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# On Kronecker's density theorem, primitive points and orbits of matrices

Michel Laurent

**Abstract:** We discuss recent quantitative results in connexion with Kronecker's theorem on the density of subgroups in  $\mathbb{R}^n$  and with Dani and Raghavan's theorem on the density of orbits in the space of frames. We also propose several related problems. The case of the natural linear action of the unimodular group  $SL_2(\mathbb{Z})$  on the real plane is investigated more closely. We then establish an intriguing link between the configuration of (discrete) orbits of primitive points and the rate of density of dense orbits.

## 1 Introduction

Let  $m$  and  $n$  be positive integers and let  $\Theta \in \text{Mat}_{n,m}(\mathbb{R})$  be an  $n \times m$  matrix with real entries. We associate to  $\Theta$  the subgroup

$$\Lambda = \Theta\mathbb{Z}^m + \mathbb{Z}^n \subset \mathbb{R}^n$$

generated over  $\mathbb{Z}$  by the  $m$  columns of  $\Theta$  and by  $\mathbb{Z}^n$  and its subset of primitive points

$$\Lambda_{\text{prim}} = \left\{ \Theta\mathbf{q} + \mathbf{p}; \mathbf{q} \in \mathbb{Z}^m, \mathbf{p} \in \mathbb{Z}^n \quad \text{with} \quad \text{gcd}(\mathbf{q}, \mathbf{p}) = 1 \right\}.$$

Put also

$$X = (\Theta, I_n) \in \text{Mat}_{n,m+n}(\mathbb{R}),$$

where  $I_n$  is the identity matrix in  $\text{Mat}_{n,n}(\mathbb{R})$ , so that we can write

$$\Lambda = X\mathbb{Z}^{m+n} \quad \text{and} \quad \Lambda_{\text{prim}} = XP(\mathbb{Z}^{m+n}).$$

Here and throughout the article,  $P(\mathbb{Z}^d)$  denotes the set of primitive points in  $\mathbb{Z}^d$ , that is the set of integer  $d$ -tuples with coprime coordinates.

Then we can state the following criterion of density:

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**Theorem** (Kronecker, Dani-Raghavan). *The following assertions are equivalent.*

- (i) *The  $n$  rows of the matrix  $\Theta$  are  $\mathbb{Z}$ -linearly independent vectors in  $\mathbb{R}^n/\mathbb{Z}^n$ .*
- (ii) *The subgroup  $\Lambda$  is dense in  $\mathbb{R}^n$ .*
- (iii) *The set  $\Lambda_{\text{prim}}$  is dense in  $\mathbb{R}^n$ .*
- (iv) *The orbit  $X\text{SL}_{m+n}(\mathbb{Z})$  of  $X$  under the action of the unimodular group  $\text{SL}_{m+n}(\mathbb{Z})$  by right matrix multiplication is dense in  $\text{Mat}_{n,m+n}(\mathbb{R})$ .*

The equivalence of properties (i) and (ii) is the content of the classical Kronecker density theorem, see for instance [2], while the equivalence of (i) and (iv) is due to Dani and Raghavan, see Theorem 3.4 in [5]. The implications (iv)  $\Rightarrow$  (iii)  $\Rightarrow$  (ii) are clear; observe that upon identifying an  $n \times (m+n)$  matrix with the  $(m+n)$ -tuple of its columns, we have the inclusions

$$X\text{SL}_{m+n}(\mathbb{Z}) \subseteq (\Lambda_{\text{prim}})^{m+n} \subset (\mathbb{R}^n)^{m+n} \simeq \text{Mat}_{n,m+n}(\mathbb{R}),$$

since the column vectors of any matrix in  $\text{SL}_{m+n}(\mathbb{Z})$  belong to  $P(\mathbb{Z}^{m+n})$ .

Quantitative results of density for the subgroup  $\Lambda$  are well understood. Assuming that the rows of  $\Theta$  satisfy some measure of  $\mathbb{Z}$ -linear independence modulo  $\mathbb{Z}^m$ , we can control by a transference principle the quality of the approximation of any point in  $\mathbb{R}^n$  by elements of  $\Lambda$ . We refer to Chapters III and V of Cassels' monograph [2] for this classical issue of Diophantine approximation, and to [1] for a formulation in terms of exponents of approximation. The inhomogeneous variants [14, 16] of the metrical Khintchine-Groshev theorem may as well be considered as quantitative versions of the assertion (ii) for a generic matrix  $\Theta$ .

The purpose of this article is to exhibit some effective density results in connexion with the assertions (iii) and (iv), both for a given matrix  $\Theta$  and for  $\Theta$  generic. Our knowledge concerning quantitative versions of (iii) and (iv) is much more limited and another goal of the paper is to formulate some related questions, or conjectures, which may lead to further improvements on these two issues. We display in Section 2 a metrical theory involving primitive points, in a generalized meaning, obtaining results refining the Khintchine-Groshev theorem. Regarding (iv), even the generic exponent of Diophantine approximation (the analogue of Dirichlet's exponent  $m/n$  for  $\Lambda$ ) remains unknown for any value of the dimensions  $m$  and  $n$ . We propose in Section 3 a conjectural value for this critical exponent which is motivated by recent works [7, 8] due to Ghosh, Gorodnik and Nevo in a much more general framework. Sections 4 and 5 are devoted to the case  $m = n = 1$ . We develop a conditional approach to reach the expected exponent  $1/2$  based on some hypothesis concerning the repartition in the plane of truncated  $\text{SL}_2(\mathbb{Z})$ -orbits of primitive integer points, a problem which may have an independent interest.

