of determining the co-efficients using the general formula

Example2: Determine  $\text{Csin}\Theta_{16}^6$  the co-efficient of  $\sin^6 \Theta$  in the expansion of  $\sin 16\Theta$  in terms of powers of  $\sin \Theta$ .

Given:  $n = 16$  and  $E = 6$ .

Step1: Determine the Term number( $T_m$ ).

$$T_m = \frac{1}{2} [16 - 6] = 5.$$

Step2: Determine the subscript number( $S$ )

$$S = \frac{E}{2} = 3.$$

Step3: Calculate the co-efficient.

Substitute Step1 and 2 into the general formula.

$$\text{Csin}\Theta_{16}^6 = 2^6 \left[ \sum_{P=0}^5 \binom{P+3}{P} + \sum_{P=0}^3 \binom{8-P}{5} (2P+3) \right]$$



$$\begin{aligned} \text{Csin}\Theta_{16}^6 &= 2^6 \left\{ \binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \binom{6}{3} + \binom{7}{4} + \binom{8}{5} + \binom{8}{5} 3 + \binom{7}{5} 5 + \binom{6}{5} 7 + \binom{5}{5} 9 \right\} \\ &= 21 \ 120 \end{aligned}$$

### 3 Simplification of the General Formulas of the Co-efficients

**Remark 3.0.2** *Recall: The general formulas in the sections above were not written in a simpler form as there are many unknowns within them. So it is essential to formulate more simpler formulas.*

#### 3.1 Sine expansion

Based on the array below, we can formulate identities relating the binomial co-efficient and this notation.

$$\begin{array}{ccccccc} \binom{0}{0} & \binom{1}{1} & \binom{2}{2} & \binom{3}{3} & \binom{4}{4} & \binom{5}{5} \\ \binom{1}{0} & \binom{2}{1} & \binom{3}{2} & \binom{4}{3} & \binom{5}{4} & \binom{6}{5} \\ \binom{2}{0} & \binom{3}{1} & \binom{4}{2} & \binom{5}{3} & \binom{6}{4} & \binom{7}{5} \\ \binom{3}{0} & \binom{4}{1} & \binom{5}{2} & \binom{6}{3} & \binom{7}{4} & \binom{8}{5} \end{array}$$

**Theorem 3.1.1** *If  $a$  and  $b$  are positive integers such that  $a, b \neq 0$  then*

$$\binom{a}{b} = \binom{a}{b} + \binom{a-1}{b} \quad (5)$$

The sum of the entries of the sine array in each row can result to a single element of the following row. These is inline with binomial co-efficient array.

**Theorem 3.1.2** *The sum of the elements in a column( $C$ ) and in a row( $R$ ) of the sine array is given by:*

$$\sum_{P=1}^n \binom{Q+P}{C} = \binom{Q+n+1}{C+1} \quad (6)$$

$$\sum_{P=\hat{0}}^n \binom{R+P}{P} = \binom{R+n+1}{n} \quad (7)$$

where  $Q, C, R$  and  $n$  are positive integers  $\geq 0$ .

**Remark 3.1.3** *Recall: If the value of  $P$  in a sigma does not have a cap on the top of 0, the answer of the first element of an array does not change.*

**Theorem 3.1.4** *If  $a$  is a positive integer such that  $a \neq 0$ : The binomial coefficient has the same value with sine array element where all the numbers in both notations are the same.*

$$\binom{a}{a} = \binom{a}{a} \quad (8)$$

### 3.1.1 Simplification of the general formula

The theorems above can be applied to the unsimplified part of the general formula of the sine expansion to understand why it equal to  $2^E \binom{T_m+E}{E+1}$ . We will use (7) to simplify the first element of the express below.

