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# LOW MACH NUMBER LIMIT OF A PRESSURE CORRECTION MAC SCHEME FOR COMPRESSIBLE BAROTROPIC FLOWS

R. HERBIN, J.-C. LATCHÉ, AND K. SALEH

ABSTRACT. We study the incompressible limit of a pressure correction MAC scheme [3] for the unstationary compressible barotropic Navier-Stokes equations. Provided the initial data are well-prepared, the solution of the numerical scheme converges, as the Mach number tends to zero, towards the solution of the classical pressure correction *inf-sup* stable MAC scheme for the incompressible Navier-Stokes equations.

## 1. INTRODUCTION

Let  $\Omega$  be parallelepiped of  $\mathbb{R}^d$ , with  $d \in \{2, 3\}$  and  $T > 0$ . The unsteady barotropic compressible Navier-Stokes equations, parametrized by the Mach number  $\varepsilon$ , read for  $(\mathbf{x}, t) \in \Omega \times (0, T)$ :

$$\begin{aligned} (1a) \quad & \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon \mathbf{u}^\varepsilon) = 0, \\ (1b) \quad & \partial_t(\rho^\varepsilon \mathbf{u}^\varepsilon) + \operatorname{div}(\rho^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) - \operatorname{div}(\boldsymbol{\tau}(\mathbf{u}^\varepsilon)) + \frac{1}{\varepsilon^2} \nabla \wp(\rho^\varepsilon) = 0, \\ (1c) \quad & \mathbf{u}^\varepsilon|_{\partial\Omega} = 0, \quad \rho^\varepsilon|_{t=0} = \rho_0^\varepsilon, \quad \mathbf{u}^\varepsilon|_{t=0} = \mathbf{u}_0^\varepsilon, \end{aligned}$$

where  $\rho^\varepsilon > 0$  and  $\mathbf{u}^\varepsilon = (u_1^\varepsilon, \dots, u_d^\varepsilon)^T$  are the density and velocity of the fluid. The pressure satisfies the ideal gas law  $\wp(\rho^\varepsilon) = (\rho^\varepsilon)^\gamma$ , with  $\gamma \geq 1$ , and

$$\operatorname{div}(\boldsymbol{\tau}(\mathbf{u})) = \mu \Delta \mathbf{u} + (\mu + \lambda) \nabla(\operatorname{div} \mathbf{u}),$$

where the real numbers  $\mu$  and  $\lambda$  satisfy  $\mu > 0$  and  $\mu + \lambda > 0$ . The smooth solutions of (1) are known to satisfy a kinetic energy balance and a renormalization identity. In addition, under assumption on the initial data, it may be inferred from these estimates that the density  $\rho^\varepsilon$  tends to a constant  $\bar{\rho}$ , and the velocity tends, in a sense to be defined, to a solution  $\bar{\mathbf{u}}$  of the incompressible Navier-Stokes equations [4]:

$$\begin{aligned} (2a) \quad & \operatorname{div} \bar{\mathbf{u}} = 0, \\ (2b) \quad & \bar{\rho} \partial_t \bar{\mathbf{u}} + \bar{\rho} \operatorname{div}(\bar{\mathbf{u}} \otimes \bar{\mathbf{u}}) - \mu \Delta \bar{\mathbf{u}} + \nabla \pi = 0, \end{aligned}$$

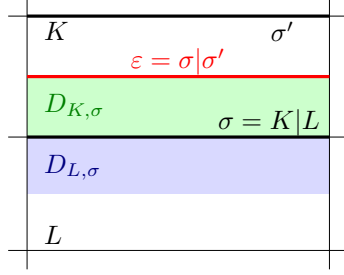
where  $\pi$  is the formal limit of  $(\wp(\rho^\varepsilon) - \wp(\bar{\rho}))/\varepsilon^2$ .

In this paper, we reproduce this theory for a pressure correction scheme, based on the Marker-And-Cell (MAC) space discretization: we first derive discrete analogues of the kinetic energy and renormalization identities, then establish from these relations that approximate solutions of (1) converge, as  $\varepsilon \rightarrow 0$ , towards the solution of the classical projection scheme for the incompressible Navier-Stokes equations (2).

