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NOTE ON BOLTHAUSEN-DEUSCHEL-ZEITOUNI'S PAPER ON THE ABSENCE OF A WETTING TRANSITION FOR A PINNED HARMONIC CRYSTAL IN DIMENSIONS THREE AND LARGER

LOREN COQUILLE AND PIOTR MIŁOŚ

ABSTRACT. The article [1] provides a proof of the absence of a wetting transition for the discrete Gaussian free field conditioned to stay positive, and undergoing a weak delta-pinning at height 0. The proof is generalized to the case of a square pinning-potential replacing the delta-pinning, but it relies on a lower bound on the probability for the field to stay above the support of the potential, the proof of which appears to be incorrect. We provide a modified proof of the absence of a wetting transition in the square-potential case, which does not require the aforementioned lower bound. An alternative approach is given in a recent paper by Giacomin and Lacoin [2].

1. DEFINITIONS AND NOTATIONS

We keep the notations of [1] except for the field which we call ϕ instead of X. Let A be a finite subset of $\mathbb{Z}^d$, let $\phi = (\phi_x)_{x\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$ and the Hamiltonian defined as

$$H_A(\phi) = \frac{1}{8d} \sum_{x,y\in A\cup\partial A\,:\,|x-y|=1} (\phi_x - \phi_y)^2 \tag{1}$$

where ∂A is the outer boundary of A. The following probability measure on $\mathbb{R}^A$ defines the discrete Gaussian free field on A (with zero boundary condition) :

$$P_A(d\phi) = \frac{1}{Z_A} e^{-H_A(\phi)} d\phi_A \delta_0(d\phi_{A^c}) \tag{2}$$

where $d\phi_A = \prod_{x\in A} d\phi_x$ and δ_0 is the Dirac mass at 0. The partition function Z_A is the normalization $Z_A = \int_{\mathbb{R}^A} \exp(-H(\phi_A))d\phi_A$. We will also need the following definition of a set A being Δ-sparse (morally meaning that it has only one pinned point per cell of side-length Δ), which we reproduce from [1, page 1215] :

Definition 1. Let $N \in \mathbb{Z}$, $\Delta > 0$, $\Lambda_N = \{-\lfloor N\rfloor/2, \dots, \lfloor N\rfloor/2\}^d$ and let $l_N^\Delta = \{z_i\}_{i=1}^{|l_N^\Delta|}$ denote a finite collection of points $z_i \in \Lambda_N$ such that for each $y \in \Lambda_N \cap \Delta\mathbb{Z}^d$ there is exacly one $z \in l_N^\Delta$ such that $|z - y| < \Delta/10$. Let $A_{l_N^\Delta} = \Lambda_N \backslash l_N^\Delta$.

2. LOWER BOUND ON THE PROBABILITY OF THE HARD WALL CONDITION

The proof of [1, Theorem 6] relies on [1, Proposition 3]. Unfortunately, the proof provided in the paper, when applied with $t > 0$ provides a lower bound which is a little bit weaker than what is claimed, namely

Proposition 2. *Correction of [1, Proposition 3] :*
Assume $d \geq 3$ and let $t \geq 0$. Then there exist three constants $c_1, c_2, c_3 > 0$ depending on t, and $c_4 > 0$ independent of t, such that, for all Δ integer large enough

$$\liminf_{N\to\infty} \inf_{l_N^\Delta} \frac{1}{(2N+1)^d} \log P_{A_{l_N^\Delta}}(X_i \geq t, i \in A_{l_N^\Delta}) \geq -\frac{d\log\Delta}{\Delta^d} + c_1 \frac{\log\log\Delta}{\Delta^d} - \frac{c_2 e^{c_4 t}\sqrt{\log\Delta}}{\Delta^d(\log\Delta)^{c_3}} \tag{3}$$

The statement of [1, Proposition 3] only contains the first two terms. The dependence in t vanishes between equations (2.4) and (2.5) in [1]. Note that for $t = 0$ the third term is irrelevant and the bound coincides with the one stated in the paper.

3. Proof of the absence of a wetting transition in the square-potential case

Let us introduce the following notations

$$\hat{\xi}_N = \sum_{x\in\Lambda_N} 1_{[|\phi_x|\leq a]}, \quad \tilde{\xi}_N = \sum_{x\in\Lambda_N} 1_{[\phi_x\in[0,a]]},$$
$$\Omega_A^+ = \{\phi_x \geq 0,\ \forall x \in A\}, \quad \Omega_N^+ = \{\phi_x \geq 0,\ \forall x \in \Lambda_N\}$$
$$\mathcal{A} = \{x \in \Lambda_N : \phi_x \in [0,a]\}$$

and the following probability measure with square-potential pinning :

$$\tilde{P}_{N,a,b}(d\phi) = \frac{1}{\tilde{Z}_{N,a,b}} \exp\left(-H(\phi) + \sum_{x\in\Lambda_N} b1_{[\phi_x\in[0,a]]}\right) d\phi_{\Lambda_N}\delta_0(d\phi_{\Lambda_N^c})$$

in contrast with the measure used in [1] :

$$\hat{P}_{N,a,b}(d\phi) = \frac{1}{\hat{Z}_{N,a,b}} \exp\left(-H(\phi) + \sum_{x\in\Lambda_N} b1_{[\phi_x\in[-a,a]]}\right) d\phi_{\Lambda_N}\delta_0(d\phi_{\Lambda_N^c}).$$

