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**NOTE ON BOLTHAUSEN-DEUSCHEL-ZEITOUNI'S PAPER ON THE
ABSENCE OF A WETTING TRANSITION FOR A PINNED HARMONIC CRYSTAL
IN DIMENSIONS THREE AND LARGER**

LOREN COQUILLE AND PIOTR MIŁOŚ

ABSTRACT. The article [1] provides a proof of the absence of a wetting transition for the discrete Gaussian free field conditioned to stay positive, and undergoing a weak delta-pinning at height 0. The proof is generalized to the case of a square pinning-potential replacing the delta-pinning, but it relies on a lower bound on the probability for the field to stay above the support of the potential, the proof of which appears to be incorrect. We provide a modified proof of the absence of a wetting transition in the square-potential case, which does not require the aforementioned lower bound. An alternative approach is given in a recent paper by Giacomin and Lacoïn [2].

1. DEFINITIONS AND NOTATIONS

We keep the notations of [1] except for the field which we call ϕ instead of X . Let A be a finite subset of \mathbb{Z}^d , let $\phi = (\phi_x)_{x \in \mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$ and the Hamiltonian defined as

$$H_A(\phi) = \frac{1}{8d} \sum_{x, y \in A \cup \partial A : |x-y|=1} (\phi_x - \phi_y)^2 \quad (1)$$

where ∂A is the outer boundary of A . The following probability measure on \mathbb{R}^A defines the discrete Gaussian free field on A (with zero boundary condition) :

$$P_A(d\phi) = \frac{1}{Z_A} e^{-H_A(\phi)} d\phi_A \delta_0(d\phi_{A^c}) \quad (2)$$

where $d\phi_A = \prod_{x \in A} d\phi_x$ and δ_0 is the Dirac mass at 0. The partition function Z_A is the normalization $Z_A = \int_{\mathbb{R}^A} \exp(-H(\phi_A)) d\phi_A$. We will also need the following definition of a set A being Δ -sparse (morally meaning that it has only one pinned point per cell of side-length Δ), which we reproduce from [1, page 1215] :

Definition 1. Let $N \in \mathbb{Z}$, $\Delta > 0$, $\Lambda_N = \{-\lfloor N \rfloor/2, \dots, \lfloor N \rfloor/2\}^d$ and let $l_N^\Delta = \{z_i\}_{i=1}^{\lfloor \Delta \rfloor}$ denote a finite collection of points $z_i \in \Lambda_N$ such that for each $y \in \Lambda_N \cap \Delta \mathbb{Z}^d$ there is exactly one $z \in l_N^\Delta$ such that $|z - y| < \Delta/10$. Let $A_{l_N^\Delta} = \Lambda_N \setminus l_N^\Delta$.

2. LOWER BOUND ON THE PROBABILITY OF THE HARD WALL CONDITION

The proof of [1, Theorem 6] relies on [1, Proposition 3]. Unfortunately, the proof provided in the paper, when applied with $t > 0$ provides a lower bound which is a little bit weaker than what is claimed, namely

Proposition 2. *Correction of [1, Proposition 3] :*

Assume $d \geq 3$ and let $t \geq 0$. Then there exist three constants $c_1, c_2, c_3 > 0$ depending on t , and $c_4 > 0$ independent of t , such that, for all Δ integer large enough

$$\liminf_{N \rightarrow \infty} \inf_{l_N^\Delta} \frac{1}{(2N+1)^d} \log P_{A_{l_N^\Delta}}(X_i \geq t, i \in A_{l_N^\Delta}) \geq -\frac{d \log \Delta}{\Delta^d} + c_1 \frac{\log \log \Delta}{\Delta^d} - \frac{c_2 e^{c_4 t} \sqrt{\log \Delta}}{\Delta^d (\log \Delta)^{c_3}} \quad (3)$$

The statement of [1, Proposition 3] only contains the first two terms. The dependence in t vanishes between equations (2.4) and (2.5) in [1]. Note that for $t = 0$ the third term is irrelevant and the bound coincides with the one stated in the paper.

3. PROOF OF THE ABSENCE OF A WETTING TRANSITION IN THE SQUARE-POTENTIAL CASE

Let us introduce the following notations

$$\begin{aligned}\hat{\xi}_N &= \sum_{x \in \Lambda_N} 1_{[\phi_x \leq a]}, & \tilde{\xi}_N &= \sum_{x \in \Lambda_N} 1_{[\phi_x \in [0, a]]}, \\ \Omega_A^+ &= \{\phi_x \geq 0, \forall x \in A\}, & \Omega_N^+ &= \{\phi_x \geq 0, \forall x \in \Lambda_N\} \\ \mathcal{A} &= \{x \in \Lambda_N : \phi_x \in [0, a]\}\end{aligned}$$

and the following probability measure with square-potential pinning :

$$\tilde{P}_{N,a,b}(d\phi) = \frac{1}{\tilde{Z}_{N,a,b}} \exp\left(-H(\phi) + \sum_{x \in \Lambda_N} b 1_{[\phi_x \in [0, a]]}\right) d\phi_{\Lambda_N} \delta_0(d\phi_{\Lambda_N^c})$$

in contrast with the measure used in [1] :

$$\hat{P}_{N,a,b}(d\phi) = \frac{1}{\hat{Z}_{N,a,b}} \exp\left(-H(\phi) + \sum_{x \in \Lambda_N} b 1_{[\phi_x \in [-a, a]]}\right) d\phi_{\Lambda_N} \delta_0(d\phi_{\Lambda_N^c}).$$