For this asymptotic analysis, we assume that the initial data is “well prepared”:  $\rho_0^\varepsilon > 0$ ,  $\rho_0^\varepsilon \in L^\infty(\Omega)$ ,  $\mathbf{u}_0^\varepsilon \in H_0^1(\Omega)^d$  and, taking without loss of generality  $\bar{\rho} = 1$ , there exists  $C$  independent of  $\varepsilon$  such that:

$$(3) \quad \|\mathbf{u}_0^\varepsilon\|_{H^1(\Omega)^d} + \frac{1}{\varepsilon} \|\operatorname{div} \mathbf{u}_0^\varepsilon\|_{L^2(\Omega)} + \frac{1}{\varepsilon^2} \|\rho_0^\varepsilon - 1\|_{L^\infty(\Omega)} \leq C.$$

Consequently,  $\rho_0^\varepsilon$  tends to 1 when  $\varepsilon \rightarrow 0$ ; moreover, we suppose that  $\mathbf{u}_0^\varepsilon$  converges in  $L^2(\Omega)^d$  towards a function  $\bar{\mathbf{u}}_0 \in L^2(\Omega)^d$  (the uniform boundedness of the sequence in the  $H^1(\Omega)^d$  norm already implies this convergence up to a subsequence).



primal cells:  $K, L$ .  
dual cell for the  $y$ -component of the velocity:  $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$ .  
primal and dual cells  $d$ -dimensional measures:  $|K|, |D_\sigma|, |D_{K,\sigma}|$ .  
faces  $(d-1)$ -dimensional measures:  $|\sigma|, |\varepsilon|$ .  
vector normal to  $\sigma$  outward  $K$ :  $\mathbf{n}_{K,\sigma}$ .

FIGURE 1. Notations for control volumes and faces.

## 2. THE NUMERICAL SCHEME

Let  $\mathcal{M}$  be a MAC mesh (see e.g. [1] and Figure 1 for the notations). The discrete density unknowns are associated with the cells of the mesh  $\mathcal{M}$ , and are denoted by  $\{\rho_K, K \in \mathcal{M}\}$ . We denote by  $\mathcal{E}$  the set of the faces of the mesh, and by  $\mathcal{E}^{(i)}$  the subset of the faces orthogonal to the  $i$ -th vector of the canonical basis of  $\mathbb{R}^d$ . The discrete  $i^{\text{th}}$  component of the velocity is located at the centre of the faces  $\sigma \in \mathcal{E}^{(i)}$ , so the whole set of discrete velocity unknowns reads  $\{u_{\sigma,i}, \sigma \in \mathcal{E}^{(i)}, 1 \leq i \leq d\}$ . We define  $\mathcal{E}_{\text{ext}} = \{\sigma \in \mathcal{E}, \sigma \subset \partial\Omega\}$ ,  $\mathcal{E}_{\text{int}} = \mathcal{E} \setminus \mathcal{E}_{\text{ext}}$ ,  $\mathcal{E}_{\text{int}}^{(i)} = \mathcal{E}_{\text{int}} \cap \mathcal{E}^{(i)}$  and  $\mathcal{E}_{\text{ext}}^{(i)} = \mathcal{E}_{\text{ext}} \cap \mathcal{E}^{(i)}$ . The boundary conditions (1c) are taken into account by setting  $u_{\sigma,i} = 0$  for all  $\sigma \in \mathcal{E}_{\text{ext}}^{(i)}, 1 \leq i \leq d$ . Let  $\delta t > 0$  be a constant time step. The approximate solution  $(\rho^n, \mathbf{u}^n)$  at time  $t_n = n\delta t$  for  $1 \leq n \leq N = \lfloor T/\delta t \rfloor$  is computed as follows: knowing  $\{\rho_K^{n-1}, \rho_K^n, K \in \mathcal{M}\} \subset \mathbb{R}$  and  $(u_{\sigma,i}^n)_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}, 1 \leq i \leq d} \subset \mathbb{R}$ , find  $(\rho_K^{n+1})_{K \in \mathcal{M}} \subset \mathbb{R}$  and  $(u_{\sigma,i}^{n+1})_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}, 1 \leq i \leq d} \subset \mathbb{R}$  by the following algorithm:

**Pressure gradient scaling step:**

$$(4a) \quad \text{For } 1 \leq i \leq d, \forall \sigma \in \mathcal{E}_{\text{int}}^{(i)}, \quad (\overline{\nabla p})_{\sigma,i}^n = \left( \frac{\rho_{D_\sigma}^n}{\rho_{D_\sigma}^{n-1}} \right)^{1/2} (\nabla p^n)_{\sigma,i}.$$

**Prediction step** – Solve for  $\tilde{\mathbf{u}}^{n+1}$ :

$$(4b) \quad \text{For } 1 \leq i \leq d, \forall \sigma \in \mathcal{E}_{\text{int}}^{(i)},$$

$$\frac{1}{\delta t} (\rho_{D_\sigma}^n \tilde{u}_{\sigma,i}^{n+1} - \rho_{D_\sigma}^{n-1} u_{\sigma,i}^n) + \text{div}(\rho^n \tilde{\mathbf{u}}_i^{n+1} \mathbf{u}^n)_\sigma - \text{div} \boldsymbol{\tau}(\tilde{\mathbf{u}}^{n+1})_{\sigma,i} + \frac{1}{\varepsilon^2} (\overline{\nabla p})_{\sigma,i}^n = 0.$$

