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# Kernels of conditional determinantal measures

Alexander I. Bufetov, Yanqi Qiu and Alexander Shamov

## Abstract

For determinantal point processes governed by self-adjoint kernels, we prove in Theorem 1.2 that conditioning on the configuration in a subset preserves the determinantal property. In Theorem 1.3 we show the tail sigma-algebra for our determinantal point processes is trivial, proving a conjecture by Lyons. If our self-adjoint kernel is a projection, then, establishing a conjecture by Lyons and Peres, we show in Theorem 1.5 that reproducing kernels corresponding to particles of almost every configuration generate the range of the projection. Our argument is based on a new local property for conditional kernels of determinantal point processes stated in Lemma 1.7.

**Keywords:** Determinantal point processes, conditional measures, tail triviality, Palm measures, measure-valued martingales, operator-valued martingales.

## 1 Introduction

### 1.1 Outline of the main results

Let  $E$  be a locally compact  $\sigma$ -compact Polish space, let  $\text{Conf}(E)$  be the space of configurations on  $E$ . A point process on  $E$  is a Borel probability measure on  $\text{Conf}(E)$ . For such a measure  $\mathbb{P}$  and any Borel subset  $C \subset E$ , the measure  $\mathbb{P}(\cdot|X;C)$  on  $\text{Conf}(E \setminus C)$  is defined as the conditional measure of  $\mathbb{P}$  with respect to the condition that the restriction of our random configuration onto  $C$  coincides with  $X \cap C$  (see §2 below for the detailed definition).

Let  $\mu$  be a sigma-finite Radon measure on  $E$ , let  $K$  be the kernel of a locally trace class positive contraction acting in the complex Hilbert space  $L^2(E, \mu)$ , and let  $\mathbb{P}_K$  be the corresponding determinantal measure on  $\text{Conf}(E)$ . Theorem 1.2 establishes that the conditional measures  $\mathbb{P}_K(\cdot|X;C)$  are themselves determinantal and governed by self-adjoint kernels. For *precompact*  $B$ , the determinantal property for  $\mathbb{P}_K(\cdot|X;B)$  follows from the characterization of Palm measures for determinantal processes due to Shirai-Takahashi [27] and the characterization of induced determinantal processes [3], [6]. For  $X \in \text{Conf}(E)$ , in Definition 1.1 below we introduce a specific self-adjoint kernel  $K^{[X,B]}$  governing the measure  $\mathbb{P}_K(\cdot|X;B)$ .

In order to prove that conditioning preserves the determinantal property, we shall prove that, along an increasing or a decreasing sequence of precompact subsets  $B$ , the kernels  $K^{[X,B]}$  form a martingale. The martingale property for spanning trees is due to Benjamini, Lyons, Peres and Schramm [1] and for processes on general discrete phase spaces to Lyons [16]. It seems to be essential for the argument of Benjamini, Lyons, Peres and Schramm [1], Lyons [16] that the phase space be discrete; we do not see how to extend their argument to continuous phase spaces.

Instead, our proof relies on a new local property for the kernels  $K^{[X,B]}$  which we now informally explain. If  $B \subset C \subset E$ , then conditioning on the restriction of the configuration onto  $B$  commutes with the natural projection map  $X \rightarrow X \cap C$  from  $\text{Conf}(E)$  to  $\text{Conf}(C)$ . This commutativity manifests itself on the level of the kernels chosen in Definition 1.1 below: we have  $\chi_C K^{[X \cap B, B]} \chi_C = (\chi_C K \chi_C)^{[X \cap C, B]}$ . Our local property claims that instead of  $\chi_C$  one can take an (almost) arbitrary projection  $Q$ , and the relation still holds. More precisely, let  $Q : L^2(E, \mu) \rightarrow L^2(E, \mu)$  be an orthogonal projection such that  $\text{Ran}(Q) \subset L^2(E \setminus B, \mu)$  and that  $QKQ$  is locally trace-class. In Lemma 1.7 below we shall see that

$$\left( (Q + \chi_B) K (Q + \chi_B) \right)^{[X, B]} = (Q + \chi_B) K^{[X, B]} (Q + \chi_B) = QK^{[X, B]} Q \quad (1.1)$$

(the second equality in (1.1) is clear since  $\chi_B K^{[X, B]} = K^{[X, B]} \chi_B = 0$ ).

Applying (1.1) to a one-dimensional projection operator  $Q$ , we obtain that, for an arbitrary  $\varphi \in L_2(E \setminus B, \mu)$ , the quantity  $\langle K^{[X, B]} \varphi, \varphi \rangle$  is a martingale indexed by  $B$ , cf. (4.3) below. Using the *Radon-Nikodym property* for the space of trace-class operators, we obtain an operator-valued martingale that converges, along an increasing sequence of bounded subsets of  $E$ , almost surely in the space of locally trace-class operators. As an immediate consequence, we prove that for determinantal point processes governed by self-adjoint kernels, conditioning on the configuration in any Borel subset preserves the determinantal property, see Theorem 1.2.

Theorem 1.3 establishes the triviality of the tail sigma-algebra for determinantal point processes governed by self-adjoint kernels. Lyons proved tail-triviality in the discrete setting in [16], extending the argument of Benjamini-Lyons-Peres-Schramm [1] for spanning trees and conjectured that tail triviality holds in full generality [17, Conjecture 3.2]. The argument of Benjamini-Lyons-Peres-Schramm [1] and of Lyons [16] relies on an estimate for the decay of the variance of the conditional kernel; using the local property of Lemma 1.7, we establish a similar variance estimate in full generality, see Lemma 6.3, and obtain the desired triviality of the tail sigma-algebra. The local property of conditional kernels thus allows us to carry out the proof of tail triviality in a unified way for both the continuous and the discrete setting.

The triviality of the tail sigma-algebra for general determinantal point processes with self-adjoint kernels is the main result of the independent and simultaneous work by Osada and Osada [20]. The argument of Osada-Osada [20] is completely different from ours: Osada-Osada [20] construct a special family of discrete approximations of continuous determinantal point processes and derive the triviality of the tail sigma-algebra in the continuous setting from the theorem of Lyons by approximation.

Theorem 1.5 establishes a conjecture by Lyons and Peres. Let  $K$  be a locally trace-class orthogonal projection onto a closed subspace  $H$  of  $L^2(E, \mu)$ ; in other words, let  $H \subset L^2(E, \mu)$  be a reproducing kernel Hilbert space, and let  $K$  be the reproducing kernel for  $H$ . For  $x \in E$ , introduce a function  $K_x \in L^2(E, \mu)$  by the formula

$$K_x(t) := K(t, x), \quad t \in E. \quad (1.2)$$

**The Lyons-Peres Conjecture** ([17, Conjecture 4.6]). *For  $\mathbb{P}_K$ -almost every  $X \in \text{Conf}(E)$ , we have*

$$\overline{\text{span}}^{L^2(E, \mu)} \{K_x; x \in X\} = H.$$

Lyons [16, Theorem 7.11] proved the Conjecture in the discrete setting. In the continuous setting, Ghosh [11] established the Conjecture under the important additional assumption that the determinantal point process  $\mathbb{P}_K$  has the Ghosh-Peres number rigidity property, which states that for a bounded Borel subset  $B \subset E$ , the number of particles  $\#_B$  of a configuration inside  $B$  is almost surely determined by the restriction of this configuration on  $E \setminus B$ . While many determinantal point processes are indeed rigid in the sense of Ghosh and Peres (cf. e.g. [11], [12], [4], [7], [8]), a natural example without number rigidity is the zero set of the Gaussian analytic function on the unit disk  $\mathbb{D}$ . By the Peres-Virág Theorem [21], our zero set is the the determinantal point process induced by the Bergman kernel

$$K_{\mathbb{D}}(z, w) = \frac{1}{\pi(1 - z\bar{w})^2}$$

corresponding to the orthogonal projection onto the Bergman space of analytic square-integrable functions on  $\mathbb{D}$ . By a theorem of Holroyd and Soo [14], the process governed by  $K_{\mathbb{D}}$  has the property of insertion and deletion tolerance, the opposite of number rigidity; in the setting of generalized Bergman spaces, insertion and deletion tolerance is established in [8].

## 1.2 Statement of the main results

Let  $E$  be a locally compact  $\sigma$ -compact Polish space, equipped with a metric such that any bounded set is relatively compact, and endowed with a positive  $\sigma$ -finite Radon measure  $\mu$ . Let  $\text{Conf}(E)$  be the space of locally finite configurations on  $E$ . Let  $K$  be a bounded self-adjoint *locally trace class* operator  $K : L^2(E, \mu) \rightarrow L^2(E, \mu)$  such that the spectrum  $\text{spec}(K) \subset [0, 1]$ . A theorem obtained by Macchi [18] and Soshnikov [30], as well as Shirai and

Takahashi [26], gives a unique point process on  $E$ , denoted by  $\mathbb{P}_K$ , such that for any compactly supported bounded measurable function  $g : E \rightarrow \mathbb{C}$ , we have

$$\mathbb{E}_{\mathbb{P}_K} \left[ \prod_{x \in X} (1 + g(x)) \right] = \det \left( 1 + \operatorname{sgn}(g) |g|^{1/2} \cdot K \cdot |g|^{1/2} \right)_{L^2(\mu)}, \quad \operatorname{sgn}(g) = \frac{g}{|g|}.$$

Here  $\det(1 + S)$  denotes the Fredholm determinant of the operator  $1 + S$ , see, e.g., Simon [29].

The locally trace class self-adjoint operator  $K$  is an integral operator. Following Soshnikov [30], we fix a Borel subset  $E_0 \subset E$  with  $\mu(E \setminus E_0) = 0$  and fix a Borel function  $K : E_0 \times E_0 \rightarrow \mathbb{C}$ , our kernel, in such a way that for any  $k \in \mathbb{N}$  and any bounded Borel subset  $B \subset E$ , we have

$$\operatorname{tr}((\chi_B K \chi_B)^k) = \int_{B^k} K(x_1, x_2) K(x_2, x_3) \cdots K(x_k, x_1) d\mu(x_1) \cdots d\mu(x_k). \quad (1.3)$$

**Definition 1.1.** For any bounded Borel subset  $B \subset E$ , we define canonical conditional kernels  $K^{[X, B]}$  with respect to the conditioning on the configuration in  $B$  as follows:

- For  $p \in E_0$ , define a kernel  $K^p$ , for  $(x, y) \in E_0 \times E_0$ , by the formula

$$K^p(x, y) := \begin{cases} K(x, y) - \frac{K(x, p)K(p, y)}{K(p, p)} & \text{if } K(p, p) > 0 \\ 0 & \text{if } K(p, p) = 0 \end{cases}.$$

- For an  $n$ -tuple  $(p_1, \dots, p_n) \in E_0^n$ , define a kernel  $K^{p_1, \dots, p_n} = (\dots (K^{p_1})^{p_2} \dots)^{p_n}$  as follows (cf. Shirai-Takahashi [27, Corollary 6.6]). Given  $x, y \in E_0$ , write  $p_0 = x, q_0 = y, q_i = p_i$  for  $1 \leq i \leq n$ , and set

$$K^{p_1, \dots, p_n}(x, y) := \begin{cases} \frac{\det[K(p_i, q_i)]_{0 \leq i, j \leq n}}{\det[K(p_i, p_j)]_{1 \leq i, j \leq n}} & \text{if } \det[K(p_i, p_j)]_{1 \leq i, j \leq n} > 0 \\ 0 & \text{if } \det[K(p_i, p_j)]_{1 \leq i, j \leq n} = 0 \end{cases} \quad (1.4)$$

- For a bounded Borel subset  $B \subset E$  and  $X \in \operatorname{Conf}(E)$  such that  $X \cap B = \{p_1, \dots, p_l\} \subset E_0$ , define

$$K^{[X, B]} = \begin{cases} \chi_{E \setminus B} K^{p_1, \dots, p_l} (1 - \chi_B K^{p_1, \dots, p_l})^{-1} \chi_{E \setminus B} & \text{if } 1 - \chi_B K^{p_1, \dots, p_l} \text{ is invertible} \\ 0 & \text{if } 1 - \chi_B K^{p_1, \dots, p_l} \text{ is not invertible} \end{cases}. \quad (1.5)$$

We will see later, from the inequalities (3.5) and (3.6), that if  $1 - \chi_B K^{p_1, \dots, p_l}$  is invertible, then the operator  $\chi_B K^{p_1, \dots, p_l}$  is strictly contractive. Therefore, the series

$$K^{[X, B]} = \chi_{E \setminus B} \sum_{n=0}^{\infty} K^{p_1, \dots, p_l} (\chi_B K^{p_1, \dots, p_l})^n \chi_{E \setminus B}$$

converges in the operator norm topology. In particular, for  $(x, y) \in E_0 \times E_0$ , we will use the formula

$$\begin{aligned} K^{[X, B]}(x, y) &= \chi_{E \setminus B}(x) \chi_{E \setminus B}(y) K^{p_1, \dots, p_l}(x, y) \\ &+ \chi_{E \setminus B}(x) \chi_{E \setminus B}(y) \left\langle \left( \sum_{n=1}^{\infty} (\chi_B K^{p_1, \dots, p_l})^{n-1} \right) (\chi_B(\cdot) K^{p_1, \dots, p_l}(\cdot, y)), K^{p_1, \dots, p_l}(\cdot, x) \right\rangle_{L^2(E, \mu)} \end{aligned} \quad (1.6)$$

as our specific Borel realization of the kernel for the operator  $K^{[X, B]}$ .

**Remark.** We will see in Proposition 2.5 below that  $K^{[X, B]}$  is the correlation kernel for the conditional measure of  $\mathbb{P}_K$ , the condition being that the configuration on  $B$  coincides with  $X \cap B$ . In particular, for  $\mathbb{P}_K$ -almost every  $X$ , we have  $X \cap B = \{p_1, \dots, p_l\} \subset E_0$  and  $1 - \chi_B K^{p_1, \dots, p_l}$  is invertible. The second case  $K^{[X, B]} = 0$  has probability zero. Note that the range of  $K^{[X, B]}$  is contained in  $L^2(E \setminus B, \mu)$  and we have

$$K^{[X, B]} = \chi_{E \setminus B} K^{[X, B]} \chi_{E \setminus B}.$$

For any Borel subset  $W \subset E$ , *not necessarily bounded*, consider the Borel surjection  $\pi_W : \text{Conf}(E) \rightarrow \text{Conf}(W)$  given by  $X \mapsto X \cap W$ . Fibres of this mapping can be identified with  $\text{Conf}(E \setminus W)$ . For a Borel probability measure  $\mathbb{P}$  on  $\text{Conf}(E)$ , the measure  $\mathbb{P}(\cdot|X;W)$  on  $\text{Conf}(E \setminus W)$  is defined as the conditional measure of  $\mathbb{P}$  with respect to the condition that the restriction of our random configuration onto  $W$  coincides with  $\pi_W(X)$ . More formally, the measures  $\mathbb{P}(\cdot|X;W)$  are conditional measures, in the sense of Rohlin [25], of our initial measure  $\mathbb{P}$  on fibres of the measurable partition induced by the surjection  $\pi_W$ .

Denote by  $\mathcal{L}_1(L^2(E, \mu))$  the space of trace class operators on  $L^2(E, \mu)$  and by  $\mathcal{L}_{1,loc}(L^2(E, \mu))$  the space of bounded and locally trace class operators on  $L^2(E, \mu)$ . The space  $\mathcal{L}_{1,loc}(L^2(E, \mu))$  is equipped with the topology induced by the semi-norms  $T \mapsto \|\chi_B T \chi_B\|_1$ , where  $\|\cdot\|_1$  is the trace class norm,  $B$  ranges over bounded Borel subsets of  $E$ .

