



## Cyclicity in weighted $\ell^p$ spaces

Florian Le Manach

### ► To cite this version:

| Florian Le Manach. Cyclicity in weighted  $\ell^p$  spaces. 2017. hal-01483933

**HAL Id: hal-01483933**

**<https://hal.science/hal-01483933>**

Preprint submitted on 6 Mar 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# CYCLICITY IN WEIGHTED $\ell^p$ SPACES

FLORIAN LE MANACH

**ABSTRACT.** We study the cyclicity in weighted  $\ell^p(\mathbb{Z})$  spaces. For  $p \geq 1$  and  $\beta \geq 0$ , let  $\ell_\beta^p(\mathbb{Z})$  be the space of sequences  $u = (u_n)_{n \in \mathbb{Z}}$  such that  $(u_n |n|^\beta) \in \ell^p(\mathbb{Z})$ . We obtain both necessary conditions and sufficient conditions for  $u$  to be cyclic in  $\ell_\beta^p(\mathbb{Z})$ , in other words, for  $\{(u_{n+k})_{n \in \mathbb{Z}}, k \in \mathbb{Z}\}$  to span a dense subspace of  $\ell_\beta^p(\mathbb{Z})$ . The conditions are given in terms of the Hausdorff dimension and the capacity of the zero set of the Fourier transform of  $u$ .

## 1. INTRODUCTION AND MAIN RESULTS

For  $p \geq 1$  and  $\beta \in \mathbb{R}$ , we define the Banach space

$$\ell_\beta^p(\mathbb{Z}) = \left\{ u = (u_n)_{n \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}, \quad \|u\|_{\ell_\beta^p}^p = \sum_{n \in \mathbb{Z}} |u_n|^p (1 + |n|)^{p\beta} < \infty \right\}$$

endowed with the norm  $\|\cdot\|_{\ell_\beta^p}$ . Notice that  $\ell_0^p(\mathbb{Z})$  is the classical  $\ell^p(\mathbb{Z})$  space.

In this work, we are going to investigate cyclic vectors for  $\ell_\beta^p(\mathbb{Z})$  when  $\beta \geq 0$ . A vector  $u \in \ell_\beta^p(\mathbb{Z})$  is called *cyclic* in  $\ell_\beta^p(\mathbb{Z})$  if the linear span of  $\{(u_{n+k})_{n \in \mathbb{Z}}, k \in \mathbb{Z}\}$  is dense in  $\ell_\beta^p(\mathbb{Z})$ .

We denote by  $\mathbb{T}$  the circle  $\mathbb{R}/2\pi\mathbb{Z}$ . The Fourier transform of  $u \in \ell^p(\mathbb{Z})$  is given by

$$\hat{u} : t \in \mathbb{T} \mapsto \sum_{n \in \mathbb{Z}} u_n e^{int}$$

and when  $\hat{u}$  is continuous, we denote by  $\mathcal{Z}(\hat{u})$  the zero set on  $\mathbb{T}$  of  $\hat{u}$ :

$$\mathcal{Z}(\hat{u}) = \{t \in \mathbb{T}, \hat{u}(t) = 0\}.$$

The case  $\beta = 0$  was already studied by Wiener, Beurling, Salem and Newman. When  $p = 1$  or  $p = 2$ , Wiener characterized the cyclic vectors  $u$  in  $\ell^p(\mathbb{Z})$  by the zeros of  $\hat{u}$ , with the following theorem.

---

2000 *Mathematics Subject Classification.* primary 43A15; secondary 28A12, 42A38.

*Key words and phrases.* Cyclicity, weighted  $\ell^p$  spaces, capacity.

**Theorem 1.1** ([16]). *Let  $u \in \ell^p(\mathbb{Z})$ .*

- (1) *If  $p = 1$  then  $u$  is cyclic in  $\ell^1(\mathbb{Z})$  if and only if  $\hat{u}$  has no zeros on  $\mathbb{T}$ .*
- (2) *If  $p = 2$  then  $u$  is cyclic in  $\ell^2(\mathbb{Z})$  if and only if  $\hat{u}$  is non-zero almost everywhere.*

Lev and Olevskii showed that, for  $1 < p < 2$  the problem of cyclicity in  $\ell^p(\mathbb{Z})$  is more complicated even for sequences in  $\ell^1(\mathbb{Z})$ . The following Theorem of Lev and Olevskii contradicts the Wiener conjecture.

**Theorem 1.2** ([8]). *If  $1 < p < 2$ , there exist  $u$  and  $v$  in  $\ell^1(\mathbb{Z})$  such that  $\mathcal{Z}(\hat{u}) = \mathcal{Z}(\hat{v})$ ,  $u$  is not cyclic in  $\ell^p(\mathbb{Z})$ , and  $v$  is cyclic in  $\ell^p(\mathbb{Z})$ .*

So we can't characterize the cyclicity of  $u$  in  $\ell^p(\mathbb{Z})$  in terms of only  $\mathcal{Z}(\hat{u})$ , the zero set of  $\hat{u}$ . However for  $u \in \ell^1(\mathbb{Z})$ , Beurling, Salem and Newman gave both necessary conditions and sufficient conditions for  $u$  to be cyclic in  $\ell^p(\mathbb{Z})$ . These conditions rely on the "size" of the set  $\mathcal{Z}(\hat{u})$  in term of it's  $h$ -measure, capacity and Hausdorff dimension.

Given  $E \subset \mathbb{T}$  and  $h$  a continuous function, non-decreasing and such that  $h(0) = 0$ , we define the  $h$ -measure of  $E$  by

$$H_h(E) = \liminf_{\delta \rightarrow 0} \left\{ \sum_{i=0}^{\infty} h(|U_i|), E \subset \bigcup_{i=0}^{\infty} U_i, |U_i| \leq \delta \right\}$$

where the  $U_i$  are open intervals of  $\mathbb{T}$  and where  $|U_i|$  denotes the length of  $U_i$ .

The Hausdorff dimension of a subset  $E \subset \mathbb{T}$  is given by

$$\dim(E) = \inf\{\alpha \in (0, 1), H_\alpha(E) = 0\} = \sup\{\alpha \in (0, 1), H_\alpha(E) = \infty\},$$

where  $H_\alpha = H_h$  for  $h(t) = t^\alpha$  (see [6], pp. 23-30).

Let  $\mu$  be a positive measure on  $\mathbb{T}$  and  $\alpha \in [0, 1)$ . We define the  $\alpha$ -energy of  $\mu$  by

$$I_\alpha(\mu) = \sum_{n \geq 1} \frac{|\hat{\mu}(n)|^2}{(1 + |n|)^{1-\alpha}}.$$

The  $\alpha$ -capacity of a Borel set  $E$  is given by

$$C_\alpha(E) = 1/\inf\{I_\alpha(\mu), \mu \in \mathcal{M}_{\mathcal{P}}(E)\},$$

where  $\mathcal{M}_{\mathcal{P}}(E)$  is the set of all probability measures on  $\mathbb{T}$  which are supported on a compact subset of  $E$ . If  $\alpha = 0$ ,  $C_0$  is called the logarithmic capacity.

An important property which connects capacity and Hausdorff dimension is that (see [6], p. 34)

$$\dim(E) = \inf\{\alpha \in (0, 1), C_\alpha(E) = 0\} = \sup\{\alpha \in (0, 1), C_\alpha(E) > 0\}. \quad (1.1)$$

In the following theorem, we summarize the results of Beurling [2], Salem [15] (see also [6] pp. 106-110) and Newman [10]. The Hölder conjugate of  $p \neq 1$  is noted by  $q = \frac{p}{p-1}$ .

**Theorem 1.3** ([2, 10, 15]). *Let  $1 \leq p \leq 2$ .*

- (1) *If  $u \in \ell^1(\mathbb{Z})$  and  $\dim(\mathcal{Z}(\hat{u})) < 2/q$  then  $u$  is cyclic in  $\ell^p(\mathbb{Z})$ .*
- (2) *For  $2/q < \alpha \leq 1$ , there exists  $E \subset \mathbb{T}$  such that  $\dim(E) = \alpha$  and every  $u \in \ell^1(\mathbb{Z})$  satisfying  $\mathcal{Z}(\hat{u}) = E$  is not cyclic in  $\ell^p(\mathbb{Z})$ .*
- (3) *There exists  $E \subset \mathbb{T}$  such that  $\dim(E) = 1$  and every  $u \in \ell^1(\mathbb{Z})$  satisfying  $\mathcal{Z}(\hat{u}) = E$  is cyclic in  $\ell^p(\mathbb{Z})$  for all  $p > 1$ .*

In this paper we give a generalization of the results of Beurling, Salem and Newman to  $\ell_\beta^p(\mathbb{Z})$  spaces.

When  $\beta q > 1$ , we have an analogue of (1) in Wiener's Theorem 1.1: a vector  $u \in \ell_\beta^p(\mathbb{Z})$  is cyclic if and only if  $\hat{u}$  has no zeros on  $\mathbb{T}$ . Indeed,  $\ell_\beta^p(\mathbb{Z})$  is a Banach algebra if and only if  $\beta q > 1$  (see [4]).

When  $p = 2$ , Richter, Ross and Sundberg gave a complete characterization of the cyclic vectors  $u$  in the weighted harmonic Dirichlet spaces  $\ell_\beta^2(\mathbb{Z})$  by showing the following result:

**Theorem 1.4** ([14]). *Let  $0 < \beta \leq \frac{1}{2}$  and  $u \in \ell_\beta^1(\mathbb{Z})$ .*

*The vector  $u$  is cyclic in  $\ell_\beta^2(\mathbb{Z})$  if and only if  $C_{1-2\beta}(\mathcal{Z}(\hat{u})) = 0$ .*

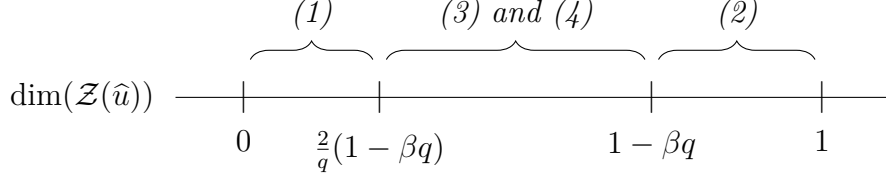
Our first main result is the following theorem.