Observe that

$$\tilde{P}_{N,a,b}(\tilde{\xi}_N < \epsilon N^d|\Omega_N^+) = \hat{P}_{N,a,b}(\hat{\xi}_N < \epsilon N^d|\Omega_N^+)$$

Theorem 3. *(Absence of wetting transition, [1, Theorem 6])*
Assume $d \geq 3$ and let $a, b > 0$ be arbitrary. Then there exists $\epsilon_{b,a}, \eta_{b,a} > 0$ such that

$$\tilde{P}_{N,a,b}(\tilde{\xi}_N > \epsilon_{b,a}N^d|\Omega_N^+) \geq 1 - \exp(-\eta_{b,a}N^d). \tag{4}$$

provided N is large enough.

Proof. Let us compute the probability of the complementary event and provide bounds on the numerator and the denominator corresponding to the conditional probability :

$$\tilde{P}_{N,a,b}(\tilde{\xi}_N < \epsilon N^d|\Omega_N^+) = \frac{\tilde{P}_{N,a,b}(\{\tilde{\xi}_N < \epsilon N^d\} \cap \Omega_N^+)}{\tilde{P}_{N,a,b}(\Omega_N^+)} \tag{5}$$

3.1. Lower bound on the denominator.

Writing

$$\exp(\sum_{x\in\Lambda_N} b1_{[\phi_x\in[0,a]]}) = \prod_{x\in\Lambda_N}((e^b-1)1_{[\phi_x\in[0,a]]}+1) \tag{6}$$

and using the FKG inequality, we get

$$\tilde{P}_{N,a,b}(\Omega_N^+) \overset{FKG}{\geq} \frac{Z_N}{\tilde{Z}_{N,a,b}} \sum_{A\subset\Lambda_N} (e^b-1)^{|A|} \underbrace{P_N(\mathcal{A}\supset A)}_{(*)} \underbrace{P_N(\Omega_{A^c}^+|\mathcal{A}\supset A)}_{(**)} \underbrace{P_N(\Omega_A^+|\mathcal{A}\supset A)}_{=1}. \tag{7}$$

Let us first bound the term $(**)$:

$$(**) = P_N(\phi \geq 0\, on\, A^c|\phi \in [0,a]\, on\, A) = \int_{[0,a]^A} P_N(\phi \geq 0\, on\, A^c|\phi = \psi\, on\, A)g(\psi)d\psi \tag{8}$$

for some density function g. Let $\tilde{\psi}$ be the harmonic extension of ψ to $\Lambda_N\backslash A$. Since $\tilde{\psi} \geq 0$, we have

$$(**) = \int_{[0,a]^A} P_N(\phi + \tilde{\psi} \geq 0\, on\, A^c|\phi = 0\, on\, A)g(\psi)d\psi \tag{9}$$
$$= \int_{[0,a]^A} P_{A^c}(\phi + \tilde{\psi} \geq 0\, on\, A^c)g(\psi)d\psi \tag{10}$$
$$\geq P_{A^c}(\Omega_{A^c}^+) \tag{11}$$

For the term $(*)$, we write $A = \{x_1, \dots, x_{|A|}\}$, and $A_i = \{x_{i+1}, \dots, x_{|A|}\}$,

$$(*) = P_N(\phi \in [0,a] \, on \, A) \tag{12}$$

$$= \prod_{i=1}^{|A|} P_N(\phi_{x_i} \in [0,a] | \phi_{x_{i+1}}, \dots, \phi_{x_{|A|}} \in [0,a]) \tag{13}$$

$$= \prod_{i=1}^{|A|} \int_{[0,a]^{A_i}} P_N(\phi_{x_i} \in [0,a] | \phi = \psi \, on \, A_i) g_i(\psi) d\psi \tag{14}$$

for some density function g_i. Let $\tilde{\psi}$ be the harmonic extension of ψ to $\Lambda_N \backslash A_i$, we have

$$(*) = \prod_{i=1}^{|A|} \int_{[0,a]^{A_i}} P_N(\phi_{x_i} + \tilde{\psi}_{x_i} \in [0,a] | \phi = 0 \, on \, A_i) g_i(\psi) d\psi \tag{15}$$

