Simultaneous global exact controllability in projection

Alessandro Duca

Simultaneous global exact controllability in projection

Alessandro Duca

Laboratoire de Mathématiques de Besançon, Université Bourgogne Franche-Comté
16, Route de Gray, 25000 Besançon, France
alessandro.duca@univ-fcomte.fr
Dipartimento di Matematica Giuseppe Peano, Università degli Studi di Torino
10, Via Carlo Alberto, 10123 Torino, Italy
aduca@unito.it
Dipartimento di Scienze Matematiche Giuseppe Luigi Lagrange, Politecnico di Torino
24, Corso Duca degli Abruzzi, 10129 Torino, Italy
alessandro.duca@polito.it
SPHINX team, Inria, 54600 Villers-lès-Nancy, France
alessandro.duca@inria.fr

ORCID: 0000-0001-7060-1723

Abstract

We consider an infinite number of one-dimensional bilinear Schrödinger equations on a segment. We prove the simultaneous local exact controllability in projection for any positive time and the simultaneous global exact controllability in projection for sufficiently large time.

AMS subject classifications: 35Q41, 93C20, 93B05, 81Q15.

Keywords: Schrödinger equation, simultaneous control, global exact controllability, moment problem, perturbation theory, density matrices.

1 Introduction

Let $\mathscr{H}$ be an infinite dimensional Hilbert space. In quantum mechanics, any statistical ensemble can be described by a wave function (pure state) or by a density matrix (mixed state) which is a positive operator of trace 1. For any density matrix ρ, there exists a sequence $\{\psi_j\}_{j\in\mathbb{N}} \subset \mathscr{H}$ such that

$$\rho = \sum_{j\in\mathbb{N}} l_j |\psi_j\rangle\langle\psi_j|, \qquad \sum_{j\in\mathbb{N}} l_j = 1, \qquad l_j \geq 0 \quad \forall j \in \mathbb{N}. \tag{1}$$

The sequence $\{\psi_j\}_{j\in\mathbb{N}}$ is a set of eigenvectors of ρ and $\{l_j\}_{j\in\mathbb{N}}$ are the corresponding eigenvalues. If there exists $j_0 \in \mathbb{N}$ such that $l_{j_0} = 1$ and $l_j = 0$ for each $j \neq j_0$, then the corresponding density matrix represents a pure state up to a phase. For this reason, the density matrices formalism is said to be an extension of the common formulation of the quantum mechanics in terms of wave function.

Let us consider $T > 0$ and a time dependent self-adjoint operator $H(t)$ (called Hamiltonian) for $t \in (0, T)$. The dynamics of a general density matrix ρ is described by the Von Neumann equation

$$\begin{cases} i\frac{d\rho}{dt}(t) = [H(t), \rho(t)], & t \in (0,T), \\ \rho(0) = \rho^0, & ([H,\rho] = H\rho - \rho H), \end{cases} \tag{2}$$

for ρ^0 the initial solution of the problem. The solution is $\rho(t) = U_t\rho(0)U_t^*$, where U_t is the unitary propagator generated by $H(t)$, *i.e.*

$$\begin{cases} \frac{d}{dt}U_t = -iH(t)U_t, & t \in (0,T), \\ U_0 = Id. \end{cases}$$

In the present work, we consider $\mathscr{H} = L^2((0,1),\mathbb{C})$ and $H(t) = A + u(t)B$, for $A = -\Delta$ the Dirichlet Laplacian (*i.e.* $D(A) = H^2 \cap H_0^1$), B a bounded symmetric operator and $u \in L^2((0,T),\mathbb{R})$ control function. From now on, we call Γ_t^u the unitary propagator U_t when it is defined. The problem (2) is said to be globally exactly controllable if, for any couple of unitarily equivalent density matrices ρ^1 and ρ^2, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that $\rho^2 = \Gamma_T^u\rho^1(\Gamma_T^u)^*$. Thanks to the decomposition (1), the controllability of (2) is equivalent (up to phases) to the simultaneous controllability of the Cauchy problems in $\mathscr{H}$

$$\begin{cases} i\partial_t\psi_j(t) = A\psi_j(t) + u(t)B\psi_j(t), & t \in (0,T), \\ \psi_j(0) = \psi_j^0, & \forall j \in \mathbb{N}. \end{cases} \tag{3}$$

