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CONVERGENCE OF THE MAC SCHEME FOR VARIABLE DENSITY FLOWS

T. GALLOUËT, R. HERBIN, J.C. LATCHÉ, AND K. MALLEM

ABSTRACT. We prove in this paper the convergence of an semi-implicit MAC scheme for the time-dependent variable density Navier-Stokes equations.

1. INTRODUCTION

Let Ω be a parallelepiped of \mathbb{R}^d , with $d \in \{2, 3\}$ and $T > 0$, and consider the following variable density Navier-Stokes equations posed on $\Omega \times (0, T)$:

$$\begin{aligned} (1a) \quad & \partial_t \bar{\rho} + \operatorname{div}(\bar{\rho} \bar{\mathbf{u}}) = 0, \\ (1b) \quad & \partial_t(\bar{\rho} \bar{\mathbf{u}}) + \operatorname{div}(\bar{\rho} \bar{\mathbf{u}} \otimes \bar{\mathbf{u}}) - \Delta \bar{\mathbf{u}} + \nabla \bar{p} = \mathbf{f}, \\ (1c) \quad & \operatorname{div} \bar{\mathbf{u}} = 0, \end{aligned}$$

where $\bar{\rho}$, $\bar{\mathbf{u}}$ and \bar{p} are the density, the velocity and the pressure of the flow and $\mathbf{f} \in L^2(0, T; L^2(\Omega)^d)$. This system is complemented with initial and boundary conditions $\bar{\mathbf{u}}|_{\partial\Omega} = 0$, $\bar{\mathbf{u}}|_{t=0} = \mathbf{u}_0$, $\bar{\rho}|_{t=0} = \rho_0$, which are such that $\rho_0 \in L^\infty(\Omega)$, $0 < \rho_{\min} < \rho_0 \leq \rho_{\max}$ and $\mathbf{u}_0 \in L^2(\Omega)^d$. A pair $(\bar{\rho}, \bar{\mathbf{u}})$ is a weak solution of problem (1) if it satisfies the following properties:

- $\bar{\rho} \in \{\rho \in L^\infty(\Omega \times (0, T)), \rho > 0 \text{ a.e. in } \Omega \times (0, T)\}$.
- $\bar{\mathbf{u}} \in \{\mathbf{u} \in L^\infty(0, T; L^2(\Omega)^d) \cap L^2(0, T; H_0^1(\Omega)^d), \operatorname{div} \mathbf{u} = 0 \text{ a.e. in } \Omega \times (0, T)\}$.
- For all φ in $C_c^\infty(\Omega \times [0, T))$,

$$(2) \quad - \int_0^T \int_\Omega \bar{\rho} \partial_t \varphi + \bar{\rho} \bar{\mathbf{u}} \cdot \nabla \varphi \, d\mathbf{x} \, dt = \int_\Omega \rho_0(\mathbf{x}) \varphi(\mathbf{x}, 0) \, d\mathbf{x}.$$

- For all \mathbf{v} in $\{\mathbf{w} \in C_c^\infty(\Omega \times [0, T))^d, \operatorname{div} \mathbf{w} = 0\}$,

$$(3) \quad \int_0^T \int_\Omega [-\bar{\rho} \bar{\mathbf{u}} \cdot \partial_t \mathbf{v} - (\bar{\rho} \bar{\mathbf{u}} \otimes \bar{\mathbf{u}}) : \nabla \mathbf{v} + \nabla \bar{\mathbf{u}} : \nabla \mathbf{v}] \, d\mathbf{x} \, dt = \int_\Omega \rho_0 \mathbf{u}_0 \cdot \mathbf{v}(\cdot, 0) \, d\mathbf{x} + \int_0^T \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} \, dt.$$

The existence of such a weak solution was proven in [9]; convergence results exist for the discontinuous Galerkin approximation [8] and for a finite volume/finite element scheme [7]. Here we prove the convergence of the MAC scheme.

2. THE NUMERICAL SCHEME

Let \mathcal{M} be a MAC mesh (see e.g. [4] and Figure 1 for the notations). The discrete pressure and density unknowns are associated with the cells of the mesh \mathcal{M} , and are denoted by $\{\rho_K, K \in \mathcal{M}\}$ and $\{p_K, K \in \mathcal{M}\}$. The discrete velocity unknowns approximate the normal velocity to the mesh faces, and are denoted $(u_\sigma)_{\sigma \in \mathcal{E}^{(i)}}$, $i \in \llbracket 1, d \rrbracket$, where \mathcal{E} is the set of the faces of the mesh, and $\mathcal{E}^{(i)}$ the subset of the faces orthogonal to the i -th vector of the canonical basis of \mathbb{R}^d . We define $\mathcal{E}_{\text{ext}} = \{\sigma \in \mathcal{E}, \sigma \subset \partial\Omega\}$, $\mathcal{E}_{\text{int}} = \mathcal{E} \setminus \mathcal{E}_{\text{ext}}$, $\mathcal{E}_{\text{int}}^{(i)} = \mathcal{E}_{\text{int}} \cap \mathcal{E}^{(i)}$ and $\mathcal{E}_{\text{ext}}^{(i)} = \mathcal{E}_{\text{ext}} \cap \mathcal{E}^{(i)}$.