**Theorem A.** *Let  $1 < p < 2$ ,  $\beta > 0$  such that  $\beta q \leq 1$ .*

- (1) *If  $u \in \ell_\beta^1(\mathbb{Z})$  and  $\dim(\mathcal{Z}(\hat{u})) < \frac{2}{q}(1 - \beta q)$  then  $u$  is cyclic in  $\ell_\beta^p(\mathbb{Z})$ .*
- (2) *If  $u \in \ell_\beta^1(\mathbb{Z})$  and  $\dim(\mathcal{Z}(\hat{u})) > 1 - \beta q$  then  $u$  is not cyclic in  $\ell_\beta^p(\mathbb{Z})$ .*
- (3) *For  $\frac{2}{q}(1 - \beta q) < \alpha \leq 1$ , there exists a closed subset  $E \subset \mathbb{T}$  such that  $\dim(E) = \alpha$  and every  $u \in \ell_\beta^1(\mathbb{Z})$  satisfying  $\mathcal{Z}(\hat{u}) = E$  is not cyclic in  $\ell_\beta^p(\mathbb{Z})$ .*
- (4) *If  $p = \frac{2k}{2k-1}$  for some  $k \in \mathbb{N}^*$  there exists a closed subset  $E \subset \mathbb{T}$  such that  $\dim(E) = 1 - \beta q$  and every  $u \in \ell_\beta^1(\mathbb{Z})$  satisfying  $\mathcal{Z}(\hat{u}) = E$  is cyclic in  $\ell_\beta^p(\mathbb{Z})$ .*

Note that in order to prove (2) and (4) we show a stronger result (see Theorem 3.4).

We can summarize Theorem A by the following diagram:



The fourth propriety shows that the bound  $1 - \beta q$  obtained in (2) is optimal in the sense that there is no cyclic vector such that  $\dim(\mathcal{Z}(\hat{u})) > 1 - q\beta$ , and, we can find some cyclic vector  $u$  with  $\dim(\mathcal{Z}(\hat{u})) = 1 - \beta q$ . However this is only proved if  $p = \frac{2k}{2k-1}$  for some positive integer  $k$ . When  $p$  is not of this form, for all positive integer  $k$ , we still prove similar results but we loose the optimality because we fail to reach the bound  $1 - \beta q$ .

The "equality case"  $\dim(\mathcal{Z}(\hat{u})) = \frac{2}{q}(1 - \beta q)$  is not treated by the previous theorem. Newman gave a partial answer to this question when  $\beta = 0$ , by showing that, under some additional conditions on  $\mathcal{Z}(\hat{u})$ ,  $\dim(\mathcal{Z}(\hat{u})) = \frac{2}{q}$  implies that  $u$  is a cyclic vector (see [10, Theorem 1]). We need the notion of strong  $\alpha$ -measure,  $\alpha \in (0, 1)$ , to state Newman's Theorem in the equality case. For  $E$  a compact subset of  $\mathbb{T}$ , we note  $(a_k, b_k)$ ,  $k \in \mathbb{N}$  its complementary intervals arranged in non-increasing order of lengths and set

$$r_n = 2\pi - \sum_{k=0}^n (b_k - a_k). \quad (1.2)$$

We will say that  $E$  has strong  $\alpha$ -measure 0 if

$$\lim_{n \rightarrow \infty} r_n n^{\frac{1}{\alpha}-1} = 0.$$

Notice that if  $E$  has strong  $\alpha$ -measure 0 then  $H_\alpha(E) = 0$ . The converse is true for some particular sets like Cantor sets but in general the converse is false (for some countable sets).

**Theorem 1.5.** *Let  $1 < p < 2$  and  $u \in \ell^1(\mathbb{Z})$ .*

*If  $\mathcal{Z}(\hat{u})$  has strong  $\alpha$ -measure 0 where  $\alpha = \frac{2}{q}$  then  $u$  is cyclic in  $\ell^p(\mathbb{Z})$ .*

Moreover, in [10], Newman asked the question :

*For  $u \in \ell^1(\mathbb{Z})$ , does  $H_{2/q}(\mathcal{Z}(\hat{u})) = 0$  imply that  $u$  is cyclic in  $\ell^p(\mathbb{Z})$  ?*

A positive answer to this question would contain Theorem 1.1 and Theorem 1.3.(1). We are not able to answer this question completely. Nevertheless, we show that if we replace  $2/q$ -measure by  $h$ -measure where  $h(t) = t^{2/q} \ln(1/t)^{-\gamma}$  with  $\gamma > \frac{2}{q}$  then the answer is negative. Moreover we extend Newman's Theorem to  $\ell_\beta^p(\mathbb{Z})$ .

**Theorem B.** *Let  $1 < p < 2$ ,  $\beta \geq 0$  such that  $\beta q < 1$ .*

- (1) *If  $u \in \ell_\beta^1(\mathbb{Z})$  and  $\mathcal{Z}(\hat{u})$  has strong  $\alpha$ -measure 0 where  $\alpha = \frac{2}{q}(1 - \beta q)$  then  $u$  is cyclic in  $\ell_\beta^p(\mathbb{Z})$ .*
- (2) *For every  $\gamma > \frac{2}{q}$ , there exists a closed subset  $E \subset \mathbb{T}$  such that every  $u \in \ell_\beta^1(\mathbb{Z})$  satisfying  $\mathcal{Z}(\hat{u}) = E$  is not cyclic in  $\ell_\beta^p(\mathbb{Z})$  and such that  $H_h(E) = 0$  where  $h(t) = t^\alpha \ln(e/t)^{-\gamma}$  with  $\alpha = \frac{2}{q}(1 - \beta q)$*

Note that the set  $E$  constructed in part (2) of Theorem B satisfy  $\dim(E) = \frac{2}{q}(1 - \beta q)$ .

## 2. PRELIMINARIES AND LEMMAS

Let  $1 \leq p < \infty$  and  $\beta \in \mathbb{R}$ . We denote by  $\mathcal{D}'(\mathbb{T})$  the set of distributions on  $\mathbb{T}$  and  $\mathcal{M}(\mathbb{T})$  the set of measures on  $\mathbb{T}$ . For  $S \in \mathcal{D}'(\mathbb{T})$ , we denote by  $\hat{S} = (\hat{S}(n))_{n \in \mathbb{Z}}$  the sequence of Fourier coefficients of  $S$  and we write  $S = \sum_n \hat{S}(n) e_n$ , where  $e_n(t) = e^{int}$ . The space  $A_\beta^p(\mathbb{T})$  will be the set of all distributions  $S \in \mathcal{D}'(\mathbb{T})$  such that  $\hat{S}$  belongs to  $\ell_\beta^p(\mathbb{Z})$ . We endow  $A_\beta^p(\mathbb{T})$  with the norm  $\|S\|_{A_\beta^p(\mathbb{T})} = \|\hat{S}\|_{\ell_\beta^p}$ . We will write  $A^p(\mathbb{T})$  for the space  $A_0^p(\mathbb{T})$ . Thus the Fourier transformation is an isometric isomorphism between  $\ell_\beta^p(\mathbb{Z})$  and  $A_\beta^p(\mathbb{T})$ . We prefer to work with  $A_\beta^p(\mathbb{T})$  rather than  $\ell_\beta^p(\mathbb{Z})$ . In this section we establish some properties of  $A_\beta^p(\mathbb{T})$  which will be needed to prove Theorems A and B.

For  $1 \leq p < \infty$  and  $\beta \geq 0$  we define the product of  $f \in A_\beta^1(\mathbb{T})$  and  $S \in A_\beta^p(\mathbb{T})$  by

$$fS = \sum_{n \in \mathbb{Z}} (\hat{f} * \hat{S})(n) e_n = \sum_{n \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} \hat{f}(k) \hat{S}(n - k) \right) e_n,$$

and we see that  $\|fS\|_{A_\beta^p(\mathbb{T})} \leq \|f\|_{A_\beta^1(\mathbb{T})} \|S\|_{A_\beta^p(\mathbb{T})}$ . Note that if  $S \in A_{-\beta}^p(\mathbb{T})$  we can also define the product  $fS \in A_{-\beta}^p(\mathbb{T})$  by the same formula and obtain a similar inequality:  $\|fS\|_{A_{-\beta}^p(\mathbb{T})} \leq \|f\|_{A_\beta^1(\mathbb{T})} \|S\|_{A_{-\beta}^p(\mathbb{T})}$ .

For  $p \neq 1$ , the dual space of  $A_\beta^p(\mathbb{T})$  can be identified with  $A_{-\beta}^q(\mathbb{T})$  ( $q = \frac{p}{p-1}$ ) by the following formula

$$\langle S, T \rangle = \sum_{n \in \mathbb{Z}} \hat{S}(n) \hat{T}(-n), \quad S \in A_\beta^p(\mathbb{T}), \quad T \in A_{-\beta}^q(\mathbb{T}).$$

We denote by  $\mathcal{P}(\mathbb{T})$  the set of trigonometric polynomials on  $\mathbb{T}$ . We rewrite the definition of cyclicity in the spaces  $A_\beta^p(\mathbb{T})$  for  $\beta \geq 0$  :  $S \in$

$A_\beta^p(\mathbb{T})$  will be a cyclic vector if the set  $\{PS, P \in \mathcal{P}(\mathbb{T})\}$  is dense in  $A_\beta^p(\mathbb{T})$ . It's clear that the cyclicity of  $S$  in  $A_\beta^p(\mathbb{T})$  is equivalent to the cyclicity of the sequence  $\hat{S}$  in  $\ell_\beta^p(\mathbb{Z})$ . Moreover for  $1 \leq p < \infty$  and  $\beta \geq 0$ ,  $S$  is cyclic in  $A_\beta^p(\mathbb{T})$  if and only if there exists a sequence  $(P_n)$  of trigonometric polynomials such that

$$\lim_{n \rightarrow \infty} \|1 - P_n S\|_{A_\beta^p(\mathbb{T})} = 0. \quad (2.1)$$

We need the following lemmas which gives us different inclusions between the  $A_\beta^p(\mathbb{T})$  spaces.