$$\begin{aligned} & \sum_{P=0}^{E-1} \binom{T_m+P-2}{P} + \sum_{P=0}^{T_m-2} \binom{T_m+E-3-P}{E-1} (2P+3) \\ &= \sum_{P=0}^{T_m-2} \binom{T_m+E-3-P}{E-1} (2P+3) + \binom{T_m+E-2}{E-1} \\ &= \binom{T_m+E-2}{E-1} + 3 \binom{T_m+E-3}{E-1} + 5 \binom{T_m+E-4}{E-1} + 7 \binom{T_m+E-5}{E-1} + 9 \binom{T_m+E-6}{E-1} + \\ & \dots + \binom{E}{E-1} (2T_m-3) + \binom{E-1}{E-1} (2T_m-1) \\ & \text{Then split the terms such that we can use (5).} \\ &= \binom{T_m+E-2}{E-1} + 3 \binom{T_m+E-3}{E-1} + 3 \binom{T_m+E-4}{E-1} + 2 \binom{T_m+E-4}{E-1} + 5 \binom{T_m+E-5}{E-1} + \\ & 2 \binom{T_m+E-5}{E-1} + 5 \binom{T_m+E-6}{E-1} + 4 \binom{T_m+E-6}{E-1} + \dots + \binom{E}{E-1} (2T_m-5) + \binom{E-1}{E-1} (2T_m+2) + \\ & \binom{E-1}{E-1} (2T_m-1) \\ &= \binom{T_m+E-2}{E-1} + 3 \left\{ \binom{T_m+E-3}{E-1} + \binom{T_m+E-4}{E-1} \right\} + 2 \left\{ \binom{T_m+E-4}{E-1} + \binom{T_m+E-5}{E-1} \right\} + \\ & 5 \left\{ \binom{T_m+E-5}{E-1} + \binom{T_m+E-6}{E-1} \right\} + \dots + \binom{E-1}{E-1} (2T_m-1) \\ & \text{When we apply (4) and (5) we get the following results.} \\ &= \binom{T_m+E-2}{E-1} + 3 \binom{T_m+E-3}{E-1} + 2 \binom{T_m+E-4}{E-1} + 5 \binom{T_m+E-5}{E-1} + \dots + \binom{E-1}{E-1} (2T_m-1) \\ &= \binom{T_m+E-2}{E-1} - \binom{T_m+E-3}{E-1} + \binom{T_m+E-4}{E-1} - \binom{T_m+E-5}{E-1} + \dots \pm \binom{E-1}{E-1} + \\ & 3 \binom{T_m+E-3}{E-1} + 2 \binom{T_m+E-4}{E-1} + 5 \binom{T_m+E-5}{E-1} + \dots + \binom{E-1}{E-1} (2T_m-1) \\ &= \binom{T_m+E-2}{E-1} + 2 \binom{T_m+E-3}{E-1} + 3 \binom{T_m+E-4}{E-1} + 4 \binom{T_m+E-5}{E-1} + \dots + \binom{E-1}{E-1} (2T_m-1 \pm 1) \\ &= \binom{T_m+E}{E+1} \end{aligned}$$

We have seen how  $\text{Csin}\Theta_{n_{\text{even}}}^E$  is simplified. So we can write the identity that is obtain on this simplification.

**Theorem 3.1.5** *If  $a, b$  are positive integer such that  $a, b \neq 0$  then:*

$$1 \binom{a}{b} + 2 \binom{a-1}{b} + 3 \binom{a-2}{b} + \dots + (a-b+1) \binom{b}{b} = \binom{a+2}{b+2} \quad (9)$$

$$\sum_{P=0}^{a-b} \binom{a-P}{b} (P+1) = \binom{a+2}{b+2} \quad (10)$$

The co-efficient of  $\sin^E \Theta$  in the expansion of  $\sin n\Theta$  in terms of powers of  $\sin \Theta$  can now be written as follows:

$$\begin{aligned} \text{Csin}\Theta_{n_{\text{even}}}^E &= 2^E \left[ \sum_{P=0}^{E-2} \binom{T_m+E-2-P}{E-1} (P+1) \right] \\ &= 2^E \binom{T_m+E}{E-1} \end{aligned}$$

### 3.2 Cosine Expansion

**Theorem 3.2.1** *If  $a, k$  and  $r$  are integers then the parallel summation of the binomial co-efficient can be written as follows [2]:*

$$\sum_{k \leq a} \binom{r+k}{k} = \binom{r+a+1}{a} \quad (11)$$

$\text{Csin}\Theta_{n_{\text{even}}}^E$  can be simplified further by simplifying the two sigmas of the co-efficient.

$$\sum_{P=0}^{S-1} \binom{P+T_m-2}{P} + \sum_{P=0}^{T_m-2} \binom{T_m+S-3-P}{S-1} (2P+3)$$

Then apply (11) on the first element.