The state ψ_j^0 is the j-th eigenfunction of ρ^0 corresponding to the eigenvalue λ_j and $\rho^0 = \sum_{j=1}^{\infty}\lambda_j|\psi_j^0\rangle\langle\psi_j^0|$. The j-th solution of (3) is $\psi_j(t) = \Gamma_t^u\psi_j^0$. To this purpose, we study the simultaneous global exact controllability of infinitely many problems (3) and we only rephrase the results in terms of the density matrices.

The controllability of the bilinear Schrödinger equation (3) has been widely studied in the literature and we start by mentioning the work on the bilinear systems of Ball, Mardsen and Slemrod [BMS82]. In the framework of the bilinear Schrödinger equation, for $B : D(A) \to D(A)$, the work shows the well-posedness of (3) in $\mathscr{H}$ for controls belonging to $L^1_{loc}(\mathbb{R},\mathbb{R})$ and an

important non-controllability result. In particular, let S be the unit sphere in $\mathscr{H}$ and

$$Z(\psi_0) := \{\psi \in D(A) |\ \exists T > 0,\ \exists r > 1, \exists u \in L^r((0,T),\mathbb{R}) :\ \psi = \Gamma_u^T \psi_0\}.$$

For every $\psi_0 \in S \cap D(A)$, the attainable set $Z(\psi_0)$ is contained in a countable union of compact sets and it has dense complement in $S \cap D(A)$.
Despite this non-controllability result, many authors have addressed the problem for weaker notions of controllability. We call M_μ the multiplication operator for a function $\mu \in \mathscr{H}$ and $H^s_{(0)} := D(|A|^{\frac{s}{2}})$ for $s > 0$.
For instance in [BL10], Beauchard and Laurent improve the work [Bea05] and they prove the local exact controllability of (3) in a neighborhood of the first eigenfunction of A in $S \cap H^3_{(0)}$ when $B = M_\mu$ for a suitable $\mu \in H^3$.
The global approximate controllability in a Hilbert space has been studied by Boscain, Caponigro, Chambrion, Mason and Sigalotti in [BCCS12] and [CMSB09]. In both, simultaneous global approximate controllability results are provided.
Morancey proves in [Mor14] the simultaneous local exact controllability in $S \cap H^3_{(0)}$ for at most three problems (3) and up to phases, when $B = M_\mu$ for suitable $\mu \in H^3$.
In [MN15], Morancey and Nersesyan extend the result. They provide the existence of a residual set of functions Q in H^4 so that, for $B = M_\mu$ and $\mu \in Q$, the simultaneous global exact controllability is verified for any finite number of (3) in $H^4_{(V)} := D(|A + V|^2)$ for $V \in H^4$.

In the present work, we use part of the notations of [BL10], [Mor14], [MN15] and we carry on the previous results. We provide explicit conditions in B that imply the simultaneous global exact controllability in projection of infinitely many problems (3) in $H^3_{(0)}$ by projecting onto suitable finite dimensional subspaces of $H^3_{(0)}$. Another goal of this work is to prove the simultaneous local exact controllability in projection for any positive time $T > 0$ up to phase-shifts. We use different techniques from the Coron's return method usually adopted for those types of results, *e.g.* [Mor14] and [MN15]. Indeed, in the appendix we develop a perturbation theory technique that we use in order to get rid of an issue appearing in the proof of the local controllability: the "eigenvalues resonances". The formulation of the controllability for orthonormal basis allows to provide the result in terms of density matrices and unitarily equivalent sets of functions.