## 2 Metrical theory for $\Lambda_{\text{prim}}$

We have obtained in [3] metrical statements in the style of the Khintchine-Groshev theorem involving primitive points in a refined sense. In this section, we describe and discuss our results.

The set-up is as follows. Let  $\Pi$  be a partition of the set  $\{1, \dots, m+n\} = \coprod_{j=1}^t \pi_j$ . We assume that all components  $\pi_j$  of  $\Pi$  have cardinality at least  $n+1$ . We then define  $P(\Pi)$  to be the set of integer  $(m+n)$ -tuples  $(v_1, \dots, v_{m+n})$  such that  $\gcd(v_i)_{i \in \pi_j} = 1$  for all  $j = 1, \dots, t$ . Note that the trivial partition (the one having only one component) satisfies the assumption and that we have  $P(\Pi) = P(\mathbb{Z}^{m+n})$  in that case. Moreover  $P(\Pi) \subseteq P(\mathbb{Z}^{m+n})$  for any relevant partition  $\Pi$ . Now let  $\psi : \mathbb{N} \mapsto \mathbb{R}^+$  be a function with positive values. We assume that the map  $x \mapsto x^{m-1}\psi(x)^n$  is non-increasing and that the series  $\sum_{\ell \geq 1} \ell^{m-1}\psi(\ell)^n$  diverges; the latter assumption being the necessary and sufficient condition occurring in the statement of the classical Khintchine-Groshev theorem.

In the following two theorems, we implicitly assume that the above hypotheses on  $\Pi$  and  $\psi$  are satisfied. The symbol  $|\cdot|$  indicates the supremum norm in  $\mathbb{R}^n$  and in  $\mathbb{R}^m$ . Our first result is a doubly metrical statement.

**Theorem 1.** *For almost every pair  $(\Theta, \mathbf{y}) \in \text{Mat}_{n,m}(\mathbb{R}) \times \mathbb{R}^n$ , there exist infinitely many points  $(\mathbf{q}, \mathbf{p}) \in P(\Pi)$  such that*

$$|\Theta \mathbf{q} + \mathbf{p} - \mathbf{y}| \leq \psi(|\mathbf{q}|).$$

**Problem 1.** *Fix arbitrarily the target point  $\mathbf{y} \in \mathbb{R}^n$ . Show that the same conclusion holds for almost every  $\Theta \in \text{Mat}_{n,m}(\mathbb{R})$ .*

Notice that the weaker inhomogeneous problem where the coprimality requirement  $(\mathbf{q}, \mathbf{p}) \in P(\Pi)$  has been removed is in fact a well-known result. That is the inhomogeneous version of the Khintchine-Groshev theorem which follows from Theorem 1 of [14] for  $m = 1$  and Theorem 15 in Chapter 1 of [16] when  $m \geq 2$ . We have established in Theorem 2 in [3] that Problem 1 holds true in the homogeneous case  $\mathbf{y} = 0$ . It is also shown in [3] that Theorem 1 is equivalent to the following result concerning the smallness of a generic system of inhomogeneous linear forms:

**Theorem 2.** *For every  $\mathbf{y} \in \mathbb{R}^n$  and for almost every matrix  $X \in \text{Mat}_{n,m+n}(\mathbb{R})$ , there exist infinitely many points  $(\mathbf{q}, \mathbf{p}) \in P(\Pi)$  such that*

$$\left| X \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} - \mathbf{y} \right| \leq \psi(|\mathbf{q}|).$$

The constant term  $\mathbf{y}$  is fixed in Theorem 2, on the contrary to Theorem 1. Finer statements of this kind, but without any coprimality constraint, have been recently

worked out by Dickinson, Fischler, Hussain, Kristensen and Levesley. We refer to [6, 10], and to the references therein, for a discussion of their results.

**Problem 2.** *Suppress, or relax, the (unnecessary ?) assumptions on  $\Pi$  and  $\psi$  occurring in the theorems 1 and 2. Namely, the lower bounds  $\text{card } \pi_j \geq n + 1$  and the monotonicity of the function  $x \mapsto x^{m-1}\psi(x)^n$ .*

It is worth noting that the condition  $\text{card } \pi_j \geq n + 1$  is needed to ensure the ergodicity of the action of some group  $\Gamma_\Pi$  used in the proof of Theorems 1 and 2. More precisely, let us denote by  $\text{SL}_{\pi_j}(\mathbb{Z})$  the subgroup of  $\text{SL}_{m+n}(\mathbb{Z})$  acting as an unimodular matrix on the coordinates with index in  $\pi_j$  and as the identity elsewhere. Let  $\Gamma_\Pi = \prod_{j=1}^t \text{SL}_{\pi_j}(\mathbb{Z})$  be the product of these commuting subgroups. Clearly,  $\Gamma_\Pi$  is a subgroup of  $\text{SL}_{m+n}(\mathbb{Z})$ . Then, the action of  $\Gamma_\Pi$  on  $\text{Mat}_{n,m+n}(\mathbb{R})$  by right matrix multiplication is ergodic, with respect to the invariant Lebesgue measure on  $\text{Mat}_{n,m+n}(\mathbb{R}) \simeq \mathbb{R}^{n(m+n)}$ , if and only if  $\text{card } \pi_j \geq n + 1$  for every index  $j = 1, \dots, t$ . Notice in particular that the action of  $\text{SL}_{m+n}(\mathbb{Z})$  on  $\text{Mat}_{n,m+n}(\mathbb{R})$  is ergodic.

