

LOW MACH NUMBER LIMIT OF A PRESSURE CORRECTION MAC SCHEME FOR COMPRESSIBLE BAROTROPIC FLOWS

R. HERBIN, J.-C. LATCHÉ, AND K. SALEH

ABSTRACT. We study the incompressible limit of a pressure correction MAC scheme [3] for the unstationary compressible barotropic Navier-Stokes equations. Provided the initial data are well-prepared, the solution of the numerical scheme converges, as the Mach number tends to zero, towards the solution of the classical pressure correction *inf-sup* stable MAC scheme for the incompressible Navier-Stokes equations.

1. INTRODUCTION

Let Ω be parallelepiped of \mathbb{R}^d , with $d \in \{2, 3\}$ and $T > 0$. The unsteady barotropic compressible Navier-Stokes equations, parametrized by the Mach number ε , read for $(\mathbf{x}, t) \in \Omega \times (0, T)$:

$$\begin{aligned} (1a) \quad & \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon \mathbf{u}^\varepsilon) = 0, \\ (1b) \quad & \partial_t(\rho^\varepsilon \mathbf{u}^\varepsilon) + \operatorname{div}(\rho^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) - \operatorname{div}(\boldsymbol{\tau}(\mathbf{u}^\varepsilon)) + \frac{1}{\varepsilon^2} \nabla \wp(\rho^\varepsilon) = 0, \\ (1c) \quad & \mathbf{u}^\varepsilon|_{\partial\Omega} = 0, \quad \rho^\varepsilon|_{t=0} = \rho_0^\varepsilon, \quad \mathbf{u}^\varepsilon|_{t=0} = \mathbf{u}_0^\varepsilon, \end{aligned}$$

where $\rho^\varepsilon > 0$ and $\mathbf{u}^\varepsilon = (u_1^\varepsilon, \dots, u_d^\varepsilon)^T$ are the density and velocity of the fluid. The pressure satisfies the ideal gas law $\wp(\rho^\varepsilon) = (\rho^\varepsilon)^\gamma$, with $\gamma \geq 1$, and

$$\operatorname{div}(\boldsymbol{\tau}(\mathbf{u})) = \mu \Delta \mathbf{u} + (\mu + \lambda) \nabla(\operatorname{div} \mathbf{u}),$$

where the real numbers μ and λ satisfy $\mu > 0$ and $\mu + \lambda > 0$. The smooth solutions of (1) are known to satisfy a kinetic energy balance and a renormalization identity. In addition, under assumption on the initial data, it may be inferred from these estimates that the density ρ^ε tends to a constant $\bar{\rho}$, and the velocity tends, in a sense to be defined, to a solution $\bar{\mathbf{u}}$ of the incompressible Navier-Stokes equations [4]:

$$\begin{aligned} (2a) \quad & \operatorname{div} \bar{\mathbf{u}} = 0, \\ (2b) \quad & \bar{\rho} \partial_t \bar{\mathbf{u}} + \bar{\rho} \operatorname{div}(\bar{\mathbf{u}} \otimes \bar{\mathbf{u}}) - \mu \Delta \bar{\mathbf{u}} + \nabla \pi = 0, \end{aligned}$$

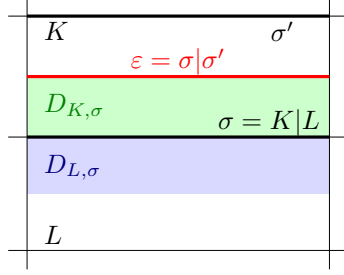
where π is the formal limit of $(\wp(\rho^\varepsilon) - \wp(\bar{\rho}))/\varepsilon^2$.

In this paper, we reproduce this theory for a pressure correction scheme, based on the Marker-And-Cell (MAC) space discretization: we first derive discrete analogues of the kinetic energy and renormalization identities, then establish from these relations that approximate solutions of (1) converge, as $\varepsilon \rightarrow 0$, towards the solution of the classical projection scheme for the incompressible Navier-Stokes equations (2).