For any Borel subset  $W \subset E$ , we denote by  $\mathcal{F}(W) := \sigma(\#_A : A \subset W)$  the  $\sigma$ -algebra on  $\text{Conf}(E)$  generated by the mappings  $\#_A : \text{Conf}(E) \rightarrow \mathbb{R}$  defined by  $\#_A(X) := \#(X \cap A)$ , where  $A$  ranges over all bounded Borel subsets of  $W$ . We are now ready to formulate our main results.

**Theorem 1.2.** *Let  $W \subset E$  be a Borel subset, let  $B_1 \subset \dots \subset B_n \subset \dots \subset W$  be an increasing exhausting sequence of bounded Borel subsets of  $W$ . For  $\mathbb{P}_K$ -almost every  $X \in \text{Conf}(E)$  there exists a positive self-adjoint contraction  $K^{[X,W]} \in \mathcal{L}_{1,loc}(L^2(E \setminus W, \mu))$  such that*

$$\chi_{E \setminus W} K^{[X, B_n]} \chi_{E \setminus W} \xrightarrow[\text{in } \mathcal{L}_{1,loc}(L^2(E \setminus W, \mu))]{n \rightarrow \infty} K^{[X,W]}$$

and

$$\mathbb{P}_K(\cdot|X, W) = \mathbb{P}_{K^{[X,W]}}.$$

**Remark.** *For fixed  $W$ , the kernel-valued function  $X \mapsto K^{[X,W]}$  almost surely does not depend on the choice of the approximating sequence  $B_1 \subset \dots \subset B_n \subset \dots \subset W$ .*

**Theorem 1.3.** *Let  $B_1 \subset \dots \subset B_n \subset \dots \subset E$  be an increasing exhausting sequence of bounded Borel subsets of  $E$ . The  $\sigma$ -algebra  $\bigcap_{n \in \mathbb{N}} \mathcal{F}(E \setminus B_n)$  is trivial with respect to  $\mathbb{P}_K$ .*

**Corollary 1.4.** *The point process  $\mathbb{P}_K$  has trivial tail  $\sigma$ -algebra.*

Assume additionally that  $K$  is an orthogonal projection onto a closed subspace  $H \subset L^2(E, \mu)$  and consider the functions  $K_x$  defined in (1.2).

**Theorem 1.5.** *Let  $K$  be a locally trace-class orthogonal projection onto a subspace  $H$  of  $L^2(E, \mu)$ . For  $\mathbb{P}_K$ -almost every  $X \in \text{Conf}(E)$ , the functions  $K_x$  defined by (1.2) satisfy*

$$\overline{\text{span}}^{L^2(E, \mu)} \{K_x; x \in X\} = H.$$

If we fix a realization for each  $h \in H$  in such a way that the equation  $h(x) = \langle h, K_x \rangle$  holds for every  $x \in E_0$  and every  $h \in H$ , then Theorem 1.5 can equivalently be reformulated as follows:

**Corollary 1.6.** *For  $\mathbb{P}_K$ -almost every  $X \in \text{Conf}(E)$ , if  $h \in H$  satisfies  $h|_X = 0$ , then  $h = 0$ .*

### 1.2.1 The local property and the martingale lemma

At the centre of our argument lies

**Lemma 1.7** (First local property of conditional kernels). *Let  $B \subset E$  be a bounded Borel subset and let  $Q$  be an orthogonal projection, acting in  $L^2(E, \mu)$ , such that  $\text{Ran}(Q) \subset L^2(E \setminus B, \mu)$  and the operator  $QKQ$  is locally trace class. For  $\mathbb{P}_K$ -almost every  $X \in \text{Conf}(E)$ , we have*

$$\left( (Q + \chi_B)K(Q + \chi_B) \right)^{[X, B]} = (Q + \chi_B)K^{[X, B]}(Q + \chi_B) = QK^{[X, B]}Q. \quad (1.7)$$

**Remark.** The formula (1.7) is a strengthening, on the level of kernels, of the general property of point processes that conditioning on the restriction to a subset commutes with the forgetting projection onto a larger subset; see Proposition 2.4 below. The local property can be interpreted in terms of Neretin’s formalism in [19]: a determinantal measure is viewed as a “determinantal state” on a specially constructed algebra, and in order that conditional states themselves be determinantal the local property must take place. The local property can thus be seen as the noncommutative analogue of the fact that the operation of conditioning commutes with the operation of restriction of a configuration onto a subset.

Let  $A, B$  be two disjoint bounded Borel subsets of  $E$ . It is a general property of point processes that conditioning first on  $A$  and then on  $B$  amounts to a single conditioning on  $A \cup B$ . A manifestation of this general property on the level of kernels is

**Lemma 1.8** (Second local property of conditional kernels). *Let  $A, B$  be two disjoint bounded Borel subsets of  $E$ . For  $\mathbb{P}_K$ -almost every  $X \in \text{Conf}(E)$ , we have*

$$(K^{[X,A]})^{[X,B]} = (K^{[X,B]})^{[X,A]} = K^{[X,A \cup B]}.$$

Using the local properties, we establish the following key martingale property of the kernels  $K^{[X,B]}$ .

**Lemma 1.9.** *Let  $W \subset E$  be a Borel subset, let  $B_1 \subset \dots \subset B_n \subset \dots \subset W$  be an increasing exhausting sequence of bounded Borel subsets of  $W$ . The sequence of random variables*

$$\left( \chi_{E \setminus W} K^{[X, B_n]} \chi_{E \setminus W} \right)_{n \in \mathbb{N}}$$

is an  $(\mathcal{F}(B_n))_{n \in \mathbb{N}}$ -adapted operator-valued martingale defined on the probability space  $(\text{Conf}(E), \mathcal{F}(E), \mathbb{P}_K)$ .

By definition, we have  $K^{[X,B]} = K^{[X \cap B, B]}$ . Hence the mapping  $X \mapsto K^{[X,B]}$  is an  $\mathcal{F}(B)$ -measurable operator-valued random variable defined on the probability space  $(\text{Conf}(E), \mathcal{F}(E), \mathbb{P}_K)$ . Lemma 1.9 is equivalent to the claim that, for any  $\varphi \in L^2(E \setminus W, \mu)$ , the sequence  $\left( \langle \chi_{E \setminus W} K^{[X, B_n]} \chi_{E \setminus W} \varphi, \varphi \rangle \right)_{n \in \mathbb{N}}$  is an  $(\mathcal{F}(B_n))_{n \in \mathbb{N}}$ -adapted real-valued martingale defined on the probability space  $(\text{Conf}(E), \mathcal{F}(E), \mathbb{P}_K)$ . This notion of being a martingale is equivalent to the general notion of Frechet space valued martingales, cf. Pisier [24].

**Remark.** The proof of Lemma 1.9 below in fact yields a stronger statement: the sequence of exterior power operators

$$\left( \left( \chi_{E \setminus W} K^{[X, B_n]} \chi_{E \setminus W} \right)^{\wedge m} \right)_{n \in \mathbb{N}}$$

is an  $(\mathcal{F}(B_n))_{n \in \mathbb{N}}$ -adapted operator-valued martingale, defined on the probability space  $(\text{Conf}(E), \mathcal{F}(E), \mathbb{P}_K)$  and almost surely convergent to  $(\chi_{E \setminus W} K^{[X, W]} \chi_{E \setminus W})^{\wedge m}$ .

## 2 Conditional processes and martingales

### 2.1 Martingales and the Radon-Nikodym property

#### 2.1.1 Vector-valued and measure-valued martingales

Let  $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n=1}^\infty, \mathbf{P})$  be a filtered probability space. Let  $\mathfrak{B}$  be a Banach space. A map  $F : \Omega \rightarrow \mathfrak{B}$  is called Bochner measurable, if there exists a sequence  $F_n$  of measurable, in the usual sense, step functions such that  $F_n(\omega) \rightarrow F(\omega)$  almost everywhere. For any  $1 \leq p < \infty$ , we denote by  $L^p(\Omega, \mathcal{F}, \mathbf{P}; \mathfrak{B})$  the set of all Bochner measurable functions  $F : \Omega \rightarrow \mathfrak{B}$ , such that  $\int_\Omega \|F(\omega)\|_{\mathfrak{B}}^p \mathbf{P}(d\omega) < \infty$ . The space  $L^p(\Omega, \mathcal{F}, \mathbf{P}; \mathfrak{B})$  is a Banach space with the norm

$$\|F\|_{L^p(\mathfrak{B})} := \left( \int_\Omega \|F(\omega)\|_{\mathfrak{B}}^p \mathbf{P}(d\omega) \right)^{1/p}.$$

The algebraic tensor product  $L^p(\Omega, \mathcal{F}, \mathbf{P}) \otimes \mathfrak{B}$  is dense in  $L^p(\Omega, \mathcal{F}, \mathbf{P}; \mathfrak{B})$ . The operator

$$\mathbb{E}[\cdot | \mathcal{F}_n] \otimes Id_{\mathfrak{B}} : L^p(\Omega, \mathcal{F}, \mathbf{P}) \otimes \mathfrak{B} \rightarrow L^p(\Omega, \mathcal{F}, \mathbf{P}) \otimes \mathfrak{B}$$

extends uniquely to a bounded linear operator on  $L^p(\Omega, \mathcal{F}, \mathbf{P}; \mathfrak{B})$ , for which we keep the name ‘‘conditional expectation’’ and the notation, thus obtaining the operator  $\mathbb{E}[\cdot | \mathcal{F}_n] : L^p(\Omega, \mathcal{F}, \mathbf{P}; \mathfrak{B}) \rightarrow L^p(\Omega, \mathcal{F}, \mathbf{P}; \mathfrak{B})$ . A sequence  $(R_n)_{n=1}^{\infty}$  in  $L^p(\Omega, \mathcal{F}, \mathbf{P}; \mathfrak{B})$  is called an  $(\mathcal{F}_n)_{n=1}^{\infty}$ -adapted martingale if  $R_n = \mathbb{E}[R_{n+1} | \mathcal{F}_n]$  for any  $n \in \mathbb{N}$ .

Assume now that  $\mathfrak{B}$  is a *separable* space. Then there exists a countable subset  $D$  of the unit ball of the dual space  $\mathfrak{B}^*$  such that for any  $x \in \mathfrak{B}$ , we have  $\|x\| = \sup_{\xi \in D} |\xi(x)|$ . We will need the Pettis measurability theorem for separable Banach spaces.

**Proposition 2.1** ([22, p. 278]). *A function  $F : \Omega \rightarrow \mathfrak{B}$  is Bochner measurable with respect to  $\mathcal{F}$  if and only if for any  $\xi \in D$ , the scalar function  $\omega \rightarrow \xi(F(\omega))$  is  $\mathcal{F}$ -measurable. A sequence  $(R_n)_{n=1}^{\infty}$  in  $L^p(\Omega, \mathcal{F}, \mathbf{P}; \mathfrak{B})$  is an  $(\mathcal{F}_n)_{n=1}^{\infty}$ -adapted martingale if and only if for any  $\xi \in D$ , the sequence  $(\xi(R_n))_{n=1}^{\infty}$  is an  $(\mathcal{F}_n)_{n=1}^{\infty}$ -adapted martingale.*

In this paper, we apply Proposition 2.1 in the particular case when  $\mathfrak{B} = \mathcal{L}_1(L^2(E, \mu))$  and  $D$  is the set of contractive finite rank operators on  $L^2(E, \mu)$ . Martingales in  $\mathcal{L}_{1,loc}(L^2(E, \mu))$  are reduced to the previous case by restricting onto  $L^2(B, \mu)$  with  $B$  a bounded Borel subset of  $E$ .

Let  $(T, \mathcal{A})$  be topological space equipped with the  $\sigma$ -algebra of Borel subsets of  $T$ . We denote by  $\mathfrak{P}(T, \mathcal{A})$  the set of probability measures on  $(T, \mathcal{A})$ . A map  $M : \Omega \rightarrow \mathfrak{P}(T, \mathcal{A})$  is called a random probability measure if for any  $A \in \mathcal{A}$ , the map  $\omega \mapsto M(\omega, A) := M(\omega)(A)$  is measurable. A sequence of random probability measures  $(M_n)_{n=1}^{\infty}$  is called an  $(\mathcal{F}_n)_{n=1}^{\infty}$ -adapted measure-valued martingale on  $(T, \mathcal{A})$  if for any  $A \in \mathcal{A}$ , the sequence  $(M_n(\cdot, A))_{n \in \mathbb{N}}$  is a usual  $(\mathcal{F}_n)_{n=1}^{\infty}$ -adapted martingale.

## 2.1.2 The Radon-Nikodym property

In proving convergence of conditional kernels, we will use the Radon-Nikodym property for the space of trace class operators. Here we briefly recall the Radon-Nikodym property for Banach spaces; see Dunford-Pettis [10], Phillips [23] and Chapter 2 in Pisier’s recent monograph [24] for a more detailed exposition.

Let  $\mathfrak{B}$  be a Banach space. Let  $(\Omega, \mathcal{F})$  be a measurable space. Any  $\sigma$ -additive map  $m : \mathcal{F} \rightarrow \mathfrak{B}$  is called a ( $\mathfrak{B}$ -valued) vector measure. A vector measure  $m$  is said to have finite total variation if

$$\sup \left\{ \sum_{i=1}^n \|m(A_i)\|_{\mathfrak{B}} \mid \Omega = \bigsqcup_{i=1}^n A_i \text{ is a measurable partition of } \Omega \right\} < \infty.$$

Given a probability measure  $\mathbf{P}$  on  $(\Omega, \mathcal{F})$ , we say that the vector measure  $m$  is absolutely continuous with respect to  $\mathbf{P}$  if there exists a non-negative function  $w \in L^1(\Omega, \mathcal{F}, \mathbf{P})$  such that

$$\|m(A)\|_{\mathfrak{B}} \leq \int_A w d\mathbf{P} \quad \text{for any } A \in \mathcal{F}.$$

**Definition 2.2.** *A Banach space  $\mathfrak{B}$  is said to have the Radon-Nikodym property if for any probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  and any  $\mathfrak{B}$ -valued measure  $m$  on  $(\Omega, \mathcal{F})$ , with  $m$  having finite total variation and being absolutely continuous with respect to  $\mathbf{P}$ , there exists a Bochner integrable function  $F_m \in L^1(\Omega, \mathcal{F}, \mathbf{P}; \mathfrak{B})$  such that*

$$m(A) = \int_A F_m d\mathbf{P} \quad \text{for any } A \in \mathcal{F}.$$

By Theorem 2.5 in Pisier [24], the Radon-Nikodym property is equivalent to either of the two requirements

1. Every  $\mathfrak{B}$ -valued martingale bounded in  $L^1(\mathfrak{B})$  converges almost surely;
2. Every uniformly integrable  $\mathfrak{B}$ -valued martingale bounded in  $L^1(\mathfrak{B})$  converges almost surely and in  $L^1(\mathfrak{B})$ .

Corollary 2.11 in Pisier [24] states that if  $\mathfrak{B}$  is separable and is a dual space of another Banach space, then  $\mathfrak{B}$  has the Radon-Nikodym property. The separable space  $\mathcal{L}_1(L^2(E, \mu))$  of trace class operators on  $L^2(E, \mu)$  is the dual space of the space of compact operators on  $L^2(E, \mu)$ , and we have

**Proposition 2.3.** *The space  $\mathcal{L}_1(L^2(E, \mu))$  has the Radon-Nikodym property.*

## 2.2 Conditional measures of point processes

Let  $E$  be a locally compact  $\sigma$ -compact Polish space, endowed with a positive  $\sigma$ -finite Radon measure  $\mu$ . We assume that the metric on  $E$  is such that any bounded set is relatively compact, see Hocking and Young [13, Theorem 2-61].