### 3 Exponent of Diophantine approximation of dense orbits

We deal in this section with quantitative versions of the assertion (iv). To that purpose, we introduce the following exponent of Diophantine approximation. Let  $X$  and  $Y$  be two matrices in  $\text{Mat}_{n,m+n}(\mathbb{R})$ . We define  $\mu(X, Y)$  as the supremum of the real numbers  $\mu$  such that there exist infinitely many  $\gamma \in \text{SL}_{m+n}(\mathbb{Z})$  satisfying the inequality

$$(1) \quad |X\gamma - Y| \leq |\gamma|^{-\mu}.$$

Few informations are known concerning the exponent  $\mu(X, Y)$ . In the case  $m = n = 1$ , the following theorem follows from [12], see also [13] for a weaker lower bound in (i). We identify  $\text{Mat}_{1,2}(\mathbb{R})$  with  $\mathbb{R}^2$ .

**Theorem 3.** *Let  $X = (x_1, x_2)$  be a non-zero point in  $\mathbb{R}^2$  with irrational slope  $x_1/x_2$ .*  
(i) *For every point  $Y$  in  $\mathbb{R}^2$ , we have the lower bound  $\mu(X, Y) \geq 1/3$ .*  
(ii) *If  $Y$  is any non-zero point in  $\mathbb{R}^2$  with rational slope (i.e. the ratio of the two coordinates is a rational number), the inequality  $\mu(X, Y) \geq 1/2$  holds, with equality for almost every point  $X$ .*

Singhal [15] has partly extended the results of Theorem 3 to the linear action of the group  $\text{SL}_2(\mathcal{O}_K)$  on  $\mathbb{C}^2$ , where  $\mathcal{O}_K$  stands for the ring of integers of an imaginary quadratic field  $K$  for which a convenient theory of continued fractions is available.

Our knowledge concerning generic values of  $\mu(X, Y)$  is a bit richer, thanks to the recent results of [7, 8]. Observe that the function  $\mu(X, Y)$  is  $\mathrm{SL}_{m+n}(\mathbb{Z}) \times \mathrm{SL}_{m+n}(\mathbb{Z})$ -invariant by componentwise right matrix multiplication. Since the action of  $\mathrm{SL}_{m+n}(\mathbb{Z})$  on  $\mathrm{Mat}_{n, m+n}(\mathbb{R})$  is ergodic, the function  $\mu(X, Y)$  is equal to a constant  $\mu_{m, n}$  (say) almost everywhere. In other words, for a fixed exponent  $\mu < \mu_{m, n}$  and for almost every pair  $(X, Y) \in \mathrm{Mat}_{n, m+n}(\mathbb{R}) \times \mathrm{Mat}_{n, m+n}(\mathbb{R})$ , the inequation (1) has infinitely many solutions  $\gamma \in \mathrm{SL}_{m+n}(\mathbb{Z})$ , while if  $\mu > \mu_{m, n}$  it has only finitely many solutions  $\gamma \in \mathrm{SL}_{m+n}(\mathbb{Z})$  almost surely. The determination of the value of  $\mu_{m, n}$  is an open problem. First, we prove the following upper bound:

**Theorem 4.** *For any integers  $m \geq 1$  and  $n \geq 1$ , we have the inequality*

$$\mu_{m, n} \leq \frac{m(m+n-1)}{n(m+n)}.$$

**Proof.** We follow the proof of the special case  $m = n = 1$  given in Section 5 of [11]. For simplicity, put  $a = m(m+n-1)$  and  $d = n(m+n)$ . Let  $\mu$  be a real number  $> a/d$ . We plan to show that the set

$$\mathcal{E}_\mu = \{(X, Y) \in \mathrm{Mat}_{n, m+n}(\mathbb{R}) \times \mathrm{Mat}_{n, m+n}(\mathbb{R}); \quad \mu(X, Y) > \mu\}$$

has null Lebesgue measure. This will imply that  $\mu_{m, n} \leq \mu$  and next that  $\mu_{m, n} \leq a/d$  by letting  $\mu$  tend to  $a/d$ .

Denote by  $\mathrm{Mat}_{n, m+n}^{\mathrm{rank} \ n}(\mathbb{R})$  the open subset of  $\mathrm{Mat}_{n, m+n}(\mathbb{R})$  consisting of the matrices with maximal rank  $n$ . The set  $\mathrm{Mat}_{n, m+n}^{\mathrm{rank} \ n}(\mathbb{R})$  is clearly stable by right multiplication by  $\mathrm{SL}_{m+n}(\mathbb{Z})$ . Let  $\Omega$  be a compact subset of  $\mathrm{Mat}_{n, m+n}^{\mathrm{rank} \ n}(\mathbb{R})$  with smooth boundary and let  $X \in \mathrm{Mat}_{n, m+n}^{\mathrm{rank} \ n}(\mathbb{R})$  be such that the orbit  $X\mathrm{SL}_{m+n}(\mathbb{Z})$  is dense in  $\mathrm{Mat}_{n, m+n}(\mathbb{R})$ . We claim that the set

$$\mathcal{E}_\mu(X, \Omega) = \{Y \in \Omega; \quad (X, Y) \in \mathcal{E}_\mu\}$$

is a null set. We naturally identify  $\mathrm{Mat}_{n, m+n}(\mathbb{R})$  with  $\mathbb{R}^d$ . For any point  $Y \in \mathrm{Mat}_{n, m+n}(\mathbb{R})$  and any positive real number  $r$ , let us denote by