**Correction step** – Solve for  $\rho^{n+1}$  and  $\mathbf{u}^{n+1}$ :

$$(4c) \quad \text{For } 1 \leq i \leq d, \forall \sigma \in \mathcal{E}_{\text{int}}^{(i)},$$

$$\frac{1}{\delta t} \rho_{D_\sigma}^n (u_{\sigma,i}^{n+1} - \tilde{u}_{\sigma,i}^{n+1}) + \frac{1}{\varepsilon^2} (\nabla p^{n+1})_{\sigma,i} - \frac{1}{\varepsilon^2} (\overline{\nabla p})_{\sigma,i}^n = 0,$$

$$(4d) \quad \forall K \in \mathcal{M}, \quad \frac{1}{\delta t} (\rho_K^{n+1} - \rho_K^n) + \text{div}(\rho^{n+1} \mathbf{u}^{n+1})_K = 0,$$

$$(4e) \quad \forall K \in \mathcal{M}, \quad p_K^{n+1} = \wp(\rho^{n+1}),$$

where the discrete densities and space operators are defined below (see also [3, 2]).

**Mass convection flux** – Given a discrete density field  $\rho = \{\rho_K, K \in \mathcal{M}\}$ , and a velocity field  $\mathbf{u} = \{u_{\sigma,i}, \sigma \in \mathcal{E}^{(i)}, 1 \leq i \leq d\}$ , the convection term in (4d) reads:

$$(5) \quad \text{div}(\rho \mathbf{u})_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}(\rho, \mathbf{u}), \quad K \in \mathcal{M},$$

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where  $F_{K,\sigma}(\rho, \mathbf{u})$  stands for the mass flux across  $\sigma$  outward  $K$ . This flux is set to 0 on external faces to account for the homogeneous Dirichlet boundary conditions; it is given on internal faces by:

$$(6) \quad F_{K,\sigma}(\rho, \mathbf{u}) = |\sigma| \rho_\sigma u_{K,\sigma}, \quad \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L,$$

where  $u_{K,\sigma} = u_{\sigma,i} \mathbf{n}_{K,\sigma} \cdot \mathbf{e}^{(i)}$ , with  $\mathbf{e}^{(i)}$  the  $i$ -th vector of the orthonormal basis of  $\mathbb{R}^d$ . The density at the face  $\sigma = K|L$  is approximated by the upwind technique, *i.e.*  $\rho_\sigma = \rho_K$  if  $u_{K,\sigma} \geq 0$  and  $\rho_\sigma = \rho_L$  otherwise.

**Pressure gradient term** – In (4a) and (4c), the term  $(\nabla p)_{\sigma,i}$  stands for the  $i^{\text{th}}$  component of the discrete pressure gradient at the face  $\sigma$ . Given a discrete density field  $\rho = \{\rho_K, K \in \mathcal{M}\}$ , this term is defined as:

$$(7) \quad (\nabla p)_{\sigma,i} = \frac{|\sigma|}{|D_\sigma|} (\wp(\rho_L) - \wp(\rho_K)) \mathbf{n}_{K,\sigma} \cdot \mathbf{e}^{(i)}, \quad 1 \leq i \leq d, \sigma \in \mathcal{E}_{\text{int}}^{(i)}, \sigma = K|L.$$

Defining for all  $K \in \mathcal{M}$ ,  $(\text{div} \mathbf{u})_K = \text{div}(1 \times \mathbf{u})_K$  (see (5)), the following discrete duality relation holds for all discrete density and velocity fields  $(\rho, \mathbf{u})$ :

$$(8) \quad \sum_{K \in \mathcal{M}} |K| \wp(\rho_K) (\text{div} \mathbf{u})_K + \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} |D_\sigma| u_{\sigma,i} (\nabla p)_{\sigma,i} = 0.$$

The MAC scheme is *inf-sup* stable: there exists  $\beta > 0$ , depending only on  $\Omega$  and the regularity of the mesh, such that, for all  $p = \{p_K, K \in \mathcal{M}\}$ , there exists  $\mathbf{u} = \{u_{\sigma,i}, \sigma \in \mathcal{E}^{(i)}, 1 \leq i \leq d\}$  satisfying homogeneous Dirichlet boundary conditions with:

$$\|\mathbf{u}\|_{1,\mathcal{M}} = 1 \text{ and } \sum_{K \in \mathcal{M}} |K| p_K (\text{div} \mathbf{u})_K \geq \beta \|p - \frac{1}{|\Omega|} \int_\Omega p \, d\mathbf{x}\|_{L^2(\Omega)},$$

where  $\|\mathbf{u}\|_{1,\mathcal{M}}$  is the usual discrete  $H^1$ -norm of  $\mathbf{u}$  (see [1]).