Observe that

$$\tilde{P}_{N,a,b}(\tilde{\xi}_N < \epsilon N^d | \Omega_N^+) = \hat{P}_{N,a,b}(\hat{\xi}_N < \epsilon N^d | \Omega_N^+)$$

Theorem 3. (Absence of wetting transition, [1, Theorem 6])

Assume $d \geq 3$ and let $a, b > 0$ be arbitrary. Then there exists $\epsilon_{b,a}, \eta_{b,a} > 0$ such that

$$\tilde{P}_{N,a,b}(\tilde{\xi}_N > \epsilon_{b,a} N^d | \Omega_N^+) \geq 1 - \exp(-\eta_{b,a} N^d). \quad (4)$$

provided N is large enough.

Proof. Let us compute the probability of the complementary event and provide bounds on the numerator and the denominator corresponding to the conditional probability :

$$\tilde{P}_{N,a,b}(\tilde{\xi}_N < \epsilon N^d | \Omega_N^+) = \frac{\tilde{P}_{N,a,b}(\{\tilde{\xi}_N < \epsilon N^d\} \cap \Omega_N^+)}{\tilde{P}_{N,a,b}(\Omega_N^+)} \quad (5)$$

3.1. Lower bound on the denominator. Writing

$$\exp\left(\sum_{x \in \Lambda_N} b 1_{[\phi_x \in [0, a]]}\right) = \prod_{x \in \Lambda_N} ((e^b - 1) 1_{[\phi_x \in [0, a]]} + 1) \quad (6)$$

and using the FKG inequality, we get

$$\tilde{P}_{N,a,b}(\Omega_N^+) \stackrel{FKG}{\geq} \frac{Z_N}{\tilde{Z}_{N,a,b}} \sum_{A \subset \Lambda_N} (e^b - 1)^{|A|} \underbrace{P_N(\mathcal{A} \supset A)}_{(*)} \underbrace{P_N(\Omega_{A^c}^+ | \mathcal{A} \supset A)}_{(**)} \underbrace{P_N(\Omega_A^+ | \mathcal{A} \supset A)}_{=1}. \quad (7)$$

Let us first bound the term (**):

$$(**) = P_N(\phi \geq 0 \text{ on } A^c | \phi \in [0, a] \text{ on } A) = \int_{[0, a]^A} P_N(\phi \geq 0 \text{ on } A^c | \phi = \psi \text{ on } A) g(\psi) d\psi \quad (8)$$

for some density function g . Let $\tilde{\psi}$ be the harmonic extension of ψ to $\Lambda_N \setminus A$. Since $\tilde{\psi} \geq 0$, we have

$$(**) = \int_{[0, a]^A} P_N(\phi + \tilde{\psi} \geq 0 \text{ on } A^c | \phi = 0 \text{ on } A) g(\psi) d\psi \quad (9)$$

$$= \int_{[0, a]^A} P_{A^c}(\phi + \tilde{\psi} \geq 0 \text{ on } A^c) g(\psi) d\psi \quad (10)$$

$$\geq P_{A^c}(\Omega_{A^c}^+) \quad (11)$$

For the term (*), we write $A = \{x_1, \dots, x_{|A|}\}$, and $A_i = \{x_{i+1}, \dots, x_{|A|}\}$,

$$(*) = P_N(\phi \in [0, a] \text{ on } A) \quad (12)$$

$$= \prod_{i=1}^{|A|} P_N(\phi_{x_i} \in [0, a] | \phi_{x_{i+1}}, \dots, \phi_{x_{|A|}} \in [0, a]) \quad (13)$$

$$= \prod_{i=1}^{|A|} \int_{[0, a]^{A_i}} P_N(\phi_{x_i} \in [0, a] | \phi = \psi \text{ on } A_i) g_i(\psi) d\psi \quad (14)$$

for some density function g_i . Let $\tilde{\psi}$ be the harmonic extension of ψ to $\Lambda_N \setminus A_i$, we have

$$(*) = \prod_{i=1}^{|A|} \int_{[0, a]^{A_i}} P_N(\phi_{x_i} + \tilde{\psi}_{x_i} \in [0, a] | \phi = 0 \text{ on } A_i) g_i(\psi) d\psi \quad (15)$$