**Lemma 2.1.** *Let  $1 \leq r, s < \infty$  and  $\beta, \gamma \in \mathbb{R}$ .*

- (1) *If  $r \leq s$  then  $A_\beta^r(\mathbb{T}) \subset A_\gamma^s(\mathbb{T}) \Leftrightarrow \gamma \leq \beta$ .*
- (2) *If  $r > s$  then  $A_\beta^r(\mathbb{T}) \subset A_\gamma^s(\mathbb{T}) \Leftrightarrow \beta - \gamma > \frac{1}{s} - \frac{1}{r}$ .*

*Proof.* (1) : We suppose that  $r \leq s$ . If  $\gamma \leq \beta$  and  $S \in A_\beta^r(\mathbb{T})$ , we have

$$\sum_{n \in \mathbb{Z}} |\hat{S}(n)|^s (1 + |n|)^{\gamma s} \leq \sum_{n \in \mathbb{Z}} |\hat{S}(n)|^s (1 + |n|)^{\beta s}.$$

Since  $\|\cdot\|_{\ell^s} \leq \|\cdot\|_{\ell^r}$ , we obtain  $S \in A_\gamma^s(\mathbb{T})$  and so  $A_\beta^r(\mathbb{T}) \subset A_\gamma^s(\mathbb{T})$ .

Now suppose  $\gamma > \beta$ . Let  $S \in \mathcal{D}'(\mathbb{T})$  be given by

$$\hat{S}(n)(1 + |n|)^\beta = \begin{cases} (1 + m)^{-2/r} & \text{if } |n| = 2^m \\ 0 & \text{otherwise.} \end{cases}$$

Then we have  $S \in A_\beta^r(\mathbb{T}) \setminus A_\gamma^s(\mathbb{T})$ .

(2) : Now suppose that  $r > s$ . If  $\beta - \gamma > \frac{1}{s} - \frac{1}{r}$ , we have by Hölder's inequality,

$$\|S\|_{A_\gamma^s(\mathbb{T})} \leq \|S\|_{A_\beta^r(\mathbb{T})} \left( \sum_{n \in \mathbb{Z}} (1 + |n|)^{\frac{rs}{r-s}(\gamma-\beta)} \right)^{1-s/r}, \quad S \in A_\beta^r(\mathbb{T}),$$

so that  $A_\beta^r(\mathbb{T}) \subset A_\gamma^s(\mathbb{T})$ .

Now suppose that  $\beta - \gamma < \frac{1}{s} - \frac{1}{r}$ . Let  $\varepsilon > 0$  such that  $\beta - \gamma + \varepsilon < \frac{1}{s} - \frac{1}{r}$ ,  $\alpha = -\frac{1}{s} - \gamma + \varepsilon$  and let  $S \in \mathcal{D}'(\mathbb{T})$  be such that  $\hat{S}(n) = n^\alpha$ . We have  $S \in A_\beta^r(\mathbb{T}) \setminus A_\gamma^s(\mathbb{T})$ .

For the case  $\beta - \gamma = \frac{1}{s} - \frac{1}{r}$  we take  $S \in \mathcal{D}'(\mathbb{T})$  such that

$$\hat{S}(n)^r (1 + |n|)^{\beta r} = \frac{1}{(1 + |n|) \ln(1 + |n|)^{1+\varepsilon}}$$

with  $\varepsilon = \frac{r}{s} - 1 > 0$ . We can show that  $S \in A_\beta^r(\mathbb{T}) \setminus A_\gamma^s(\mathbb{T})$  which proves that  $A_\beta^r(\mathbb{T}) \not\subset A_\gamma^s(\mathbb{T})$ .  $\square$

For  $E \subset \mathbb{T}$ , we denote by  $A_\beta^p(E)$  the set of  $S \in A_\beta^p(\mathbb{T})$  such that  $\text{supp}(S) \subset E$ , where  $\text{supp}(S)$  denotes the support of the distribution  $S$ . The following lemma is a direct consequence of the definition of capacity (see [6]) and the inclusion  $A_{-\beta}^q(\mathbb{T}) \subset A_{\frac{\alpha-1}{2}}^2(\mathbb{T})$  when  $q \geq 2$  and  $0 \leq \alpha < \frac{2}{q}(1 - \beta q)$ .

**Lemma 2.2.** *Let  $E$  a Borel set,  $\beta \geq 0$  and  $q \geq 2$ . If there exists  $\alpha$ ,  $0 \leq \alpha < \frac{2}{q}(1 - \beta q)$ , such that  $C_\alpha(E) = 0$  then  $A_{-\beta}^q(E) = \{0\}$ .*

We obtain the first results about cyclicity for the spaces  $A_\beta^p(\mathbb{T})$ , when  $A_\beta^p(\mathbb{T})$  is a Banach algebra. More precisely, we have (see [4])

**Proposition 2.3.** *Let  $1 \leq p < \infty$  and  $\beta \geq 0$ .  $A_\beta^p(\mathbb{T})$  is a Banach algebra if and only if  $\beta q > 1$ . Moreover when  $\beta q > 1$ , a vector  $f \in A_\beta^p(\mathbb{T})$  is cyclic in  $A_\beta^p(\mathbb{T})$  if and only if  $f$  has no zeros on  $\mathbb{T}$ .*

Let  $f \in A_\beta^1(\mathbb{T})$  and  $S \in \mathcal{D}'(\mathbb{T})$ . We denote by  $\mathcal{Z}(f)$  the zero set of the function  $f$ . Recall that  $e_n : t \mapsto e^{int}$ .

**Lemma 2.4.** *Let  $1 \leq p < \infty$  and  $\beta \geq 0$ . Let  $f \in A_\beta^1(\mathbb{T})$  and  $S \in A_{-\beta}^p(\mathbb{T})$ . If for all  $n \in \mathbb{Z}$ ,  $\langle S, e_n f \rangle = 0$  then  $\text{supp}(S) \subset \mathcal{Z}(f)$ .*

*Proof.* We have

$$\langle S, e_n f \rangle = \langle f S, e_n \rangle = 0.$$

Hence  $f S = 0$ . Let  $\varphi \in C^\infty(\mathbb{T})$  such that  $\text{supp}(\varphi) \subset \mathbb{T} \setminus \mathcal{Z}(f)$ . We claim that  $\frac{\varphi}{f} \in A_\beta^1(\mathbb{T}) \subset A_\beta^q(\mathbb{T})$  where  $q = \frac{p}{p-1}$ . So we obtain

$$\langle S, \varphi \rangle = \langle f S, \frac{\varphi}{f} \rangle = 0$$

which proves that  $\text{supp}(S) \subset \mathcal{Z}(f)$ .

Now we prove the claim. Let  $\varepsilon = \min\{|f(t)|, t \in \text{supp}(\varphi)\} > 0$  and  $P \in \mathcal{P}(\mathbb{T})$  such that  $\|f - P\|_{A_\beta^1(\mathbb{T})} \leq \varepsilon/3$ .

By the Cauchy-Schwarz and Parseval inequalities, for every  $g \in C^1(\mathbb{T})$ , we get

$$\|g\|_{A_\beta^1(\mathbb{T})} \leq \|g\|_\infty + 2\sqrt{\frac{2-2\beta}{1-2\beta}}\|g'\|_\infty. \quad (2.2)$$

Now, as in [11], by applying (2.2) to  $\frac{\varphi}{P^n}$  we see that

$$\frac{\varphi}{f} = \sum_{n \geq 1} \varphi \frac{(P - f)^{n-1}}{P^n} \in A_\beta^1(\mathbb{T}),$$

which finishes the proof.  $\square$



**Proposition 2.5.** *Let  $1 \leq p < \infty$  and  $f \in A_\beta^1(\mathbb{T})$  with  $\beta \geq 0$ . We have*

- (1) *If  $f$  is not cyclic in  $A_\beta^p(\mathbb{T})$  then there exists  $S \in A_{-\beta}^q(\mathbb{T}) \setminus \{0\}$  such that  $\text{supp}(S) \subset \mathcal{Z}(f)$ .*
- (2) *If there exists a nonzero measure  $\mu \in A_{-\beta}^q(\mathbb{T})$  such that  $\text{supp}(\mu) \subset \mathcal{Z}(f)$  then  $f$  is not cyclic in  $A_\beta^p(\mathbb{T})$ .*

*Proof.* (1) If  $f$  is not cyclic in  $A_\beta^p(\mathbb{T})$ , by duality there exists  $S \in A_{-\beta}^q(\mathbb{T}) \setminus \{0\}$  such that

$$\langle S, e_n f \rangle = 0, \quad \forall n \in \mathbb{Z}.$$

Thus, by lemma 2.4, we have  $\text{supp}(S) \subset \mathcal{Z}(f)$ .

(2) Let  $\mu \in A^q(\mathbb{T}) \cap \mathcal{M}(\mathbb{T}) \setminus \{0\}$  such that  $\text{supp}(\mu) \subset \mathcal{Z}(f)$ . Since  $\mu$  is a measure on  $\mathbb{T}$  we have  $\langle \mu, e_n f \rangle = 0$ , for all  $n \in \mathbb{Z}$ . So  $f$  is not cyclic in  $A_\beta^p(\mathbb{T})$ .  $\square$

Recall that  $A_\beta^1(\mathbb{T})$  is a Banach algebra. Let  $I$  be a closed ideal in  $A_\beta^1(\mathbb{T})$ . We denote by  $\mathcal{Z}_I$  the set of common zeros of the functions of  $I$ ,

$$\mathcal{Z}_I = \bigcap_{f \in I} \mathcal{Z}(f).$$

We have the following result about spectral synthesis in  $A_\beta^1(\mathbb{T})$ .