$$= \sum_{P=0}^{T_m-2} \binom{T_m+S-3-P}{S-1} (2P+3) + \binom{T_m+S-2}{S-1}$$

$$\begin{aligned}
&= 1 \binom{T_m+S-2}{S-1} + 3 \binom{T_m+S-3}{S-1} + 5 \binom{T_m+S-4}{S-1} + 7 \binom{T_m+S-5}{S-1} + 9 \binom{T_m+S-6}{S-1} + \\
&\dots + \binom{S-1}{S-1} (2T_m - 1) \\
&= 1 \binom{T_m+S-2}{S-1} + 2 \binom{T_m+S-3}{S-1} + 3 \binom{T_m+S-4}{S-1} + 4 \binom{T_m+S-5}{S-1} + 5 \binom{T_m+S-6}{S-1} + \\
&\dots + T_m \binom{S-1}{S-1} + 1 \binom{T_m+S-3}{S-1} + 2 \binom{T_m+S-4}{S-1} + 3 \binom{T_m+S-5}{S-1} + 4 \binom{T_m+S-6}{S-1} + \\
&\dots + \binom{S-1}{S-1} (T_m - 1)
\end{aligned}$$

Then we can apply (9) and get the following results:

$$= \binom{T_m+S}{S+1} + \binom{T_m+S-1}{S+1}$$

The co-efficient of  $\cos^E \Theta$  in the expansion of  $\cos n\Theta$  in terms of powers of  $\cos \Theta$  can now be written as follows:

$$\begin{aligned}
C_{\cos \Theta_n^E} &= 2^S \left[ \sum_{P=0}^{S-2} \binom{T_m+S-2-P}{S-1} (P+1) + \sum_{P=0}^{S-2} \binom{T_m+S-3-P}{S-1} (P+1) \right] \\
&= 2^S \left[ \binom{T_m+S}{S+1} + \binom{T_m+S-1}{S+1} \right]
\end{aligned}$$

## 4 The general formula of the expansion of $\cos n\Theta$ and $\sin n\Theta$ in the powers of $\cos \Theta$ and $\sin \Theta$ respectively

The general formula for the co-efficients can be used to find the general formula of  $\cos n\Theta$  and  $\sin n\Theta$ . We have seen that the expansions have different formulas for  $n_{odd}$  and  $n_{even}$ . So we will use  $n_{odd}$  and  $n_{even}$  to represent expansions where  $n$  is an odd and even respectively.

### 4.1 Cosine expansion

$$\begin{aligned}
\cos n\Theta &= \sum_{P=0}^A \left\{ (-1)^{P+1} 2^{(n-3-2P)} \left[ \binom{n-1-P}{n-2-2P} + \binom{n-2-P}{n-2-2P} \right] \cos^{(n-2-2P)} \theta \right\} \\
&+ 2^{n-1} \cos^n \Theta + (-1)^{A+2} n_{odd} \cos \Theta / 1_{even} \\
&\text{where } A = \left[ \frac{n_{odd}-1}{2} - 2 \right] \text{ and } \left[ \frac{n_{even}}{2} - 2 \right]
\end{aligned}$$

Example3: Expand  $\cos 10\Theta$  in terms of powers of  $\cos \Theta$ .

Sustitute the value of  $n$  into  $A$  and the general formula.

$$A = \left[ \frac{10}{2} - 2 \right] = 3.$$

$$\begin{aligned}
\cos 10\Theta &= \sum_{P=0}^3 \left\{ (-1)^{(P+1)} 2^{(7-2P)} \left[ \binom{9-P}{8-2P} + \binom{8-P}{8-2P} \right] \cos^{(8-2P)} \theta \right\} \\
&+ 2^9 \cos^{10} \Theta + (-1)^5 1_{even} \\
&= 2^9 \cos^{10} \Theta + (-1)^1 . 2^7 \left[ \binom{9}{8} + \binom{8}{8} \right] \cos^8 \theta + (-1)^2 . 2^5 \left[ \binom{8}{6} + \binom{7}{6} \right] \cos^6 \theta + \\
&(-1)^3 . 2^3 \left[ \binom{7}{4} + \binom{6}{4} \right] \cos^4 \theta + (-1)^4 . 2^1 \left[ \binom{6}{2} + \binom{5}{2} \right] \cos^2 \theta + (-1)^5 \\
&= 2^9 \cos^{10} \Theta - 2^7 [9+1] \cos^8 \theta + 2^5 [28+7] \cos^6 \theta - 2^3 [35+15] \cos^4 \theta + 2^1 [15+10] \cos^2 \theta - \\
&(-1)^5 \\
&= 2^9 \cos^{10} \Theta - 2^7 [10] \cos^8 \theta + 2^5 [35] \cos^6 \theta - 2^3 [50] \cos^4 \theta + 2^1 [25] \cos^2 \theta - 1 \\
&= 512 \cos^{10} \Theta - 1280 \cos^8 \theta + 1120 \cos^6 \theta - 400 \cos^4 \theta + 50 \cos^2 \theta - 1
\end{aligned}$$

## 4.2 Sine Expansion

### 4.2.1 The general formula of $\sin n\theta$ for the n of even numbers

$$\sin n\Theta = \left[ \frac{n}{2} + \sum_{P=0}^A \left\{ (-1)^B . 2^B \binom{\frac{n}{2}+B}{2B+1} \sin^{2B} \theta \right\} \right] \sin 2\Theta$$

where  $A = \left[ \frac{n_{even}}{2} - 2 \right]$  and  $B = P + 1$ .