For this asymptotic analysis, we assume that the initial data is “well prepared”: $\rho_0^\varepsilon > 0$, $\rho_0^\varepsilon \in L^\infty(\Omega)$, $\mathbf{u}_0^\varepsilon \in H_0^1(\Omega)^d$ and, taking without loss of generality $\bar{\rho} = 1$, there exists C independent of ε such that:

$$(3) \quad \|\mathbf{u}_0^\varepsilon\|_{H^1(\Omega)^d} + \frac{1}{\varepsilon} \|\operatorname{div} \mathbf{u}_0^\varepsilon\|_{L^2(\Omega)} + \frac{1}{\varepsilon^2} \|\rho_0^\varepsilon - 1\|_{L^\infty(\Omega)} \leq C.$$

Consequently, ρ_0^ε tends to 1 when $\varepsilon \rightarrow 0$; moreover, we suppose that \mathbf{u}_0^ε converges in $L^2(\Omega)^d$ towards a function $\bar{\mathbf{u}}_0 \in L^2(\Omega)^d$ (the uniform boundedness of the sequence in the $H^1(\Omega)^d$ norm already implies this convergence up to a subsequence).



primal cells: K, L .
dual cell for the y -component of the velocity: $D_\sigma = D_{K,\sigma} \cup D_{L,\sigma}$.
primal and dual cells d -dimensional measures: $|K|, |D_\sigma|, |D_{K,\sigma}|$.
faces $(d-1)$ -dimensional measures: $|\sigma|, |\varepsilon|$.
vector normal to σ outward K : $\mathbf{n}_{K,\sigma}$.

FIGURE 1. Notations for control volumes and faces.

2. THE NUMERICAL SCHEME

Let \mathcal{M} be a MAC mesh (see e.g. [1] and Figure 1 for the notations). The discrete density unknowns are associated with the cells of the mesh \mathcal{M} , and are denoted by $\{\rho_K, K \in \mathcal{M}\}$. We denote by \mathcal{E} the set of the faces of the mesh, and by $\mathcal{E}^{(i)}$ the subset of the faces orthogonal to the i -th vector of the canonical basis of \mathbb{R}^d . The discrete i^{th} component of the velocity is located at the centre of the faces $\sigma \in \mathcal{E}^{(i)}$, so the whole set of discrete velocity unknowns reads $\{u_{\sigma,i}, \sigma \in \mathcal{E}^{(i)}, 1 \leq i \leq d\}$. We define $\mathcal{E}_{\text{ext}} = \{\sigma \in \mathcal{E}, \sigma \subset \partial\Omega\}$, $\mathcal{E}_{\text{int}} = \mathcal{E} \setminus \mathcal{E}_{\text{ext}}$, $\mathcal{E}_{\text{int}}^{(i)} = \mathcal{E}_{\text{int}} \cap \mathcal{E}^{(i)}$ and $\mathcal{E}_{\text{ext}}^{(i)} = \mathcal{E}_{\text{ext}} \cap \mathcal{E}^{(i)}$. The boundary conditions (1c) are taken into account by setting $u_{\sigma,i} = 0$ for all $\sigma \in \mathcal{E}_{\text{ext}}^{(i)}, 1 \leq i \leq d$. Let $\delta t > 0$ be a constant time step. The approximate solution (ρ^n, \mathbf{u}^n) at time $t_n = n\delta t$ for $1 \leq n \leq N = \lfloor T/\delta t \rfloor$ is computed as follows: knowing $\{\rho_K^{n-1}, \rho_K^n, K \in \mathcal{M}\} \subset \mathbb{R}$ and $(u_{\sigma,i}^n)_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}, 1 \leq i \leq d} \subset \mathbb{R}$, find $(\rho_K^{n+1})_{K \in \mathcal{M}} \subset \mathbb{R}$ and $(u_{\sigma,i}^{n+1})_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}, 1 \leq i \leq d} \subset \mathbb{R}$ by the following algorithm:

Pressure gradient scaling step:

$$(4a) \quad \text{For } 1 \leq i \leq d, \forall \sigma \in \mathcal{E}_{\text{int}}^{(i)}, \quad (\overline{\nabla p})_{\sigma,i}^n = \left(\frac{\rho_{D_\sigma}^n}{\rho_{D_\sigma}^{n-1}} \right)^{1/2} (\nabla p^n)_{\sigma,i}.$$

Prediction step – Solve for $\tilde{\mathbf{u}}^{n+1}$:

$$(4b) \quad \text{For } 1 \leq i \leq d, \forall \sigma \in \mathcal{E}_{\text{int}}^{(i)},$$

$$\frac{1}{\delta t} (\rho_{D_\sigma}^n \tilde{u}_{\sigma,i}^{n+1} - \rho_{D_\sigma}^{n-1} u_{\sigma,i}^n) + \text{div}(\rho^n \tilde{\mathbf{u}}_i^{n+1} \mathbf{u}^n)_\sigma - \text{div} \boldsymbol{\tau}(\tilde{\mathbf{u}}^{n+1})_{\sigma,i} + \frac{1}{\varepsilon^2} (\overline{\nabla p})_{\sigma,i}^n = 0.$$