**Lemma 2.6.** *Let  $0 \leq \beta < 1/2$ . Let  $I$  be a closed ideal in  $A_\beta^1(\mathbb{T})$ . If  $g$  is a Lipschitz function which vanishes on  $\mathcal{Z}_I$  then  $g \in I$ .*

*Proof.* The proof is similar to the one given in [6] pp. 121-123. For the sake of completeness we give the important steps. Let  $I^\perp$  be the set of all  $S$  in the dual space of  $A_\beta^1(\mathbb{T})$  satisfying  $\langle S, f \rangle = 0$  for all  $f \in I$ . Let  $g$  be a Lipschitz function which vanishes on  $\mathcal{Z}_I$  and  $S \in I^\perp$ . By Lemma 2.4,  $\text{supp}(S) \subset \mathcal{Z}_I$ . For  $h > 0$ , we set  $S_h = \widehat{S} * \Delta_h$  where  $\Delta_h : t \mapsto \frac{-|t|}{h^2} + \frac{1}{h}$  if  $t \in [-h, h]$  and 0 otherwise. We have  $\widehat{\Delta_h}(0) = 1/2\pi$  and  $\widehat{\Delta_h}(n) = \frac{1}{2\pi} \frac{4 \sin(nh/2)^2}{(nh)^2}$  for  $n \neq 0$ . Since  $S$  is in the dual of  $A_\beta^1(\mathbb{T})$ ,  $S_h \in A^1(\mathbb{T})$ . Moreover we have  $\text{supp}(S_h) \subset \text{supp}(S) + \text{supp}(\Delta_h) \subset \mathcal{Z}_I^h := \mathcal{Z}_I + [-h, h]$ . We have

$$\begin{aligned} |\langle S_h, g \rangle|^2 &= \left| \int_{\mathcal{Z}_I^h \setminus \mathcal{Z}(g)} S_h(x) g(x) dx \right|^2 \\ &\leq \left( \sum_{n \in \mathbb{Z}} |\widehat{S}(n) \widehat{\Delta_h}(n)|^2 \right) \left( \int_{\mathcal{Z}_I^h \setminus \mathcal{Z}(g)} |g(x)|^2 dx \right) \\ &\leq C \left( \sum_{n \in \mathbb{Z}} \frac{\widehat{S}(n)^2}{n^2} \right) (|\mathcal{Z}_I^h \setminus \mathcal{Z}(g)|) \end{aligned}$$

where  $C$  is a positive constant and where  $|E|$  denotes the Lebesgue measure of  $E$ . So  $\lim_{h \rightarrow 0} \langle S_h, g \rangle = 0$ . By the dominated convergence theorem, we obtain that

$$\lim_{h \rightarrow 0} \langle S_h, g \rangle = \lim_{h \rightarrow 0} \sum_{n \in \mathbb{Z}} \widehat{S}_h(n) \widehat{g}(-n) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \widehat{S}(n) \widehat{g}(-n) = \frac{1}{2\pi} \langle S, g \rangle.$$

So  $\langle S, g \rangle = 0$ . Therefore  $g \in I$ .  $\square$

We also need the following lemma which is a consequence of Lemma 2.6. Newman gave a proof of this when  $\beta = 0$  (see [10, Lemma 2]).

**Lemma 2.7.** *Let  $0 \leq \beta < 1/2$  and a closed set  $E \subset \mathbb{T}$ . There exists  $(f_n)$  a sequence of Lipschitz functions which are zero on  $E$  and such that*

$$\lim_{n \rightarrow \infty} \|f_n - 1\|_{A_\beta^p(\mathbb{T})} = 0$$

*if and only if every  $f \in A_\beta^1(\mathbb{T})$  satisfying  $\mathcal{Z}(f) = E$  is cyclic in  $A_\beta^p(\mathbb{T})$ .*

### 3. PROOF OF THEOREM A

Before proving Theorem A, let us recall Salem's Theorem (see [15] and [6] pp. 106-110).

**Theorem 3.1.** *Let  $0 < \alpha < 1$  and  $q > \frac{2}{\alpha}$ . There exists a compact set  $E \subset \mathbb{T}$  which satisfies  $\dim(E) = \alpha$  and there exists a positive measure  $\mu \in A^q(\mathbb{T}) \setminus \{0\}$  such that  $\text{supp}(\mu) \subset E$ .*

To prove Theorem A, we also need the following result. The case  $\beta = 0$  was considered by Newman in [10]. For  $k \in \mathbb{N}$  and  $E \subset \mathbb{T}$ , we denote

$$k \times E = E + E + \dots + E = \left\{ \sum_{n=1}^k x_n, x_n \in E \right\}.$$

**Theorem 3.2.** *Let  $1 < p < 2$  and  $\beta > 0$  such that  $\beta q \leq 1$ , and let  $f \in A_\beta^1(\mathbb{T})$ .*

- (a) *Let  $k \in \mathbb{N}^*$  be such that  $k \leq q/2$ . If  $C_\alpha(k \times \mathcal{Z}(f)) = 0$  for some  $\alpha < \frac{2}{q}(1 - \beta q)k$ , then  $f$  is cyclic in  $A_\beta^p(\mathbb{T})$ .*
- (b) *Let  $k \in \mathbb{N}^*$  be such that  $q/2 \leq k \leq 1/(2\beta)$ . If  $C_\alpha(k \times \mathcal{Z}(f)) = 0$  where  $\alpha = 1 - 2k\beta$ , then  $f$  is cyclic in  $A_\beta^p(\mathbb{T})$ .*

*Proof.* Let  $k \in \mathbb{N}^*$ . Suppose that  $f$  is not cyclic in  $A_\beta^p(\mathbb{T})$ . Then there exists  $L \in A_{-\beta}^q(\mathbb{T})$ , the dual of  $A_\beta^p(\mathbb{T})$ , such that  $L(1) = 1$  and  $L(Pf) = 0$ , for all  $P \in \mathcal{P}(\mathbb{T})$ .

Since  $\beta < \frac{1}{2}$ , by (2.2), we get  $C^1(\mathbb{T}) \subset A_\beta^1(\mathbb{T}) \subset A_\beta^p(\mathbb{T})$ , and by [9] (see also [10, Lemma 5]), there exists  $\phi \in L^2(\mathbb{T})$  such that

$$L(g) = \int_{\mathbb{T}} (g'(x)\phi(x) + g(x)) \, dx, \quad \forall g \in C^1(\mathbb{T}).$$

Since  $L \in A_{-\beta}^q(\mathbb{T})$  which implies  $(L(e_n))_{n \in \mathbb{Z}} \in \ell_{-\beta}^q(\mathbb{Z})$ , we obtain

$$\sum_{n \in \mathbb{Z}} |n \widehat{\phi}(n)|^q (1 + |n|)^{-\beta q} < \infty. \quad (3.1)$$

Moreover we have,

$$\int_{\mathbb{T}} ((e_n f)'(x)\phi(x) + (e_n f)(x)) \, dx = 0, \quad n \in \mathbb{Z},$$

and so  $\langle \phi' - 1, e_n f \rangle = 0$  where  $\phi'$  is defined in terms of distribution. By (3.1),  $\phi' - 1 \in A_{-\beta}^q(\mathbb{T})$ , so by lemma 2.4, we get  $\text{supp}(\phi' - 1) \subset \mathcal{Z}(f)$ .

For  $m \in \mathbb{N}$ , we denote by  $\phi^{*m}$  the result of convolving  $\phi$  with itself  $m$  times. Using the fact that  $S' * T = S * T'$  and  $1 * S' = 0$  for any distributions  $S$  and  $T$ , we have

$$(\phi' - 1) * \left( (\phi^{*(m-1)})^{(m-1)} + (-1)^{m-1} \right) = (\phi^{*m})^{(m)} + (-1)^m.$$

So we can show by induction on  $m \geq 1$  and by the formula  $\text{supp}(T * S) \subset \text{supp}(T) + \text{supp}(S)$  that

$$\text{supp} \left( (\phi^{*m})^{(m)} + (-1)^m \right) \subset m \times \mathcal{Z}(f), \quad \forall m \geq 1. \quad (3.2)$$

Note that  $\widehat{(\phi^{*k})^{(k)}}(n) = i^k n^k \widehat{\phi}(n)^k$  for  $k \geq 1$  and  $n \in \mathbb{Z}$ .

(a) : Suppose that  $0 < k \leq q/2$  and  $C_\alpha(k \times \mathcal{Z}(f)) = 0$  for some  $\alpha < \frac{2}{q}(1 - \beta q)k$ . We rewrite (3.1) as

$$\sum_{n \in \mathbb{Z}} \left( |n \widehat{\phi}(n)|^k \right)^{\frac{q}{k}} (1 + |n|)^{-\frac{q}{k}\beta k} < \infty.$$

So, if we set  $q' = \frac{q}{k} \geq 2$  and  $\beta' = \beta k$ , we have  $(\phi^{*k})^{(k)} \in A_{-\beta'}^{q'}(\mathbb{T})$ . By (3.2) and by Lemma 2.2 we obtain that  $(\phi^{*k})^{(k)} = (-1)^{k-1}$ . This contradicts the fact that  $\widehat{(\phi^{*k})^{(k)}}(0) = 0$ .

(b) : Now suppose that  $k \geq q/2$  and  $C_\alpha(k \times \mathcal{Z}(f)) = 0$  where  $\alpha = 1 - 2k\beta$ . Since  $q \leq 2k$ , we have by (3.1),

$$\sum_{n \in \mathbb{Z}} |n \widehat{\phi}(n)|^{2k} (1 + |n|)^{-2k\beta} < \infty.$$

So  $(\phi^{*k})^{(k)} \in A_{-k\beta}^2(\mathbb{T})$  and  $(\phi^{*k})^{(k)} = (-1)^{k-1}$ . Again this is absurd since  $(\widehat{\phi^{*k}})^{(k)}(0) = 0$ .  $\square$

We need to compute the capacity of the Minkowski sum of some Cantor type subset of  $\mathbb{T}$ . We denote by  $[x]$  the integer part of  $x \in \mathbb{R}$ . For  $\lambda \in [0, 1]$  and  $k \in \mathbb{N}^*$ , we define

$$K_\lambda^k = \{m \in \mathbb{N}, \exists j \in \mathbb{N}, m \in [2^j, 2^j(1 + \lambda + 1/j) - k + 1]\}$$

and we set in  $\mathbb{R}/\mathbb{Z} \simeq [0, 1[$ ,

$$S_\lambda^k = \left\{ x = \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}}, (x_i) \in \{0, 1\}^{\mathbb{N}} \text{ such that } i \in K_\lambda^k \Rightarrow x_i = 0 \right\}.$$

We denote  $K_\lambda = K_\lambda^1$  and  $S_\lambda = S_\lambda^1$ .