A configuration  $X = \{x_i\}$  on  $E$  is by definition a *locally finite* countable subset of  $E$ , possibly with multiplicities. A configuration is called simple if all points in it have multiplicity one. Let  $\text{Conf}(E)$  denote the set of all configurations on  $E$ . The mapping  $X \mapsto N_X := \sum_i \delta_{x_i}$  embeds  $\text{Conf}(E)$  into the space of Radon measures on  $E$ . Under the *vague topology*,  $\text{Conf}(E)$  is a Polish space, see, e.g., Daley and Vere-Jones [9, Theorem 9.1. IV]. By definition, a *point process* on  $E$  is a Borel probability measure  $\mathbb{P}$  on  $\text{Conf}(E)$ . We call  $\mathbb{P}$  simple if  $\mathbb{P}(\{X : X \text{ is simple}\}) = 1$ .

For a Borel subset  $W \subset E$ , let  $\mathcal{F}(W)$  be the  $\sigma$ -algebra on  $\text{Conf}(E)$  generated by all mappings  $X \mapsto \#_B(X) := \#(X \cap B)$ , where  $B \subset W$  are bounded Borel subsets; the algebra  $\mathcal{F}(W)$  coincides with the Borel  $\sigma$ -algebra.

Take a Borel subset  $W \subset E$ . A Borel probability measure  $\mathbb{P}$  on  $\text{Conf}(E)$  can be viewed as a measure on  $\text{Conf}(W) \times \text{Conf}(W^c)$ ; we shall sometimes write  $\mathbb{P} = \mathbb{P}_{W, W^c}$  to stress dependence on  $W$ .

Denote by  $(\pi_W)_*(\mathbb{P})$  the image measure of  $\mathbb{P}$  under the surjective mapping  $\pi_W : \text{Conf}(E) \rightarrow \text{Conf}(W)$  defined by  $\pi_W(X) = X \cap W$ . By disintegrating the probability measure  $\mathbb{P}_{W, W^c}$ , for  $(\pi_W)_*(\mathbb{P})$ -almost every configuration  $X_0 \in \text{Conf}(W)$ , there exists a probability measure, denoted by  $\mathbb{P}(\cdot | X_0, W)$ , supported on  $\{X_0\} \times \text{Conf}(W^c) \subset \text{Conf}(E)$ , such that

$$\mathbb{P}_{W, W^c} = \int_{\text{Conf}(W)} \mathbb{P}(\cdot | X_0, W) (\pi_W)_*(\mathbb{P})(dX_0).$$

The measure  $\mathbb{P}(\cdot | X_0, W)$  is referred to as the conditional measure on  $\text{Conf}(W^c)$  or conditional point process on  $W^c$  of  $\mathbb{P}$ , the condition being that the configuration on  $W$  coincides with  $X_0$ . In what follows, we denote also

$$\mathbb{P}(\cdot | X, W) := \mathbb{P}(\cdot | X \cap W, W), \quad \text{for } \mathbb{P}\text{-almost every } X \in \text{Conf}(E).$$

Moreover, for a random variable  $f \in L^1(\text{Conf}(E), \mathbb{P})$ , we will denote by

$$\mathbb{E}_{\mathbb{P}}(f | X, W) := \mathbb{E}_{\mathbb{P}}[f | \mathcal{F}(W)](X \cap W).$$

**Proposition 2.4.** *Let  $W_1, W_2$  be two disjoint Borel subsets of  $E$ . For  $\mathbb{P}$ -almost every  $X \in \text{Conf}(E)$ , we have*

$$(\pi_{W_1 \cup W_2})_*[\mathbb{P}](\cdot | X, W_1) = (\pi_{W_1 \cup W_2})_*[\mathbb{P}](\cdot | X, W_1). \quad (2.1)$$

*Proof.* First we have

$$\mathbb{P} = \int_{\text{Conf}(E)} \mathbb{P}(\cdot | X, W_1) \mathbb{P}(dX) \quad \text{and} \quad (\pi_{W_1 \cup W_2})_*[\mathbb{P}] = \int_{\text{Conf}(E)} (\pi_{W_1 \cup W_2})_*[\mathbb{P}(\cdot | X, W_1)] \mathbb{P}(dX).$$

Since  $\mathbb{P}(\cdot | X, W_1)$  is supported on the subset  $\{Y \in \text{Conf}(E) : Y \cap W_1 = X \cap W_1\}$ , and  $(\pi_{W_1 \cup W_2})_*[\mathbb{P}(\cdot | X, W_1)]$  is supported on  $\{Z \in \text{Conf}(C \cup B) : Z \cap B = X \cap B\}$ , by the uniqueness of conditional measures, we get (2.1).  $\square$

Since  $\mathbb{P}(\cdot | X, W)$  is by definition supported on  $\{X \cap W\} \times \text{Conf}(W^c)$ , we consider  $\mathbb{P}(\cdot | X, W)$  as a measure on  $\text{Conf}(W^c)$ . Further identifying  $\text{Conf}(W^c)$  with the subset  $\text{Conf}(E, W^c) := \{X \in \text{Conf}(E) : X \cap W = \emptyset\} \subset \text{Conf}(E)$ , when it is necessary, we may also view  $\mathbb{P}(\cdot | X, W)$  as a measure on  $\text{Conf}(E)$  supported on the subset  $\text{Conf}(E, W^c)$ .

## 2.3 Palm measures

The  $n$ -th correlation measure  $\rho_{n, \mathbb{P}}$  of a point process  $\mathbb{P}$  on  $E$ , if it exists, is the unique  $\sigma$ -finite Borel measure on  $E^n$  satisfying

$$\rho_{n, \mathbb{P}}(A_1^{k_1} \times \cdots \times A_j^{k_j}) = \int_{\text{Conf}(E)} \prod_{i=1}^j \frac{\#(X \cap A_i)!}{(\#(X \cap A_i) - k_i)!} d\mathbb{P}(X),$$

for all bounded disjoint Borel subsets  $A_1, \dots, A_j \subset E$  and positive integers  $k_1, \dots, k_j$  with  $k_1 + \cdots + k_j = n$ . Here if  $\#(X \cap A_i) < k_i$ , we set  $\#(X \cap A_i)! / (\#(X \cap A_i) - k_i)! = 0$ .



For example, the  $n$ -th correlation measure of a determinantal process  $\mathbb{P}_K$  is given by

$$\rho_{n,\mathbb{P}_K}(\mathrm{d}x_1 \cdots \mathrm{d}x_n) = \det(K(x_i, x_j))_{1 \leq i, j \leq n} \cdot \mu^{\otimes n}(\mathrm{d}x_1 \cdots \mathrm{d}x_n),$$

where  $K(x, y)$  is the integral kernel of the operator  $K$  satisfying (1.3).

Assume that  $\mathbb{P}$  is a simple point process on  $E$  such that  $\rho_{n,\mathbb{P}}$  exists for any  $n \in \mathbb{N}$ . The reduced  $n$ -th order Campbell measure  $\mathcal{C}_{n,\mathbb{P}}^!$  of  $\mathbb{P}$  is a  $\sigma$ -finite measure on  $E^n \times \mathrm{Conf}(E)$  satisfying

$$\int_{E^n \times \mathrm{Conf}(E)} F(x, X) \mathcal{C}_{n,\mathbb{P}}^!(\mathrm{d}x \times \mathrm{d}X) = \int_{\mathrm{Conf}(E)} \left[ \sum_{x \in X^n}^{\#} F(x, X \setminus \{x_1, \dots, x_n\}) \right] \mathbb{P}(\mathrm{d}X),$$

for any Borel function  $F : E^n \times \mathrm{Conf}(E) \rightarrow \mathbb{R}^+$ . Here  $\sum^{\#}$  is the summation over all ordered  $n$ -tuples  $(x_1, \dots, x_n)$  with distinct coordinates  $x_1, \dots, x_n \in X$ . Disintegrating  $\mathcal{C}_{n,\mathbb{P}}^!(\mathrm{d}x \times \mathrm{d}X)$ , we obtain

$$\int_{E^n \times \mathrm{Conf}(E)} F(x, X) \mathcal{C}_{n,\mathbb{P}}^!(\mathrm{d}x \times \mathrm{d}X) = \int_{E^n} \rho_{n,\mathbb{P}}(\mathrm{d}x) \int_{\mathrm{Conf}(E)} F(x, X) \mathbb{P}^x(\mathrm{d}X), \quad (2.2)$$

where the probability measures  $\mathbb{P}^x$  are defined for  $\rho_{n,\mathbb{P}}$ -almost every  $x \in E^n$  and are called *reduced Palm measures* of  $\mathbb{P}$ . In what follows, by Palm measures we always mean reduced Palm measures. Since  $\mathbb{P}^{x_1, \dots, x_n}$  is invariant under permutation of the coordinates in  $(x_1, \dots, x_n)$ , we may write

$$\mathbb{P}^X := \mathbb{P}^{x_1, \dots, x_n}, \quad \text{if } X = \{x_1, \dots, x_n\}.$$

## 2.4 Determinantal point processes, conditioning on bounded subsets

Let  $W \subset E$  be a Borel subset. Recall that, by definition, the push-forward  $(\pi_W)_*(\mathbb{P}_K)$  is a determinantal point process on  $W$ , induced by a correlation kernel  $\chi_W K \chi_W$ . We next recall, for determinantal point processes, the form of conditional measures with respect to restricting the configuration on a bounded subset. For a point process  $\mathbb{P}$  on  $E$ , set

$$\overline{\mathbb{P}} \upharpoonright_{\mathrm{Conf}(W)} := \begin{cases} \frac{\mathbb{P} \upharpoonright_{\mathrm{Conf}(W)}}{\mathbb{P}(\mathrm{Conf}(W))}, & \text{if } \mathbb{P}(\mathrm{Conf}(W)) > 0 \\ 0, & \text{if } \mathbb{P}(\mathrm{Conf}(W)) = 0 \end{cases}. \quad (2.3)$$

Let  $B \subset E$  be a bounded Borel subset. If  $\mathbb{P}_K(\mathrm{Conf}(B^c)) > 0$ , then, by [6, Proposition 2.1],  $\overline{\mathbb{P}_K} \upharpoonright_{\mathrm{Conf}(B^c)}$  is a determinantal point process on  $B^c$  induced by the correlation kernel  $\chi_{B^c} K (1 - \chi_B K)^{-1} \chi_{B^c}$ ; in the discrete setting, cf. also Borodin and Rains [2], Lyons [16]. Next, By a Theorem of Shirai and Takahashi [27, Theorem 1.7], for  $\mathbb{P}_K$ -almost every  $X \in \mathrm{Conf}(E)$ , the Palm measure  $\mathbb{P}_K^{X \cap B}$  is a determinantal point process on  $E$ , induced by the correlation kernel

$$K^{X \cap B} = K^{p_1, \dots, p_n}, \quad \text{if } X \cap B = \{p_1, \dots, p_n\};$$

Summing up, we obtain

**Proposition 2.5.**  $\mathbb{P}_K(\cdot | X, B)$  is a determinantal point process on  $B^c$  for  $\mathbb{P}_K$ -almost every  $X \in \mathrm{Conf}(E)$ , induced by a correlation kernel  $K^{[X, B]}$  defined in (1.5).

*Proof.* Indeed, by Proposition 8.1 in the Appendix below, for  $\mathbb{P}_K$ -almost every  $X \in \mathrm{Conf}(E)$ , we have

$$\mathbb{P}_K(\cdot | X, B) = \overline{\mathbb{P}_K^{X \cap B}} \upharpoonright_{\mathrm{Conf}(B^c)} = \mathbb{P}_K^{[X, B]}.$$

□

If  $K$  is the orthogonal projection onto a closed subspace  $H \subset L^2(E, \mu)$ , then the kernel  $K^{p_1, \dots, p_n}$  corresponds to the orthogonal projection from  $L^2(E, \mu)$  onto the subspace  $H(p_1, \dots, p_n) := \{h \in H : h(p_1) = \dots = h(p_n) = 0\}$ , and, for a bounded Borel subset  $B \subset E$ , the operator  $K^{[X, B]}$  is the orthogonal projection onto the closure of the subspace

$$\chi_{E \setminus B} H(X \cap B) = \{\chi_{E \setminus B} h : h \in H(X \cap B)\}.$$

### 3 The local property: proof of Lemmata 1.7, 1.8.

#### 3.1 Proof of Lemma 1.7.

Let  $B \subset E$  be a bounded Borel subset and let  $Q : L^2(E, \mu) \rightarrow L^2(E, \mu)$  be an orthogonal projection whose range satisfies  $\text{Ran}(Q) \subset L^2(E \setminus B, \mu)$  and such that  $QKQ$  is locally trace-class. Introduce a positive contractive locally trace-class operator  $R$  by the formula

$$R = R(K, B, Q) := (Q + \chi_B)K(Q + \chi_B). \quad (3.1)$$

Recall that from the introduction, we fixed a Borel subset  $E_0 \subset E$ , such that  $\mu(E \setminus E_0) = 0$  and the kernel  $K(x, y)$  is well-defined on  $E_0 \times E_0$ . Recall also the notation introduced in Definition 1.1.

**Lemma 3.1.** *Let  $R$  be the operator introduced in (3.1). For any  $p \in B \cap E_0$ , we have  $R^p = (Q + \chi_B)K^p(Q + \chi_B)$ . More generally, for  $(p_1, \dots, p_n) \in (B \cap E_0)^n$ , we have*

$$R^{p_1, \dots, p_n} = (Q + \chi_B)K^{p_1, \dots, p_n}(Q + \chi_B).$$

In particular,

$$R^{X \cap B} = (Q + \chi_B)K^{X \cap B}(Q + \chi_B), \quad \text{for } \mathbb{P}_K\text{-almost every } X \in \text{Conf}(E).$$

*Proof.* Take an orthonormal basis  $\varphi_i$  of the range  $\text{Ran}(Q) \subset L^2(E \setminus B, \mu)$  of  $Q$  and write

$$Q = \sum_{i \in \mathbb{N}} \varphi_i \otimes \overline{\varphi_i}.$$

We may assume that the values  $\varphi_i(x)$  are well-defined for any index  $i \in \mathbb{N}$  and any  $x \in E_0$ . Observe that for any  $p \in B \cap E_0$ , we have

$$R(\cdot, p) = (Q + \chi_B)[K(\cdot, p)]. \quad (3.2)$$

Indeed, write

$$R = \left( \sum_{i \in \mathbb{N}} \varphi_i \otimes \overline{\varphi_i} \right) K \left( \sum_{j \in \mathbb{N}} \varphi_j \otimes \overline{\varphi_j} \right) + \left( \sum_{i \in \mathbb{N}} \varphi_i \otimes \overline{\varphi_i} \right) K \chi_B + \chi_B K \left( \sum_{j \in \mathbb{N}} \varphi_j \otimes \overline{\varphi_j} \right) + \chi_B K \chi_B,$$

since  $p \in B \cap E_0$ , we get for any  $x \in E_0$ :

$$\begin{aligned} R(x, p) &= \sum_{i \in \mathbb{N}} \varphi_i(x) \int_E \overline{\varphi_i(y)} K(y, p) \mu(dy) + \chi_B(x) K(x, p) \\ &= \sum_{i \in \mathbb{N}} \varphi_i(x) \langle K(\cdot, p), \varphi_i \rangle + \chi_B(x) K(x, p), \end{aligned}$$

which is equivalent to (3.2). Since  $R(p, p) = K(p, p)$ , we have

$$\begin{aligned} R^p &= R - \frac{R(\cdot, p) \otimes \overline{R(\cdot, p)}}{R(p, p)} = (Q + \chi_B)K(Q + \chi_B) - \frac{(Q + \chi_B)[K(\cdot, p)] \otimes \overline{(Q + \chi_B)[K(\cdot, p)]}}{K(p, p)} \\ &= (Q + \chi_B) \left[ K - \frac{K(\cdot, p) \otimes \overline{K(\cdot, p)}}{K(p, p)} \right] (Q + \chi_B) = (Q + \chi_B)K^p(Q + \chi_B). \end{aligned}$$

The formula for  $R^{p_1, \dots, p_n}$  follows immediately by induction on  $n$ . □

Recall that, by our discussion in §2.4, the kernel  $\chi_{E \setminus B} K (1 - \chi_B K)^{-1} \chi_{E \setminus B}$  is a correlation kernel for the determinantal point process  $\overline{\mathbb{P}_K} \upharpoonright_{\text{Conf}(B^c)}$ , provided that  $\mathbb{P}_K(\text{Conf}(B^c)) > 0$ .