$$B(Y, r) = \{Z \in \mathrm{Mat}_{n, m+n}(\mathbb{R}); \quad |Z - Y| \leq r\}$$

the closed ball centered at  $Y$  with radius  $r$ , whose Lebesgue measure  $\lambda(B(Y, r))$  is equal to  $(2r)^d$ . Now, let  $Y$  belongs to  $\mathcal{E}_\mu(X, \Omega)$ . This means that there exist infinitely many  $\gamma \in \mathrm{SL}_{m+n}(\mathbb{Z})$  satisfying (1). It follows that  $Y$  belongs to infinitely many balls  $B(X\gamma, |\gamma|^{-\mu})$ . Since  $Y \in \Omega$ , the center  $X\gamma$  belongs to the neighborhood  $\Omega_{|\gamma|^{-\mu}}$  of points whose distance to  $\Omega$  is  $\leq |\gamma|^{-\mu}$ . Therefore, for any  $\eta > 0$ , we have the inclusions

$$\mathcal{E}_\mu(X, \Omega) \subseteq \limsup_{\substack{\gamma \in \mathrm{SL}_{m+n}(\mathbb{Z}) \\ X\gamma \in \Omega_{|\gamma|^{-\mu}}} B(X\gamma, |\gamma|^{-\mu}) \subseteq \limsup_{\substack{\gamma \in \mathrm{SL}_{m+n}(\mathbb{Z}) \\ X\gamma \in \Omega_\eta}} B(X\gamma, |\gamma|^{-\mu}) =: \mathcal{B}_\eta.$$

Fix an  $\eta > 0$  sufficiently small so that  $\Omega_\eta$  remains contained in the open set  $\text{Mat}_{n,m+n}^{\text{rank } n}(\mathbb{R})$ . We now use a fundamental counting result due to Gorodnik. Setting

$$N_R = \text{Card}\{\gamma \in \text{SL}_{m+n}(\mathbb{Z}); \quad X\gamma \in \Omega_\eta, |\gamma| = R\},$$

we have the upper bound

$$(2) \quad N_1 + \cdots + N_R = \text{Card}\{\gamma \in \text{SL}_{m+n}(\mathbb{Z}); \quad X\gamma \in \Omega_\eta, |\gamma| \leq R\} \leq cR^a$$

for some positive constant  $c$  depending only on  $X$  and  $\Omega_\eta$  and any positive integer  $R$ . Indeed, assuming that the orbit  $X\text{SL}_{m+n}(\mathbb{Z})$  is dense in  $\text{Mat}_{n,m+n}(\mathbb{R})$ , it is established in Theorem 3 of [9] that we have an asymptotic equivalence of the form

$$\text{Card}\{\gamma \in \text{SL}_{m+n}(\mathbb{Z}); \quad X\gamma \in \Omega_\eta, |\gamma| \leq R\} \sim \delta R^a$$

as  $R$  tends to infinity, with an explicit formula for the coefficient  $\delta$  depending only on  $X$  and  $\Omega_\eta$ . Using (2), we majorize

$$\begin{aligned} \sum_{\substack{\gamma \in \text{SL}_{m+n}(\mathbb{Z}) \\ |\gamma| \leq R, X\gamma \in \Omega_\eta}} \lambda(B(X\gamma, |\gamma|^{-\mu})) &= 2^d \sum_{k=1}^R \frac{N_k}{k^{d\mu}} \\ &= 2^d \left( \sum_{k=1}^{R-1} (N_1 + \cdots + N_k) \left( \frac{1}{k^{d\mu}} - \frac{1}{(k+1)^{d\mu}} \right) + \frac{N_1 + \cdots + N_R}{R^{d\mu}} \right) \\ &\leq 2^d c \left( \sum_{k=1}^{R-1} k^a \left( \frac{1}{k^{d\mu}} - \frac{1}{(k+1)^{d\mu}} \right) + \frac{R^a}{R^{d\mu}} \right) \\ &\leq 2^d c \mu d \left( \sum_{k=1}^{R-1} k^{a-d\mu-1} \right) + 2^d c R^{a-d\mu}. \end{aligned}$$

We deduce from the above inequality that the sum

$$\sum_{\substack{\gamma \in \text{SL}_{m+n}(\mathbb{Z}) \\ X\gamma \in \Omega_\eta}} \lambda(B(X\gamma, |\gamma|^{-\mu}))$$

converges, since we have assumed that  $\mu > a/d$ . By Borel-Cantelli lemma, the limsup set  $\mathcal{B}_\eta$  has null Lebesgue measure, as well as its subset  $\mathcal{E}_\mu(X, \Omega)$ . We have proved the claim.

Selecting a countable family of compacts  $\Omega$  covering  $\text{Mat}_{n,m+n}^{\text{rank } n}(\mathbb{R})$ , we deduce from the claim that the fiber  $\{Y \in \text{Mat}_{n,m+n}(\mathbb{R}); (X, Y) \in \mathcal{E}_\mu\}$  of  $\mathcal{E}_\mu$  over  $X$  has null Lebesgue measure whenever the orbit  $X\text{SL}_{m+n}(\mathbb{Z})$  is dense. Now, Fubini theorem

yields that  $\mathcal{E}_\mu$  is a null set, since the set of  $X \in \text{Mat}_{n,m+n}(\mathbb{R})$  for which  $X\text{SL}_{m+n}(\mathbb{Z})$  is not dense has null Lebesgue measure by the criterion of Dani and Raghavan.  $\square$