Correction step – Solve for ρ^{n+1} and \mathbf{u}^{n+1} :

$$(4c) \quad \text{For } 1 \leq i \leq d, \forall \sigma \in \mathcal{E}_{\text{int}}^{(i)},$$

$$\frac{1}{\delta t} \rho_{D_\sigma}^n (u_{\sigma,i}^{n+1} - \tilde{u}_{\sigma,i}^{n+1}) + \frac{1}{\varepsilon^2} (\nabla p^{n+1})_{\sigma,i} - \frac{1}{\varepsilon^2} (\overline{\nabla p})_{\sigma,i}^n = 0,$$

$$(4d) \quad \forall K \in \mathcal{M}, \quad \frac{1}{\delta t} (\rho_K^{n+1} - \rho_K^n) + \text{div}(\rho^{n+1} \mathbf{u}^{n+1})_K = 0,$$

$$(4e) \quad \forall K \in \mathcal{M}, \quad p_K^{n+1} = \wp(\rho^{n+1}),$$

where the discrete densities and space operators are defined below (see also [3, 2]).

Mass convection flux – Given a discrete density field $\rho = \{\rho_K, K \in \mathcal{M}\}$, and a velocity field $\mathbf{u} = \{u_{\sigma,i}, \sigma \in \mathcal{E}^{(i)}, 1 \leq i \leq d\}$, the convection term in (4d) reads:

$$(5) \quad \text{div}(\rho \mathbf{u})_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma}(\rho, \mathbf{u}), \quad K \in \mathcal{M},$$

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where $F_{K,\sigma}(\rho, \mathbf{u})$ stands for the mass flux across σ outward K . This flux is set to 0 on external faces to account for the homogeneous Dirichlet boundary conditions; it is given on internal faces by:

$$(6) \quad F_{K,\sigma}(\rho, \mathbf{u}) = |\sigma| \rho_\sigma u_{K,\sigma}, \quad \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L,$$

where $u_{K,\sigma} = u_{\sigma,i} \mathbf{n}_{K,\sigma} \cdot \mathbf{e}^{(i)}$, with $\mathbf{e}^{(i)}$ the i -th vector of the orthonormal basis of \mathbb{R}^d . The density at the face $\sigma = K|L$ is approximated by the upwind technique, *i.e.* $\rho_\sigma = \rho_K$ if $u_{K,\sigma} \geq 0$ and $\rho_\sigma = \rho_L$ otherwise.

Pressure gradient term – In (4a) and (4c), the term $(\nabla p)_{\sigma,i}$ stands for the i^{th} component of the discrete pressure gradient at the face σ . Given a discrete density field $\rho = \{\rho_K, K \in \mathcal{M}\}$, this term is defined as:

$$(7) \quad (\nabla p)_{\sigma,i} = \frac{|\sigma|}{|D_\sigma|} (\wp(\rho_L) - \wp(\rho_K)) \mathbf{n}_{K,\sigma} \cdot \mathbf{e}^{(i)}, \quad 1 \leq i \leq d, \sigma \in \mathcal{E}_{\text{int}}^{(i)}, \sigma = K|L.$$

Defining for all $K \in \mathcal{M}$, $(\text{div} \mathbf{u})_K = \text{div}(1 \times \mathbf{u})_K$ (see (5)), the following discrete duality relation holds for all discrete density and velocity fields (ρ, \mathbf{u}) :

$$(8) \quad \sum_{K \in \mathcal{M}} |K| \wp(\rho_K) (\text{div} \mathbf{u})_K + \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} |D_\sigma| u_{\sigma,i} (\nabla p)_{\sigma,i} = 0.$$

The MAC scheme is *inf-sup* stable: there exists $\beta > 0$, depending only on Ω and the regularity of the mesh, such that, for all $p = \{p_K, K \in \mathcal{M}\}$, there exists $\mathbf{u} = \{u_{\sigma,i}, \sigma \in \mathcal{E}^{(i)}, 1 \leq i \leq d\}$ satisfying homogeneous Dirichlet boundary conditions with:

$$\|\mathbf{u}\|_{1,\mathcal{M}} = 1 \text{ and } \sum_{K \in \mathcal{M}} |K| p_K (\text{div} \mathbf{u})_K \geq \beta \|p - \frac{1}{|\Omega|} \int_\Omega p \, d\mathbf{x}\|_{L^2(\Omega)},$$

where $\|\mathbf{u}\|_{1,\mathcal{M}}$ is the usual discrete H^1 -norm of \mathbf{u} (see [1]).