**Velocity convection operator** – Given a density field  $\rho = \{\rho_K, K \in \mathcal{M}\}$ , and two velocity fields  $\mathbf{u} = \{u_{\sigma,i}, \sigma \in \mathcal{E}^{(i)}, 1 \leq i \leq d\}$  and  $\mathbf{v} = \{v_{\sigma,i}, \sigma \in \mathcal{E}^{(i)}, 1 \leq i \leq d\}$ , we build for each  $\sigma \in \mathcal{E}_{\text{int}}$  the following quantities:

- an approximation of the density on the dual cell  $\rho_{D_\sigma}$  defined as:

$$(9) \quad |D_\sigma| \rho_{D_\sigma} = |D_{K,\sigma}| \rho_K + |D_{L,\sigma}| \rho_L, \quad \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L,$$

- a discrete divergence for the convection on the dual cell  $D_\sigma$ :

$$\text{div}(\rho v_i \mathbf{u})_\sigma = \sum_{\varepsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\varepsilon}(\rho, \mathbf{u}) v_{i,\varepsilon}, \quad \sigma \in \mathcal{E}_{\text{int}}^{(i)}, 1 \leq i \leq d.$$

For  $i \in \{1, \dots, d\}$ , and  $\sigma \in \mathcal{E}_{\text{int}}^{(i)}$ ,  $\sigma = K|L$ ,

- If the vector  $\mathbf{e}^{(i)}$  is normal to  $\varepsilon$ ,  $\varepsilon$  is included in a primal cell  $K$ , and we denote by  $\sigma'$  the second face of  $K$  which, in addition to  $\sigma$ , is normal to  $\mathbf{e}^{(i)}$ . We thus have  $\varepsilon = D_\sigma|D_{\sigma'}$ . Then the mass flux through  $\varepsilon$  is given by:

$$(10) \quad F_{\sigma,\varepsilon}(\rho, \mathbf{u}) = \frac{1}{2} (F_{K,\sigma}(\rho, \mathbf{u}) \mathbf{n}_{D_\sigma,\varepsilon} \cdot \mathbf{n}_{K,\sigma} + F_{K,\sigma'}(\rho, \mathbf{u}) \mathbf{n}_{D_\sigma,\varepsilon} \cdot \mathbf{n}_{K,\sigma'}).$$

- If the vector  $\mathbf{e}^{(i)}$  is tangent to  $\varepsilon$ ,  $\varepsilon$  is the union of the halves of two primal faces  $\tau$  and  $\tau'$  such that  $\tau \in \mathcal{E}(K)$  and  $\tau' \in \mathcal{E}(L)$ . The mass flux through  $\varepsilon$  is then given by:

$$(11) \quad F_{\sigma,\varepsilon}(\rho, \mathbf{u}) = \frac{1}{2} (F_{K,\tau}(\rho, \mathbf{u}) + F_{L,\tau'}(\rho, \mathbf{u})).$$

With this definition, the dual fluxes are locally conservative through dual faces  $\varepsilon = D_\sigma|D_{\sigma'}$  (*i.e.*  $F_{\sigma,\varepsilon}(\rho, \mathbf{u}) = -F_{\sigma',\varepsilon}(\rho, \mathbf{u})$ ), and vanish through a dual face included in the boundary of  $\Omega$ . For this reason, the values  $v_{\varepsilon,i}$  are only needed at the internal dual faces, and are chosen centered, *i.e.*, for  $\varepsilon = D_\sigma|D_{\sigma'}$ ,  $v_{\varepsilon,i} = (v_{\sigma,i} + v_{\sigma',i})/2$ .

As a result, a finite volume discretization of the mass balance (1a) holds over the internal dual cells. Indeed, if  $\rho^{n+1} = \{\rho_K^{n+1}, K \in \mathcal{M}\}$ ,  $\rho^n = \{\rho_K^n, K \in \mathcal{M}\}$  and  $\mathbf{u}^{n+1} = \{u_{\sigma,i}^{n+1}, \sigma \in \mathcal{E}^{(i)}, 1 \leq i \leq d\}$  are density and velocity fields satisfying (4d), then, the dual quantities  $\{\rho_{D_\sigma}^{n+1}, \rho_{D_\sigma}^n, \sigma \in \mathcal{E}_{\text{int}}\}$  and the dual fluxes