**Remark:** The set  $\text{Mat}_{n,m+n}^{\text{rank } n}(\mathbb{R})$  is an homogeneous space  $H \backslash \text{SL}_{m+n}(\mathbb{R})$ , where  $H$  is the semi-direct product  $H = \text{SL}_m(\mathbb{R}) \ltimes \text{Mat}_{m,n}(\mathbb{R})$  and  $\text{SL}_m(\mathbb{R})$  acts on the  $\mathbb{R}$ -vector space  $\text{Mat}_{m,n}(\mathbb{R})$  by left matrix multiplication. One can also prove Theorem 4 as part of deep results obtained by Ghosh, Gorodnik and Nevo in [7, 8]. They give estimates of generic exponents of Diophantine approximation associated to the action of lattice orbits on homogeneous spaces. In fact, the above upper bound corresponds to the easiest part of their results. See Section 3.1 of [8] and Section 4.1 of [7] for more details in the case  $n = 1$ . For comparison, notice however that the exponent  $\kappa$  occurring in [7, 8] is a *uniform* exponent, according to the terminology of [1], while our exponent  $\mu$  is an ordinary one. Following [12], let us define the uniform variant  $\hat{\mu}(X, Y)$  of the exponent  $\mu(X, Y)$  as the supremum of the real numbers  $\mu$  such that for any large real number  $R$ , there exists  $\gamma \in \text{SL}_{m+n}(\mathbb{Z})$  satisfying

$$|\gamma| \leq R \quad \text{and} \quad |X\gamma - Y| \leq R^{-\mu}.$$

Obviously  $\hat{\mu}(X, Y) \leq \mu(X, Y)$  and it is expected that  $\hat{\mu}(X, Y)$  and  $\mu(X, Y)$  have the same generic value  $\mu_{m,n}$ . The exponent  $\kappa(X, Y)$  studied in [7, 8] is the infimum of the real numbers  $\kappa$  such the inequalities

$$|\gamma| \leq R^\kappa \quad \text{and} \quad |X\gamma^{-1} - Y| \leq R^{-1}.$$

have a solution  $\gamma \in \text{SL}_{m+n}(\mathbb{Z})$  for any large real number  $R$ . It is now expected that  $\kappa(X, Y)$  has the generic value  $1/\mu_{m,n}$ .

**Problem 3.** *Show that the formula*

$$(3) \quad \mu_{m,n} = \frac{m(m+n-1)}{n(m+n)}$$

*holds for every (at least one) pair of positive integers  $m$  and  $n$ .*

In view of the proof of Theorem 4, the conjecture may be considered as an optimistic version of the box principle. Fix a compact subset  $\Omega \subset \text{Mat}_{n,m+n}^{\text{rank } n}(\mathbb{R})$ . For large values of the radius  $R$ , the set  $\{\gamma \in \text{SL}_{m+n}(\mathbb{Z}); X\gamma \in \Omega, |\gamma| \leq R\}$  has cardinality  $\sim \delta R^a$  whenever the orbit  $X\text{SL}_{m+n}(\mathbb{Z})$  is dense. It is expected that the images  $X\gamma$  are well distributed in  $\Omega$  when  $\gamma$  ranges over this set, at least for a generic matrix  $X$ . However, Theorem 3 only yields the lower bound  $\mu_{1,1} \geq 1/3$  and it follows from Proposition 4.1 of [7] that  $\mu_{m,1} \geq m/(m+1)$  when  $m \geq 2$ . Speculating further on the validity of formula (3), we shall prove in Section 5 that  $\mu_{1,1}$  has the expected value  $1/2$ , assuming some strong properties on the distribution of truncated  $\text{SL}_2(\mathbb{Z})$ -orbits of primitive points in the plane, which remain unproved but seem plausible. This is the topic of the next section.

## 4 Discrete orbits in the plane

The set of primitive points  $P(\mathbb{Z}^2)$  is an  $\mathrm{SL}_2(\mathbb{Z})$ -orbit in the plane. In other words, we have  $X\mathrm{SL}_2(\mathbb{Z}) = P(\mathbb{Z}^2)$  for any  $X \in P(\mathbb{Z}^2)$ . We are interested in the repartition of the finite set of primitive points  $X\gamma$  when  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  ranges over a ball  $|\gamma| \leq R$ . Let us first recall a result due to Erdős [4] on the distribution of primitive points in the real plane:

**Theorem** (Erdős).

(i) For any  $Y$  in  $\mathbb{R}^2$  with  $|Y|$  large enough, there exists  $Z \in P(\mathbb{Z}^2)$  such that

$$|Z - Y| \leq \frac{\log |Y|}{\log \log |Y|}.$$

(ii) There exists  $Y$  in  $\mathbb{R}^2$  with  $|Y|$  arbitrarily large such that we have the lower bound

$$|Z - Y| \geq \frac{1}{2} \left( \frac{\log |Y|}{\log \log |Y|} \right)^{1/2},$$

for any  $Z \in P(\mathbb{Z}^2)$ .

Let  $Y = (y_1, y_2)$  be the coordinates of  $Y$ . Erdős' results are in fact expressed differently in term of the quantity  $\min(|y_1|, |y_2|)$  rather than  $|Y|$ . The above formulation involving the norm  $|Y| = \max(|y_1|, |y_2|)$  is a straightforward consequence of the statements (2) and (3) of [4] and their proof. An interesting, but difficult, open question is to sharpen (i) and/or (ii):

**Problem 4.** Determine the smallest possible upper bound in the assertion (i) of Erdős Theorem as a function of the norm  $|Y|$ .