**Lemma 3.2.** *Let  $B$  be a bounded Borel subset of  $E$  such that  $\mathbb{P}_K(\#_B = 0) > 0$ . Let  $R$  be the operator introduced in (3.1). Then*

$$\chi_{E \setminus B} R (1 - \chi_B R)^{-1} \chi_{E \setminus B} = Q (\chi_{E \setminus B} K (1 - \chi_B K)^{-1} \chi_{E \setminus B}) Q. \quad (3.3)$$

*Proof.* The gap probability  $\mathbb{P}_K(\#_B = 0)$  is given by

$$\mathbb{P}_K(\#_B = 0) = \mathbb{P}_K(\{X : X \cap B = \emptyset\}) = \det(1 - \chi_B K \chi_B) > 0. \quad (3.4)$$

It follows that  $1 - \chi_B K \chi_B$  is invertible and hence 1 is not an eigenvalue of  $\chi_B K \chi_B$ . But since  $\chi_B K \chi_B$  is a priori a positive contraction and  $\chi_B K \chi_B$  is compact, its norm coincides with its maximal eigenvalue. Hence  $\chi_B K \chi_B$  is strictly contractive. But we also have

$$\|\chi_B K \chi_B\| = \|(\chi_B K^{1/2})(\chi_B K^{1/2})^*\| = \|\chi_B K^{1/2}\|^2 < 1. \quad (3.5)$$

Hence

$$\|\chi_B K\| \leq \|\chi_B K^{1/2}\| \|K^{1/2}\| < 1. \quad (3.6)$$

Therefore, both  $\chi_B K$  and  $\chi_B R = \chi_B K(Q + \chi_B)$  are strictly contractive. In particular, the operators on both the left hand side and the right hand side of (3.3) are well-defined.

Since  $Q$  commutes with  $\chi_{E \setminus B}$ , we have

$$\chi_{E \setminus B} R \chi_{E \setminus B} = Q \chi_{E \setminus B} K \chi_{E \setminus B} Q \text{ and } \chi_{E \setminus B} R \chi_B = Q \chi_{E \setminus B} K \chi_B.$$

Since  $\chi_B R \chi_B = \chi_B K \chi_B$ , for  $n \geq 1$ , we have

$$\begin{aligned} \chi_{E \setminus B} R (\chi_B R)^n \chi_{E \setminus B} &= \chi_{E \setminus B} R (\chi_B R) \cdots (\chi_B R) \chi_{E \setminus B} = \chi_{E \setminus B} R \chi_B (\chi_B R \chi_B)^{n-1} \chi_B R \chi_{E \setminus B} = \\ &= Q \chi_{E \setminus B} K \chi_B (\chi_B K \chi_B)^{n-1} \chi_B K \chi_{E \setminus B} Q = Q \chi_{E \setminus B} K (\chi_B K)^n \chi_{E \setminus B} Q. \end{aligned} \quad (3.7)$$

Now since  $\chi_B R$  and  $\chi_B K$  are both strictly contractive, we finally write

$$\begin{aligned} \chi_{E \setminus B} R (1 - \chi_B R)^{-1} \chi_{E \setminus B} &= \sum_{n=0}^{\infty} \chi_{E \setminus B} R (\chi_B R)^n \chi_{E \setminus B} = \\ &= \sum_{n=0}^{\infty} Q \chi_{E \setminus B} K (\chi_B K)^n \chi_{E \setminus B} Q = Q \chi_{E \setminus B} K (1 - \chi_B K)^{-1} \chi_{E \setminus B} Q. \end{aligned} \quad (3.8)$$

□

*Conclusion of the proof of Lemma 1.7.* By Proposition 8.1 and Proposition 2.5,

$$\mathbb{P}_K(\cdot | X, B) = \overline{(\mathbb{P}_K)^{X \cap B}} \upharpoonright_{\text{Conf}(B^c)} = \overline{\mathbb{P}_{K^{X \cap B}}} \upharpoonright_{\text{Conf}(B^c)}, \quad \text{for } \mathbb{P}\text{-almost every } X \in \text{Conf}(E).$$

By definition (2.3) of the normalized restriction measure  $\overline{\mathbb{P}_{K^{X \cap B}}} \upharpoonright_{\text{Conf}(B^c)}$ , we must have

$$\mathbb{P}_{K^{X \cap B}}(\#_B = 0) = \mathbb{P}_{K^{X \cap B}}(\text{Conf}(B^c)) > 0, \quad \text{for } \mathbb{P}_K\text{-almost every } X \in \text{Conf}(E). \quad (3.9)$$

Lemma 3.2 applied to the operators  $K^{X \cap B}$  and  $R^{X \cap B}$  and Lemma 3.1 now imply Lemma 1.7. □

### 3.2 Proof of Lemma 1.8

Choose an arbitrary unit vector  $\varphi \in L^2(E \setminus (A \cup B), \mu)$ , let  $Q$  be the orthogonal projection from  $L^2(E, \mu)$  onto the one dimensional subspace spanned by  $\varphi$ . Define

$$R = R_\varphi := (\chi_A + \chi_B + Q)K(\chi_A + \chi_B + Q).$$

Arguing as in the proof of Lemma 1.7, we obtain the  $\mathbb{P}_K$ -almost sure equalities

$$R^{[X,A]} = (\chi_B + Q)K^{[X,A]}(\chi_B + Q); \quad R^{[X,A \cup B]} = QK^{[X,A \cup B]}Q; \quad (R^{[X,A]})^{[X,B]} = Q(K^{[X,A]})^{[X,B]}Q. \quad (3.10)$$

We also have the following description of conditional measures:

$$\mathbb{P}_R(\cdot | X, A) = \mathbb{P}_{R^{[X,A]}} \text{ and } \mathbb{P}_R(\cdot | X, A \cup B) = \mathbb{P}_{R^{[X,A \cup B]}}, \quad \text{for } \mathbb{P}_R\text{-almost every } X \in \text{Conf}(E).$$

The above first equality implies that

$$[\mathbb{P}_R(\cdot | X, A)](\cdot | X, B) = \mathbb{P}_{R^{[X,A]}(\cdot | X, B)} = \mathbb{P}_{(R^{[X,A]})^{[X,B]}}, \quad \text{for } \mathbb{P}_R\text{-almost every } X \in \text{Conf}(E).$$

Now we may apply the measure-theoretic identity

$$[\mathbb{P}_R(\cdot | X, A)](\cdot | X, B) = \mathbb{P}_R(\cdot | X, A \cup B), \quad \text{for } \mathbb{P}_R\text{-almost every } X \in \text{Conf}(E)$$

and obtain

$$\mathbb{P}_{R^{[X,A \cup B]}} = \mathbb{P}_{(R^{[X,A]})^{[X,B]}}, \quad \text{for } \mathbb{P}_R\text{-almost every } X \in \text{Conf}(E). \quad (3.11)$$

It follows that for  $\mathbb{P}_R$ -almost every  $X \in \text{Conf}(E)$ , we have

$$\mathbb{E}_{\mathbb{P}_R} [\#(X \cap (E \setminus (A \cup B))) | X, A \cup B] = \text{tr} \left( \chi_{E \setminus (A \cup B)} R^{[X,A \cup B]} \chi_{E \setminus (A \cup B)} \right) = \text{tr} \left( \chi_{E \setminus (A \cup B)} (R^{[X,A]})^{[X,B]} \chi_{E \setminus (A \cup B)} \right).$$

Combining with (3.10), we obtain the  $\mathbb{P}_R$ -almost sure equality

$$\text{tr} \left( \chi_{E \setminus (A \cup B)} Q K^{[X,A \cup B]} Q \chi_{E \setminus (A \cup B)} \right) = \text{tr} \left( \chi_{E \setminus (A \cup B)} Q (K^{[X,A]})^{[X,B]} Q \chi_{E \setminus (A \cup B)} \right).$$

That is,

$$\langle K^{[X,A \cup B]} \varphi, \varphi \rangle = \langle (K^{[X,A]})^{[X,B]} \varphi, \varphi \rangle, \quad \text{for } \mathbb{P}_R\text{-almost every } X \in \text{Conf}(E).$$

Since  $\varphi$  is arbitrary and since  $L^2(E \setminus (A \cup B), \mu)$  is separable and both  $K^{[X,A \cup B]}$  and  $(K^{[X,A]})^{[X,B]}$  are supported on  $L^2(E \setminus (A \cup B), \mu)$ , we obtain

$$K^{[X,A \cup B]} = (K^{[X,A]})^{[X,B]}, \quad \text{for } \mathbb{P}_R\text{-almost every } X \in \text{Conf}(E). \quad (3.12)$$

Observe that the equality  $\chi_{A \cup B} R \chi_{A \cup B} = \chi_{A \cup B} K \chi_{A \cup B}$  implies the equality  $(\pi_{A \cup B})_*(\mathbb{P}_R) = (\pi_{A \cup B})_*(\mathbb{P}_K)$ . Combining with (3.12) and the fact that  $K^{[X,A \cup B]}$  and  $(K^{[X,A]})^{[X,B]}$  are  $\mathcal{F}(A \cup B)$ -measurable, we get the desired equality

$$K^{[X,A \cup B]} = (K^{[X,A]})^{[X,B]}, \quad \text{for } \mathbb{P}_K\text{-almost every } X \in \text{Conf}(E).$$

□

## 4 The martingale property: proof of Lemma 1.9.

**Proposition 4.1.** *For any bounded Borel subset  $B \subset E$ , we have*

$$\mathbb{E}_{\mathbb{P}_K}(K^{[X,B]}) = \int_{\text{Conf}(E)} K^{[X,B]} \mathbb{P}_K(dX) = \chi_{E \setminus B} K \chi_{E \setminus B}. \quad (4.1)$$

**Remark.** *Extending the argument of Benjamini, Lyons, Peres and Schramm [1] for the case of spanning trees, Lyons [16, Lemma 7.17] proved (4.1) when  $E$  is discrete and  $K$  is an orthogonal projection on  $\ell^2(E)$ . Our proof, based on the local property, is quite different and works both in the continuous and the discrete setting.*

*Proof of Lemma 1.9 assuming Proposition 4.1.* Applying Proposition 4.1 to the kernel  $K^{[X,B_n]}$  and the bounded Borel subset  $B_{n+1} \setminus B_n \subset E \setminus B_n$ , we obtain

$$\mathbb{E}_{\mathbb{P}_K^{[X,B_n]}} \left[ (K^{[X,B_n]})^{[X,B_{n+1} \setminus B_n]} \right] = \chi_{E \setminus B_{n+1}} K^{[X,B_n]} \chi_{E \setminus B_{n+1}}, \quad \text{for } \mathbb{P}_K\text{-almost every } X.$$

The equality  $\mathbb{P}_K^{[X,B_n]} = \mathbb{P}_K(\cdot | X, B_n)$  now yields

$$\mathbb{E}_{\mathbb{P}_K^{[X,B_n]}} \left[ (K^{[X,B_n]})^{[X,B_{n+1} \setminus B_n]} \right] = \mathbb{E}_{\mathbb{P}_K} \left[ (K^{[X,B_n]})^{[X,B_{n+1} \setminus B_n]} \middle| \mathcal{F}(B_n) \right], \quad \text{for } \mathbb{P}_K\text{-almost every } X.$$

Combining with Lemma 1.8, we get

$$\mathbb{E}_{\mathbb{P}_K} \left[ K^{[X,B_{n+1}]} \middle| \mathcal{F}(B_n) \right] = \chi_{E \setminus B_{n+1}} K^{[X,B_n]} \chi_{E \setminus B_{n+1}}, \quad \text{for } \mathbb{P}_K\text{-almost every } X.$$

By linearity of the composition on the left and on the right with the operator of multiplication by  $\chi_{E \setminus W}$  and the elementary equalities  $\chi_{E \setminus W} \cdot \chi_{E \setminus B_{n+1}} = \chi_{E \setminus W}$ , we get the desired martingale property:

$$\mathbb{E} \left[ \chi_{E \setminus W} K^{[X,B_{n+1}]} \chi_{E \setminus W} \middle| \mathcal{F}(B_n) \right] = \chi_{E \setminus W} K^{[X,B_n]} \chi_{E \setminus W}, \quad \text{for } \mathbb{P}_K\text{-almost every } X.$$

□

*Proof of Proposition 4.1.* Let  $\varphi \in L^2(E \setminus B, \mu)$  be such that  $\|\varphi\|_2 = 1$ . We use (3.1) for  $Q = \varphi \otimes \bar{\varphi}$ , the orthogonal projection onto the one-dimensional space spanned by  $\varphi$ , and thus set

$$R = (\varphi \otimes \bar{\varphi} + \chi_B) K (\varphi \otimes \bar{\varphi} + \chi_B).$$

We have the clear identity

$$(\pi_B)_*(\mathbb{P}_R) = \mathbb{P}_{\chi_B R \chi_B} = \mathbb{P}_{\chi_B K \chi_B} = (\pi_B)_*(\mathbb{P}_K). \quad (4.2)$$

By Lemma 1.7, for  $\mathbb{P}_K$ -almost every  $X \in \text{Conf}(E)$ , we have

$$R^{[X,B]} = Q K^{[X,B]} Q = (\varphi \otimes \bar{\varphi}) K^{[X,B]} (\varphi \otimes \bar{\varphi}).$$

Since clearly  $K^{[X,B]} = K^{[X \cap B, B]}$  and  $R^{[X,B]} = R^{[X \cap B, B]}$ , the above equality holds for  $\mathbb{P}_R$ -almost every  $X \in \text{Conf}(E)$ . Now recall that  $\mathbb{P}_R(\cdot | X, B) = \mathbb{P}_{R^{[X,B]}}$ , for  $\mathbb{P}_R$ -almost every  $X \in \text{Conf}(E)$ . Hence

$$\mathbb{E}_{\mathbb{P}_R} [\#_{E \setminus B} | X, B] = \mathbb{E}_{\mathbb{P}_{R^{[X,B]}}} [\#_{E \setminus B}] = \text{tr}(\chi_{E \setminus B} R^{[X,B]} \chi_{E \setminus B}) = \langle K^{[X,B]} \varphi, \varphi \rangle, \quad \text{for } \mathbb{P}_R\text{-almost every } X \in \text{Conf}(E).$$