To prove (4) of Theorem A we need the following lemma.

**Lemma 3.3.** *For all  $k \geq 1$ , we have*

- (1)  $k \times S_\lambda \subset S_\lambda^k$  ;
- (2)  $C_\alpha(S_\lambda^k) = 0$  if and only if  $\alpha \geq \frac{1-\lambda}{1+\lambda}$  ;
- (3)  $\dim(k \times S_\lambda) = \frac{1-\lambda}{1+\lambda}$  and  $C_{\frac{1-\lambda}{1+\lambda}}(k \times S_\lambda) = 0$ .

*Proof.* (1) : We prove this by induction. If  $k = 1$  we have  $S_\lambda = S_\lambda^1$ . We suppose the result true for  $k - 1$  for some  $k \geq 2$  and we will show  $k \times S_\lambda \subset S_\lambda^k$ . We have  $k \times S_\lambda \subset (k - 1) \times S_\lambda + S_\lambda \subset S_\lambda^{k-1} + S_\lambda$ . Let  $x \in S_\lambda^{k-1}$ ,  $y \in S_\lambda$  and  $z = x + y$ . Denote by  $(x_i)$ ,  $(y_i)$  and  $(z_i)$  their binary decomposition. Let  $m \in K_\lambda^k$ . There exists  $j \in \mathbb{N}$  such that  $m \in [2^j, 2^j(1 + \lambda + 1/j) - k + 1]$ . Since  $m \in K_\lambda^k$ ,  $m$  and  $m + 1$  are contained in  $K_\lambda^{k-1} \subset K_\lambda$ , we have  $x_m = y_m = x_{m+1} = y_{m+1} = 0$ . Therefore we write

$$z = x + y = \sum_{i=0}^{m-1} \frac{x_i + y_i}{2^{i+1}} + \sum_{i=m+2}^{\infty} \frac{x_i + y_i}{2^{i+1}}.$$

Note that for infinitely many  $i \geq m + 2$ ,  $x_i + y_i < 2$ , so we see that

$$\sum_{i=m+2}^{\infty} \frac{x_i + y_i}{2^{i+1}} < \frac{1}{2^{m+1}}.$$

Therefore, we obtain by uniqueness of the decomposition that  $z_m = 0$ . This proves that  $x + y \in S_\lambda^k$  and  $k \times S_\lambda \subset S_\lambda^k$ .

(2) : We will study the capacity of  $S_\lambda^k$  by decomposing it. First we show that the set  $S_\lambda^k$  is a generalized Cantor set in the sense of [3, 13].

Let  $\nu_j = [2^j(1 + \lambda + 1/j) - k + 1] + 1$  and  $N_0$  (depending only on  $k$  and  $\lambda$ ) such that for all  $j \geq N_0$ ,  $2^j < \nu_j < 2^{j+1}$ . We set for  $N \geq N_0$ ,

$$l_N = \sum_{j=N}^{\infty} \frac{1}{2^{\nu_j}} - \frac{1}{2^{2^{j+1}}}.$$

Since  $2^j(1 + \lambda + 1/j) - k + 1 < \nu_j \leq 2^j(1 + \lambda + 1/j) - k + 2$ , we have

$$\begin{aligned} \sum_{j=N}^{\infty} \frac{1}{2^{2^j(1+\lambda+\frac{1}{j})}} \left( \frac{1}{2^{2-k}} - \frac{1}{2^{2^j(1-\lambda-\frac{1}{j})}} \right) &\leq l_N \\ &\leq \sum_{j=N}^{\infty} \frac{1}{2^{2^j(1+\lambda+\frac{1}{j})}} \left( \frac{1}{2^{1-k}} - \frac{1}{2^{2^j(1-\lambda-\frac{1}{j})}} \right) \end{aligned}$$

On one hand, there exists  $C \geq 1$  such that for all  $j \geq N$ ,

$$\frac{1}{C} \leq \frac{1}{2^{2-k}} - \frac{1}{2^{2^j(1-\lambda-\frac{1}{j})}} \leq \frac{1}{2^{1-k}} - \frac{1}{2^{2^j(1-\lambda-\frac{1}{j})}} \leq C.$$

On the other hand, for  $N \geq N_0$ ,

$$\begin{aligned} \frac{1}{2^{2^N(1+\lambda+\frac{1}{N})}} &\leq \sum_{j=N}^{\infty} \frac{1}{2^{2^j(1+\lambda+\frac{1}{j})}} \leq \frac{1}{2^{2^N(1+\lambda+\frac{1}{N})}} + \sum_{j=0}^{\infty} \left( \frac{1}{2^{2^{N+1}(1+\lambda)}} \right)^{2^j} \\ &\leq \frac{1}{2^{2^N(1+\lambda+\frac{1}{N})}} + \sum_{j=0}^{\infty} \left( \frac{1}{2^{2^{N+1}(1+\lambda)}} \right)^{j+1} \\ &\leq \frac{1}{2^{2^N(1+\lambda+\frac{1}{N})}} + \frac{2}{2^{2^{N+1}(1+\lambda)}} \\ &\leq \frac{3}{2^{2^N(1+\lambda+\frac{1}{N})}}. \end{aligned}$$

Hence we obtain that  $l_N$  is comparable to  $2^{-2^N(1+\lambda+1/N)}$ , that is:

$$\frac{1}{C 2^{2^N(1+\lambda+\frac{1}{N})}} \leq l_N \leq \frac{3C}{2^{2^N(1+\lambda+\frac{1}{N})}}. \quad (3.3)$$

Moreover we have

$$l_N = \frac{1}{2^{\nu_N}} - \sum_{j=N+1}^{\infty} \frac{1}{2^{2^j}} - \frac{1}{2^{\nu_j}} < \frac{1}{2^{\nu_N}} \leq \frac{1}{2^{2^N}}. \quad (3.4)$$

We set

$$E_N = \left\{ \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + l_N z, \ z \in [0, 1], \ x_i \in \{0, 1\}, \ i \in K_{\lambda}^k \Rightarrow x_i = 0 \right\}.$$

We can see  $E_N$  as a union of disjoint intervals by writing

$$E_N = \bigcup_{\substack{(x_i) \in \{0,1\}^{2^N} \\ i \in K_\lambda^k \Rightarrow x_i = 0}} E_N^{(x_i)},$$

where

$$E_N^{(x_i)} = \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + l_N[0, 1].$$

Note that the intervals  $E_N^{(x_i)}$  are disjoint since by (3.4),  $l_N < \frac{1}{2^{2^N}}$ . For fixed  $N \geq N_0$ , let  $(x_i)_{0 \leq i \leq 2^{N-1}} \in \{0,1\}^{2^N}$  and  $(y_i)_{0 \leq i \leq 2^{N+1}-1} \in \{0,1\}^{2^{N+1}}$ .

Claim :  $E_{N+1}^{(y_i)} \subset E_N^{(x_i)}$  if and only if  $x_i = y_i$  for all  $0 \leq i < 2^N$  and  $y_i = 0$  for all  $2^N \leq i < \nu_N$ .

Indeed, suppose that  $E_{N+1}^{(y_i)} \subset E_N^{(x_i)}$  and let  $u \in E_{N+1}^{(y_i)}$ . We have

$$u = \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + l_N z_1 = \sum_{i=0}^{2^{N+1}-1} \frac{y_i}{2^{i+1}} + l_{N+1} z_2,$$

where  $z_1$  and  $z_2$  are in  $[0, 1]$ . By (3.4),  $l_N < \frac{1}{2^{\nu_N}}$ , and using the uniqueness of the binary representation, we obtain  $x_i = y_i$  for all  $0 \leq i < 2^N$  and  $y_i = 0$  for all  $2^N \leq i < \nu_N$ .

Now suppose  $x_i = y_i$  for all  $0 \leq i < 2^N$  and  $y_i = 0$  for all  $2^N \leq i < \nu_N$ . Let  $u \in E_{N+1}^{(y_i)}$ . We write

$$u = \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + \sum_{i=\nu_N}^{2^{N+1}-1} \frac{y_i}{2^{i+1}} + l_{N+1} z,$$

where  $z \in [0, 1]$ . Note that

$$\sum_{i=\nu_N}^{2^{N+1}-1} \frac{1}{2^{i+1}} + l_{N+1} = \frac{1}{2^{\nu_N}} - \frac{1}{2^{2^{N+1}}} + l_{N+1} = l_N. \quad (3.5)$$

So we have

$$\sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} \leq \sum_{i=0}^{2^N-1} \frac{x_i}{2^{i+1}} + \sum_{i=\nu_N}^{2^{N+1}-1} \frac{y_i}{2^{i+1}} + l_{N+1} z \leq \frac{1}{2} \sum_{i=0}^{2^N-1} \frac{x_i}{2^i} + l_N,$$

and  $u \in E_N^{(x_i)}$ . This conclude the proof of the claim.

By the claim, for fixed  $(x_i)$  and for  $N \geq N_0$ , we have the following properties :

(i) the interval  $E_N^{(x_i)}$  contains precisely

$$p_N = \#\{(y_i)_{\nu_N \leq i \leq 2^{N+1}-1} : y_i \in \{0, 1\}\} = 2^{2^{N+1}-\nu_N}$$

intervals of the form  $E_{N+1}^{(y_i)}$ ,

(ii) the intervals of the form  $E_{N+1}^{(y_i)}$  contained in  $E_N^{(x_i)}$  are equidistant intervals of length  $l_{N+1}$ : the distance of two consecutive intervals of the form  $E_{N+1}^{(y_i)}$  is equal to  $\frac{1}{2^{2^{N+1}}-l_{N+1}}$ ,

(iii) if we denote  $E_N^{(x_i)} = [a, b]$  then there exist  $(y_i)$  and  $(z_i)$  such that  $E_{N+1}^{(y_i)} = [a, a + l_{N+1}]$  and  $E_{N+1}^{(z_i)} = [b - l_{N+1}, b]$ .