We now come to the study of the orbit  $X\mathrm{SL}_2(\mathbb{Z})$  where  $X$  is a given primitive point. For each point  $Z \in P(\mathbb{Z}^2)$ , we choose a matrix  $\mathbf{Z} \in \mathrm{SL}_2(\mathbb{Z})$  whose first row is equal to  $Z$  having norm  $|\mathbf{Z}| = |Z|$ . Using Bézout, it is easily seen that there are two such matrices  $\mathbf{Z}$ , unless  $Z = (\pm 1, 0)$  or  $Z = (0, \pm 1)$  in which cases there are three. Then, we can obviously reformulate (i) in the matricial form

$$|(1, 0)\mathbf{Z} - Y| \leq \frac{\log |Y|}{\log \log |Y|}.$$

Writing  $(1, 0) = X\mathbf{X}^{-1}$ , we replace in the above inequality the base point  $(1, 0)$  by an arbitrary primitive point  $X \in P(\mathbb{Z}^2)$ . Setting  $\gamma = \mathbf{X}^{-1}\mathbf{Z} \in \mathrm{SL}_2(\mathbb{Z})$  and  $R = 3|X||Y|$ , we thus obtain the estimates

$$(4) \quad |\gamma| \leq 2|X| \left( |Y| + \frac{\log |Y|}{\log \log |Y|} \right) \leq R \quad \text{and} \quad |X\gamma - Y| \leq \frac{\log R}{\log \log R},$$

when  $|Y|$  is large enough. We wish to find analogous estimates for smaller values of  $R$ , enlarging possibly the upper bound  $\log R/\log \log R$  for the distance between  $X\gamma$  and  $Y$ . More precisely we propose the following

**Problem 5.** *Let  $\varepsilon$  be a positive real number. Show that for any large real number  $R$ , any primitive point  $X \in P(\mathbb{Z}^2)$  and any real point  $Y \in \mathbb{R}^2$  with norm  $|Y| \leq (|X|R)^{\frac{1}{1+\varepsilon}}$ , there exists  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  such that*

$$(5) \quad |\gamma| \leq R \quad \text{and} \quad |X\gamma - Y| \leq |X|^{1+\varepsilon} R^\varepsilon.$$

The above hypothesis may be viewed as a property of uniform repartition of the truncated orbit  $\{X\gamma; \gamma \in \mathrm{SL}_2(\mathbb{Z}), |\gamma| \leq R\}$  in the following way. The number of  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  whose norm  $|\gamma|$  is  $\leq R$  is  $\asymp R^2$ , while  $|X\gamma| \leq 2|X||\gamma| \leq 2|X|R$ . If we divide the square  $\{Y \in \mathbb{R}^2; |Y| \leq 2|X|R\}$  into  $\asymp R^2$  small squares with side  $\asymp |X|$ , each small square should ideally contain a point of the set  $\{X\gamma; \gamma \in \mathrm{SL}_2(\mathbb{Z}), |\gamma| \leq R\}$ . On the other hand, the correspondence  $\gamma \mapsto X\gamma$  is not one to one, but the objection is irrelevant here. Indeed, putting  $Z = X\gamma \in P(\mathbb{Z}^2)$ , we can express  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  in the form

$$\gamma = \begin{pmatrix} x'_2 z_1 - x_2 z'_1 - u x_2 z_1 & x'_2 z_2 - x_2 z'_2 - u x_2 z_2 \\ -x'_1 z_1 + x_1 z'_1 + u x_1 z_1 & -x'_1 z_2 + x_1 z'_2 + u x_1 z_2 \end{pmatrix}$$

for some  $u \in \mathbb{Z}$ , where

$$\mathbf{X} = \begin{pmatrix} x_1 & x_2 \\ x'_1 & x'_2 \end{pmatrix} \quad \text{and} \quad \mathbf{Z} = \begin{pmatrix} z_1 & z_2 \\ z'_1 & z'_2 \end{pmatrix}$$

are the respective liftings of  $X$  and  $Z$  in  $\mathrm{SL}_2(\mathbb{Z})$  previously considered. It follows that there exist at most  $2R/(|X||Z|) + 1$  matrices  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  for which  $|\gamma| \leq R$  and  $X\gamma = Z$ . The number of such matrices  $\gamma$  is thus bounded by 3 when  $R \leq |X||Z|$ . Now, the number of  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  for which  $|\gamma| \leq R$  and  $|X\gamma| \leq R/|X|$  is at most

$$\sum_{\substack{Z \in P(\mathbb{Z}^2) \\ |Z| \leq R/|X|}} \frac{2R}{|X||Z|} + 1 \leq \frac{3R}{|X|} \sum_{\substack{Z \in P(\mathbb{Z}^2) \\ |Z| \leq R/|X|}} \frac{1}{|Z|} = \frac{3R}{|X|} \sum_{k=1}^{\lfloor R/|X| \rfloor} \frac{8\varphi(k)}{k} \leq \frac{24R^2}{|X|^2}.$$

We easily deduce from the preceding considerations that the cardinality of the set of points  $\{X\gamma; \gamma \in \mathrm{SL}_2(\mathbb{Z}), |\gamma| \leq R\}$  is  $\asymp R^2$  when  $R$  is sufficiently large independently of  $X$ , where the two multiplicative coefficients implicitly involved in the symbol  $\asymp$  are absolute constants.

Notice that we have relaxed somehow the constraints by assuming that

$$|Y| \leq (|X|R)^{1/(1+\varepsilon)} \leq 2|X|R$$

and requiring only the weaker upper bound  $|X\gamma - Y| \leq |X|^{1+\varepsilon}R^\varepsilon$ , because these estimates are sufficient for our purpose. Of course, sharper estimates may reveal useful as well.