Consequently,

$$\mathbb{E}_{\mathbb{P}_R} [\#_{E \setminus B}] = \text{tr}(\chi_{E \setminus B} R \chi_{E \setminus B}) = \text{tr}(Q K Q) = \langle K \varphi, \varphi \rangle.$$

On the other hand,

$$\mathbb{E}_{\mathbb{P}_R} [\#_{E \setminus B}] = \mathbb{E}_{\mathbb{P}_R} \left( \mathbb{E}_{\mathbb{P}_R} [\#_{E \setminus B} | X, B] \right) = \mathbb{E}_{\mathbb{P}_R} \left( \langle K^{[X,B]} \varphi, \varphi \rangle \right),$$

whence

$$\mathbb{E}_{\mathbb{P}_R} \left( \langle K^{[X,B]} \varphi, \varphi \rangle \right) = \langle K \varphi, \varphi \rangle.$$

The relation  $K^{[X,B]} = K^{[X \cap B, B]}$  and the identity (4.2) together give

$$\mathbb{E}_{\mathbb{P}_R} \left( \langle K^{[X,B]} \varphi, \varphi \rangle \right) = \mathbb{E}_{\mathbb{P}_R} \left( \langle K^{[X \cap B, B]} \varphi, \varphi \rangle \right) = \mathbb{E}_{\mathbb{P}_K} \left( \langle K^{[X \cap B, B]} \varphi, \varphi \rangle \right) = \mathbb{E}_{\mathbb{P}_K} \left( \langle K^{[X,B]} \varphi, \varphi \rangle \right)$$

and

$$\mathbb{E}_{\mathbb{P}_K} \left( \langle K^{[X,B]} \varphi, \varphi \rangle \right) = \langle K \varphi, \varphi \rangle. \quad (4.3)$$

Since  $\varphi$  is an arbitrarily chosen unit function in  $L^2(E \setminus B)$  and since  $K^{[X,B]} = \chi_{E \setminus B} K^{[X,B]} \chi_{E \setminus B}$ , we obtain (4.1).  $\square$

## 5 Proof of Theorem 1.2

**Proposition 5.1.** *Let  $W \subset E$  be a Borel subset, and let  $B_1 \subset B_2 \subset \dots \subset B_n \subset \dots \subset W$  be an increasing exhausting sequence of bounded Borel subsets of  $W$ . The sequence  $(\chi_{E \setminus W} K^{[X, B_n]} \chi_{E \setminus W})_{n \in \mathbb{N}}$  converges  $\mathbb{P}_K$ -almost surely in the space of locally trace class operators.*

*Proof.* Since  $K$  is locally of trace class, there exists a positive function  $\psi : E \setminus W \rightarrow (0, 1]$  such that  $\psi^{1/2} K \psi^{1/2}$  is of trace class and for any bounded subset  $B \subset E$ , we have

$$\inf_{x \in B} \psi(x) > 0. \quad (5.1)$$

Then

$$\mathbb{E}_{\mathbb{P}_K} \left( \sum_{x \in X} \psi(x) \right) = \int_E \psi(x) K(x, x) \mu(dx) = \text{tr}(\psi^{1/2} K \psi^{1/2}) = M_\psi < \infty.$$

Denote

$$G(X, n) := \chi_{E \setminus W} K^{[X, B_n]} \chi_{E \setminus W}.$$

Then for any  $n \in \mathbb{N}$ , we have

$$M_\psi = \mathbb{E}_{\mathbb{P}_K} \left( \sum_{x \in X} \psi(x) \right) = \mathbb{E}_{\mathbb{P}_K} \left[ \mathbb{E}_{\mathbb{P}_K} \left( \sum_{x \in X} \psi(x) \mid \mathcal{F}(B_n) \right) \right] = \mathbb{E}_{\mathbb{P}_K} \left[ \text{tr}(\psi^{1/2} G(X, n) \psi^{1/2}) \right]. \quad (5.2)$$

By the martingale property of the sequence  $(G(X, n))_{n \in \mathbb{N}}$  and the equality (5.2), the sequence  $(\psi^{1/2} G(X, n) \psi^{1/2})_{n \in \mathbb{N}}$  forms a *bounded* martingale in  $L^1(\mathbb{P}_K, \mathcal{L}_1(L^2(E, \mu)))$ . By Proposition 2.3, the Banach space  $\mathcal{L}_1(L^2(E, \mu))$  has the Radon-Nikodym property. Therefore there exists a measurable function  $F(X, \infty)$  with values in  $\mathcal{L}_1(L^2(E, \mu))$ , such that

$$\psi^{1/2} G(X, n) \psi^{1/2} \xrightarrow[\mathbb{P}_K\text{-a.s.}]{\text{in } \mathcal{L}_1(L^2(E, \mu))} F(X, \infty).$$

The assumption (5.1) implies that  $\psi^{-1/2} F(X, \infty) \psi^{-1/2} \in \mathcal{L}_{1,loc}(L^2(E, \mu))$  and we have

$$\chi_{E \setminus W} K^{[X, B_n]} \chi_{E \setminus W} = G(X, n) \xrightarrow[\mathbb{P}_K\text{-a.s.}]{\text{in } \mathcal{L}_{1,loc}(L^2(E, \mu))} \psi^{-1/2} F(X, \infty) \psi^{-1/2}. \quad (5.3)$$

$\square$

*Proof of Theorem 1.2.* By (8.6), for  $\mathbb{P}_K$ -almost every  $X \in \text{Conf}(E)$ , we have

$$(\pi_{W^c})_* [\mathbb{P}_K(\cdot \mid X, B_n)] \xrightarrow[\text{weakly}]{n \rightarrow \infty} \mathbb{P}_K(\cdot \mid X, W). \quad (5.4)$$

By items (i) and (iv) of Proposition 2.5, for  $\mathbb{P}_K$ -almost every  $X \in \text{Conf}(E)$ , we have

$$(\pi_{W^c})_* [\mathbb{P}_K(\cdot \mid X, B_n)] = \mathbb{P}_{\chi_{E \setminus W} K^{[X, B_n]} \chi_{E \setminus W}}. \quad (5.5)$$

Combining (5.3), (5.4) and (5.5) with the fact that the convergence of correlation kernels in  $\mathcal{L}_{1,loc}(L^2(E, \mu))$  implies the weak convergence of the corresponding determinantal measures, we complete the proof of Theorem 1.2.  $\square$

We conclude this section with a simple general proposition that allows us to construct bounded martingales from the sequence  $(K^{[X, B_n]})_{n \in \mathbb{N}}$ .

**Proposition 5.2.** *Let  $W \subset E$  be a Borel subset, and let  $B_1 \subset B_2 \subset \dots \subset B_n \subset \dots \subset W$  be an increasing exhausting sequence of bounded Borel subsets of  $W$ . Fix any positive function  $\psi : E \setminus W \rightarrow (0, 1]$  such that  $\psi^{1/2} K \psi^{1/2}$  is of trace class and for any bounded subset  $B \subset E$ , we have  $\inf_{x \in B} \psi(x) > 0$ . Then*

$$(\psi^{1/2} K^{[X, B_n]} \psi^{1/2})_{n \in \mathbb{N}} \quad (5.6)$$

is an  $\mathcal{L}_1(L^2(E \setminus W, \mu))$ -valued martingale which is bounded in  $L^2(\text{Conf}(E), \mathbb{P}; \mathcal{L}_1(L^2(E \setminus W, \mu)))$ . In particular, the sequence converges in  $L^1(\text{Conf}(E), \mathbb{P}; \mathcal{L}_1(L^2(E \setminus W, \mu)))$ .

*Proof.* It suffices to show that the sequence (5.6) is bounded in  $L^2(\text{Conf}(E), \mathbb{P}; \mathcal{L}_1(L^2(E \setminus W, \mu)))$ . Indeed, we have

$$\|\psi^{1/2} K^{[X, B_n]} \psi^{1/2}\|_{\mathcal{L}_1(L^2(E \setminus W, \mu))} = \text{tr}(\psi^{1/2} K^{[X, B_n]} \psi^{1/2}) = \mathbb{E}_{\mathbb{P}_K} \left( \sum_{x \in X} \psi(x) \mid X, B_n \right).$$

By Proposition 8.4, we get the desired  $L^2(\text{Conf}(E), \mathbb{P}; \mathcal{L}_1(L^2(E \setminus W, \mu)))$ -boundedness of the sequence (5.6).  $\square$

**Remark.** Let  $\mathcal{B}(W)$  be the directed set of bounded measurable subsets of  $W$ , ordered by set-inclusion. Then the set-indexed family  $(\chi_{E \setminus W} K^{[X, B]} \chi_{E \setminus W})_{B \in \mathcal{B}(W)}$  is a set-indexed martingale adapted to the filtration  $(\mathcal{F}(B))_{B \in \mathcal{B}(W)}$ . By virtue of Proposition 5.2, for any positive function  $\psi : E \setminus W \rightarrow (0, 1]$  such that  $\psi^{1/2} K \psi^{1/2}$  is of trace class and for any bounded subset  $B \subset E$ , we have  $\inf_{x \in B} \psi(x) > 0$ , the set-indexed martingale

$$\left( \psi^{1/2} K^{[X, B]} \psi^{1/2} \right)_{B \in \mathcal{B}(W)}$$

converges in  $L^1(\text{Conf}(E), \mathbb{P}_K; \mathcal{L}_1(L^2(E \setminus W, \mu)))$ .

## 6 Triviality of the tail $\sigma$ -algebra: proof of Theorem 1.3

We start by fixing a specific sequence  $D_n$ , independent of  $W$ , which allows us then, for a given  $W$ , to choose the approximating sequence  $W_n$  in a specific way.

**Definition 6.1.** Fix any increasing exhausting sequence  $D_1 \subset \dots \subset D_n \subset \dots \subset E$  of bounded Borel subsets of  $E$ . For any Borel subset  $W \subset E$ , set

$$K^{[X, W]} := \lim_{n \rightarrow \infty} \chi_{E \setminus W} K^{[X, W \cap D_n]} \chi_{E \setminus W}.$$

The convergence takes place in  $\mathcal{L}_{1,loc}(L^2(E, \mu))$  by Proposition 5.1. The kernel  $K^{[X, W]}$  is well-defined for  $\mathbb{P}_K$ -almost every  $X$ . For fixed  $W$ , the limit almost surely is independent of the choice of the sequence  $(D_n)_{n=1}^\infty$ .

**Proposition 6.2.** Fix a bounded Borel subset  $B \subset E$  and let  $E \setminus B \supset W_1 \supset \dots \supset W_n \supset \dots$  be any decreasing sequence of Borel subsets. Then  $(\chi_B K^{[X, W_n]} \chi_B)_{n \in \mathbb{N}}$  is an  $(\mathcal{F}(W_n))_{n \in \mathbb{N}}$ -adapted reverse martingale defined on the probability space  $(\text{Conf}(E), \mathcal{F}(E), \mathbb{P}_K)$ .

*Proof.* It suffices to prove that for any  $\phi \in L^2(B, \mu)$ , the sequence  $(\langle K^{[X, W_n]} \phi, \phi \rangle)_{n \in \mathbb{N}}$  is an  $(\mathcal{F}(W_n))_{n \in \mathbb{N}}$ -adapted reverse martingale defined on the probability space  $(\text{Conf}(E), \mathcal{F}(E), \mathbb{P}_K)$ . By definition, for any  $n \in \mathbb{N}$ , we have

$$\langle K^{[X, W_n]} \phi, \phi \rangle = \lim_{k \rightarrow \infty} \langle K^{[X, W_n \cap D_k]} \phi, \phi \rangle, \quad \mathbb{P}_K\text{-almost surely.} \quad (6.1)$$

Since all the operators  $K^{[X, W_n]}$  are contractive, by the bounded convergence theorem, the convergence (6.1) takes place in  $L^1(\mathbb{P}_K)$  as well. Fix an natural number  $n \in \mathbb{N}$ . For any  $\varepsilon > 0$ , let  $k \in \mathbb{N}$  be large enough in such a way that

$$\left\| \langle K^{[X, W_n]} \phi, \phi \rangle - \langle K^{[X, W_n \cap D_k]} \phi, \phi \rangle \right\|_{L^1(\mathbb{P}_K)} \leq \varepsilon; \quad \left\| \langle K^{[X, W_{n+1}]} \phi, \phi \rangle - \langle K^{[X, W_{n+1} \cap D_k]} \phi, \phi \rangle \right\|_{L^1(\mathbb{P}_K)} \leq \varepsilon. \quad (6.2)$$

For fixed  $n \in \mathbb{N}$ , the sequence

$$\left( \mathbb{E}_{\mathbb{P}_K} \left[ \langle K^{[X, W_n]} \phi, \phi \rangle \middle| \mathcal{F}(W_{n+1} \cap D_k) \right] \right)_{k=1}^{\infty}$$

is a martingale that converges in  $L^1$ -norm to  $\mathbb{E}_{\mathbb{P}_K} \left[ \langle K^{[X, W_n]} \phi, \phi \rangle \middle| \mathcal{F}(W_{n+1}) \right]$ . We can therefore choose  $k$  large enough in such a way that

$$\left\| \mathbb{E}_{\mathbb{P}_K} \left[ \langle K^{[X, W_n]} \phi, \phi \rangle \middle| \mathcal{F}(W_{n+1}) \right] - \mathbb{E}_{\mathbb{P}_K} \left[ \langle K^{[X, W_n]} \phi, \phi \rangle \middle| \mathcal{F}(W_{n+1} \cap D_k) \right] \right\|_{L^1(\mathbb{P}_K)} \leq \varepsilon.$$

Since  $W_{n+1} \cap D_k \subset W_n \cap D_k$  and  $D_k$  is bounded, Lemma 1.9 implies

$$\mathbb{E}_{\mathbb{P}_K} \left[ \langle K^{[X, W_n \cap D_k]} \phi, \phi \rangle \middle| \mathcal{F}(W_{n+1} \cap D_k) \right] = \langle K^{[X, W_{n+1} \cap D_k]} \phi, \phi \rangle,$$

whence

$$\begin{aligned} & \left\| \mathbb{E}_{\mathbb{P}_K} \left[ \langle K^{[X, W_n]} \phi, \phi \rangle \middle| \mathcal{F}(W_{n+1}) \right] - \langle K^{[X, W_{n+1}]} \phi, \phi \rangle \right\|_{L^1(\mathbb{P}_K)} \leq \\ & \leq 2\varepsilon + \left\| \mathbb{E}_{\mathbb{P}_K} \left[ \langle K^{[X, W_n]} \phi, \phi \rangle \middle| \mathcal{F}(W_{n+1} \cap D_k) \right] - \langle K^{[X, W_{n+1} \cap D_k]} \phi, \phi \rangle \right\|_{L^1(\mathbb{P}_K)} \leq \\ & \leq 3\varepsilon + \left\| \mathbb{E}_{\mathbb{P}_K} \left[ \langle K^{[X, W_n \cap D_k]} \phi, \phi \rangle \middle| \mathcal{F}(W_{n+1} \cap D_k) \right] - \langle K^{[X, W_{n+1} \cap D_k]} \phi, \phi \rangle \right\|_{L^1(\mathbb{P}_K)} = 3\varepsilon, \end{aligned} \quad (6.3)$$

and we obtain the desired reverse martingale relation  $\mathbb{E}_{\mathbb{P}_K} \left[ \langle K^{[X, W_n]} \phi, \phi \rangle \middle| \mathcal{F}(W_{n+1}) \right] = \langle K^{[X, W_{n+1}]} \phi, \phi \rangle$ .  $\square$

**Lemma 6.3.** *For any bounded Borel subset  $B \subset E$  and  $\phi \in L^2(B^c, \mu)$ , we have*

$$\text{Var}_{\mathbb{P}_K} \left[ \langle K^{[X, B]} \phi, \phi \rangle \right] \leq \|\phi\|_2^2 \cdot \|\chi_B K \phi\|_2^2, \quad (6.4)$$

where  $\|\cdot\|_2$  is the Hilbert norm on  $L^2(E, \mu)$ .