The regularity of the mesh is defined by:

$$\eta_{\mathcal{M}} = \max \left\{ \frac{|\sigma|}{|\sigma'|}, \sigma \in \mathcal{E}^{(i)}, \sigma' \in \mathcal{E}^{(j)}, i, j \in \llbracket 1, d \rrbracket, i \neq j \right\},$$

and we denote by $h_{\mathcal{M}}$ the space step. The discrete space $L_{\mathcal{M}}$ for the scalar unknowns (*i.e.* the pressure and the density) is defined as the set of piecewise constant functions over each of the grid cells K of \mathcal{M} , and the

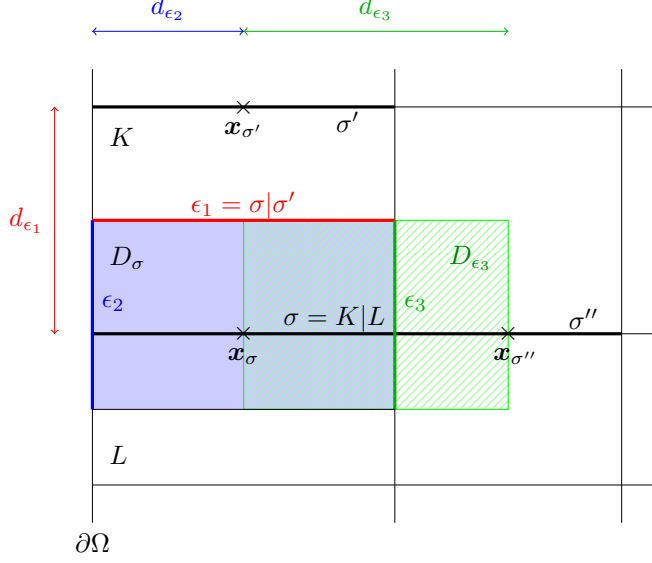


FIGURE 1. Notations for control volumes and dual cells.

discrete space for the i^{th} velocity component, $H_{\mathcal{E}^{(i)}}$, as the set of piecewise constant functions over each of the grid cells D_σ , $\sigma \in \mathcal{E}^{(i)}$. The set of functions of $L_{\mathcal{M}}$ with zero mean value is denoted by $L_{\mathcal{M},0}$. As in the continuous case, the Dirichlet boundary conditions are (partly) incorporated into the definition of the velocity spaces:

$$H_{\mathcal{E}^{(i)},0} = \left\{ u \in H_{\mathcal{E}^{(i)}}, u(\mathbf{x}) = 0 \ \forall \mathbf{x} \in D_\sigma, \ \sigma \in \mathcal{E}_{\text{ext}}^{(i)} \right\}, \quad \text{for } 1 \leq i \leq d$$

(i.e. we impose $u_\sigma = 0$ for all $\sigma \in \mathcal{E}_{\text{ext}}$). We then set $\mathbf{H}_{\mathcal{E},0} = \prod_{i=1}^d H_{\mathcal{E}^{(i)},0}$.

Let $0 = t_0 < t_1 < \dots < t_N = T$ be a partition of the time interval $(0, T)$, with $\delta t = t_{n+1} - t_n$. Let $\{u_\sigma^{n+1}, \sigma \in \mathcal{E}^{(i)}, 0 \leq n \leq N-1, 1 \leq i \leq d\}$, $\{p_K^{n+1}, K \in \mathcal{M}, 0 \leq n \leq N-1\}$ and $\{\rho_K^{n+1}, K \in \mathcal{M}, 0 \leq n \leq N-1\}$ be the sets of discrete velocity, pressure and density unknowns. Defining the characteristic function $\mathbb{1}_A$ of any subset $A \subset \Omega$ by $\mathbb{1}_A(\mathbf{x}) = 1$ if $\mathbf{x} \in A$ and $\mathbb{1}_A(\mathbf{x}) = 0$ otherwise, the corresponding piecewise constant functions for the velocities are of the form:

$$u_i = \sum_{n=0}^{N-1} \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} u_\sigma^{n+1} \mathbb{1}_{D_\sigma} \mathbb{1}_{[t_n, t_{n+1}]},$$