Finally we can write  $S_\lambda^k$  as

$$S_\lambda^k = \bigcap_{N \geq N_0} E_N.$$

This shows that  $S_\lambda^k$  is a generalized Cantor set in the sense of [3, 13]. So, by [3, 13], we have for  $0 < \alpha < 1$  that  $C_\alpha(S_\lambda^k) = 0$  if and only if

$$\sum_{N=N_0}^{\infty} \frac{1}{(p_{N_0} \cdots p_{N-1}) l_N^\alpha} = \infty.$$

Since

$$\begin{aligned} 2^{(k-2)(N-N_0)+(2^N-2^{N_0})(1-\lambda)-\sigma_N} &\leq p_{N_0} \cdots p_{N-1} \\ &\leq 2^{(k-1)(N-N_0)+(2^N-2^{N_0})(1-\lambda)-\sigma_N}, \end{aligned}$$

where

$$\sigma_N = \sum_{j=N_0}^{N-1} \frac{2^j}{j},$$

we have, by (3.3),  $C_\alpha(S_\lambda^k) = 0$  if and only if

$$\sum_{N=N_0}^{\infty} 2^{2^N(\alpha(1+\lambda)-(1-\lambda))+\alpha 2^N/N+\sigma_N-(k-1)(N-N_0)+2^{N_0}(1-\lambda)} = \infty.$$

Therefore  $C_\alpha(S_\lambda^k) = 0$  if and only if  $\alpha \geq \frac{1-\lambda}{1+\lambda}$ .

(3) immediately follows from (1) and (2) by the capacity property (1.1).  $\square$

We are ready to prove Theorem A. The following Theorem is a reformulation of Theorem A in  $A_\beta^p(\mathbb{T})$  spaces.

**Theorem 3.4.** *Let  $1 < p < 2$ ,  $\beta > 0$  such that  $\beta q \leq 1$ .*

*(1) If  $f \in A_\beta^1(\mathbb{T})$  and  $\dim(\mathcal{Z}(f)) < \frac{2}{q}(1 - \beta q)$  then  $f$  is cyclic in  $A_\beta^p(\mathbb{T})$ .*

- (2) If  $f \in A_\beta^1(\mathbb{T})$  and  $C_{1-\beta q}(\mathcal{Z}(f)) > 0$  then  $f$  is not cyclic in  $A_\beta^p(\mathbb{T})$ .
- (3) For  $\frac{2}{q}(1 - \beta q) < \alpha \leq 1$ , there exists a closed set  $E \subset \mathbb{T}$  such that  $\dim(E) = \alpha$  and every  $f \in A_\beta^1(\mathbb{T})$  satisfying  $\mathcal{Z}(f) = E$  is not cyclic in  $A_\beta^p(\mathbb{T})$ .
- (4) Let  $k = [q/2]$ . For all  $\varepsilon > 0$ , there exists a closed set  $E \subset \mathbb{T}$  such that

$$\dim(E) \geq \max \left( \frac{2}{q}(1 - \beta q)k - \varepsilon, 1 - 2(k + 1)\beta \right)$$

and such that every  $f \in A_\beta^1(\mathbb{T})$  satisfying  $\mathcal{Z}(f) = E$  is cyclic in  $A_\beta^p(\mathbb{T})$ .

Furthermore, if  $p = \frac{2k}{2k-1}$  for some  $k \in \mathbb{N}^*$ ,  $E$  can be chosen such that  $\dim(E) = 1 - \beta q$ .

*Proof.* (1) : Note that, by (1.1),  $\dim(\mathcal{Z}(f)) < \frac{2}{q}(1 - \beta q)$  if and only if there exists  $\alpha < \frac{2}{q}(1 - \beta q)$  such that  $C_\alpha(\mathcal{Z}(f)) = 0$ . If  $C_\alpha(\mathcal{Z}(f)) = 0$ , by Lemma 2.2, there is no  $S \in A_{-\beta}^q(\mathbb{T}) \setminus \{0\}$  such that  $\text{supp}(S) \subset \mathcal{Z}(f)$ . So, by Proposition 2.5 (1),  $f$  is cyclic in  $A_\beta^p(\mathbb{T})$ .

(2) : Suppose that  $C_{1-\beta q}(\mathcal{Z}(f)) > 0$ . There exists a probability measure  $\mu$  of energy  $I_{1-\beta q}(\mu) < \infty$ , such that  $\text{supp}(\mu) \subset \mathcal{Z}(f)$ . So  $\mu \in A_{-\beta q/2}^2(\mathbb{T}) \setminus \{0\}$ . Since  $|\hat{\mu}(n)| \leq 1$  for all  $n \in \mathbb{Z}$  and  $q \geq 2$ , we have  $\mu \in A_{-\beta}^q(\mathbb{T})$ . By proposition 2.5 (2),  $f$  is not cyclic in  $A_\beta^p(\mathbb{T})$ .

(3) : Let  $\frac{2}{q}(1 - \beta q) < \alpha \leq 1$ . There exists  $\varepsilon > 0$  such that  $\frac{2}{q}(1 - \beta q) + \varepsilon < \alpha$ . Let  $q'$  such that  $\frac{2}{q} - 2\beta + \varepsilon = \frac{2}{q'}$ . Since  $\beta > \frac{1}{q} - \frac{1}{q'}$ , by Lemma 2.1,  $A^{q'}(\mathbb{T}) \subset A_{-\beta}^q(\mathbb{T})$ . By Theorem 3.1, as  $q'$  satisfies  $q' > \frac{2}{\alpha}$ , there exists a closed subset  $E \subset \mathbb{T}$  such that  $\dim(E) = \alpha$  and a non zero positive measure  $\mu \in A^{q'}(\mathbb{T}) \subset A_{-\beta}^q(\mathbb{T})$  such that  $\text{supp}(\mu) \subset E$ . Now (3) follows from proposition 2.5.(2).

(4) : Let  $k = [q/2]$ . Suppose first  $\frac{2}{q}(1 - \beta q)k > 1 - 2(k + 1)\beta$  and let  $0 < \varepsilon' < \varepsilon$  satisfying  $1 - 2(k + 1)\beta \leq \frac{2}{q}(1 - \beta q)k - \varepsilon'$ . Consider the set  $S_\lambda$  where  $\lambda$  verifies

$$\frac{2}{q}(1 - \beta q)k - \varepsilon' < \frac{1 - \lambda}{1 + \lambda} < \frac{2}{q}(1 - \beta q)k.$$

By Lemma 3.3.(3) we have  $\dim(S_\lambda) = \frac{1-\lambda}{1+\lambda}$  and  $C_{\frac{1-\lambda}{1+\lambda}}(k \times S_\lambda) = 0$ . Therefore by Theorem 3.2.(a), every  $f \in A_\beta^1(\mathbb{T})$  such that  $\mathcal{Z}(f) = S_\lambda$  is cyclic in  $A_\beta^p(\mathbb{T})$ .



Now suppose  $\frac{2}{q}(1 - \beta q)k \leq 1 - 2(k + 1)\beta$ . We consider  $S_\lambda$  where  $\frac{1-\lambda}{1+\lambda} = 1 - 2(k + 1)\beta$ . By lemma 3.3.(3) we have  $\dim(S_\lambda) = \frac{1-\lambda}{1+\lambda} = 1 - 2(k + 1)\beta$  and  $C_{\frac{1-\lambda}{1+\lambda}}((k + 1) \times S_\lambda) = 0$ . So by Theorem 3.2.(b), every  $f \in A_\beta^1(\mathbb{T})$  such that  $\mathcal{Z}(f) = S_\lambda$  is cyclic in  $A_\beta^p(\mathbb{T})$ .

Suppose now that  $p = \frac{2k}{2k-1}$  for some  $k \in \mathbb{N}^*$ . As before, we consider  $S_\lambda$  where  $\frac{1-\lambda}{1+\lambda} = 1 - 2k\beta = 1 - \beta q$ . So again by Theorem 3.2.(b), every  $f \in A_\beta^1(\mathbb{T})$  such that  $\mathcal{Z}(f) = S_\lambda$  is cyclic in  $A_\beta^p(\mathbb{T})$ .  $\square$

Note that the set  $E$  which is considered in 3.4.(4) verifies  $C_\alpha(E) = 0$  where

$$\alpha \geq \max \left( \frac{2}{q}(1 - \beta q)k - \varepsilon, 1 - 2(k + 1)\beta \right).$$

#### 4. PROOF OF THEOREM B

In this section we investigate the sharpness of the constant  $\frac{2}{q}(1 - \beta q)$  in Theorem A.

Before proving Theorem B, we need the following two results. The following Lemma is an extension of Newman's Lemma 3 (see [10] pp 654-655).