Example4: Expand  $\sin 10\Theta$  in terms of  $\sin 2\Theta$  powers of  $\sin \Theta$ .

$A = \left[ \frac{10}{2} - 2 \right] = 3$  and  $B = P + 1$ .

$$\begin{aligned}
\sin 10\Theta &= \left[ \frac{10}{2} + \sum_{P=0}^3 \left\{ (-1)^B . 2^B \binom{\frac{10}{2}+B}{2B+1} \sin^{2B} \theta \right\} \right] \sin 2\Theta \\
&= \left[ 5 + (-1)^1 . 2^2 \binom{6}{3} \sin^2 \theta + (-1)^2 . 2^4 \binom{7}{5} \sin^4 \theta + (-1)^3 . 2^6 \binom{8}{7} \sin^6 \theta + (-1)^4 . 2^8 \binom{9}{9} \sin^8 \theta \right] \sin 2\theta \\
&= [5 - 4[20] \sin^2 \theta + 16[21] \sin^4 \theta - 64[8] \sin^6 \theta + 256[1] \sin^8 \theta] \sin 2\theta \\
&= [5 - 80 \sin^2 \theta + 336 \sin^4 \theta - 512 \sin^6 \theta + 256 \sin^8 \theta] \sin 2\theta
\end{aligned}$$

### 4.2.2 The general formula of $\sin n\theta$ for the n of odd numbers

It have been shown above that the expansion of  $\sin n\theta$  for the n of odd numbers share the same general formula with  $\cos n\theta$  of the same kind. However the signs of the co-efficients are not same the between these two expansion.

$$\begin{aligned}
\sin n\Theta &= \sum_{P=0}^A \left\{ (-1)^{\left(\frac{n-1}{2}-1-P\right)} . 2^{(n-3-2P)} \left[ \binom{n-1-P}{n-2-2P} + \binom{n-2-P}{n-2-2P} \right] \sin^{(n-2-2P)} \theta \right\} \\
&+ (-1)^{\left(\frac{n-1}{2}\right)} . 2^{n-1} \sin^n \Theta + n_{odd} \sin \Theta
\end{aligned}$$

where  $A = \left[ \frac{n_{odd}-1}{2} - 2 \right]$

Example5: Expand  $\sin 9\Theta$  in terms of powers of  $\sin \Theta$ .

$$A = \left[ \frac{8-1}{2} - 2 \right] = 2$$

$$\begin{aligned} \sin 9\Theta &= \sum_{P=0}^2 \left\{ (-1)^{(3-P)} \cdot 2^{(6-2P)} \left[ \binom{8-P}{7-2P} + \binom{7-P}{7-2P} \right] \sin^{(7-2P)} \theta \right\} \\ &+ (-1)^4 \cdot 2^8 \sin^9 \Theta + 9 \sin \Theta \\ &= 9 \sin \Theta + 2^8 \sin^9 \Theta + (-1)^3 \cdot 2^6 \left[ \binom{8}{7} + \binom{7}{7} \right] \sin^7 \theta + (-1)^2 \cdot 2^4 \left[ \binom{7}{5} + \binom{6}{5} \right] \sin^5 \theta + \\ &(-1)^1 \cdot 2^2 \left[ \binom{6}{3} + \binom{5}{3} \right] \sin^3 \theta \\ &= 9 \sin \Theta + 2^8 \sin^9 \Theta - 2^6 [8+1] \sin^7 \theta + 2^4 [21+6] \sin^5 \theta - 2^2 [20+10] \sin^3 \theta \\ &= 9 \sin \Theta - 2^2 [30] \sin^3 \theta + 2^4 [27] \sin^5 \theta - 2^6 [9] \sin^7 \theta + 2^8 \sin^9 \Theta \\ &= 9 \sin \Theta - 120 \sin^3 \theta + 432 \sin^5 \theta - 576 \sin^7 \theta + 256 \sin^9 \Theta \end{aligned}$$

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