and $X_{i,\mathcal{E},\delta t}$ denotes the set of such piecewise constant functions on time intervals and dual cells; we then set $\mathbf{X}_{\mathcal{E},\delta t} = \prod_{i=1}^d X_{i,\mathcal{E},\delta t}$. The pressure and density discrete functions are defined by:

$$p = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} p_K^{n+1} \mathbb{1}_K \mathbb{1}_{[t_n, t_{n+1}]}, \quad \rho = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{M}} \rho_K^{n+1} \mathbb{1}_K \mathbb{1}_{[t_n, t_{n+1}]},$$

and $Y_{\mathcal{M},\delta t}$ denotes the space of such piecewise constant functions. The numerical scheme reads:

$$(4a) \quad \textbf{Initialization:} \quad \mathbf{u}^{(0)} = \tilde{\mathcal{P}}_{\mathcal{E}} \mathbf{u}_0, \quad \rho^{(0)} = \mathcal{P}_{\mathcal{M}} \rho_0.$$

For $0 \leq n \leq N-1$, solve for $\mathbf{u}^{n+1} \in \mathbf{H}_{\mathcal{E},0}$, $\rho^{n+1} \in L_{\mathcal{M}}$ and $p^{n+1} \in L_{\mathcal{M},0}$:

$$(4b) \quad \partial_t \rho^{n+1} + \text{div}_{\mathcal{M}}(\rho^{n+1} \mathbf{u}^n) = 0,$$

$$(4c) \quad \partial_t (\rho \mathbf{u})^{n+1} + \mathbf{C}_{\mathcal{E}}(\rho^{n+1} \mathbf{u}^n) \mathbf{u}^{n+1} - \Delta_{\mathcal{E}} \mathbf{u}^{n+1} + \nabla_{\mathcal{E}} p^{n+1} = \mathbf{f}_{\mathcal{E}}^{n+1},$$

$$(4d) \quad \text{div}_{\mathcal{M}} \mathbf{u}^{n+1} = 0,$$

with the interpolators and discrete operators defined as follows.

Grid interpolators – The Fortin interpolator is defined by $\tilde{\mathcal{P}}_{\mathcal{E}} \mathbf{u} = (\tilde{\mathcal{P}}_{\mathcal{E}^{(i)}})_{i=1,\dots,d}$ with $\tilde{\mathcal{P}}_{\mathcal{E}^{(i)}} : H_0^1(\Omega) \longrightarrow H_{\mathcal{E}^{(i)},0}$ and

$$v_i \longmapsto \tilde{\mathcal{P}}_{\mathcal{E}^{(i)}} v_i = \sum_{\sigma \in \mathcal{E}^{(i)}} v_{\sigma} \mathbf{1}_{D_{\sigma}} \text{ with } v_{\sigma} = \frac{1}{|\sigma|} \int_{\sigma} v_i \, d\gamma(\mathbf{x}), \sigma \in \mathcal{E}^{(i)}.$$

For $q \in L^2(\Omega)$, $\mathcal{P}_{\mathcal{M}} q \in L_{\mathcal{M}}$ is defined by $\mathcal{P}_{\mathcal{M}} q(\mathbf{x}) = \frac{1}{|K|} \int_K q \, d\mathbf{x}$ for $\mathbf{x} \in K$.

Discrete time derivative – For $\rho \in Y_{\mathcal{M},\delta t}$, $\tilde{\partial}_t \rho \in Y_{\mathcal{M},\delta t}$ is defined by:

$$\tilde{\partial}_t \rho(\mathbf{x}, t) = \sum_{n=0}^{N-1} \tilde{\partial}_t \rho^{n+1}(\mathbf{x}) \mathbf{1}_{[t_n, t_{n+1}]}(t) \text{ with } \tilde{\partial}_t \rho^{n+1} = \sum_{K \in \mathcal{M}} \frac{1}{\delta t} (\rho_K^{n+1} - \rho_K^n) \mathbf{1}_K.$$

Discrete divergence – Let $u_{K,\sigma}$ be defined as $u_{K,\sigma} = u_{\sigma} \mathbf{n}_{K,\sigma} \cdot \mathbf{e}_i$ for any face $\sigma \in \mathcal{E}^{(i)}$, $i = 1, \dots, d$. The discrete (upwind finite volume) divergence operator $\text{div}_{\mathcal{M}}$ is defined by:

$$\text{div}_{\mathcal{M}} : L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0} \longrightarrow L_{\mathcal{M}}, \quad (\rho, \mathbf{u}) \mapsto \text{div}_{\mathcal{M}}(\rho \mathbf{u}) = \sum_{K \in \mathcal{M}} \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma} \mathbf{1}_K,$$

with $F_{K,\sigma} = |\sigma| \rho_{\sigma} u_{K,\sigma}$ for $K \in \mathcal{M}$, $\sigma = K|L \in \mathcal{E}(K)$, and $\rho_{\sigma} = \rho_K$ if $u_{K,\sigma} \geq 0$, $\rho_{\sigma} = \rho_L$ otherwise. For all $K \in \mathcal{M}$, we set $(\text{div}_{\mathcal{M}} \mathbf{u})_K = \text{div}(1 \times \mathbf{u})_K$.