We first prove Lemma 6.3 when  $K$  is an orthogonal projection. This part of the proof is similar to the argument of Benjamini, Lyons, Peres and Schramm [1, Lemma 8.6] and Lyons [16, Lemma 7.17]. The proof of Lemma 6.3 in full generality proceeds by reduction to the case of projections (the usual argument of extending the phase space must be slightly modified in the continuous setting) and is postponed to the end of the section.

*Proof of Lemma 6.3 when  $K$  is an orthogonal projection.* By homogeneity, we may assume that  $\|\phi\|_2 \leq 1$ . Since  $K$  is an orthogonal projection, by [5, Proposition 2.4], so is  $K^{[X, B]}$  for  $\mathbb{P}_K$ -almost every  $X \in \text{Conf}(E)$ . By Proposition 4.1, we have

$$\begin{aligned} \text{Var}_{\mathbb{P}_K} \left[ \langle K^{[X, B]} \phi, \phi \rangle \right] &= \mathbb{E}_{\mathbb{P}_K} \left| \langle (K^{[X, B]} - \chi_{B^c} K \chi_{B^c}) \phi, \phi \rangle \right|^2 \leq \mathbb{E}_{\mathbb{P}_K} \left( \|(K^{[X, B]} - \chi_{B^c} K \chi_{B^c}) \phi\|_2^2 \right) = \\ &= \mathbb{E}_{\mathbb{P}_K} \left( \|K^{[X, B]} \phi\|_2^2 - \langle K^{[X, B]} \phi, \chi_{B^c} K \chi_{B^c} \phi \rangle - \langle \chi_{B^c} K \chi_{B^c} \phi, K^{[X, B]} \phi \rangle + \|\chi_{B^c} K \chi_{B^c} \phi\|_2^2 \right) = \\ &= \mathbb{E}_{\mathbb{P}_K} \left( \langle K^{[X, B]} \phi, \phi \rangle - \langle K^{[X, B]} \phi, \chi_{B^c} K \chi_{B^c} \phi \rangle - \langle \chi_{B^c} K \chi_{B^c} \phi, K^{[X, B]} \phi \rangle + \|\chi_{B^c} K \chi_{B^c} \phi\|_2^2 \right) = \\ &= \langle \chi_{B^c} K \chi_{B^c} \phi, \phi \rangle - \|\chi_{B^c} K \chi_{B^c} \phi\|_2^2 = \langle K \phi, \phi \rangle - \|\chi_{B^c} K \phi\|_2^2 = \|K \phi\|_2^2 - \|\chi_{B^c} K \phi\|_2^2 = \|\chi_B K \phi\|_2^2. \end{aligned} \quad (6.5)$$

$\square$

**Proposition 6.4.** *Fix any  $\ell \in \mathbb{N}$ . Then  $(\chi_{D_\ell} K^{[X, E \setminus D_{n+\ell}]} \chi_{D_\ell})_{n \in \mathbb{N}}$  is an  $(\mathcal{F}(E \setminus D_{n+\ell}))_{n \in \mathbb{N}}$ -adapted reverse martingale defined on the probability space  $(\text{Conf}(E), \mathcal{F}(E), \mathbb{P}_K)$ , and we have*

$$\chi_{D_\ell} K^{[X, E \setminus D_{n+\ell}]} \chi_{D_\ell} \xrightarrow[\mathbb{P}_K\text{-a.s. in } \mathcal{L}_1(L^2(E, \mu) \text{ and in } L^2(\mathbb{P}_K; \mathcal{L}_1(L^2(E, \mu)))]{n \rightarrow \infty} \chi_{D_\ell} K \chi_{D_\ell}. \quad (6.6)$$



For any  $\ell \in \mathbb{N}$ , we have

$$\mathbb{E}_{\mathbb{P}_K} \left[ \mathbb{P}_K(\cdot | X, E \setminus D_\ell) \Big| \bigcap_{n=1}^{\infty} \mathcal{F}(E \setminus D_{n+\ell}) \right] = \pi_{D_\ell}(\mathbb{P}_K), \quad \mathbb{P}_K\text{-almost surely} \quad (6.7)$$

and, for any  $\mathcal{A} \in \mathcal{F}(D_\ell)$ , we have

$$\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}_K} \left[ \mathbb{E}_{\mathbb{P}_K} [\chi_{\mathcal{A}} | \mathcal{F}(E \setminus D_{n+\ell})] - \mathbb{P}_K(\mathcal{A}) \right] = 0. \quad (6.8)$$

*Proof.* The reverse martingale property of the sequence follows from Proposition 6.2. Set

$$\mathcal{G} := \bigcap_{n=1}^{\infty} \mathcal{F}(E \setminus D_{n+\ell}). \quad (6.9)$$

Since a Banach space valued reverse martingale converges (see, e.g., Pisier [24, p. 34]), we obtain

$$\chi_{D_\ell} K^{[X, E \setminus D_{n+\ell}]} \chi_{D_\ell} \xrightarrow[\mathbb{P}_K\text{-a.s. in } \mathcal{L}_1(L^2(E, \mu) \text{ and in } L^2(\mathbb{P}_K; \mathcal{L}_1(L^2(E, \mu))))]{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}_K} [\chi_{D_\ell} K^{[X, E \setminus D_{1+\ell}]} \chi_{D_\ell} | \mathcal{G}].$$

Set

$$G_\infty(X) = \mathbb{E}_{\mathbb{P}_K} [\chi_{D_\ell} K^{[X, E \setminus D_{1+\ell}]} \chi_{D_\ell} | \mathcal{G}].$$

In particular, for any  $\phi \in L^2(D_\ell, \mu)$  with  $\|\phi\|_2 \leq 1$ , we have

$$\langle G_\infty(X)\phi, \phi \rangle = \mathbb{E}_{\mathbb{P}_K} [\langle K^{[X, E \setminus D_{1+\ell}]} \phi, \phi \rangle | \mathcal{G}], \quad \mathbb{P}_K\text{-almost surely.}$$

By Definition 6.1 and the inequality  $|\langle K^{[X, (E \setminus D_{n+\ell}) \cap D_k]} \phi, \phi \rangle| \leq 1$ , which holds  $\mathbb{P}_K$ -almost surely, for any  $n \in \mathbb{N}$ , we have

$$\langle K^{[X, (E \setminus D_{n+\ell}) \cap D_k]} \phi, \phi \rangle \xrightarrow{k \rightarrow \infty} \langle K^{[X, E \setminus D_{n+\ell}]} \phi, \phi \rangle, \quad \mathbb{P}_K\text{-almost surely and in } L^2(\mathbb{P}_K). \quad (6.10)$$

Similarly,

$$\langle K^{[X, E \setminus D_{n+\ell}]} \phi, \phi \rangle \xrightarrow{n \rightarrow \infty} \langle G_\infty(X)\phi, \phi \rangle, \quad \mathbb{P}_K\text{-almost surely and in } L^2(\mathbb{P}_K). \quad (6.11)$$

In particular, since  $(E \setminus D_{1+\ell}) \cap D_k$  are bounded for all  $k \in \mathbb{N}$ , we can apply Proposition 4.1 to obtain

$$\mathbb{E}_{\mathbb{P}_K} \langle G_\infty(X)\phi, \phi \rangle = \mathbb{E}_{\mathbb{P}_K} [\langle K^{[X, E \setminus D_{1+\ell}]} \phi, \phi \rangle] = \lim_{k \rightarrow \infty} \mathbb{E}_{\mathbb{P}_K} [\langle K^{[X, (E \setminus D_{1+\ell}) \cap D_k]} \phi, \phi \rangle] = \langle K\phi, \phi \rangle.$$

Now by Lemma 6.3, we have

$$\text{Var}_{\mathbb{P}_K} \left( \langle K^{[X, (E \setminus D_{n+\ell}) \cap D_k]} \phi, \phi \rangle \right) \leq \|\chi_{(E \setminus D_{n+\ell}) \cap D_k} K\phi\|_2^2 \leq \|\chi_{E \setminus D_{n+\ell}} K\phi\|_2^2.$$

The convergence (6.10), (6.11) yields

$$\begin{aligned} \text{Var}_{\mathbb{P}_K} \left( \langle K^{[X, E \setminus D_{n+\ell}]} \phi, \phi \rangle \right) &= \lim_{k \rightarrow \infty} \text{Var}_{\mathbb{P}_K} \left( \langle K^{[X, (E \setminus D_{n+\ell}) \cap D_k]} \phi, \phi \rangle \right) \leq \|\chi_{E \setminus D_{n+\ell}} K\phi\|_2^2; \\ \text{Var}_{\mathbb{P}_K} \left( \langle G_\infty(X)\phi, \phi \rangle \right) &= \lim_{n \rightarrow \infty} \text{Var}_{\mathbb{P}_K} \left( \langle K^{[X, E \setminus D_{n+\ell}]} \phi, \phi \rangle \right) \leq \limsup_{n \rightarrow \infty} \|\chi_{E \setminus D_{n+\ell}} K\phi\|_2^2 = 0. \end{aligned}$$

Consequently, we have  $\langle G_\infty(X)\phi, \phi \rangle = \langle K\phi, \phi \rangle$ ,  $\mathbb{P}_K$ -almost surely. Since  $\chi_{D_\ell} G_\infty(X) \chi_{D_\ell} = G_\infty(X)$  and since  $\phi$  is arbitrarily chosen from the separable unit sphere in  $L^2(D_\ell, \mu)$ , we obtain the desired equality

$$G_\infty(X) = \chi_{D_\ell} K \chi_{D_\ell}, \quad \mathbb{P}_K\text{-almost surely.}$$

Finally, Proposition 8.2 implies that

$$\pi_{D_\ell}[\mathbb{P}_K(\cdot|X, E \setminus D_{n+\ell})] = \mathbb{E}_{\mathbb{P}_K} \left[ \mathbb{P}_K(\cdot|X, E \setminus D_\ell) \Big| \mathcal{F}(E \setminus D_{n+\ell}) \right], \quad \mathbb{P}_K\text{-almost surely,}$$

and

$$\pi_{D_\ell}[\mathbb{P}_K(\cdot|X, E \setminus D_{n+\ell})] \xrightarrow[\text{weakly}]{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}_K} \left[ \mathbb{P}_K(\cdot|X, E \setminus D_\ell) \Big| \mathcal{F} \right], \quad \mathbb{P}_K\text{-almost surely.} \quad (6.12)$$

But the convergence (6.6) implies that

$$\pi_{D_\ell}[\mathbb{P}_K(\cdot|X, E \setminus D_{n+\ell})] = \mathbb{P}_{\chi_{D_\ell} K^{[X, E \setminus D_{n+\ell}]} \chi_{D_\ell}} \xrightarrow[\text{weakly}]{n \rightarrow \infty} \mathbb{P}_{\chi_{D_\ell} K \chi_{D_\ell}} = \pi_{D_\ell}(\mathbb{P}_K), \quad \mathbb{P}_K\text{-almost surely.} \quad (6.13)$$

Now (6.12) and (6.13) yield (6.7). Martingale convergence for a bounded random variable implies (6.8).  $\square$

*Proof of Theorem 1.3.* Take  $D_n := B_n$ . We prove that the  $\sigma$ -algebra  $\mathcal{F}$  in (6.9) is trivial with respect to  $\mathbb{P}_K$ . Take an event  $\mathcal{A} \in \mathcal{F}$ . For  $\varepsilon > 0$ , find  $\ell \in \mathbb{N}$  large enough and  $\mathcal{A}_1 \in \mathcal{F}(D_\ell)$  such that  $\mathbb{P}_K(\mathcal{A}_1 \Delta \mathcal{A}) < \varepsilon/3$ . By (6.8), we have

$$\lim_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}_K} \left| \mathbb{E}_{\mathbb{P}_K} [\chi_{\mathcal{A}_1} | \mathcal{F}(E \setminus D_{n+\ell})] - \mathbb{P}_K(\mathcal{A}_1) \right| = 0.$$

Now find  $n \in \mathbb{N}$  large enough in such a way that

$$\mathbb{E}_{\mathbb{P}_K} \left| \mathbb{E}_{\mathbb{P}_K} [\chi_{\mathcal{A}_1} | \mathcal{F}(E \setminus D_{n+\ell})] - \mathbb{P}_K(\mathcal{A}_1) \right| \leq \varepsilon/3.$$

It follows that for any  $\mathcal{A}_2 \in \mathcal{F}(E \setminus D_{n+\ell})$ , we have

$$\begin{aligned} & |\mathbb{P}_K(\mathcal{A}_1 \cap \mathcal{A}_2) - \mathbb{P}_K(\mathcal{A}_1)\mathbb{P}_K(\mathcal{A}_2)| = \left| \mathbb{E}_{\mathbb{P}_K} \left( \chi_{\mathcal{A}_2} \mathbb{E}_{\mathbb{P}_K} [\chi_{\mathcal{A}_1} | \mathcal{F}(E \setminus D_{n+\ell})] \right) - \mathbb{E}_{\mathbb{P}_K} \left( \chi_{\mathcal{A}_2} \mathbb{P}_K(\mathcal{A}_1) \right) \right| = \\ & = \left| \mathbb{E}_{\mathbb{P}_K} \left( \chi_{\mathcal{A}_2} \left[ \mathbb{E}_{\mathbb{P}_K} [\chi_{\mathcal{A}_1} | \mathcal{F}(E \setminus D_{n+\ell})] - \mathbb{P}_K(\mathcal{A}_1) \right] \right) \right| \leq \mathbb{E}_{\mathbb{P}_K} \left( \left| \mathbb{E}_{\mathbb{P}_K} [\chi_{\mathcal{A}_1} | \mathcal{F}(E \setminus D_{n+\ell})] - \mathbb{P}_K(\mathcal{A}_1) \right| \right) \leq \varepsilon/3. \end{aligned} \quad (6.14)$$

Finally, we obtain

$$|\mathbb{P}_K(\mathcal{A} \cap \mathcal{A}_2) - \mathbb{P}_K(\mathcal{A})\mathbb{P}_K(\mathcal{A}_2)| \leq 2\mathbb{P}_K(\mathcal{A}_1 \Delta \mathcal{A}) + |\mathbb{P}_K(\mathcal{A}_1 \cap \mathcal{A}_2) - \mathbb{P}_K(\mathcal{A}_1)\mathbb{P}_K(\mathcal{A}_2)| \leq \varepsilon.$$

Taking  $\mathcal{A}_2 = \mathcal{A}$ , we obtain  $\mathbb{P}_K(\mathcal{A}) = (\mathbb{P}_K(\mathcal{A}))^2$ , whence  $\mathbb{P}_K(\mathcal{A})$  is either 0 or 1, as desired.  $\square$

*Proof of Lemma 6.3 in the general case.* Fix a bounded Borel subset  $B \subset E$  and a function  $\phi \in L^2(E \setminus B, \mu)$  such that  $\|\phi\|_2 = 1$ . Recalling (3.1), set

$$R(K, B, \phi) = (\phi \otimes \bar{\phi} + \chi_B)K(\phi \otimes \bar{\phi} + \chi_B).$$

By Lemma 1.7,

$$\langle R(K, B, \phi)^{[X, B]} \phi, \phi \rangle = \langle K^{[X, B]} \phi, \phi \rangle, \quad \text{for } \mathbb{P}_K\text{-almost every } X \in \text{Conf}(E).$$

By definition, we have  $K^{[X, B]} = K^{[X \cap B, B]}$  and similarly  $R(K, B, \phi)^{[X, B]} = R(K, B, \phi)^{[X \cap B, B]}$ . In particular, we have

$$\langle R(K, B, \phi)^{[X, B]} \phi, \phi \rangle = \langle K^{[X, B]} \phi, \phi \rangle \quad \text{for } (\pi_B)_*(\mathbb{P}_K) = \mathbb{P}_{\chi_B K \chi_B}\text{-almost every } X \in \text{Conf}(B);$$

$$\text{Var}_{\mathbb{P}_K} [\langle K^{[X, B]} \phi, \phi \rangle] = \text{Var}_{\mathbb{P}_{\chi_B K \chi_B}} [\langle K^{[X, B]} \phi, \phi \rangle] = \text{Var}_{\mathbb{P}_{\chi_B R(K, B, \phi) \chi_B}} \langle R(K, B, \phi)^{[X, B]} \phi, \phi \rangle. \quad (6.15)$$

Let  $\mathbf{m}$  be the counting measure on  $\mathbb{N}$ .