Velocity convection operator – Given a density field $\rho = \{\rho_K, K \in \mathcal{M}\}$, and two velocity fields $\mathbf{u} = \{u_{\sigma,i}, \sigma \in \mathcal{E}^{(i)}, 1 \leq i \leq d\}$ and $\mathbf{v} = \{v_{\sigma,i}, \sigma \in \mathcal{E}^{(i)}, 1 \leq i \leq d\}$, we build for each $\sigma \in \mathcal{E}_{\text{int}}$ the following quantities:

- an approximation of the density on the dual cell ρ_{D_σ} defined as:

$$(9) \quad |D_\sigma| \rho_{D_\sigma} = |D_{K,\sigma}| \rho_K + |D_{L,\sigma}| \rho_L, \quad \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L,$$

- a discrete divergence for the convection on the dual cell D_σ :

$$\text{div}(\rho v_i \mathbf{u})_\sigma = \sum_{\varepsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\varepsilon}(\rho, \mathbf{u}) v_{i,\varepsilon}, \quad \sigma \in \mathcal{E}_{\text{int}}^{(i)}, 1 \leq i \leq d.$$

For $i \in \{1, \dots, d\}$, and $\sigma \in \mathcal{E}_{\text{int}}^{(i)}$, $\sigma = K|L$,

- If the vector $\mathbf{e}^{(i)}$ is normal to ε , ε is included in a primal cell K , and we denote by σ' the second face of K which, in addition to σ , is normal to $\mathbf{e}^{(i)}$. We thus have $\varepsilon = D_\sigma|D_{\sigma'}$. Then the mass flux through ε is given by:

$$(10) \quad F_{\sigma,\varepsilon}(\rho, \mathbf{u}) = \frac{1}{2} (F_{K,\sigma}(\rho, \mathbf{u}) \mathbf{n}_{D_\sigma,\varepsilon} \cdot \mathbf{n}_{K,\sigma} + F_{K,\sigma'}(\rho, \mathbf{u}) \mathbf{n}_{D_\sigma,\varepsilon} \cdot \mathbf{n}_{K,\sigma'}).$$

- If the vector $\mathbf{e}^{(i)}$ is tangent to ε , ε is the union of the halves of two primal faces τ and τ' such that $\tau \in \mathcal{E}(K)$ and $\tau' \in \mathcal{E}(L)$. The mass flux through ε is then given by:

$$(11) \quad F_{\sigma,\varepsilon}(\rho, \mathbf{u}) = \frac{1}{2} (F_{K,\tau}(\rho, \mathbf{u}) + F_{L,\tau'}(\rho, \mathbf{u})).$$

With this definition, the dual fluxes are locally conservative through dual faces $\varepsilon = D_\sigma|D_{\sigma'}$ (*i.e.* $F_{\sigma,\varepsilon}(\rho, \mathbf{u}) = -F_{\sigma',\varepsilon}(\rho, \mathbf{u})$), and vanish through a dual face included in the boundary of Ω . For this reason, the values $v_{\varepsilon,i}$ are only needed at the internal dual faces, and are chosen centered, *i.e.*, for $\varepsilon = D_\sigma|D_{\sigma'}$, $v_{\varepsilon,i} = (v_{\sigma,i} + v_{\sigma',i})/2$.

As a result, a finite volume discretization of the mass balance (1a) holds over the internal dual cells. Indeed, if $\rho^{n+1} = \{\rho_K^{n+1}, K \in \mathcal{M}\}$, $\rho^n = \{\rho_K^n, K \in \mathcal{M}\}$ and $\mathbf{u}^{n+1} = \{u_{\sigma,i}^{n+1}, \sigma \in \mathcal{E}^{(i)}, 1 \leq i \leq d\}$ are density and velocity fields satisfying (4d), then, the dual quantities $\{\rho_{D_\sigma}^{n+1}, \rho_{D_\sigma}^n, \sigma \in \mathcal{E}_{\text{int}}\}$ and the dual fluxes

$\{F_{\sigma,\varepsilon}(\rho^{n+1}, \mathbf{u}^{n+1}), \sigma \in \mathcal{E}_{\text{int}}, \varepsilon \in \bar{\mathcal{E}}(D_\sigma)\}$ satisfy a finite volume discretization of the mass balance (1a) over the internal dual cells:

$$(12) \quad \frac{|D_\sigma|}{\delta t}(\rho_{D_\sigma}^{n+1} - \rho_{D_\sigma}^n) + \sum_{\varepsilon \in \bar{\mathcal{E}}(D_\sigma)} F_{\sigma,\varepsilon}(\rho^{n+1}, \mathbf{u}^{n+1}) = 0, \quad \sigma \in \mathcal{E}_{\text{int}}.$$