Pressure gradient operator – The discrete pressure gradient operator is defined as the transpose of the divergence operator, so $\nabla_{\mathcal{E}} : L_{\mathcal{M}} \longrightarrow \mathbf{H}_{\mathcal{E},0}$, $p \mapsto \nabla_{\mathcal{E}}(p)$ with:

$$(5) \quad \nabla_{\mathcal{E}} p = \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}^{(i)}} (\tilde{\partial} p)_{\sigma} \mathbf{n}_{K,\sigma} \mathbf{1}_{D_{\sigma}}, \text{ with } (\tilde{\partial} p)_{\sigma} = \frac{|\sigma|}{|D_{\sigma}|} (p_L - p_K).$$

Discrete Laplace operator – The discrete diffusion operator $\Delta_{\mathcal{E}}$ is defined in [4] and is coercive in the sense that $-\int_{\Omega} \Delta_{\mathcal{E}} \mathbf{v} \cdot \mathbf{v} \, d\mathbf{x} = \|\mathbf{v}\|_{1,\mathcal{E},0}^2$ for any $\mathbf{v} \in \mathbf{H}_{\mathcal{E},0}$, where $\|\cdot\|_{1,\mathcal{E},0}$ is the usual discrete H^1 -norm of \mathbf{u} (see [4]). This inner product may also be formulated as the L^2 -inner product of adequately chosen discrete gradients [4].

Discrete convection operator – The numerical convection fluxes and the approximations of ρ in the momentum equation are chosen so as ensure that a discrete mass balance holds on the dual cells, in order to recover a discrete kinetic energy inequality. This idea was first introduced in [3, 1] for the Crouzeix-Raviart and Rannacher-Turek scheme, in [6] for the MAC scheme and was adapted to a DDFV scheme [5]. For $\epsilon = \sigma|\sigma'$, the convection flux $\int_{\epsilon} \rho u_i \mathbf{u} \cdot \mathbf{n}_{\sigma,\epsilon} \, d\gamma(\mathbf{x})$ is approximated by $F_{\sigma,\epsilon} u_{\epsilon}$, where $u_{\epsilon} = (u_{\sigma} + u_{\sigma'})/2$ and $F_{\sigma,\epsilon}$ is the numerical mass flux through ϵ outward D_{σ} defined as follows:

- First case – The vector \mathbf{e}_i is normal to ϵ , and ϵ is included in a primal cell K . Then the mass flux through $\epsilon = \sigma|\sigma'$ is given by:

$$F_{\sigma,\epsilon} = \frac{1}{2} (F_{K,\sigma} \mathbf{n}_{D_{\sigma},\epsilon} \cdot \mathbf{n}_{K,\sigma} + F_{K,\sigma'} \mathbf{n}_{D_{\sigma},\epsilon} \cdot \mathbf{n}_{K,\sigma'}).$$

- Second case – The vector \mathbf{e}_i is tangent to ϵ , and ϵ is the union of the halves of two primal faces τ and τ' such that $\sigma = K|L$ with $\tau \in \mathcal{E}(K)$ and $\tau' \in \mathcal{E}(L)$. Then:

$$F_{\sigma,\epsilon} = \frac{1}{2} (F_{K,\tau} + F_{L,\tau'}).$$

Remark 2.1. In both cases, for $\epsilon = \sigma|\sigma'$, the mass flux $F_{\sigma,\epsilon}$ may be written as $F_{\sigma,\epsilon} = |\epsilon| \rho_{\epsilon} \tilde{u}_{\epsilon}$, with $\rho_{\epsilon} = (\rho_{\sigma} + \rho_{\sigma'})$ and $\tilde{u}_{\epsilon} = (\rho_{\sigma} u_{\sigma} + \rho_{\sigma'} u_{\sigma'})/(\rho_{\sigma} + \rho_{\sigma'})$ in the first case, and $\rho_{\epsilon} = (|\tau| \rho_{\tau} + |\tau'| \rho_{\tau'})/(|\tau| + |\tau'|)$ and $\tilde{u}_{\epsilon} = (|\tau| \rho_{\tau} u_{\tau} + |\tau'| \rho_{\tau'} u_{\tau'})/(|\tau| \rho_{\tau} + |\tau'| \rho_{\tau'})$ in the second case.