$\{F_{\sigma,\varepsilon}(\rho^{n+1}, \mathbf{u}^{n+1}), \sigma \in \mathcal{E}_{\text{int}}, \varepsilon \in \bar{\mathcal{E}}(D_\sigma)\}$  satisfy a finite volume discretization of the mass balance (1a) over the internal dual cells:

$$(12) \quad \frac{|D_\sigma|}{\delta t}(\rho_{D_\sigma}^{n+1} - \rho_{D_\sigma}^n) + \sum_{\varepsilon \in \bar{\mathcal{E}}(D_\sigma)} F_{\sigma,\varepsilon}(\rho^{n+1}, \mathbf{u}^{n+1}) = 0, \quad \sigma \in \mathcal{E}_{\text{int}}.$$

**Diffusion term** – The discrete diffusion term in (4b) is defined in [2] and is coercive in the following sense: for every discrete velocity field  $\mathbf{u}$  satisfying the homogeneous Dirichlet boundary conditions, one has:

$$(13) \quad - \sum_{i=1}^d \sum_{\varepsilon \in \mathcal{E}_{\text{int}}^{(i)}} |D_\sigma| u_{\sigma,i} \operatorname{div} \boldsymbol{\tau}(\mathbf{u})_{\sigma,i} \geq \mu \|\mathbf{u}\|_{1,\mathcal{M}}^2.$$

The initialization of the scheme (4) is performed by setting

$$\forall K \in \mathcal{M}, \rho_K^0 = \frac{1}{|K|} \int_K \rho_0^\varepsilon(\mathbf{x}) \, d\mathbf{x} \text{ and } \forall \sigma \in \mathcal{E}_{\text{int}}^{(i)}, 1 \leq i \leq d, u_{\sigma,i}^0 = \frac{1}{|\sigma|} \int_\sigma \mathbf{u}_0^\varepsilon(\mathbf{x}) \cdot \mathbf{e}^{(i)} \, d\mathbf{x},$$

and computing  $\rho^{-1}$  by solving the backward mass balance equation (4d) for  $n = -1$  where the unknown is  $\rho^{-1}$  and not  $\rho^0$ . This allows to perform the first prediction step with  $\{\rho_{D_\sigma}^0, \rho_{D_\sigma}^{-1}, \sigma \in \mathcal{E}_{\text{int}}\}$  and the dual mass fluxes  $\{F_{\sigma,\varepsilon}(\rho^0, \mathbf{u}^0), \sigma \in \mathcal{E}_{\text{int}}, \varepsilon \in \bar{\mathcal{E}}(D_\sigma)\}$  satisfying the mass balance (12). Moreover, since  $\rho_0^\varepsilon > 0$ , one clearly has  $\rho_K^0 > 0$  for all  $K \in \mathcal{M}$  and therefore  $\rho_{D_\sigma}^0 > 0$  for all  $\sigma \in \mathcal{E}_{\text{int}}$ . The positivity of  $\rho^{-1}$  is a consequence of the following Lemma.

**Lemma 2.1.** *If  $(\rho_0^\varepsilon, \mathbf{u}_0^\varepsilon)$  satisfies (3), then there exists  $C$ , depending on the mesh but independent of  $\varepsilon$  such that:*

$$(14) \quad \frac{1}{\varepsilon^2} \max_{K \in \mathcal{M}} |\rho_K^0 - 1| + \frac{1}{\varepsilon^2} \max_{1 \leq i \leq d} \max_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} |(\nabla p)_\sigma^0| + \frac{1}{\varepsilon} \max_{K \in \mathcal{M}} |\rho_K^{-1} - 1| \leq C.$$

*Proof.* We sketch the proof. The boundedness of the first two terms is a straightforward consequence of (3). For the third term we remark that, again by (3):

$$\forall K \in \mathcal{M}, \quad \rho_K^{-1} - 1 = \underbrace{\rho_K^0 - 1}_{=\mathcal{O}(\varepsilon^2)} + \underbrace{\delta t \rho_K^0 (\operatorname{div} \mathbf{u}^0)_K}_{=\mathcal{O}(\varepsilon)} + \underbrace{\delta t \sum_{\sigma \in \mathcal{E}(K)} |\sigma| (\rho_\sigma^0 - \rho_K^0) \mathbf{u}_{K,\sigma}^0}_{=\mathcal{O}(\varepsilon^2)}.$$

□

### 3. ASYMPTOTIC ANALYSIS OF THE ZERO MACH LIMIT

By the results of [3], there exists a solution  $(\rho^n, \mathbf{u}^n)_{0 \leq n \leq N}$  to the scheme (4) and any solution satisfies the following relations:

- a discrete kinetic energy balance: for all  $\sigma \in \mathcal{E}_{\text{int}}^{(i)}, 1 \leq i \leq d, 0 \leq n \leq N - 1$ :

$$(15) \quad \frac{1}{2\delta t} \left( \rho_{D_\sigma}^n |u_{\sigma,i}^{n+1}|^2 - \rho_{D_\sigma}^{n-1} |u_{\sigma,i}^n|^2 \right) + \frac{1}{2|D_\sigma|} \sum_{\substack{\varepsilon \in \bar{\mathcal{E}}(D_\sigma) \\ \varepsilon = D_\sigma | D_{\sigma'}}} F_{\sigma,\varepsilon}(\rho^n, \mathbf{u}^n) \tilde{u}_{\sigma,i}^{n+1} \tilde{u}_{\sigma',i}^{n+1} \\ - \operatorname{div} \boldsymbol{\tau}(\tilde{\mathbf{u}}^{n+1})_{\sigma,i} \tilde{u}_{\sigma,i}^{n+1} + \frac{1}{\varepsilon^2} (\nabla p)_{\sigma,i}^{n+1} u_{\sigma,i}^{n+1} + \frac{\delta t^2}{\varepsilon^4} \left( \frac{|(\nabla p)_{\sigma,i}^{n+1}|^2}{\rho_{D_\sigma}^n} - \frac{|(\nabla p)_{\sigma,i}^n|^2}{\rho_{D_\sigma}^{n-1}} \right) \\ + R_{\sigma,i}^{n+1} = 0, \quad \text{with } R_{\sigma,i}^{n+1} = \frac{1}{2\delta t} \rho_{D_\sigma}^{n-1} (\tilde{u}_{\sigma,i}^{n+1} - u_{\sigma,i}^n)^2.$$

- a discrete renormalization identity: for all  $K \in \mathcal{M}, 0 \leq n \leq N - 1$ :

$$(16) \quad \frac{1}{\delta t} \left( \Pi_\gamma(\rho_K^{n+1}) - \Pi_\gamma(\rho_K^n) \right) + \operatorname{div} (b_\gamma(\rho^{n+1}) \mathbf{u}^{n+1} - b'_\gamma(1) \rho^{n+1} \mathbf{u}^{n+1})_K + p_K^{n+1} \operatorname{div}(\mathbf{u}^{n+1})_K + R_K^{n+1} = 0,$$

with  $R_K^{n+1} \geq 0$ , where the function  $b_\gamma$  is defined by  $b_\gamma(\rho) = \rho \log \rho$  if  $\gamma = 1$ ,  $b_\gamma(\rho) = \rho^\gamma / (\gamma - 1)$  if  $\gamma > 1$  and satisfies  $\rho b'_\gamma(\rho) - b_\gamma(\rho) = \rho^\gamma = \wp(\rho)$  for all  $\rho > 0$ , and  $\Pi_\gamma(\rho) = b_\gamma(\rho) - 1 - b'_\gamma(1)(\rho - 1)$ .

Summing (15) and (16) over the primal cells from one side, and over the dual cells and the components on the other side, and invoking the grad-div duality relation (8), we obtain a local-in-time discrete entropy inequality, for  $0 \leq n \leq N-1$ :

$$(17) \quad \frac{1}{2} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} |D_\sigma| \left( \rho_{D_\sigma}^n |u_{\sigma,i}^{n+1}|^2 - \rho_{D_\sigma}^{n-1} |u_{\sigma,i}^n|^2 \right) + \frac{1}{\varepsilon^2} \sum_{K \in \mathcal{M}} |K| \left( \Pi_\gamma(\rho_K^{n+1}) - \Pi_\gamma(\rho_K^n) \right) \\ + \mu \delta t \|\tilde{\mathbf{u}}^{n+1}\|_{1,\mathcal{M}}^2 + \frac{1}{\varepsilon^4} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} |D_\sigma| \delta t^2 \left( \frac{|(\nabla p)_{\sigma,i}^{n+1}|^2}{\rho_{D_\sigma}^n} - \frac{|(\nabla p)_{\sigma,i}^n|^2}{\rho_{D_\sigma}^{n-1}} \right) + \mathcal{R}^{n+1} \leq 0$$

$$\text{where } \mathcal{R}^{n+1} = \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} R_{\sigma,i}^{n+1} + \frac{1}{\varepsilon^2} \sum_{K \in \mathcal{M}} R_K^{n+1} \geq 0.$$

The function  $\Pi_\gamma$  has some important properties:

- (18a) • For all  $\gamma \geq 1$  there exists  $C_\gamma$  such that:  $\Pi_\gamma(\rho) \leq C_\gamma |\rho - 1|^2$ ,  $\forall \rho \in (0, 2)$ .
- (18b) • If  $\gamma \geq 2$  then  $\Pi_\gamma(\rho) \geq |\rho - 1|^2$ ,  $\forall \rho > 0$ .
- (18c) • If  $\gamma \in [1, 2)$  then for all  $R \in (2, +\infty)$ , there exists  $C_{\gamma,R}$  such that:
 
$$\Pi_\gamma(\rho) \geq C_{\gamma,R} |\rho - 1|^2, \quad \forall \rho \in (0, R),$$

$$\Pi_\gamma(\rho) \geq C_{\gamma,R} |\rho - 1|^\gamma, \quad \forall \rho \in [R, \infty).$$

**Lemma 3.1** (Global discrete entropy inequality). *Under assumption (3), there exists  $C_0 > 0$  independent of  $\varepsilon$  such that the solution  $(\rho^n, \mathbf{u}^n)_{0 \leq n \leq N}$  to the scheme (4) satisfies, for  $\varepsilon$  small enough, and for  $1 \leq n \leq N$ :*

$$(19) \quad \frac{1}{2} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} |D_\sigma| \rho_{D_\sigma}^{n-1} |u_{\sigma,i}^n|^2 + \mu \sum_{k=1}^n \delta t \|\tilde{\mathbf{u}}^k\|_{1,\mathcal{M}}^2 \\ + \frac{1}{\varepsilon^2} \sum_{K \in \mathcal{M}} |K| \Pi_\gamma(\rho_K^n) + \frac{1}{\varepsilon^4} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \frac{|D_\sigma| \delta t^2}{\rho_{D_\sigma}^{n-1}} |(\nabla p)_{\sigma,i}^n|^2 \leq C_0.$$

*Proof.* Summing (17) over  $n$  yields the inequality (19) with

$$(20) \quad C_0 = \frac{1}{2} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} |D_\sigma| \rho_{D_\sigma}^{-1} |u_{\sigma,i}^0|^2 + \frac{1}{\varepsilon^2} \sum_{K \in \mathcal{M}} |K| \Pi_\gamma(\rho_K^0) + \frac{1}{\varepsilon^4} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \frac{|D_\sigma| \delta t^2}{\rho_{D_\sigma}^{-1}} |(\nabla p)_{\sigma,i}^0|^2.$$

By (14), for  $\varepsilon$  small enough, one has  $\rho_K^{-1} \leq 2$  for all  $K \in \mathcal{M}$  and therefore  $\rho_{D_\sigma}^{-1} \leq 2$  for all  $\sigma \in \mathcal{E}_{\text{int}}^{(i)}$  and  $1 \leq i \leq d$ . Hence, since  $\mathbf{u}_0^\varepsilon$  is uniformly bounded in  $H^1(\Omega)^d$  by (3), a classical trace inequality yields the boundedness of the first term. Again by (14), one has  $|\rho_K^0 - 1| \leq C\varepsilon^2$  for all  $K \in \mathcal{M}$ . Hence, by (18a), the second term vanishes as  $\varepsilon \rightarrow 0$ . The third term is also uniformly bounded with respect to  $\varepsilon$  thanks to (14).  $\square$

**Lemma 3.2** (Control of the pressure). *Assume that  $(\rho_0^\varepsilon, \mathbf{u}_0^\varepsilon)$  satisfies (3) and let  $(\rho^n, \mathbf{u}^n)_{0 \leq n \leq N}$  satisfy (4). Let  $p^n = \wp(\rho^n)$  and define  $\delta p^n = \{\delta p_K^n, K \in \mathcal{M}\}$  where  $\delta p_K^n = (p_K^n - |\Omega|^{-1} \int_\Omega p^n \, d\mathbf{x})/\varepsilon^2$ . Then, one has, for all  $1 \leq n \leq N$ :*

$$\|\delta p^n\| \leq C_{\mathcal{M}},$$

where  $C_{\mathcal{M}} \geq 0$  depends on the mesh but not on  $\varepsilon$ , and  $\|\cdot\|$  stands for any norm on the space of discrete functions.

*Proof.* By (19), the discrete pressure gradient is controlled in  $L^\infty$  by  $C_{\mathcal{M}} \varepsilon^2$ , so that  $\nabla(\delta p^n)$  is bounded in any norm independently of  $\varepsilon$ . Using the discrete  $(H^{-1})^d$ -norm (see e.g. [1]), invoking the gradient divergence duality (8) and the *inf-sup* stability of the scheme,  $\|\nabla(\delta p^n)\|_{-1,\mathcal{M}} \leq C_{\mathcal{M}}$  implies that  $\|\delta p^n\|_{L^2} \leq \beta^{-1} C_{\mathcal{M}}$ .  $\square$

**Theorem 3.3** (Incompressible limit of the MAC pressure correction scheme).

Let  $(\varepsilon^{(m)})_{m \in \mathbb{N}}$  be a sequence of positive real numbers tending to zero, and let  $(\rho^{(m)}, \mathbf{u}^{(m)})$  be a corresponding sequence of solutions of the scheme (4). Then the sequence  $(\rho^{(m)})_{m \in \mathbb{N}}$  converges to the constant function  $\rho = 1$  when  $m$  tends to  $+\infty$  in  $L^\infty((0, T), L^q(\Omega))$ , for all  $q \in [1, \min(\gamma, 2)]$ .