Diffusion term – The discrete diffusion term in (4b) is defined in [2] and is coercive in the following sense: for every discrete velocity field \mathbf{u} satisfying the homogeneous Dirichlet boundary conditions, one has:

$$(13) \quad - \sum_{i=1}^d \sum_{\varepsilon \in \mathcal{E}_{\text{int}}^{(i)}} |D_\sigma| u_{\sigma,i} \operatorname{div} \boldsymbol{\tau}(\mathbf{u})_{\sigma,i} \geq \mu \|\mathbf{u}\|_{1,\mathcal{M}}^2.$$

The initialization of the scheme (4) is performed by setting

$$\forall K \in \mathcal{M}, \rho_K^0 = \frac{1}{|K|} \int_K \rho_0^\varepsilon(\mathbf{x}) \, d\mathbf{x} \text{ and } \forall \sigma \in \mathcal{E}_{\text{int}}^{(i)}, 1 \leq i \leq d, u_{\sigma,i}^0 = \frac{1}{|\sigma|} \int_\sigma \mathbf{u}_0^\varepsilon(\mathbf{x}) \cdot \mathbf{e}^{(i)} \, d\mathbf{x},$$

and computing ρ^{-1} by solving the backward mass balance equation (4d) for $n = -1$ where the unknown is ρ^{-1} and not ρ^0 . This allows to perform the first prediction step with $\{\rho_{D_\sigma}^0, \rho_{D_\sigma}^{-1}, \sigma \in \mathcal{E}_{\text{int}}\}$ and the dual mass fluxes $\{F_{\sigma,\varepsilon}(\rho^0, \mathbf{u}^0), \sigma \in \mathcal{E}_{\text{int}}, \varepsilon \in \bar{\mathcal{E}}(D_\sigma)\}$ satisfying the mass balance (12). Moreover, since $\rho_0^\varepsilon > 0$, one clearly has $\rho_K^0 > 0$ for all $K \in \mathcal{M}$ and therefore $\rho_{D_\sigma}^0 > 0$ for all $\sigma \in \mathcal{E}_{\text{int}}$. The positivity of ρ^{-1} is a consequence of the following Lemma.

Lemma 2.1. *If $(\rho_0^\varepsilon, \mathbf{u}_0^\varepsilon)$ satisfies (3), then there exists C , depending on the mesh but independent of ε such that:*

$$(14) \quad \frac{1}{\varepsilon^2} \max_{K \in \mathcal{M}} |\rho_K^0 - 1| + \frac{1}{\varepsilon^2} \max_{1 \leq i \leq d} \max_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} |(\nabla p)_{\sigma,i}^0| + \frac{1}{\varepsilon} \max_{K \in \mathcal{M}} |\rho_K^{-1} - 1| \leq C.$$

Proof. We sketch the proof. The boundedness of the first two terms is a straightforward consequence of (3). For the third term we remark that, again by (3):

$$\forall K \in \mathcal{M}, \quad \rho_K^{-1} - 1 = \underbrace{\rho_K^0 - 1}_{=\mathcal{O}(\varepsilon^2)} + \underbrace{\delta t \rho_K^0 (\operatorname{div} \mathbf{u}^0)_K}_{=\mathcal{O}(\varepsilon)} + \underbrace{\delta t \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} (\rho_\sigma^0 - \rho_K^0) \mathbf{u}_{K,\sigma}^0}_{=\mathcal{O}(\varepsilon^2)}.$$

□

3. ASYMPTOTIC ANALYSIS OF THE ZERO MACH LIMIT

By the results of [3], there exists a solution $(\rho^n, \mathbf{u}^n)_{0 \leq n \leq N}$ to the scheme (4) and any solution satisfies the following relations:

- a discrete kinetic energy balance: for all $\sigma \in \mathcal{E}_{\text{int}}^{(i)}, 1 \leq i \leq d, 0 \leq n \leq N - 1$:

$$(15) \quad \frac{1}{2\delta t} \left(\rho_{D_\sigma}^n |u_{\sigma,i}^{n+1}|^2 - \rho_{D_\sigma}^{n-1} |u_{\sigma,i}^n|^2 \right) + \frac{1}{2|D_\sigma|} \sum_{\substack{\varepsilon \in \bar{\mathcal{E}}(D_\sigma) \\ \varepsilon = D_\sigma | D_{\sigma'}}} F_{\sigma,\varepsilon}(\rho^n, \mathbf{u}^n) \tilde{u}_{\sigma,i}^{n+1} \tilde{u}_{\sigma',i}^{n+1} \\ - \operatorname{div} \boldsymbol{\tau}(\tilde{\mathbf{u}}^{n+1})_{\sigma,i} \tilde{u}_{\sigma,i}^{n+1} + \frac{1}{\varepsilon^2} (\nabla p)_{\sigma,i}^{n+1} u_{\sigma,i}^{n+1} + \frac{\delta t}{\varepsilon^4} \left(\frac{|(\nabla p)_{\sigma,i}^{n+1}|^2}{2 \rho_{D_\sigma}^n} - \frac{|(\nabla p)_{\sigma,i}^n|^2}{2 \rho_{D_\sigma}^{n-1}} \right) \\ + R_{\sigma,i}^{n+1} = 0, \quad \text{with } R_{\sigma,i}^{n+1} = \frac{1}{2\delta t} \rho_{D_\sigma}^{n-1} (\tilde{u}_{\sigma,i}^{n+1} - u_{\sigma,i}^n)^2.$$

- a discrete renormalization identity: for all $K \in \mathcal{M}, 0 \leq n \leq N - 1$:

$$(16) \quad \frac{1}{\delta t} \left(\Pi_\gamma(\rho_K^{n+1}) - \Pi_\gamma(\rho_K^n) \right) + \operatorname{div} (b_\gamma(\rho^{n+1}) \mathbf{u}^{n+1} - b'_\gamma(1) \rho^{n+1} \mathbf{u}^{n+1})_K + p_K^{n+1} \operatorname{div}(\mathbf{u}^{n+1})_K + R_K^{n+1} = 0,$$

with $R_K^{n+1} \geq 0$, where the function b_γ is defined by $b_\gamma(\rho) = \rho \log \rho$ if $\gamma = 1$, $b_\gamma(\rho) = \rho^\gamma / (\gamma - 1)$ if $\gamma > 1$ and satisfies $\rho b'_\gamma(\rho) - b_\gamma(\rho) = \rho^\gamma = \wp(\rho)$ for all $\rho > 0$, and $\Pi_\gamma(\rho) = b_\gamma(\rho) - b_\gamma(1) - b'_\gamma(1)(\rho - 1)$.

Summing (15) and (16) over the primal cells from one side, and over the dual cells and the components on the other side, and invoking the grad-div duality relation (8), we obtain a local-in-time discrete entropy inequality, for $0 \leq n \leq N-1$:

$$(17) \quad \frac{1}{2} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} |D_\sigma| \left(\rho_{D_\sigma}^n |u_{\sigma,i}^{n+1}|^2 - \rho_{D_\sigma}^{n-1} |u_{\sigma,i}^n|^2 \right) + \frac{1}{\varepsilon^2} \sum_{K \in \mathcal{M}} |K| \left(\Pi_\gamma(\rho_K^{n+1}) - \Pi_\gamma(\rho_K^n) \right) \\ + \mu \delta t \|\tilde{\mathbf{u}}^{n+1}\|_{1,\mathcal{M}}^2 + \frac{1}{\varepsilon^4} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} |D_\sigma| \delta t^2 \left(\frac{|(\nabla p)_{\sigma,i}^{n+1}|^2}{2 \rho_{D_\sigma}^n} - \frac{|(\nabla p)_{\sigma,i}^n|^2}{2 \rho_{D_\sigma}^{n-1}} \right) + \mathcal{R}^{n+1} \leq 0$$

where $\mathcal{R}^{n+1} = \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} R_{\sigma,i}^{n+1} + \frac{1}{\varepsilon^2} \sum_{K \in \mathcal{M}} R_K^{n+1} \geq 0$.

The function Π_γ has some important properties:

- (18a) • For all $\gamma \geq 1$ there exists C_γ such that: $\Pi_\gamma(\rho) \leq C_\gamma |\rho - 1|^2$, $\forall \rho \in (0, 2)$.
(18b) • If $\gamma \geq 2$ then $\Pi_\gamma(\rho) \geq |\rho - 1|^2$, $\forall \rho > 0$.
(18c) • If $\gamma \in [1, 2)$ then for all $R \in (2, +\infty)$, there exists $C_{\gamma,R}$ such that:
 $\Pi_\gamma(\rho) \geq C_{\gamma,R} |\rho - 1|^2$, $\forall \rho \in (0, R)$,
 $\Pi_\gamma(\rho) \geq C_{\gamma,R} |\rho - 1|^\gamma$, $\forall \rho \in [R, \infty)$.