With this expression of the flux, we may define a discrete divergence operator on the dual cells:

$$\operatorname{div}_{\mathcal{E}^{(i)}}(\rho, \mathbf{v}) : \left\{ \begin{array}{l} L_{\mathcal{M}} \times \mathbf{H}_{\mathcal{E},0} \longrightarrow L_{\mathcal{E}} \\ (\rho, \mathbf{v}) \longmapsto \operatorname{div}_{\mathcal{E}^{(i)}}(\rho, \mathbf{v}) = \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \operatorname{div}_{D_{\sigma}}(\rho \mathbf{v}) \mathbf{1}_{D_{\sigma}}, \\ \text{with } \operatorname{div}_{D_{\sigma}}(\rho, \mathbf{v}) = \frac{1}{|D_{\sigma}|} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_{\sigma})} F_{\sigma,\epsilon}, \quad \forall \sigma \in \mathcal{E}_{\text{int}}^{(i)}. \end{array} \right.$$

For the definition of the time-derivative $\partial_t(\rho \mathbf{u})$, an approximation of the density on the dual cell $\rho_{D_{\sigma}}$ is defined as:

$$|D_{\sigma}| \rho_{D_{\sigma}} = |D_{K,\sigma}| \rho_K + |D_{L,\sigma}| \rho_L, \quad \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L.$$

With the above definitions, if $(\rho, \mathbf{u}) \in L_{\mathcal{M}} \times \mathbf{X}_{\mathcal{E},\delta t}$ satisfies the mass balance equation (4b), then the following mass balance on the dual cells holds:

$$(6) \quad \frac{1}{\delta t}(\rho_{D_{\sigma}}^{n+1} - \rho_{D_{\sigma}}^n) + \operatorname{div}_{D_{\sigma}}(\rho^{n+1} \mathbf{u}^n) = 0.$$

Note that a discrete duality property also holds, in the sense that, for $1 \leq i \leq d$,

$$(7) \quad \forall \rho \in L_{\mathcal{M}}, \forall \mathbf{v} \in \mathbf{H}_{\mathcal{E},0}, \forall w \in H_{\mathcal{E}^{(i)},0}, \int_{\Omega} \operatorname{div}_{\mathcal{E}^{(i)}}(\rho, \mathbf{v}) w \, d\mathbf{x} = \int_{\Omega} \rho \mathbf{v} \cdot \nabla_{\mathcal{E}^{(i)}} w \, d\mathbf{x},$$

where $(\rho \mathbf{v})_{\mathcal{E}^{(i)}}$ and $\nabla_{\mathcal{E}^{(i)}} w$ are vector valued functions of components:

$$[(\rho \mathbf{v})_{\mathcal{E}^{(i)}}]_j = \sum_{\epsilon \in \tilde{\mathcal{E}}^{(i,j)}} \rho_{\epsilon} \tilde{v}_{\epsilon} \mathbf{1}_{D_{\epsilon}}, \quad [(\nabla w)_{\mathcal{E}^{(i)}}]_j = \sum_{\epsilon \in \tilde{\mathcal{E}}^{(i,j)}, \sigma = \sigma' | \sigma} \frac{u_{\sigma'} - u_{\sigma}}{d_{\epsilon}} \mathbf{1}_{D_{\epsilon}},$$

with ρ_{ϵ} and \tilde{v}_{ϵ} defined in Remark 2.1 and $\tilde{\mathcal{E}}^{(i,j)} = \{\epsilon \in \tilde{\mathcal{E}}^{(i)}; \epsilon \perp \mathbf{e}^{(j)}\}$. We finally define the i -th component $C_{\mathcal{E}}^{(i)}(\rho \mathbf{u})$ of the non linear convection operator by:

$$C_{\mathcal{E}}^{(i)}(\rho, \mathbf{u}) : \left\{ \begin{array}{l} H_{\mathcal{E}^{(i)},0} \longrightarrow H_{\mathcal{E}^{(i)},0} \\ v \longmapsto C_{\mathcal{E}}^{(i)}(\rho, \mathbf{u}) v = \sum_{\sigma \in \tilde{\mathcal{E}}_{\text{int}}^{(i)}} \frac{1}{|D_{\sigma}|} \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}(D_{\sigma}) \\ \epsilon = \sigma | \sigma'}} F_{\sigma,\epsilon} \frac{v_{\sigma} + v_{\sigma'}}{2} \mathbf{1}_{D_{\sigma}}. \end{array} \right.$$