**Proposition 6.5.** *There exists a locally trace class orthogonal projection operator  $\tilde{K} \in \mathcal{L}_{1,loc}(L^2(E \sqcup \mathbb{N}, \mu \oplus \mathbf{m}))$  such that  $K = \chi_E \tilde{K} \chi_E$ .*

*Proof.* The canonical orthogonal projection dilation of  $K$  on  $L^2(E, \mu) \oplus L^2(E, \mu)$  is given by the formula

$$\begin{bmatrix} K & \sqrt{K-K^2} \\ \sqrt{K-K^2} & 1-K \end{bmatrix},$$

but it is not in general locally trace class. Since  $L^2(E, \mu)$  is separable and all infinite dimensional separable Hilbert spaces are isometrically isomorphic, there exists a unitary operator  $U : L^2(E, \mu) \rightarrow \ell^2(\mathbb{N}) = L^2(\mathbb{N}, \mathbf{m})$ , and we set

$$\tilde{K} := \begin{bmatrix} 1 & 0 \\ 0 & U^{-1} \end{bmatrix} \begin{bmatrix} K & \sqrt{K-K^2} \\ \sqrt{K-K^2} & 1-K \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix}.$$

□

Since  $\tilde{K}$  is an orthogonal projection, for any bounded Borel subset  $B \subset E$ , which is of course also a subset of  $E \sqcup \mathbb{N}$ , and any  $\phi \in L^2(E \setminus B, \mu)$ , which of course also lies in  $L^2((E \sqcup \mathbb{N}) \setminus B, \mu \oplus \mathbf{m})$ , we have

$$\text{Var}_{\mathbb{P}_{\tilde{K}}} [\langle \tilde{K}^{[X, B]} \phi, \phi \rangle] \leq \|\chi_B \tilde{K} \phi\|_2^2.$$

For the term on the right hand side, we have

$$\chi_B \tilde{K} \phi = \chi_B K \phi. \quad (6.16)$$

Since  $\phi \otimes \bar{\phi} + \chi_B = (\phi \otimes \bar{\phi} + \chi_B) \chi_E$ , we have

$$R(\tilde{K}, B, \phi) = (\phi \otimes \bar{\phi} + \chi_B) \tilde{K} (\phi \otimes \bar{\phi} + \chi_B) = (\phi \otimes \bar{\phi} + \chi_B) K (\phi \otimes \bar{\phi} + \chi_B) = R(K, B, \phi).$$

It follows that

$$\langle \tilde{K}^{[X, B]} \phi, \phi \rangle \stackrel{\mathbb{P}_{\chi_B \tilde{K} \chi_B} \text{-a.s.}}{=} \langle R(\tilde{K}, B, \phi)^{[X, B]} \phi, \phi \rangle = \langle R(K, B, \phi)^{[X, B]} \phi, \phi \rangle \stackrel{\mathbb{P}_{\chi_B K \chi_B} \text{-a.s.}}{=} \langle K^{[X, B]} \phi, \phi \rangle.$$

The equality  $\chi_B \tilde{K} \chi_B = \chi_B K \chi_B$  implies the equality  $\mathbb{P}_{\chi_B \tilde{K} \chi_B} = \mathbb{P}_{\chi_B K \chi_B}$ , and we have

$$\text{Var}_{\mathbb{P}_K} [\langle K^{[X, B]} \phi, \phi \rangle] = \text{Var}_{\mathbb{P}_{\chi_B K \chi_B}} [\langle K^{[X, B]} \phi, \phi \rangle] = \text{Var}_{\mathbb{P}_{\chi_B \tilde{K} \chi_B}} [\langle \tilde{K}^{[X, B]} \phi, \phi \rangle] = \text{Var}_{\mathbb{P}_{\tilde{K}}} [\langle \tilde{K}^{[X, B]} \phi, \phi \rangle]. \quad (6.17)$$

Combining (6.16) and (6.17), we obtain the desired inequality (6.4). □

## 7 Proof of Theorem 1.5

Recall that we have fixed a realization of our kernel, namely, a Borel function  $K(x, y)$  defined on the set  $E_0 \times E_0$ , where  $\mu(E \setminus E_0) = 0$ . In this section, we make the additional assumption that  $K$  is an orthogonal projection onto a subspace  $H \subset L^2(E, \mu)$ . Recalling (1.2), we fix a realization also for each  $h \in H$ : namely, in such a way that the equation  $h(x) = \langle h, K_x \rangle$  holds for every  $x \in E_0$  and every  $h \in H$ . Given any configuration  $X \in \text{Conf}(E)$  and a bounded Borel subset  $B \subset E$ , we set  $L(X) := \{h \in H : h|_X \equiv 0\}$  and  $\chi_B L(X) := \{\chi_B h : h \in L(X)\} \subset L^2(E, \mu)$ . The subspace  $L(X)$  is of course closed, but  $\chi_B L(X)$  need not be closed.

Fix an exhausting sequence  $E_1 \subset \dots \subset E_n \subset \dots \subset E \setminus B$  of bounded Borel subsets of  $E \setminus B$ , and denote

$$F_n = E \setminus (B \cup E_n).$$

Since  $B$  is bounded, we have  $\mathcal{L}_{1, \text{loc}}(L^2(B, \mu)) = \mathcal{L}_1(L^2(B, \mu))$ . By Theorem 1.2, for  $\mathbb{P}_K$ -almost every  $X \in \text{Conf}(E)$ , there exists a positive contraction  $K^{[X, E \setminus B]} \in \mathcal{L}_1(L^2(B, \mu))$ , such that

$$\chi_B K^{[X, E_n]} \chi_B \xrightarrow[\text{in } \mathcal{L}_1(L^2(B, \mu))]{n \rightarrow \infty} K^{[X, E \setminus B]} \quad (7.1)$$

and

$$\mathbb{P}_K(\cdot | X, E \setminus B) = \mathbb{P}_{K^{[X, E \setminus B]}}. \quad (7.2)$$

**Lemma 7.1.** For  $\mathbb{P}_K$ -almost every  $X \in \text{Conf}(E)$ , we have  $K^{[X, E \setminus B]}(\chi_B h) = \chi_B h$  for any  $h \in L(X \cap (E \setminus B))$ .

*Proof.* Fix any  $h \in L(X \cap (E \setminus B)) \subset H$ . For any  $n \in \mathbb{N}$ , since  $E_n \subset E \setminus B$ , by definition, we have

$$L(X \cap (E \setminus B)) \subset L(X \cap E_n).$$

Since  $E_n$  is bounded and  $E \setminus E_n = B \cup F_n$ , the operator  $K^{[X, E_n]}$  is the orthogonal projection from  $L^2(E, \mu)$  onto the closure of the subspace  $\chi_{E \setminus E_n} L(X \cap E_n) = \chi_{B \cup F_n} L(X \cap E_n)$ . By (7.1), we have

$$K^{[X, E \setminus B]}(\chi_B h) = \lim_{n \rightarrow \infty} (\chi_B K^{[X, E_n]} \chi_B)(\chi_B h) = \lim_{n \rightarrow \infty} \chi_B K^{[X, E_n]}(\chi_B h).$$

Using the equalities  $\chi_B h = \chi_{B \cup F_n} h - \chi_{F_n} h$ ,  $K^{[X, E_n]}(\chi_{B \cup F_n} h) = \chi_{B \cup F_n} h$ , and the relation

$$\|\chi_B K^{[X, E_n]}(\chi_{F_n} h)\|_2 \leq \|\chi_{F_n} h\|_2 \xrightarrow{n \rightarrow \infty} 0,$$

we obtain  $K^{[X, E \setminus B]}(\chi_B h) = \lim_{n \rightarrow \infty} \chi_B K^{[X, E_n]}(\chi_{B \cup F_n} h - \chi_{F_n} h) = \chi_B h - \lim_{n \rightarrow \infty} \chi_B K^{[X, E_n]}(\chi_{F_n} h) = \chi_B h$ .  $\square$

**Lemma 7.2.** Let  $\mathbb{P}$  be a point process on  $E$ . Then for any bounded Borel subset  $B \subset E$ , we have

$$\mathbb{P}(\#_B = \#(X \cap B) | X, B^c) > 0 \quad \text{for } \mathbb{P}\text{-almost every } X \in \text{Conf}(E). \quad (7.3)$$

*Proof.* First of all, decomposing  $X = Y \cup Z$ ,  $Y \in \text{Conf}(B)$ ,  $Z \in \text{Conf}(B^c)$ , we can rewrite the statement as follows:

$$\mathbb{P}(\{W \in \text{Conf}(B) : \#(W) = \#(Y)\} | Z, B^c) > 0 \quad (7.4)$$

for  $(\pi_{B^c})_*(\mathbb{P})$ -almost every  $Z \in \text{Conf}(B^c)$  and  $\mathbb{P}(\cdot | Z, B^c)$ -almost every  $Y \in \text{Conf}(B)$ . We make a simple general claim: given an integer-valued measurable function  $f$  on a probability space  $(\Omega, \mathbf{P})$ , for  $\mathbf{P}$ -almost every  $y \in \Omega$  we have  $\mathbf{P}\{x : f(x) = f(y)\} > 0$ . Indeed, if  $N = \{n \in \mathbb{Z} : \mathbf{P}\{x : f(x) = n\} = 0\}$ , then the relation  $\mathbf{P}\{x : f(x) = f(y)\} > 0$  fails only if  $f(y) \in N$ , and

$$\mathbf{P}\{y : f(y) \in N\} = \sum_{n \in N} \mathbf{P}\{y : f(y) = n\} = 0.$$

Taking  $\Omega = \text{Conf}(B)$ ,  $\mathbf{P} = \mathbb{P}(\cdot | Z, B^c)$ ,  $f = \#_B$ , we obtain (7.4).  $\square$

*Proof of Theorem 1.5.* Fix a countable dense subset  $T$  of  $E$  and let  $S_n$  be an enumeration of balls with rational radii centred at  $T$ :

$$\{S_n : n \in \mathbb{N}\} = \{B(x, q) : x \in T, q \in \mathbb{Q}\}.$$

Fix a measurable subset  $\mathcal{A} \subset \text{Conf}(E)$  with  $\mathbb{P}_K(\mathcal{A}) = 1$ , such that for all  $X \in \mathcal{A}$  and all  $n \in \mathbb{N}$ , the conditional measures  $\mathbb{P}_K(\cdot | X, S_n^c)$  and conditional kernels  $K^{[X, S_n^c]}$  are defined and satisfy

$$\mathbb{P}_K(\cdot | X, S_n^c) = \mathbb{P}_{K^{[X, S_n^c]}}, \quad (7.5)$$

and, moreover, the inequality (7.3) holds:

$$\mathbb{P}_K(\#_{S_n} = \#(X \cap S_n) | X, S_n^c) > 0 \quad \text{for any } X \in \mathcal{A}, n \in \mathbb{N}. \quad (7.6)$$

We now show that  $L(X) = \{0\}$  for any  $X \in \mathcal{A}$ . Take  $X \in \mathcal{A}$  and assume, by contradiction, that there exists  $h \in L(X)$ ,  $h \neq 0$ . Choose a small ball  $S_n$  in such a way that  $h|_{S_n} \neq 0$  and  $X \cap S_n = \emptyset$ . We have  $0 \neq \chi_{S_n} h \in \chi_{S_n} L(X \cap S_n^c)$ . By Lemma 7.1, the function  $\chi_{S_n} h$  satisfies  $K^{[X, S_n^c]}(\chi_{S_n} h) = \chi_{S_n} h$ , whence the operator  $1 - K^{[X, S_n^c]}$  has a kernel. In particular,  $\det(1 - K^{[X, S_n^c]}) = 0$ . On the other hand, the relations (7.5), (7.6) together with the gap probability formula (3.4) imply that

$$\det(1 - K^{[X, S_n^c]}) = \mathbb{P}_{K^{[X, S_n^c]}}(\#_{S_n} = 0) = \mathbb{P}_K(\#_{S_n} = 0 | X, S_n^c) = \mathbb{P}_K(\#_{S_n} = \#(X \cap S_n) | X, S_n^c) > 0.$$

We thus obtain a contradiction and Theorem 1.5 is proved completely.  $\square$

## 8 Appendix: Martingales corresponding to conditional processes

**Proposition 8.1.** *Let  $B \subset E$  be a bounded Borel subset. If  $\mathbb{P}$  is a simple point process on  $E$  admitting correlation measures of all orders, then  $\mathbb{P}(\cdot|X, B) = \overline{\mathbb{P}^{X \cap B} \upharpoonright_{\text{Conf}(B^c)}}$  for  $\mathbb{P}$ -almost every  $X \in \text{Conf}(E)$ .*

*Proof.* Let  $\text{Conf}_n(E) = \{X \in \text{Conf}(E) : \#X = n\}$  and similarly define  $\text{Conf}_n(B)$ . By the natural map  $E^n \rightarrow \text{Conf}_n(E)$  defined by  $(x_1, \dots, x_n) \mapsto \{x_1, \dots, x_n\}$ , we define a measure  $\rho_{n, \mathbb{P}}^\#$  on  $\text{Conf}_n(E)$  as the push-forward measure of the correlation measure  $\rho_{n, \mathbb{P}}$  and define a  $\sigma$ -finite measure  $\mathcal{C}_{n, \mathbb{P}}^\#$  on  $\text{Conf}_n(E) \times \text{Conf}(E)$  as the push-forward measure of  $n$ -th order Campbell measure  $\mathcal{C}_{n, \mathbb{P}}^1$ . The formula (2.2) implies that

$$\mathcal{C}_{n, \mathbb{P}}^\#(\mathbf{dp} \times \mathbf{dX}_1) = \rho_{n, \mathbb{P}}^\#(\mathbf{dp}) \mathbb{P}^\#(\mathbf{dX}_1). \quad (8.1)$$