Lemma 3.1 (Global discrete entropy inequality). *Under assumption (3), there exists $C_0 > 0$ independent of ε such that the solution $(\rho^n, \mathbf{u}^n)_{0 \leq n \leq N}$ to the scheme (4) satisfies, for ε small enough, and for $1 \leq n \leq N$:*

$$(19) \quad \frac{1}{2} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} |D_\sigma| \rho_{D_\sigma}^{n-1} |u_{\sigma,i}^n|^2 + \mu \sum_{k=1}^n \delta t \|\tilde{\mathbf{u}}^k\|_{1,\mathcal{M}}^2 \\ + \frac{1}{\varepsilon^2} \sum_{K \in \mathcal{M}} |K| \Pi_\gamma(\rho_K^n) + \frac{1}{\varepsilon^4} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \frac{|D_\sigma| \delta t^2}{2 \rho_{D_\sigma}^{n-1}} |(\nabla p)_{\sigma,i}^n|^2 \leq C_0.$$

Proof. Summing (17) over n yields the inequality (19) with

$$(20) \quad C_0 = \frac{1}{2} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} |D_\sigma| \rho_{D_\sigma}^{-1} |u_{\sigma,i}^0|^2 + \frac{1}{\varepsilon^2} \sum_{K \in \mathcal{M}} |K| \Pi_\gamma(\rho_K^0) + \frac{1}{\varepsilon^4} \sum_{i=1}^d \sum_{\sigma \in \mathcal{E}_{\text{int}}^{(i)}} \frac{|D_\sigma| \delta t^2}{2 \rho_{D_\sigma}^{-1}} |(\nabla p)_{\sigma,i}^0|^2.$$

By (14), for ε small enough, one has $\rho_K^{-1} \leq 2$ for all $K \in \mathcal{M}$ and therefore $\rho_{D_\sigma}^{-1} \leq 2$ for all $\sigma \in \mathcal{E}_{\text{int}}^{(i)}$ and $1 \leq i \leq d$. Hence, since \mathbf{u}_0^ε is uniformly bounded in $H^1(\Omega)^d$ by (3), a classical trace inequality yields the boundedness of the first term. Again by (14), one has $|\rho_K^0 - 1| \leq C\varepsilon^2$ for all $K \in \mathcal{M}$. Hence, by (18a), the second term vanishes as $\varepsilon \rightarrow 0$. The third term is also uniformly bounded with respect to ε thanks to (14). \square

Lemma 3.2 (Control of the pressure). *Assume that $(\rho_0^\varepsilon, \mathbf{u}_0^\varepsilon)$ satisfies (3) and let $(\rho^n, \mathbf{u}^n)_{0 \leq n \leq N}$ satisfy (4). Let $p^n = \varphi(\rho^n)$ and define $\delta p^n = \{\delta p_K^n, K \in \mathcal{M}\}$ where $\delta p_K^n = (p_K^n - |\Omega|^{-1} \int_\Omega p^n \, d\mathbf{x}) / \varepsilon^2$. Then, one has, for all $1 \leq n \leq N$:*

$$\|\delta p^n\| \leq C_{\mathcal{M},\delta t},$$

where $C_{\mathcal{M},\delta t} \geq 0$ depends on the mesh and δt but not on ε , and $\|\cdot\|$ stands for any norm on the space of discrete functions.

Proof. By (19), the discrete pressure gradient is controlled in L^∞ by $C_{\mathcal{M},\delta t} \varepsilon^2$, so that $\nabla(\delta p^n)$ is bounded in any norm independently of ε . Using the discrete $(H^{-1})^d$ -norm (see e.g. [1]), invoking the gradient divergence duality (8) and the *inf-sup* stability of the scheme, $\|\nabla(\delta p^n)\|_{-1,\mathcal{M}} \leq C_{\mathcal{M},\delta t}$ implies that $\|\delta p^n\|_{L^2} \leq \beta^{-1} C_{\mathcal{M},\delta t}$. \square

Theorem 3.3 (Incompressible limit of the MAC pressure correction scheme).

Let $(\varepsilon^{(m)})_{m \in \mathbb{N}}$ be a sequence of positive real numbers tending to zero, and let $(\rho^{(m)}, \mathbf{u}^{(m)})$ be a corresponding sequence of solutions of the scheme (4). Then the sequence $(\rho^{(m)})_{m \in \mathbb{N}}$ converges to the constant function $\rho = 1$ when m tends to $+\infty$ in $L^\infty((0, T), L^q(\Omega))$, for all $q \in [1, \min(\gamma, 2)]$.