and the full (*i.e.* for all the velocity components) discrete convection operator $\mathbf{C}_{\mathcal{E}}(\rho, \mathbf{u})$, $\mathbf{H}_{\mathcal{E},0} \longrightarrow \mathbf{H}_{\mathcal{E},0}$ by $\mathbf{C}_{\mathcal{E}}(\rho, \mathbf{u}) \mathbf{v} = (C_{\mathcal{E}}^{(1)}(\rho, \mathbf{u}) v_1, \dots, C_{\mathcal{E}}^{(d)}(\rho, \mathbf{u}) v_d)^t$. Let $\mathbf{E}_{\mathcal{E}}$ be the subspace of $\mathbf{H}_{\mathcal{E},0}$ of divergence-free functions (with respect to the discrete divergence operator). By Hölder's inequality and [4, Lemma 3.9], there exists $C_{\eta_{\mathcal{M}}} > 0$ (depending only on $\eta_{\mathcal{M}}$) such that, $\forall (\rho, \mathbf{u}, \mathbf{v}, \mathbf{w}) \in L_{\mathcal{M}} \times \mathbf{E}_{\mathcal{E}} \times \mathbf{H}_{\mathcal{E},0}^2$,

$$\begin{aligned} |\mathbf{C}_{\mathcal{E}}(\rho \mathbf{u}) \mathbf{v} \cdot \mathbf{w}| &\leq C_{\eta_{\mathcal{M}}} \|\rho\|_{L^{\infty}(\Omega)} \|\mathbf{u}\|_{L^4(\Omega)^d} \|\mathbf{v}\|_{L^4(\Omega)^d} \|\mathbf{w}\|_{1,\mathcal{E},0} \\ \text{and} \quad |\mathbf{C}_{\mathcal{E}}(\rho \mathbf{u}) \mathbf{v} \cdot \mathbf{w}| &\leq C_{\eta_{\mathcal{M}}} \|\rho\|_{L^{\infty}(\Omega)} \|\mathbf{u}\|_{1,\mathcal{E},0} \|\mathbf{v}\|_{1,\mathcal{E},0} \|\mathbf{w}\|_{1,\mathcal{E},0}. \end{aligned}$$

3. ESTIMATES AND CONVERGENCE ANALYSIS

Since the velocity is divergence-free, the mass equation is a transport equation on ρ , so that, thanks to the upwind choice, the following estimate holds:

$$(8) \quad \rho_{\min} \leq \rho^{n+1} \leq \rho_{\max},$$

and the L^2 -norm of ρ^{n+1} is lower than the L^2 -norm of the initial data ρ_0 , for $0 \leq n \leq N-1$. In addition, thanks to (6), any solution to the scheme (4) satisfies the following discrete kinetic energy balance, for $1 \leq i \leq d$, $\sigma \in \mathcal{E}^{(i)}$, $0 \leq n \leq N-1$,

$$(9) \quad \frac{1}{2\delta t} [\rho_{D_{\sigma}}^{n+1} (u_{\sigma}^{n+1})^2 - \rho_{D_{\sigma}}^n (u_{\sigma}^n)^2] + \frac{1}{2|D_{\sigma}|} \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}(D_{\sigma}) \\ \epsilon = \sigma | \sigma'}} F_{\sigma,\epsilon} (\rho^{n+1}, u^n) u_{\sigma}^{n+1} u_{\sigma'}^{n+1} \\ - (\Delta u)_{\sigma}^{n+1} u_{\sigma}^{n+1} + (\partial p)_{\sigma}^{n+1} u_{\sigma}^{n+1} - f_{\sigma}^{n+1} u_{\sigma}^{n+1} = -\frac{1}{2\delta t} \rho_{D_{\sigma}}^n (u_{\sigma}^{n+1} - u_{\sigma}^n)^2.$$

From this inequality, we obtain estimates on the velocity. For $\mathbf{u} \in \mathbf{X}_{\varepsilon, \delta t}$ satisfying (4), there exists $C > 0$ depending on \mathbf{u}_0 , ρ_0 and \mathbf{f} such that,

$$(10) \quad \|\mathbf{u}\|_{L^2(\mathbf{H}_{\varepsilon,0})} = \sum_{n=0}^{N-1} \delta t \|\mathbf{u}^{n+1}\|_{1,\varepsilon,0}^2 \leq C \text{ and } \|\mathbf{u}\|_{L^\infty(L^2)} = \max_{0 \leq n \leq N-1} \|\mathbf{u}^{n+1}\|_{L^2(\Omega)^d} \leq C.$$

These estimates yields the existence of a unique solution to the scheme: indeed, the first equation may be solved separately for ρ^{n+1} and is linear with respect to this unknown and, once ρ^{n+1} is known, the last two equations are a linear generalized Oseen problem for \mathbf{u}^{n+1} and p^{n+1} , which is uniquely solvable thanks to the *inf-sup* stability of the MAC discretization. The convergence of the scheme requires some time compactness. Contrary to the constant density case [4], there is no uniform estimate on the time derivative, and compactness is obtained thanks to the following lemma together with the Fréchet-Kolmogorov theorem.