**Lemma 4.1.** *Let  $p \in [1, 2]$ ,  $\beta \geq 0$  such that  $\beta q \leq 1$ . There exists  $C > 0$  such that for all  $f \in A_1^2(\mathbb{T})$ ,*

$$\|f\|_{A_\beta^p(\mathbb{T})} \leq C^{\frac{1}{p}} \|f\|_{A^2(\mathbb{T})}^{\frac{3}{2} - \frac{1}{p} - \beta} (\|f\|_{A^2(\mathbb{T})} + \|f'\|_{A^2(\mathbb{T})})^{\frac{1}{p} - \frac{1}{2} + \beta}.$$

*Proof.* It suffices to show that there exists  $C > 0$  such that for all sequences  $(c_n) \in \mathbb{C}^{\mathbb{N}^*}$ ,

$$\sum_{n=1}^{\infty} |c_n|^p (1 + |n|)^{p\beta} \leq C \left( \sum_{n=1}^{\infty} |c_n|^2 \right)^{\frac{3}{4}p - \frac{1}{2} - \frac{\beta p}{2}} \left( \sum_{n=1}^{\infty} n^2 |c_n|^2 \right)^{\frac{1}{2} - \frac{p}{4} + \frac{\beta p}{2}}.$$

Then we apply this inequality to  $(\hat{f}(n))_{n \geq 1}$  and  $(\hat{f}(-n))_{n \geq 1}$ . Let  $x^2 = \sum_{n \geq 1} |c_n|^2$  and  $x^2 y^2 = \sum_{n \geq 1} n^2 |c_n|^2$ . Note that  $y \geq 1$ . On one hand, by the Hölder inequality,

$$\begin{aligned} \sum_{1 \leq n \leq y} |c_n|^p (1 + n)^{p\beta} &\leq \left( \sum_{n=1}^y |c_n|^2 \right)^{p/2} \left( \sum_{n=1}^y (1 + n)^{\frac{2p\beta}{2-p}} \right)^{1-p/2} \\ &\leq (x^2)^{\frac{p}{2}} \left( y(1 + y)^{\frac{2p\beta}{2-p}} \right)^{1-\frac{p}{2}} \leq 2^{\beta p} x^p y^{1-\frac{p}{2}+p\beta}. \end{aligned}$$

On the other hand we set  $\gamma = \frac{2p(1-\beta)}{2-p}$ . Since  $\beta q \leq 1$ ,  $\gamma > 1$  and again by the Hölder inequality we obtain,

$$\begin{aligned} \sum_{n>y} |c_n|^p (1+n)^{p\beta} &\leq 2^p \left( \sum_{n>y} n^2 |c_n|^2 \right)^{\frac{p}{2}} \left( \sum_{n>y} (1+n)^{\frac{2p(\beta-1)}{2-p}} \right)^{1-\frac{p}{2}} \\ &\leq 2^p (x^2 y^2)^{\frac{p}{2}} \left( \frac{1}{\gamma-1} \right)^{1-p/2} (y^{1-\gamma})^{1-\frac{p}{2}} \\ &\leq 2^p \left( \frac{1}{\gamma-1} \right)^{1-p/2} x^p y^{1-\frac{p}{2}+p\beta} \end{aligned}$$

So the conclusion of the Lemma holds with

$$C = \max \left( 2^{\beta p}, 2^p \left( \frac{1}{\gamma-1} \right)^{1-p/2} \right)$$

which is a positive constant depending only on  $p$  and  $\beta$ .  $\square$

The following theorem is due to Körner (see [7, Theorem 1.2]).

**Theorem 4.2.** *Let  $h : [0, \infty) \rightarrow [0, \infty)$  be an increasing continuous function with  $h(0) = 0$  and let  $\phi : [0, \infty) \rightarrow [0, \infty)$  be a decreasing function. Suppose that*

- (1)  $\int_1^\infty \phi(x)^2 dx = \infty$ ;
- (2) there exist  $K_1, K_2 > 1$  such that for all  $1 \leq x \leq y \leq 2x$ ,

$$K_1 \phi(2x) \leq \phi(x) \leq K_2 \phi(y);$$

- (3) there exists  $\gamma > 0$  such that

$$\lim_{x \rightarrow \infty} x^{1-\gamma} \phi(x) = \infty;$$

- (4) there exist  $0 < K_2 < K_3 < 1$  such that for all  $t > 0$ ,

$$K_2 h(2t) \leq h(t) \leq K_3 h(2t).$$

Then there exists a probability measure  $\mu$  with support of Hausdorff  $h$ -measure zero such that

$$|\hat{\mu}(n)| \leq \phi \left( \frac{1}{h(|n|^{-1})} \right) \left( \ln \left( \frac{1}{h(|n|^{-1})} \right) \right)^{1/2}, \quad \forall n \neq 0.$$

Recall Theorem B reformulated in  $A_\beta^p(\mathbb{T})$  space.

**Theorem 4.3.** *Let  $1 < p < 2$  and  $\beta \geq 0$  such that  $\beta q < 1$ .*

- (1) *If  $f \in A_\beta^1(\mathbb{T})$  and  $\mathcal{Z}(f)$  has strong  $\alpha$ -measure 0 where  $\alpha = \frac{2}{q}(1 - \beta q)$  then  $f$  is cyclic in  $A_\beta^p(\mathbb{T})$ .*

- (2) For every  $\gamma > \frac{2}{q}$ , there exists a closed subset  $E \subset \mathbb{T}$  such that every  $f \in A_\beta^1(\mathbb{T})$  satisfying  $\mathcal{Z}(f) = E$  is not cyclic in  $A_\beta^p(\mathbb{T})$  and such that  $H_h(E) = 0$  where  $h(t) = \frac{t^\alpha}{\ln(e/t)^\gamma}$  with  $\alpha = \frac{2}{q}(1 - \beta q)$ .

Note that in (2),  $H_h$  is closed to  $H_\alpha$ .

*Proof.* (1) : The proof of this result holds by using arguments analogous to those of Newman for  $\beta = 0$  (see [10, Theorem 1]). Denote by  $(a_k, b_k)$  the complementary intervals of  $\mathcal{Z}(f)$  arranged in non-increasing order of lengths and set

$$r_n = 2\pi - \sum_{k=0}^n (b_k - a_k).$$

The set  $\mathcal{Z}(f)$  has strong  $\alpha$ -measure 0 where  $\alpha = \frac{2}{q}(1 - \beta q)$  so

$$\lim_{n \rightarrow \infty} r_n n^{\frac{1}{\alpha}-1} = 0.$$

Let  $\varepsilon > 0$  and  $n \in \mathbb{N}$  such that  $r_n < \varepsilon n^{1-\frac{1}{\alpha}}$  and  $\varepsilon n^{-\frac{1}{\alpha}} < 1$ . Let the function  $\psi$  be given by

$$\psi(x) = \max \left( 1 - \frac{n^{\frac{1}{\alpha}}}{\varepsilon} \rho(x), 0 \right), \quad x \in \mathbb{T},$$

where

$$\rho(x) = \text{dist} \left( x, \mathbb{T} \setminus \bigcup_{k=1}^n ]a_k, b_k[ \right).$$

Then

$$\begin{aligned} \|\psi\|_{A^2(\mathbb{T})}^2 &= \int_{\mathbb{T} \setminus \bigcup_{k=1}^n ]a_k, b_k[} \psi(t)^2 \, dt + \sum_{k=1}^n \int_{a_k}^{b_k} \psi(t)^2 \chi_{\{\rho(x) \leq \varepsilon n^{-\frac{1}{\alpha}}\}}(t) \, dt \\ &\leq r_n + \sum_{k=1}^n 2\varepsilon n^{-\frac{1}{\alpha}} \leq 3\varepsilon n^{1-\frac{1}{\alpha}}. \end{aligned}$$

Moreover

$$\begin{aligned} \|\psi'\|_{A^2(\mathbb{T})}^2 &= \int_{\mathbb{T}} \psi'(t)^2 \, dt = \sum_{k=1}^n \int_{a_k}^{b_k} \psi'(t)^2 \chi_{\{\rho(x) \leq \varepsilon n^{-\frac{1}{\alpha}}\}}(t) \, dt \\ &\leq \sum_{k=1}^n \left( \frac{n^{\frac{1}{\alpha}}}{\varepsilon} \right)^2 2\varepsilon n^{-\frac{1}{\alpha}} \leq 2 \frac{n^{1+\frac{1}{\alpha}}}{\varepsilon}. \end{aligned}$$

Since  $\varepsilon n^{-\frac{1}{\alpha}} < \frac{n^{\frac{1}{\alpha}}}{\varepsilon}$  and  $\alpha = \frac{2}{q}(1 - \beta q)$ , by Lemma 4.1,

$$\|\psi\|_{A_\beta^p(\mathbb{T})} \leq C^{\frac{1}{p}} \left( 3\varepsilon n^{1-\frac{1}{\alpha}} \right)^{\frac{3}{4}-\frac{1}{2p}-\frac{\beta}{2}} \left( 5 \frac{n^{1+\frac{1}{\alpha}}}{\varepsilon} \right)^{\frac{1}{2p}-\frac{1}{4}+\frac{\beta}{2}} \leq C' \varepsilon^{1-\frac{1}{p}-\beta}$$

where  $C$  and  $C'$  depend only on  $\beta$  and  $p$ . Note that  $1 - \psi$  is a Lipschitz function and  $\mathcal{Z}(f) \subset \mathcal{Z}(1 - \psi)$ . We conclude by Lemma 2.7.

(2) : Let  $\alpha = \frac{2}{q}(1 - \beta q)$  and  $\gamma > \frac{2}{q}$ . By Theorem 4.2 with  $\phi(t) = (t \ln(et))^{-1/2}$  for  $t \geq 1$  and  $h(t) = \frac{t^\alpha}{\ln(e/t)^\gamma}$  for  $t \in [0, \infty)$ , there exists a probability measure  $\mu$  with support of Hausdorff  $h$ -measure zero such that

$$|\hat{\mu}(n)| \leq \phi\left(\frac{1}{h(|n|^{-1})}\right) \left(\ln\left(\frac{1}{h(|n|^{-1})}\right)\right)^{1/2} \leq (|n|^\alpha \ln(e|n|)^\gamma)^{-1/2},$$

for  $n \neq 0$ . So

$$\begin{aligned} \sum_{n \neq 0} |\hat{\mu}(n)|^q (1 + |n|)^{-\beta q} &\leq C \sum_{n \neq 0} |n|^{-\alpha q/2 - \beta q} \ln(e|n|)^{-\gamma q/2} \\ &\leq C \sum_{n \neq 0} \frac{1}{|n| \ln(e|n|)^{\gamma q/2}} < \infty \end{aligned}$$

with  $C$  a positive constant. Hence  $\mu \in A_{-\beta}^q(\mathbb{T})$ . We set  $E = \text{supp}(\mu)$ . By lemma 2.5 the result is proved.  $\square$

## 5. REMARKS

We say that  $(\omega_n) \in \mathbb{R}^{\mathbb{Z}}$  is a weight if  $\omega_n \geq 1$  and  $\omega_{n+k} \leq C\omega_n\omega_k$  for all  $k, n \in \mathbb{Z}$  and  $C$  a positive constant. For  $\omega$  a weight and  $1 \leq p < \infty$  we set

$$A_\omega^p(\mathbb{T}) = \left\{ f \in C(\mathbb{T}), \quad \|f\|_{A_\omega^p(\mathbb{T})}^p = \sum_{n \in \mathbb{Z}} |\hat{f}(n)|^p \omega_n^p < \infty \right\}.$$

Note that  $\|fS\|_{A_\omega^p(\mathbb{T})} \leq \|f\|_{A_\omega^1(\mathbb{T})} \|S\|_{A_\omega^p(\mathbb{T})}$  for  $f \in A_\omega^1(\mathbb{T})$  and  $S \in A_\omega^p(\mathbb{T})$ . So we have the same result as (2.1) to characterize cyclicity in  $A_\omega^p(\mathbb{T})$  by norm.