In addition, the sequence $(\mathbf{u}^{(m)}, \delta p^{(m)})_{m \in \mathbb{N}}$ tends, in any discrete norm, to the solution $(\mathbf{u}, \delta p)$ of the usual MAC pressure correction scheme for the incompressible Navier-Stokes equations, which reads:

Prediction step – Solve for $\tilde{\mathbf{u}}^{n+1}$:

$$\text{For } 1 \leq i \leq d, \forall \sigma \in \mathcal{E}_{\text{int}}^{(i)}, \quad \frac{1}{\delta t} (\tilde{u}_{\sigma,i}^{n+1} - u_{\sigma,i}^n) + \text{div}(\tilde{u}_i^{n+1} \mathbf{u}^n)_\sigma - \text{div} \boldsymbol{\tau}(\tilde{\mathbf{u}}^{n+1})_{\sigma,i} + (\nabla(\delta p)^n)_{\sigma,i} = 0.$$

Correction step – Solve for $(\delta p)^{n+1}$ and \mathbf{u}^{n+1} :

$$\begin{aligned} \text{For } 1 \leq i \leq d, \forall \sigma \in \mathcal{E}_{\text{int}}^{(i)}, \quad & \frac{1}{\delta t} (u_{\sigma,i}^{n+1} - \tilde{u}_{\sigma,i}^{n+1}) + (\nabla(\delta p)^{n+1})_{\sigma,i} - (\nabla(\delta p)^n)_{\sigma,i} = 0, \\ \forall K \in \mathcal{M}, \quad & \text{div}(\mathbf{u}^{n+1})_K = 0. \end{aligned}$$

Proof. By (18b) and the global entropy estimate (19), one has for $\gamma \geq 2$,

$$\|\rho^{(m)}(t) - 1\|_{L^2(\Omega)}^2 \leq \int_{\Omega} \Pi_\gamma(\rho^{(m)}(t)) \leq C_0 \varepsilon^2, \forall t \in (0, T).$$

For $1 \leq \gamma \leq 2$, invoking (18c) and estimate (19), we obtain for all $t \in (0, T)$ and for all $R \in (2, +\infty)$:

$$\begin{aligned} (i) \quad & \|(\rho^{(m)}(t) - 1) \mathbb{1}_{\{\rho^{(m)}(t) \leq R\}}\|_{L^2(\Omega)}^2 \leq \frac{1}{C_{\gamma,R}} \int_{\Omega} \Pi_\gamma(\rho^{(m)}(t)) \leq C \varepsilon^2, \forall t \in (0, T), \\ (ii) \quad & \|(\rho^{(m)}(t) - 1) \mathbb{1}_{\{\rho^{(m)}(t) \geq R\}}\|_{L^\gamma(\Omega)}^\gamma \leq \frac{1}{C_{\gamma,R}} \int_{\Omega} \Pi_\gamma(\rho^{(m)}(t)) \leq C \varepsilon^2, \forall t \in (0, T), \end{aligned}$$

which proves the convergence of $(\rho^{(m)})_{m \in \mathbb{N}}$ to the constant function $\rho = 1$ as $m \rightarrow +\infty$ in $L^\infty((0, T), L^q(\Omega))$ for all $q \in [1, \min(\gamma, 2)]$. Using again (19), the sequence $(\mathbf{u}^{(m)})_{m \in \mathbb{N}}$ is bounded in any discrete norm and the same holds for the sequence $(\delta p^{(m)})_{m \in \mathbb{N}}$ by Lemma 3.2. By the Bolzano-Weierstrass theorem and a norm equivalence argument, there exists a subsequence of $(\mathbf{u}^{(m)}, \delta p^{(m)})_{m \in \mathbb{N}}$ which tends, in any discrete norm, to a limit $(\mathbf{u}, \delta p)$. Passing to the limit cell-by-cell in (4), one obtains that $(\mathbf{u}, \delta p)$ is a solution to (21). Since this solution is unique, the whole sequence converges, which concludes the proof. \square

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I2M UMR 7373, AIX-MARSEILLE UNIVERSITÉ, CNRS, ÉCOLE CENTRALE DE MARSEILLE.
E-mail address: raphael.e.herbin@univ-amu.fr

INSTITUT DE RADIOPROTECTION ET DE SÛRETÉ NUCLÉAIRE (IRSN), SAINT-PAUL-LEZ-DURANCE, 13115, FRANCE.
E-mail address: jean-claude.latche@irsn.fr

UNIVERSITÉ DE LYON, CNRS UMR 5208, UNIVERSITÉ LYON 1, INSTITUT CAMILLE JORDAN. 43 BD 11 NOVEMBRE 1918;
F-69622 VILLEURBANNE CEDEX, FRANCE.
E-mail address: saleh@math.univ-lyon1.fr