Lemma 3.1 (Estimate on the time translates of the velocity). *Let $\mathbf{u} \in X_{\varepsilon, \delta t}$ and $\rho \in Y_{\mathcal{M}, \delta t}$ and let $\tau > 0$ then*

$$(11) \quad \int_0^{T-\tau} \int_\Omega |\mathbf{u}(\mathbf{x}, t + \tau) - \mathbf{u}(\mathbf{x}, t)|^2 \, d\mathbf{x} \, dt \leq C_{\eta_{\mathcal{M}}, T} \frac{\rho_{\max}}{\rho_{\min}} (\|\mathbf{u}\|_{L^2(\mathbf{H}_{\varepsilon,0})}^3 + 1) \sqrt{\tau + \delta t}$$

where $C_{\eta_{\mathcal{M}}, T} > 0$ only depends on Ω , T , \mathbf{f} and on the regularity of the mesh $\eta_{\mathcal{M}}$.

Proof. In the continuous case, see e.g. [2, pages 444-452], the estimate (11) is obtained by bounding the term $\int_0^{T-\tau} \int_\Omega (\rho(\mathbf{x}, t) \mathbf{u}(\mathbf{x}, t + \tau) - \rho(\mathbf{x}, t) \mathbf{u}(\mathbf{x}, t)) \cdot \mathbf{w}(\mathbf{x}, t) \, d\mathbf{x} \, dt$ with $\mathbf{w}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}, t + \tau) - \mathbf{u}(\mathbf{x}, t)$. However, in the context of the MAC scheme, the components of \mathbf{u} are piecewise constant on different meshes so we need to treat the space indices separately. For a given $i = 1, \dots, d$, we denote by u and w the i -th component of \mathbf{u} and \mathbf{w} , and by $\tilde{\rho}$ the piecewise constant function defined by $\tilde{\rho}(\mathbf{x}, t) = \rho_{D_\sigma}^{n+1}$ for $(\mathbf{x}, t) \in D_\sigma \times [t_n, t_{n+1})$. We then wish to bound the terms

$$\begin{aligned} A^{(i)} &= \int_0^{T-\tau} (A_1^{(i)}(t) + A_2^{(i)}(t)) \, dt, \text{ with} \\ A_1^{(i)}(t) &= \int_\Omega (\tilde{\rho}(\mathbf{x}, t + \tau) u(\mathbf{x}, t + \tau) - \tilde{\rho}(\mathbf{x}, t) u(\mathbf{x}, t)) w(\mathbf{x}, t) \, d\mathbf{x}, \\ A_2^{(i)}(t) &= \int_\Omega (\tilde{\rho}(\mathbf{x}, t) - \tilde{\rho}(\mathbf{x}, t + \tau)) u(\mathbf{x}, t + \tau) w(\mathbf{x}, t) \, d\mathbf{x}. \end{aligned}$$

For lack of space, we only deal here with the term $A_2^{(i)}(t)$. Thanks to the mass balance on the dual cells (6) and to the discrete duality formula (7) we have:

$$\begin{aligned} A_2^{(i)}(t) &= \sum_{n=1}^{N-1} \delta t \, \mathbf{1}_{(t, t+\tau)}(t_n) \int_\Omega \operatorname{div}_{\mathcal{E}^{(i)}}(\tilde{\rho}^{n+1} \mathbf{u}^n) u(\cdot, t + \tau) w(\cdot, t) \, d\mathbf{x} \\ &= \sum_{n=1}^{N-1} \delta t \, \mathbf{1}_{(t, t+\tau)}(t_n) \int_\Omega (\rho^{n+1} \mathbf{u}^n)_{\mathcal{E}^{(i)}} \nabla_{\mathcal{E}^{(i)}}(u(\cdot, t + \tau) w(\cdot, t)) \, d\mathbf{x}. \end{aligned}$$

Using Hölder's inequalities and the fact that $\sum_{n=1}^{N-1} \delta t \, \mathbf{1}_{(t, t+\tau)}(t_n) \leq \tau + \delta t$,

$$\begin{aligned} A_2^{(i)}(t) &\leq \rho_{\max} \delta t \left(\sum_{n=1}^{N-1} \|\mathbf{u}^n\|_{L^6} \right)^{\frac{1}{2}} \left(\sum_{n=1}^{N-1} \mathbf{1}_{(t, t+\tau)}(t_n) \right)^{\frac{1}{2}} \|\nabla_{\mathcal{E}^{(i)}}(u(t + \tau) w(t))\|_{L^{\frac{6}{5}}} \\ &\leq |\Omega|^{\frac{1}{6}} \rho_{\max} \|u\|_{L^2(L^6)}^{\frac{1}{2}} (\delta t + \tau)^{\frac{1}{2}} \|\nabla_{\mathcal{E}^{(i)}}(u(t + \tau) w(t))\|_{L^{\frac{3}{2}}}. \end{aligned}$$