**Remark.** The finite set  $\{X\gamma; \gamma \in \mathrm{SL}_2(\mathbb{Z}), |\gamma| \leq R\}$  does not look like a grid of step  $|X|$  in  $\mathbb{R}^2$ , as one might believe at first glance. Any point  $Y \in P(\mathbb{Z}^2)$  with norm  $|Y| \leq R/(2|X|)$  belongs to this set (take  $\gamma = \mathbf{X}^{-1}\mathbf{Y}$  as in (4)). Therefore, we have a concentration of points around the origin when  $R \gg |X|$ . However, the analogy with a grid remains meaningful inside some annulus centered at the origin.

## 5 A corollary

We relate the distribution of truncated integral orbits to the rate of density of dense orbits, showing that a positive answer to Problem 5 implies that  $\mu_{1,1} = 1/2$ .

We say that a real number  $y$  is *very well approximated by rationals* if for some exponent  $\omega > 2$ , the inequation  $|y - p/q| \leq q^{-\omega}$  has infinitely many rational solutions  $p/q$ . An irrational number  $y$  is not very well approximated by rationals if and only if the sequence  $(s_j)_{j \geq 0}$  of denominators of its convergents satisfies the asymptotic growth condition  $s_{j+1} \leq s_j^{1+\varepsilon}$  for every  $\varepsilon > 0$ . It is well-known that almost every real number  $y$  is not very well approximated by rationals. It then suffices to prove the

**Proposition.** *Let  $X = (x_1, x_2)$  and  $Y = (y_1, y_2)$  be two points in  $\mathbb{R}^2$ . We suppose that the point  $Y$  does not belong to the orbit  $X\mathrm{SL}_2(\mathbb{Z})$ , that the slope  $x_1/x_2$  of  $X$  is an irrational number and that the slope  $y_1/y_2$  of  $Y$  is an irrational number which is not very well approximated by rationals. Assume that Problem 5 has been affirmatively resolved for any  $\varepsilon > 0$ . Then  $\mu(X, Y) \geq 1/2$ .*

**Proof.** We have to show that for any  $\eta > 0$ , there exist infinitely many  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  such that

$$(6) \quad |X\gamma - Y| \leq |\gamma|^{-1/2+\eta}.$$

The proof is a variant of that of Theorem 1 in [12]: we replace the unipotent matrix of the form  $\begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix}$  occurring there by a matrix  $G$  whose transposed matrix is a solution to Problem 5.

Put  $\xi = x_1/x_2$  and  $y = y_1/y_2$ . Let  $(p_k/q_k)_{k \geq 0}$  and  $(t_j/s_j)_{j \geq 0}$  be the sequences of convergents of  $\xi$  and  $y$  respectively and set

$$M_k = \begin{pmatrix} q_k & q_{k-1} \\ -p_k & -p_{k-1} \end{pmatrix}, \quad N_j = \begin{pmatrix} t_j & s_j \\ t_{j-1} & s_{j-1} \end{pmatrix}.$$

The indices  $k$  and  $j$  will be restricted to odd integers, so that both  $M_k$  and  $N_j$  belong to  $\mathrm{SL}_2(\mathbb{Z})$ . We construct  $\gamma$  in the form

$$\gamma = M_k G N_j$$

where

$$G = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \quad \text{and} \quad |G| \leq R.$$

The indices  $j, k$  and the norm upper bound  $R$  will be chosen later.

Put  $\epsilon_n = q_n \xi - p_n$  and recall the estimate

$$\frac{1}{2q_{n+1}} \leq |\epsilon_n| \leq \frac{1}{q_{n+1}}$$

derived from the theory of continued fractions. Our starting point is the identity  $q_k \epsilon_{k-1} - q_{k-1} \epsilon_k = 1$ . Multiplying by  $y_2/x_2$ , we find

$$(7) \quad \frac{y_2}{x_2} = \left(q_k \frac{y_2}{x_2}\right) \epsilon_{k-1} + \left(-q_{k-1} \frac{y_2}{x_2}\right) \epsilon_k$$

and on the other hand, we have the obvious equality

$$(8) \quad 0 = \left(q_k \frac{y_2}{x_2}\right) q_{k-1} + \left(-q_{k-1} \frac{y_2}{x_2}\right) q_k.$$

Now write

$$X\gamma - Y = x_2(\xi, 1)M_k G N_j - Y = x_2(\epsilon_k, \epsilon_{k-1})G N_j - Y = (z_1, z_2)$$

with

$$(9) \quad \begin{aligned} z_1 &= x_2 \left( (at_j + bt_{j-1})\epsilon_k + (ct_j + dt_{j-1})\epsilon_{k-1} \right) - y_1 \\ z_2 &= x_2 \left( (as_j + bs_{j-1})\epsilon_k + (cs_j + ds_{j-1})\epsilon_{k-1} \right) - y_2 \\ &= x_2 \left( \left( as_j + bs_{j-1} + q_{k-1} \frac{y_2}{x_2} \right) \epsilon_k + \left( cs_j + ds_{j-1} - q_k \frac{y_2}{x_2} \right) \epsilon_{k-1} \right), \end{aligned}$$

using (7) for the last equality. Put

$$\Delta = \max \left( \left| as_j + bs_{j-1} + q_{k-1} \frac{y_2}{x_2} \right|, \left| cs_j + ds_{j-1} - q_k \frac{y_2}{x_2} \right| \right).$$