By convention, we set  $\rho_{0, \mathbb{P}}^\#(\mathbf{dp}) := \delta_\emptyset$  and  $\mathcal{C}_{0, \mathbb{P}}^\# := \delta_\emptyset \otimes \mathbb{P}$ , where  $\delta_\emptyset$  is the Dirac measure at the empty configuration  $\emptyset$ , i.e., the unique element  $\emptyset \in \text{Conf}_0(E)$ . Equivalently, for any positive Borel function  $H : \text{Conf}_n(E) \times \text{Conf}(E) \rightarrow \mathbb{R}^+$ :

$$\int_{\text{Conf}_n(E) \times \text{Conf}(E)} H(\mathbf{p}, X_1) \mathcal{C}_{n, \mathbb{P}}^\#(\mathbf{dX}_0 \times \mathbf{dX}_1) = \int_{\text{Conf}(E)} \left[ \sum_{x \in X^n}^\# H(\{x_1, \dots, x_n\}, X \setminus \{x_1, \dots, x_n\}) \right] \mathbb{P}(\mathbf{dX}),$$

where the summation  $\sum^\#$  is taken over all *ordered*  $n$ -tuples  $(x_1, \dots, x_n)$  with distinct coordinates  $x_1, \dots, x_n \in X$ . In particular, when  $n = 0$ , this equality reads as: for any  $H : \text{Conf}_0(E) \times \text{Conf}(E) \rightarrow \mathbb{R}^+$ , we have

$$\int_{\text{Conf}_0(E) \times \text{Conf}(E)} H(\mathbf{p}, X_1) \mathcal{C}_{0, \mathbb{P}}^\#(\mathbf{dp} \times \mathbf{dX}_1) = \int_{\text{Conf}(E)} H(\emptyset, X) \mathbb{P}(\mathbf{dX}).$$

The boundedness of  $B \subset E$  implies that  $\text{Conf}(B) = \bigsqcup_{n=0}^\infty \text{Conf}_n(B)$ . Hence

$$\text{Conf}(E) \simeq \text{Conf}(B) \times \text{Conf}(B^c) = \left( \bigsqcup_{n=0}^\infty \text{Conf}_n(B) \right) \times \text{Conf}(B^c) = \bigsqcup_{n=0}^\infty \left( \text{Conf}_n(B) \times \text{Conf}(B^c) \right).$$

For any  $n = 0, 1, 2, \dots$ , let  $H : \text{Conf}_n(E) \times \text{Conf}(E) \rightarrow \mathbb{R}^+$  be any non-negative Borel function supported on the subset  $\text{Conf}_n(B) \times \text{Conf}(B^c) \subset \text{Conf}_n(E) \times \text{Conf}(E)$ . Then for any configuration  $X \in \text{Conf}(E)$ , we have

$$\sum_{x \in X^n}^\# H(\{x_1, \dots, x_n\}, X \setminus \{x_1, \dots, x_n\}) = n! \cdot \chi_{\{\#(X \cap B) = n\}} \cdot H(X \cap B, X \cap B^c).$$

When  $n = 0$ , this equality reads as  $H(\emptyset, X) = \chi_{\{X \cap B = \emptyset\}} \cdot H(X \cap B, X \cap B^c)$ . By definition of  $\mathcal{C}_{n, \mathbb{P}}^\#$ , we get

$$\begin{aligned} \int_{\text{Conf}_n(E) \times \text{Conf}(E)} H(\mathbf{p}, X_1) \mathcal{C}_{n, \mathbb{P}}^\#(\mathbf{dp} \times \mathbf{dX}_1) &= \int_{\text{Conf}(E)} \left[ \sum_{x \in X^n}^\# H(\{x_1, \dots, x_n\}, X \setminus \{x_1, \dots, x_n\}) \right] \mathbb{P}(\mathbf{dX}) \\ &= n! \cdot \int_{\text{Conf}(E)} \chi_{\{\#(X \cap B) = n\}} \cdot H(X \cap B, X \cap B^c) \mathbb{P}(\mathbf{dX}) = n! \cdot \int_{\text{Conf}_n(B) \times \text{Conf}(B^c)} H(\mathbf{p}, X_1) \mathbb{P}_{B, B^c}(\mathbf{dp} \times \mathbf{dX}_1). \end{aligned}$$

The above equality, combined with (8.1), yields

$$\begin{aligned} \mathbb{P}_{B, B^c} \upharpoonright_{\text{Conf}_n(B) \times \text{Conf}(B^c)}(\mathbf{dp} \times \mathbf{dX}_1) &= \frac{1}{n!} \mathcal{C}_{n, \mathbb{P}}^\# \upharpoonright_{\text{Conf}_n(B) \times \text{Conf}(B^c)}(\mathbf{dp} \times \mathbf{dX}_1) \\ &= \frac{1}{n!} \rho_{n, \mathbb{P}}^\# \upharpoonright_{\text{Conf}(B)}(\mathbf{dp}) \mathbb{P}^\# \upharpoonright_{\text{Conf}(B^c)}(\mathbf{dX}_1) = \frac{\mathbb{P}^\#(\text{Conf}(B^c))}{n!} \rho_{n, \mathbb{P}}^\# \upharpoonright_{\text{Conf}(B)}(\mathbf{dp}) \overline{\mathbb{P}^\# \upharpoonright_{\text{Conf}(B^c)}}(\mathbf{dX}_1). \end{aligned}$$

Consequently,

$$\mathbb{P}_{B, B^c}(\mathbf{dp} \times \mathbf{dX}_1) = \left( \sum_{n=0}^\infty \frac{\mathbb{P}^\#(\text{Conf}(B^c))}{n!} \rho_{n, \mathbb{P}}^\# \upharpoonright_{\text{Conf}(B)}(\mathbf{dp}) \right) \overline{\mathbb{P}^\# \upharpoonright_{\text{Conf}(B^c)}}(\mathbf{dX}_1).$$

This implies both the formula for  $\pi_B(\mathbb{P})(\mathbf{dp})$  and the formula for  $\mathbb{P}(\mathbf{dX}_1 | \mathbf{p}, B) = \mathbb{P}_{B, B^c}(\mathbf{dX}_1 | \mathbf{p}, B)$ :

$$\pi_B(\mathbb{P})(\mathbf{dp}) = \sum_{n=0}^\infty \frac{\mathbb{P}^\#(\text{Conf}(B^c))}{n!} \rho_{n, \mathbb{P}}^\# \upharpoonright_{\text{Conf}(B)}(\mathbf{dp}); \quad (8.2)$$

$$\mathbb{P}(\mathbf{dX}_1 | \mathbf{p}, B) = \overline{\mathbb{P}^\# \upharpoonright_{\text{Conf}(B^c)}}(\mathbf{dX}_1), \quad \text{for } \pi_B(\mathbb{P})\text{-almost every } \mathbf{p} \in \text{Conf}(B). \quad (8.3)$$

Hence we get the desired relation  $\mathbb{P}(\cdot|X, B) = \overline{\mathbb{P}^{X \cap B} \upharpoonright_{\text{Conf}(B^c)}}$ , for  $\mathbb{P}$ -almost every  $X \in \text{Conf}(E)$ .  $\square$

**Remark.** Kallenberg [15, Section 12.3] defined the compound Campbell measure of  $\mathbb{P}$  on  $\text{Conf}_{fin}(E) \times \text{Conf}(E)$  by

$$\mathcal{C}_{\mathbb{P}}^{\#}(\mathbf{dp} \times \mathbf{dX}_1) := \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{C}_{n, \mathbb{P}}^{\#}(\mathbf{dp} \times \mathbf{dX}_1),$$

where  $\text{Conf}_{fin}(E) = \sqcup_{n=0}^{\infty} \text{Conf}_n(E)$ .

Let  $\mathbb{P}$  be a point process on  $E$  and let  $W \subset E$  be a Borel subset of  $E$ . Let  $W_1 \subset \dots \subset W_n \subset \dots \subset W$  be an increasing sequence of Borel subsets of  $W$  such that  $W = \bigcup_{n=1}^{\infty} W_n$ .

**Proposition 8.2.** *The sequence  $((\pi_{W^c})_*[\mathbb{P}(\cdot|X, W_n)])_{n \in \mathbb{N}}$  is an  $(\mathcal{F}(W_n))_{n \in \mathbb{N}}$ -adapted martingale defined on the probability space  $(\text{Conf}(E), \mathcal{F}(E), \mathbb{P})$ . Moreover, we have*

$$(\pi_{W^c})_*[\mathbb{P}(\cdot|X, W_n)] = \mathbb{E}_{\mathbb{P}} \left[ \mathbb{P}(\cdot|X, W) \middle| \mathcal{F}(W_n) \right], \quad \text{for } \mathbb{P}\text{-almost every } X \in \text{Conf}(E). \quad (8.4)$$

In particular, by martingale convergence theorem, for all Borel subsets  $\mathcal{A} \subset \text{Conf}(W^c)$  and any  $1 \leq p < \infty$ , we have

$$\left( (\pi_{W^c})_*[\mathbb{P}(\cdot|X, W_n)] \right) (\mathcal{A}) \xrightarrow[\mathbb{P}\text{-a.s. and in } L^p(\text{Conf}(E), \mathbb{P})]{n \rightarrow \infty} \mathbb{P}(\mathcal{A}|X, W). \quad (8.5)$$

Moreover, for  $\mathbb{P}$ -almost every  $X \in \text{Conf}(E)$ , we have

$$(\pi_{W^c})_*[\mathbb{P}(\cdot|X, W_n)] \xrightarrow[\text{weakly}]{n \rightarrow \infty} \mathbb{P}(\cdot|X, W). \quad (8.6)$$

**Remark.** In general, the statement (8.5) cannot be strengthened to the claim that for  $\mathbb{P}$ -almost every  $X \in \text{Conf}(E)$ , we have  $\left( (\pi_{W^c})_*[\mathbb{P}(\cdot|X, W_n)] \right) (\mathcal{A}) \xrightarrow{n \rightarrow \infty} \mathbb{P}(\mathcal{A}|X, W)$ , for all Borel subsets  $\mathcal{A} \subset \text{Conf}(W^c)$ .

We prepare a simple lemma. Let  $\Omega_i$ ,  $i = 1, \dots, n, \dots$ , and  $\Omega^*$  be standard Borel spaces. Fix  $n \in \mathbb{N}$  and denote

$$x := (x_i)_{i=1}^{\infty} \quad \text{and} \quad t := (x_i)_{i \geq n+1},$$

while  $z$  will stand for the coordinate on  $\Omega^*$ . Let  $Q(\mathbf{dx} \times \mathbf{dz})$  be a Borel probability measure on  $(\prod_{i=1}^{\infty} \Omega_i) \times \Omega^*$ . For any  $n \in \mathbb{N}$ , let  $q_n(x_1, \dots, x_n; \mathbf{dz})$  be the marginal on  $\Omega^*$  of the conditional measure  $Q(\mathbf{dt} \times \mathbf{dz}|x_1, \dots, x_n)$ .

**Lemma 8.3.** *We have*

$$q_n(x_1, \dots, x_n; \mathbf{dz}) = \mathbb{E}[Q(\mathbf{dz}|x_1, \dots, x_n, t)|x_1, \dots, x_n].$$

*Proof.* Denote by  $Q_n$  the marginal measure of  $Q$  on  $\Omega_1 \times \dots \times \Omega_n$ . Let  $Q_{\infty}$  be the marginal measure of  $Q$  on  $\prod_{i=1}^{\infty} \Omega_i$ . By definition of conditional measures, we have

$$\begin{aligned} Q(\mathbf{dx} \times \mathbf{dz}) &= Q_{\infty}(\mathbf{dx})Q(\mathbf{dz}|x_1, \dots, x_n, t); \\ Q(\mathbf{dx} \times \mathbf{dz}) &= Q_n(\mathbf{dx}_1 \cdots \mathbf{dx}_n)Q(\mathbf{dt} \times \mathbf{dz}|x_1, \dots, x_n). \end{aligned}$$

And also

$$\mathbb{E}[Q(\mathbf{dz}|x_1, \dots, x_n, t)|x_1, \dots, x_n] = \int_{t \in \prod_{i=n+1}^{\infty} \Omega_i} Q(\mathbf{dz}|x_1, \dots, x_n, t)Q_{\infty}(\mathbf{dt}|x_1, \dots, x_n).$$

Since

$$Q_{\infty}(\mathbf{dx}) = Q_n(\mathbf{dx}_1 \cdots \mathbf{dx}_n)Q_{\infty}(\mathbf{dt}|x_1, \dots, x_n),$$

we get

$$Q(\mathbf{dx} \times \mathbf{dz}) = Q_n(\mathbf{dx}_1 \cdots \mathbf{dx}_n)Q_{\infty}(\mathbf{dt}|x_1, \dots, x_n)Q(\mathbf{dz}|x_1, \dots, x_n, t).$$

Consequently,

$$Q(dt \times dz|x_1, \dots, x_n) = Q_\infty(dt|x_1, \dots, x_n)Q(dz|x_1, \dots, x_n, t).$$

By definition, we have

$$\begin{aligned} q_n(x_1, \dots, x_n; dz) &= \int_{t \in \prod_{i=n+1}^\infty \Omega_i} Q(dt \times dz|x_1, \dots, x_n) \\ &= \int_{t \in \prod_{i=n+1}^\infty \Omega_i} Q_\infty(dt|x_1, \dots, x_n)Q(dz|x_1, \dots, x_n, t) \\ &= \mathbb{E}[Q(dz|x_1, \dots, x_n, t)|x_1, \dots, x_n]. \end{aligned}$$

□

*Proof of Proposition 8.2.* Apply Lemma 8.3 to  $\Omega_i = \text{Conf}(W_i \setminus W_{i-1})$ . □

Given a bounded non-negative Borel function  $g : E \rightarrow \mathbb{R}^+$ , let  $S_g : \text{Conf}(E) \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  denote the linear statistics defined, for  $Z \in \text{Conf}(E)$ , by the formula  $S_g(Z) = \sum_{x \in Z} g(x)$ . Denote by  $\mathbb{E}_{\mathbb{P}}(S_g|X, W)$  the conditional expectation of  $S_g$  with respect to the sigma-algebra  $\mathcal{F}(W)$ .

**Proposition 8.4.** *If  $g \upharpoonright_W \equiv 0$  and  $\mathbb{E}_{\mathbb{P}}(S_g^2) < \infty$ , then the sequence*

$$\left( \mathbb{E}_{\mathbb{P}}(S_g|X, W_n) \right)_{n \in \mathbb{N}} \tag{8.7}$$

*is an  $(\mathcal{F}(W_n))_{n \in \mathbb{N}}$ -adapted  $L^2(\text{Conf}(E), \mathbb{P})$ -bounded martingale defined on the probability space  $(\text{Conf}(E), \mathcal{F}(E), \mathbb{P})$ .*

*Proof.* Since  $g \upharpoonright_W \equiv 0$ , by (8.4), we have

$$\mathbb{E}_{\mathbb{P}}(S_g|X, W_n) = \mathbb{E}_{\mathbb{P}} \left[ \mathbb{E}_{\mathbb{P}}(S_g|X, W) \middle| \mathcal{F}(W_n) \right], \quad \text{for } \mathbb{P}\text{-almost every } X \in \text{Conf}(E).$$

By Jensen's inequality, we have

$$[\mathbb{E}_{\mathbb{P}}(S_g|X, W_n)]^2 \leq \mathbb{E}_{\mathbb{P}}(S_g^2|X, W_n), \quad \text{for } \mathbb{P}\text{-almost every } X \in \text{Conf}(E).$$

Therefore, for any  $n \in \mathbb{N}$ ,

$$\mathbb{E}_{\mathbb{P}}[\mathbb{E}_{\mathbb{P}}(S_g|X, W_n)]^2 \leq \mathbb{E}_{\mathbb{P}}(S_g^2) < \infty.$$

□

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