1.1 Framework and main results

We denote $\mathscr{H} = L^2((0,1),\mathbb{C})$, its norm $\|\cdot\|$ and its scalar product $\langle\cdot,\cdot\rangle$. The operator A is the Dirichlet Laplacian, *i.e.* $A = -\frac{d^2}{dx^2}$ and $D(A) = H^1_0((0,1),\mathbb{C}) \cap H^2((0,1),\mathbb{C})$. The control function u belongs to $L^2((0,T),\mathbb{R})$ and B is a bounded symmetric operator.

We consider an Hilbert basis $\{\phi_j\}_{j\in\mathbb{N}}$ composed by eigenfunctions of A related to the eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ and we have

$$\phi_j(t) = e^{-iAt}\phi_j = e^{-i\lambda_j t}\phi_j. \tag{4}$$

Let us define the spaces for $s > 0$

$$H^s_{(0)} = H^s_{(0)}((0,1),\mathbb{C}) := D(A^{\frac{s}{2}}), \qquad \|\cdot\|_{(s)} = \|\cdot\|_{H^s_{(0)}} = \Big(\sum_{k=1}^{\infty} |k^s\langle\cdot,\phi_k\rangle|^2\Big)^{\frac{1}{2}},$$

$$\ell^\infty(\mathscr{H}) = \big\{\{\psi_j\}_{j\in\mathbb{N}} \subset \mathscr{H}\big|\ \sup_{j\in\mathbb{N}} \|\psi_j\|_{\mathscr{H}} < \infty\big\},$$

$$h^s(\mathscr{H}) = \Big\{\{\psi_j\}_{j\in\mathbb{N}} \subset \mathscr{H}\big|\ \sum_{j=1}^{\infty} (j^s\|\psi_j\|)^2 < \infty\Big\}.$$

We call $H^s := H^s((0,1),\mathbb{C})$, $H^s_0 := H^s_0((0,1),\mathbb{C})$ and, for $N \in \mathbb{N}$

$$I^N := \{(j,k) \in \mathbb{N}\times\{1,...,N\} : j \neq k\}. \tag{5}$$

Assumptions (I)**.** The bounded symmetric operator B satisfies the following conditions.

1. For any $N \in \mathbb{N}$, there exists $C_N > 0$ so that for every $j \leq N$ and $k \in \mathbb{N}$
$$|\langle\phi_k, B\phi_j\rangle| \geq C_N/k^3.$$
2. $Ran(B|_{H^2_{(0)}}) \subseteq H^2_{(0)}$ and $Ran(B|_{H^3_{(0)}}) \subseteq H^3 \cap H^1_0$.
3. For every $N \in \mathbb{N}$ and $(j,k),(l,m) \in I^N$ such that $(j,k) \neq (l,m)$ and $j^2 - k^2 - l^2 + m^2 = 0$, there holds $\langle\phi_j, B\phi_j\rangle - \langle\phi_k, B\phi_k\rangle - \langle\phi_l, B\phi_l\rangle + \langle\phi_m, B\phi_m\rangle \neq 0$.

Remark 1.1. *If a bounded operator B satisfies Assumptions I, then $B \in L(H^2_{(0)}, H^2_{(0)})$. Indeed, B is closed in $\mathscr{H}$, so for every $\{u_n\}_{n\in\mathbb{N}} \subset \mathscr{H}$ such that $u_n \xrightarrow{\mathscr{H}} u$ and $Bu_n \xrightarrow{\mathscr{H}} v$, we have $Bu = v$. Now, for every $\{u_n\}_{n\in\mathbb{N}} \subset H^2_{(0)}$ such that $u_n \xrightarrow{H^2_{(0)}} u$ and $Bu_n \xrightarrow{H^2_{(0)}} v$, the convergences with respect to the $\mathscr{H}$-norm are implied and $Bu = v$. Hence, the operator B is closed in $H^2_{(0)}$ and $B \in L(H^2_{(0)}, H^2_{(0)})$. The same argument leads to $B \in L(H^3_{(0)}, H^3 \cap H^1_0)$ since $Ran(B|_{H^3_{(0)}}) \subseteq H^3 \cap H^1_0$.*