$$= \prod_{i=1}^{|A|} \int_{[0, a]^{A_i}} P_{A_i^c}(\phi_{x_i} + \tilde{\psi}_{x_i} \in [0, a]) g_i(\psi) d\psi \quad (16)$$

$$\geq \prod_{i=1}^{|A|} P_{A_i^c}(\phi_{x_i} \in [0, a]) \quad (17)$$

$$\geq [c(1/2 \wedge a)]^{|A|} \quad (18)$$

for some $c = c(d) > 0$, since the variance of the free field is bounded in $d \geq 3$. The inequality (17) comes from the fact that $P_{A_i^c}(\phi_{x_i} + \tilde{\psi}_{x_i} \in [0, a]) \geq P_{A_i^c}(\phi_{x_i} \in [0, a])$ since $\tilde{\psi}_{x_i} \in [0, a]$ and ϕ_{x_i} is a centered Gaussian variable.

Hence,

$$\tilde{P}_{N,a,b}(\Omega_N^+) \geq \frac{Z_N}{\tilde{Z}_{N,a,b}} \sum_{A \subset \Lambda_N} \exp(J'|A|) P_{A^c}(\Omega_{A^c}^+) \quad (19)$$

with $J' = \log(e^b - 1) + \log c + \log(1/2 \wedge a)$.

3.2. Upper bound on the numerator.

$$\tilde{P}_{N,a,b}(\{\tilde{\xi}_N < \epsilon N^d\} \cap \Omega_N^+) = \frac{Z_N}{\tilde{Z}_{N,a,b}} \sum_{A: |A| < \epsilon N^d} (e^b - 1)^{|A|} \underbrace{P_N(\mathcal{A} \supset A)}_{\leq (1/2 \wedge a)^{|A|}} \underbrace{P_N(\Omega_N^+ | \mathcal{A} \supset A)}_{\leq 1} \quad (20)$$

$$\leq \frac{Z_N}{\tilde{Z}_{N,a,b}} \#\{A : |A| < \epsilon N^d\} \exp(J\epsilon N^d) \quad (21)$$

with $J = \log(e^b - 1) + \log(1/2 \wedge a)$, where $\#X$ denotes the cardinality of the set X .

3.3. Upper bound on (5).

$$\tilde{P}_{N,a,b}(\tilde{\xi}_N < \epsilon N^d | \Omega_N^+) \leq \frac{\exp(J\epsilon N^d) \#\{A : |A| < \epsilon N^d\}}{\sum_{A \subset \Lambda_N} \exp(J'|A|) P_{A^c}(\Omega_{A^c}^+)} \quad (22)$$

And now we proceed similarly as for the proof with δ -pinning potential:

$$\frac{1}{N^d} \log \tilde{P}_{N,a,b}(\tilde{\xi}_N < \epsilon N^d | \Omega_N^+) \leq \frac{1}{N^d} \log \left(\exp(J\epsilon N^d) \#\{A : |A| < \epsilon N^d\} \right) \quad (23)$$

$$- \frac{1}{N^d} \log \sum_{A \subset \Lambda_N} \exp(J'|A|) P_{A^c}(\Omega_{A^c}^+) \quad (24)$$

The right hand side of (23) can be bounded by $\epsilon(J + 1 - \log \epsilon)$ as N tends to infinity (by a rough approximation and the Stirling formula), which in turn can be made as close to 0 as we want by choosing $\epsilon = \epsilon(J)$ sufficiently small. See [1].

To bound (24) we use [1, Proposition 3] with $t = 0$ which matches to our Proposition 2 :

$$(24) \leq -\frac{1}{N^d} \log \sum_{A \subset \Lambda_N : A \text{ is } \Delta\text{-sparse}} \exp(J'|A|) P_{A^c}(\Omega_{A^c}^+) \quad (25)$$

$$\leq -\frac{1}{N^d} \left(\left(\frac{N}{\Delta} \right)^d [(d \log \Delta + c_0) + J' - d \log \Delta + c_1 \log \log \Delta] \right) \quad (26)$$

$$= -\frac{J' + c_0 + c_1 \log \log \Delta}{\Delta^d} < 0 \text{ for } \Delta = \Delta(J') \text{ large enough.} \quad (27)$$

where Δ -sparseness corresponds to Definition 1 : a set $A \subset \Lambda_N$ is Δ -sparse if it equals $A_{I_N^\Delta}$, for some set I_N^Δ . □

REFERENCES

- [1] E. Bolthausen, J. D. Deuschel, and O. Zeitouni. Absence of a wetting transition for a pinned harmonic crystal in dimensions three and larger. *J. Math. Phys.*, 41(3):1211–1223, 2000. Probabilistic techniques in equilibrium and nonequilibrium statistical physics.
- [2] G. Giacomin, and H. Lacoin, Disorder and wetting transition: the pinned harmonic crystal in dimension three or larger. ArXiv:1607.03859 [math-ph]

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