In addition, the sequence  $(\mathbf{u}^{(m)}, \delta p^{(m)})_{m \in \mathbb{N}}$  tends, in any discrete norm, to the solution  $(\mathbf{u}, \delta p)$  of the usual MAC pressure correction scheme for the incompressible Navier-Stokes equations, which reads:

**Prediction step** – Solve for  $\tilde{\mathbf{u}}^{n+1}$ :

$$\text{For } 1 \leq i \leq d, \forall \sigma \in \mathcal{E}_{\text{int}}^{(i)}, \quad \frac{1}{\delta t} (\tilde{u}_{\sigma,i}^{n+1} - u_{\sigma,i}^n) + \text{div}(\tilde{u}_i^{n+1} \mathbf{u}^n)_\sigma - \text{div} \boldsymbol{\tau}(\tilde{\mathbf{u}}^{n+1})_{\sigma,i} + (\nabla(\delta p)^n)_{\sigma,i} = 0.$$

**Correction step** – Solve for  $(\delta p)^{n+1}$  and  $\mathbf{u}^{n+1}$ :

$$\begin{aligned} \text{For } 1 \leq i \leq d, \forall \sigma \in \mathcal{E}_{\text{int}}^{(i)}, \quad & \frac{1}{\delta t} (u_{\sigma,i}^{n+1} - \tilde{u}_{\sigma,i}^{n+1}) + (\nabla(\delta p)^{n+1})_{\sigma,i} - (\nabla(\delta p)^n)_{\sigma,i} = 0, \\ \forall K \in \mathcal{M}, \quad & \text{div}(\mathbf{u}^{n+1})_K = 0. \end{aligned}$$

*Proof.* By (18b) and the global entropy estimate (19), one has for  $\gamma \geq 2$ ,

$$\|\rho^{(m)}(t) - 1\|_{L^2(\Omega)}^2 \leq \int_{\Omega} \Pi_\gamma(\rho^{(m)}(t)) \leq C_0 \varepsilon^2, \forall t \in (0, T).$$

For  $1 \leq \gamma \leq 2$ , invoking (18c) and estimate (19), we obtain for all  $t \in (0, T)$  and for all  $R \in (2, +\infty)$ :

$$\begin{aligned} (i) \quad & \|(\rho^{(m)}(t) - 1) \mathbb{1}_{\{\rho^{(m)}(t) \leq R\}}\|_{L^2(\Omega)}^2 \leq \frac{1}{C_{\gamma,R}} \int_{\Omega} \Pi_\gamma(\rho^{(m)}(t)) \leq C \varepsilon^2, \forall t \in (0, T), \\ (ii) \quad & \|(\rho^{(m)}(t) - 1) \mathbb{1}_{\{\rho^{(m)}(t) \geq R\}}\|_{L^\gamma(\Omega)}^\gamma \leq \frac{1}{C_{\gamma,R}} \int_{\Omega} \Pi_\gamma(\rho^{(m)}(t)) \leq C \varepsilon^2, \forall t \in (0, T), \end{aligned}$$

which proves the convergence of  $(\rho^{(m)})_{m \in \mathbb{N}}$  to the constant function  $\rho = 1$  as  $m \rightarrow +\infty$  in  $L^\infty((0, T), L^q(\Omega))$  for all  $q \in [1, \min(\gamma, 2)]$ . Using again (19), the sequence  $(\mathbf{u}^{(m)})_{m \in \mathbb{N}}$  is bounded in any discrete norm and the same holds for the sequence  $(\delta p^{(m)})_{m \in \mathbb{N}}$  by Lemma 3.2. By the Bolzano-Weierstrass theorem and a norm equivalence argument, there exists a subsequence of  $(\mathbf{u}^{(m)}, \delta p^{(m)})_{m \in \mathbb{N}}$  which tends, in any discrete norm, to a limit  $(\mathbf{u}, \delta p)$ . Passing to the limit cell-by-cell in (4), one obtains that  $(\mathbf{u}, \delta p)$  is a solution to (21). Since this solution is unique, the whole sequence converges, which concludes the proof.  $\square$

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