We immediately deduce from (9) that

$$(10) \quad |z_2| \leq 2|x_2| \frac{\Delta}{q_k}.$$

Using again the expressions (9), observe now that

$$(11) \quad |z_1 - yz_2| = |x_2| \left| (t_j - s_j y)(a\epsilon_k + c\epsilon_{k-1}) + (t_{j-1} - s_{j-1} y)(b\epsilon_k + d\epsilon_{k-1}) \right| \leq 4|x_2| \frac{R}{s_j q_k}.$$

Combining (10) and (11), we deduce from the triangle inequality the upper bound

$$(12) \quad |X\gamma - Y| = \max(|z_1|, |z_2|) \ll \frac{\Delta}{q_k} + \frac{R}{s_j q_k}.$$

We now bound the norm of  $\gamma = M_k G N_j$  and claim that

$$(13) \quad |\gamma| \ll \Delta q_k + \frac{s_j R}{q_k} + \frac{q_k R}{s_j}.$$

Expand the product

$$\gamma = \begin{pmatrix} q_k & q_{k-1} \\ -p_k & -p_{k-1} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t_j & s_j \\ t_{j-1} & s_{j-1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where we have set

$$\begin{aligned} A &= q_k(at_j + bt_{j-1}) + q_{k-1}(ct_j + dt_{j-1}), \\ B &= q_k(as_j + bs_{j-1}) + q_{k-1}(cs_j + ds_{j-1}), \\ C &= -p_k(at_j + bt_{j-1}) - p_{k-1}(ct_j + dt_{j-1}), \\ D &= -p_k(as_j + bs_{j-1}) - p_{k-1}(cs_j + ds_{j-1}). \end{aligned}$$

Since

$$(t_j, t_{j-1}) = y(s_j, s_{j-1}) + \mathcal{O}\left(\frac{1}{s_j}\right) \quad \text{and} \quad (p_k, p_{k-1}) = \xi(q_k, q_{k-1}) + \mathcal{O}\left(\frac{1}{q_k}\right),$$

we have the estimates

$$A = yB + \mathcal{O}\left(\frac{Rq_k}{s_j}\right), \quad D = -\xi B + \mathcal{O}\left(\frac{Rs_j}{q_k}\right), \quad C = -\xi yB + \mathcal{O}\left(\frac{Rq_k}{s_j}\right) + \mathcal{O}\left(\frac{Rs_j}{q_k}\right).$$

Using (8), we can express  $B$  in the form

$$B = q_k \left( as_j + bs_{j-1} + q_{k-1} \frac{y_2}{x_2} \right) + q_{k-1} \left( cs_j + ds_{j-1} - q_k \frac{y_2}{x_2} \right)$$

which yields the bound  $|B| \leq 2q_k \Delta$ . The inequality (13) is thus established.

At this stage, we use the (conjectural) estimate (5) of the preceding section applied to the pair of points

$$(s_j, s_{j-1}) \in P(\mathbb{Z}^2) \quad \text{and} \quad \left(-q_{k-1} \frac{y_2}{x_2}, q_k \frac{y_2}{x_2}\right) \in \mathbb{R}^2$$

in order to majorize  $\Delta$  in a non-trivial way. Fix arbitrarily  $\varepsilon > 0$  and set

$$(14) \quad R = \frac{(|y_2|q_k/|x_2|)^{1+\varepsilon}}{s_j}.$$

The condition

$$\frac{q_k|y_2|}{|x_2|} = \left| \left(-q_{k-1} \frac{y_2}{x_2}, q_k \frac{y_2}{x_2}\right) \right| \leq (|(s_j, s_{j-1})|R)^{\frac{1}{1+\varepsilon}} = (s_j R)^{\frac{1}{1+\varepsilon}}$$

occurring in Problem 5 is obviously satisfied. It then follows from (5) that there exists a matrix  $G = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$  with norm  $|G| \leq R$  such that

$$(15) \quad \Delta = \max \left( \left| a s_j + b s_{j-1} + q_{k-1} \frac{y_2}{x_2} \right|, \left| c s_j + d s_{j-1} - q_k \frac{y_2}{x_2} \right| \right) \leq s_j^{1+\varepsilon} R^\varepsilon.$$

Now fix an odd index  $k$  and choose  $j$  odd so that  $s_j$  should be located in the interval

$$(16) \quad q_k^{1/3} \leq s_j \leq q_k^{1/3+\varepsilon}.$$

This is possible when  $k$  is large enough, since we have assumed that  $y$  is not very well approximated by rationals. Then, combining the estimates (12) to (16) easily yields the upper bounds

$$|X\gamma - Y| \ll q_k^{-2/3+2\varepsilon+2\varepsilon^2} \quad \text{and} \quad |\gamma| \ll q_k^{4/3+2\varepsilon+2\varepsilon^2}.$$

It follows that

$$|X\gamma - Y| \ll q_k^{-2/3+2\varepsilon+2\varepsilon^2} \ll |\gamma|^{-1/2+\eta},$$

where  $\eta = 9(\varepsilon + \varepsilon^2)/(4 + 6\varepsilon + 6\varepsilon^2)$  is arbitrarily small when  $\varepsilon$  is sufficiently small. We have thus established (6).

The above construction produces infinitely many solutions  $\gamma$  to (6) when  $k$  ranges over the odd integers, because the distance  $|X\gamma - Y|$  tends to 0 as  $k$  tends to infinity, while remaining positive (recall that we have assumed that  $Y$  does not belong to the orbit  $X\text{SL}_2(\mathbb{Z})$ ). The proposition is proved.  $\square$

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