When  $\omega_n = O((1 + |n|)^\varepsilon)$  for all  $\varepsilon > 0$ , for example  $\omega_n = \ln(e + |n|)^\beta$  where  $\beta \geq 0$ , we can show the same result as Lemma 2.7. So by noting that for all  $p \geq 1$  and  $\delta > 0$ ,

$$A_\delta^p(\mathbb{T}) \subset A_\omega^p(\mathbb{T}) \subset A^p(\mathbb{T})$$

we obtain by Theorem A the following result:

**Theorem 5.1.** *Let  $1 < p < 2$  and  $\omega = (\omega_n)_{n \in \mathbb{Z}}$  a weight satisfying  $\omega_n = O((1 + |n|)^\varepsilon)$  for all  $\varepsilon > 0$ .*

(1) *If  $f \in A_\omega^1(\mathbb{T})$  and  $\dim(\mathcal{Z}(f)) < \frac{2}{q}$  then  $f$  is cyclic in  $A_\omega^p(\mathbb{T})$ .*

- (2) For  $\frac{2}{q} < \alpha \leq 1$ , there exists a closed subset  $E \subset \mathbb{T}$  such that  $\dim(E) = \alpha$  and every  $f \in A_\omega^1(\mathbb{T})$  satisfying  $\mathcal{Z}(f) = E$  is not cyclic in  $A_\omega^p(\mathbb{T})$ .
- (3) For all  $0 < \varepsilon < 1$ , there exists a closed subset  $E \subset \mathbb{T}$  such that  $\dim(E) = 1 - \varepsilon$  and every  $f \in A_\omega^1(\mathbb{T})$  satisfying  $\mathcal{Z}(f) = E$  is cyclic in  $A_\omega^p(\mathbb{T})$ .

*Proof.* (1) : Let  $f \in A_\omega^1(\mathbb{T})$  such that  $\dim(\mathcal{Z}(f)) < \frac{2}{q}$ . There exists  $0 < \delta < 1/2$  such that  $\dim(\mathcal{Z}(f)) < \frac{2}{q}(1 - \delta q)$ . By Theorem 3.4.(1), every  $g \in A_\delta^1(\mathbb{T})$  satisfying  $\mathcal{Z}(g) = \mathcal{Z}(f)$  is cyclic in  $A_\delta^p(\mathbb{T})$ . Therefore by Lemma 2.7, there exist  $(f_n)$  a sequence of Lipschitz functions which are zero on  $\mathcal{Z}(f)$  and such that

$$\lim_{n \rightarrow \infty} \|f_n - 1\|_{A_\delta^p(\mathbb{T})} = 0.$$

Moreover  $\omega_n = O((1 + |n|)^\delta)$  so

$$\lim_{n \rightarrow \infty} \|f_n - 1\|_{A_\omega^p(\mathbb{T})} = 0.$$

Again by Lemma 2.7 in  $A_\omega^p(\mathbb{T})$ , we obtain that  $f$  is cyclic in  $A_\omega^p(\mathbb{T})$ .

(2) : By the theorem of Salem (see Theorem 3.1 and Theorem 1.3.(2)), there exists a closed set  $E \subset \mathbb{T}$  such that  $\dim(E) = \alpha$  and every  $f \in A^1(\mathbb{T})$  satisfying  $\mathcal{Z}(f) = E$  is not cyclic in  $A^p(\mathbb{T})$ . Let  $f \in A_\omega^1(\mathbb{T})$  such that  $\mathcal{Z}(f) = E$ . Since  $f \in A^1(\mathbb{T})$ ,  $f$  is not cyclic in  $A^p(\mathbb{T})$ . However  $\|\cdot\|_{A^p(\mathbb{T})} \leq \|\cdot\|_{A_\omega^p(\mathbb{T})}$  therefore  $f$  is not cyclic in  $A_\omega^p(\mathbb{T})$ .

(3) : Let  $0 < \varepsilon < 1$  and  $\beta > 0$  such that  $1 - 2([q/2] + 1)\beta \geq 1 - \varepsilon$ . By Theorem 3.4.(4), there exists a closed set  $E \subset \mathbb{T}$  such that

$$\dim(E) \geq 1 - 2([q/2] + 1)\beta \geq 1 - \varepsilon$$

and such that every  $f \in A_\beta^1(\mathbb{T})$  satisfying  $\mathcal{Z}(f) = E$  is cyclic in  $A_\beta^p(\mathbb{T})$ . Since  $A_\beta^p(\mathbb{T}) \subset A_\omega^p(\mathbb{T})$ , we obtain, by Lemma 2.7, that every  $f \in A_\omega^1(\mathbb{T})$  satisfying  $\mathcal{Z}(f) = E$  is cyclic in  $A_\omega^p(\mathbb{T})$ .  $\square$

When  $p > 2$  the search for cyclic vectors in  $A^p(\mathbb{T})$  seems extremely difficult. Newman in [10] shows that for all  $\alpha < 2\pi$  there exists  $E \subset \mathbb{T}$  which has a Lebesgue measure  $|E| > \alpha$  and such that every  $f \in A^1(\mathbb{T})$  satisfying  $\mathcal{Z}(f) = E$  is cyclic in  $A^p(\mathbb{T})$ . See also [10, Theorem 6] for the existence of non cyclic functions under some conditions. We also have a characterization of the cyclic vectors in term of the zeros of the Fourier transform when  $p > 2$  but it's not very effective : A function  $f \in A^1(\mathbb{T})$  is cyclic in  $A^p(\mathbb{T})$  if and only if  $\mathcal{Z}(f)$  does not support any non-zero function  $g \in A^q(\mathbb{T})$  where  $q = \frac{p}{p-1}$ .

When  $\omega_n = \log(e + |n|)^\beta$  where  $0 < \beta < 1$ , for all  $p > \frac{2}{1-\beta}$  and for all  $\alpha < 2\pi$ , Nikolskii shows in [12, Corollary 6], there exists  $E \subset \mathbb{T}$  which has a Lebesgue measure  $|E| > \alpha$  and such that every  $f \in A_\beta^1(\mathbb{T})$  satisfying  $\mathcal{Z}(f) = E$  is cyclic in  $A_\beta^p(\mathbb{T})$ .

## ACKNOWLEDGEMENTS

I would like to acknowledge my doctoral advisors, K. Kellay and M. Zarrabi.

## REFERENCES

- [1] E. ABAKUMOV, A. ATZMON, S. GRIVAUX, *Cyclicity of bicyclic operators*, Comptes Rendus Mathématique Vol. 344, 447-452 (2007)
- [2] A. BEURLING, *On a closure problem*, Ark. Mat. 1 (1951), 301-303
- [3] V.YA. EIDERMAN, *Capacities of Generalized Cantor Sets*, Operator Theory: Advances and Applications, Vol. 158, 131-139 (2005)
- [4] O. EL-FALLAH, N. K. NIKOLSKI, M. ZARRABI, *Estimates for resolvents in Beurling-Sobolev algebras*, St. Petersburg Math. J. 10 (1999), no. 6, 901-964
- [5] J-P. KAHANE, *Séries de Fourier absolument convergentes*, Springer-Verlag, Berlin-Heidenberg-New York, (1970)
- [6] J-P. KAHANE, R. SALEM, *Ensembles parfaits et séries trigonométriques*, Hermann (1963)
- [7] T.W. KÖRNER, *On the theorem of ivašev-musatov iii*, Proc. Lond. Math. Soc. (1986) 53(3) 143-192
- [8] N. LEV, A. OLEVSKII, *Wiener's 'closure of translates' problem and Piatetski-Shapiro's uniqueness phenomenon*, Ann. Math. (2) 174, No. 1, 519-541 (2011)
- [9] D. J. NEWMAN, *Some results in spectral synthesis*, Duke Mathematical Journal, vol. 27 (1960), pp. 359-362
- [10] D. J. NEWMAN, *The closure of translates in  $\ell^p$* , Amer. J. Math. 86 (1964), pp. 651-667
- [11] D.J. NEWMAN, *A simple proof of Wiener's  $1/f$  theorem*. Proc. Amer. Math. Soc. 48 (1975), 264-265.
- [12] N. K. NIKOLSKII, *Lectures on the shift operator IV*, Zapiski Nauchnykh Seminarov LOMI Vol 65 (1976) pp. 103-132
- [13] M. OHTSUKA, *Capacité d'ensembles de Cantor généralisés*, Nagoya Math. J. 11 151-160 (1957)
- [14] W. T. ROSS, S. RICHTER, C. SUNDBERG, *Hyperinvariant subspaces of the harmonic Dirichlet space*, J. Reine Angew. Math. 448 (1994) 1-26.
- [15] R. SALEM, *On singular monotonic functions whose spectrum has a given Hausdorff dimension*, Ark. Mat., 1: 353-365, (1950)
- [16] N. WIENER, *Tauberian theorems*, Ann. of Math. (2) 33 (1932), 1-100

IMB, UNIVERSITÉ DE BORDEAUX, 351 COURS DE LA LIBÉRATION, 33405  
TALENCE, FRANCE

*E-mail address:* `florian.le-manach@math.u-bordeaux.fr`