$$= \prod_{i=1}^{|A|} \int_{[0,a]^{A_i}} P_{A_i^c}(\phi_{x_i} + \tilde{\psi}_{x_i} \in [0,a]) g_i(\psi) d\psi \tag{16}$$

$$\geq \prod_{i=1}^{|A|} P_{A_i^c}(\phi_{x_i} \in [0,a]) \tag{17}$$

$$\geq [c(1/2 \wedge a)]^{|A|} \tag{18}$$

for some $c = c(d) > 0$, since the variance of the free field is bounded in $d \geq 3$. The inequality (17) comes from the fact that $P_{A_i^c}(\phi_{x_i} + \tilde{\psi}_{x_i} \in [0,a]) \geq P_{A_i^c}(\phi_{x_i} \in [0,a])$ since $\tilde{\psi}_{x_i} \in [0,a]$ and ϕ_{x_i} is a centered Gaussian variable.
Hence,

$$\tilde{P}_{N,a,b}(\Omega_N^+) \geq \frac{Z_N}{\tilde{Z}_{N,a,b}} \sum_{A \subset \Lambda_N} \exp(J'|A|) P_{A^c}(\Omega_{A^c}^+) \tag{19}$$

with $J' = \log(e^b - 1) + \log c + \log(1/2 \wedge a)$.

3.2. **Upper bound on the numerator.**

$$\tilde{P}_{N,a,b}(\{\tilde{\xi}_N < \epsilon N^d\} \cap \Omega_N^+) = \frac{Z_N}{\tilde{Z}_{N,a,b}} \sum_{A : |A| < \epsilon N^d} (e^b - 1)^{|A|} \underbrace{P_N(\mathcal{A} \supset A)}_{\leq (1/2 \wedge a)^{|A|}} \underbrace{P_N(\Omega_N^+ | \mathcal{A} \supset A)}_{\leq 1} \tag{20}$$

$$\leq \frac{Z_N}{\tilde{Z}_{N,a,b}} \sharp\{A : |A| < \epsilon N^d\} \exp(J \epsilon N^d) \tag{21}$$

with $J = \log(e^b - 1) + \log(1/2 \wedge a)$, where $\sharp X$ denotes the cardinality of the set X.

3.3. **Upper bound on** (5)**.**

$$\tilde{P}_{N,a,b}(\tilde{\xi}_N < \epsilon N^d | \Omega_N^+) \leq \frac{\exp(J \epsilon N^d) \sharp\{A : |A| < \epsilon N^d\}}{\sum_{A \subset \Lambda_N} \exp(J'|A|) P_{A^c}(\Omega_{A^c}^+)} \tag{22}$$

And now we proceed similarily as for the proof with δ-pinning potential:

$$\frac{1}{N^d} \log \tilde{P}_{N,a,b}(\tilde{\xi}_N < \epsilon N^d | \Omega_N^+) \leq \frac{1}{N^d} \log \Big(\exp(J \epsilon N^d) \sharp\{A : |A| < \epsilon N^d\} \Big) \tag{23}$$

$$- \frac{1}{N^d} \log \sum_{A \subset \Lambda_N} \exp(J'|A|) P_{A^c}(\Omega_{A^c}^+) \tag{24}$$

The right hand side of (23) can be bounded by $\epsilon(J + 1 - \log \epsilon)$ as N tends to infinity (by a rough approximation and the Stirling formula), which in turn can be made as close to 0 as we want by choosing $\epsilon = \epsilon(J)$ sufficiently small. See [1].

To bound (24) we use [1, Proposition 3] with $t = 0$ which matches to our Proposition 2 :

$$(24) \leq -\frac{1}{N^d} \log \sum_{A \subset \Lambda_N \,:\, A \, is \, \Delta - sparse} \exp(J'|A|) P_{A^c}(\Omega^+_{A^c}) \tag{25}$$

$$\leq -\frac{1}{N^d} \left(\left(\frac{N}{\Delta} \right)^d [(d \log \Delta + c_0) + J' - d \log \Delta + c_1 \log \log \Delta] \right) \tag{26}$$

$$= -\frac{J' + c_0 + c_1 \log \log \Delta}{\Delta^d} < 0 \text{ for } \Delta = \Delta(J') \text{ large enough.} \tag{27}$$

where Δ-sparseness corresponds to Definition 1 : a set $A \subset \Lambda_N$ is Δ-sparse if it equals $A_{l_N^\Delta}$, for some set l_N^Δ.

□

References

[1] E. Bolthausen, J. D. Deuschel, and O. Zeitouni. Absence of a wetting transition for a pinned harmonic crystal in dimensions three and larger. *J. Math. Phys.*, 41(3):1211–1223, 2000. Probabilistic techniques in equilibrium and nonequilibrium statistical physics.

[2] G. Giacomin, and H. Lacoin, Disorder and wetting transition: the pinned harmonic crystal in dimension three or larger. ArXiv:1607.03859 [math-ph]

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