Now, by Hölder's inequality,

$$\begin{aligned} \|\nabla_{\mathcal{E}^{(i)}}(u(\cdot, t + \tau) w(\cdot, t))\|_{L^{\frac{3}{2}}} &\leq \|(\nabla_{\mathcal{E}^{(i)}} u(\cdot, t + \tau)) w(\cdot, t)\|_{L^{\frac{3}{2}}} + \|u(\cdot, t + \tau) \nabla_{\mathcal{E}^{(i)}}(w(\cdot, t))\|_{L^{\frac{3}{2}}} \\ &\leq \|\nabla_{\mathcal{E}^{(i)}} u(\cdot, t + \tau)\|_{L^2}^2 + \|w(\cdot, t)\|_{L^6}^2 + \|\nabla_{\mathcal{E}^{(i)}} w(\cdot, t)\|_{L^2}^2 + \|u(\cdot, t + \tau)\|_{L^6}^2. \end{aligned}$$

Therefore, integrating over $(0, T - \tau)$ yields that

$$\int_0^{T-\tau} A_2^{(i)}(t) \, dt \leq |\Omega|^{\frac{1}{6}} \rho_{\max} [\tau + \delta]^{\frac{1}{2}} \|u\|_{L^2(L^6)} [\|u\|_{L^2(L^6)} + \|w\|_{L^2(H_{\varepsilon^{(i)},0})} + \|w\|_{L^2(L^6)} + \|u\|_{L^2(H_{\varepsilon^{(i)},0})}].$$

Similar computations for the term $\int_0^{T-\tau} A_1^{(i)}(t) \, dt$ yield the result. \square

Theorem 3.2 (Convergence of the scheme). *Let $(\delta t_m)_{m \in \mathbb{N}}$ and $(\mathcal{M}_m)_{m \in \mathbb{N}}$ be a sequence of time steps and MAC grids such that $\delta t_m \rightarrow 0$ and $h_{\mathcal{M}_m} \rightarrow 0$ as $m \rightarrow +\infty$; assume that there exists $\eta > 0$ such that $\eta_{\mathcal{M}_m} \leq \eta$ for any $m \in \mathbb{N}$. Let (ρ_m, \mathbf{u}_m) be a solution to (4) for $\delta t = \delta t_m$ and $\mathcal{M} = \mathcal{M}_m$. Then there exists $\bar{\rho}$ with $\rho_{\min} \leq \bar{\rho} \leq \rho_{\max}$ and $\bar{\mathbf{u}} \in L^2(0, T; \mathbf{E}(\Omega))$ such that, up to a subsequence:*

- the sequence $(\mathbf{u}_m)_{m \in \mathbb{N}}$ converges to $\bar{\mathbf{u}}$ in $L^2(0, T; L^2(\Omega)^d)$,
- the sequence $(\rho_m)_{m \in \mathbb{N}}$ converges to $\bar{\rho}$ in $L^2(0, T; L^2(\Omega))$,
- $(\bar{\rho}, \bar{\mathbf{u}})$ is a solution to the weak formulation (2) and (3).

Sketch of proof:

- Thanks to (8), there exists a subsequence of $(\rho_m)_{m \in \mathbb{N}}$ star-weakly converging to some $\bar{\rho}$ in $L^\infty(\Omega \times (0, T))$; thanks to (10) and (11), there exists a subsequence of $(u_m)_{m \in \mathbb{N}}$ converging to some \bar{u} in $L^2(0, T; (L^2(\Omega))^d)$.
- Passing to the limit in (4b) yields that $(\bar{\rho}, \bar{u})$ satisfies (2).
- The strong convergence of the approximate densities is then obtained thanks to the L^2 estimates for ρ in both the discrete and continuous case [7, Proposition 8.7].
- Passing to the limit in (4c) yields that $(\bar{\rho}, \bar{u})$ satisfies (3).
- We finally obtain that $\bar{\mathbf{u}} \in L^2(0, T; \mathbf{E}(\Omega))$, where $\mathbf{E}(\Omega) = \{\mathbf{v} \in H_0^1(\Omega) \text{ s.t. } \operatorname{div} \mathbf{v} = 0\}$, as in [4, Proof of Theorem 4.3].

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