Example 1.2. *Assumptions I are satisfied for $B : \psi \mapsto x^2\psi$. Indeed, the condition 2) is trivially verified, while the first directly follows by considering*

$$\begin{cases} |\langle\phi_j, x^2\phi_k\rangle| = \left|\frac{(-1)^{j-k}}{(j-k)^2\pi^2} - \frac{(-1)^{j+k}}{(j+k)^2\pi^2}\right|, & j \neq k, \\ |\langle\phi_k, x^2\phi_k\rangle| = \left|\frac{1}{3} - \frac{1}{2k^2\pi^2}\right|, & k \in \mathbb{N}. \end{cases}$$

The point 3) holds since for $(j,k),(l,m) \in I^N$ so that $(j,k) \neq (l,m)$

$$j^2 - k^2 - l^2 + m^2 = 0 \quad \Longrightarrow \quad j^{-2} - k^{-2} - l^{-2} + m^{-2} \neq 0.$$

Let $\Psi := \{\psi_j\}_{j\in\mathbb{N}} \subset \mathscr{H}$ and $\mathscr{H}_N(\Psi) := \text{span}\{\psi_j : j \leq N\}$. We define $\pi_N(\Psi)$ the orthogonal projector onto $\mathscr{H}_N(\Psi)$.

Definition 1.3. The problems (3) are simultaneously globally exactly controllable in projection in $H^3_{(0)}$ if there exist $T > 0$ and $\Psi := \{\psi_j\}_{j\in\mathbb{N}} \subset \mathscr{H}$ such that the following property is verified. For every $\{\psi^1_j\}_{j\in\mathbb{N}}$, $\{\psi^2_j\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ unitarily equivalent, there exists $u \in L^2((0,T),\mathbb{R})$ such that

$$\pi_N(\Psi)\psi^2_j = \pi_N(\Psi)\Gamma^u_T\psi^1_j, \qquad \forall j \in \mathbb{N}. \tag{6}$$

In other words, $\langle\psi_k, \psi^2_j\rangle = \langle\psi_k, \Gamma^u_T\psi^1_j\rangle$ for every $j, k \in \mathbb{N}$ and $k \leq N$.

Definition 1.4. Let us define

$$O_{\epsilon,T} := \Big\{ \{\psi_j\}_{j\in\mathbb{N}} \subset H^3_{(0)} \big|\ \langle\psi_j, \psi_k\rangle = \delta_{j,k};\ \sup_{j\in\mathbb{N}} \|\psi_j - \phi_j(T)\|_{(3)} < \epsilon \Big\}.$$

The problems (3) are simultaneously locally exactly controllable in projection in $O_{\epsilon,T} \subset H^3_{(0)}$ up to phases if there exist $\epsilon > 0$, $T > 0$ and $\Psi := \{\psi_j\}_{j\in\mathbb{N}} \in O_{\epsilon,T}$ such that the following property is verified. For every $\{\psi^1_j\}_{j\in\mathbb{N}} \in O_{\epsilon,T}$, there exist $\{\theta_j\}_{j\in\mathbb{N}} \subset \mathbb{R}$ and $u \in L^2((0,T),\mathbb{R})$ such that

$$\pi_N(\Psi)\psi^1_j = \pi_N(\Psi)e^{i\theta_j}\Gamma^u_T\psi_j, \qquad \forall j \in \mathbb{N}.$$

In other words, $\langle\psi_k, \psi^1_j\rangle = e^{i\theta_j}\langle\psi_k, \Gamma^u_T\psi_j\rangle$ for every $j, k \in \mathbb{N}$ and $k \leq N$.

Let $U(\mathscr{H})$ be the space of the unitary operators on $\mathscr{H}$. We present the simultaneous local exact controllability in projection for any $T > 0$ up to phases.

Theorem 1.5. *Let B satisfy Assumptions I. For every $T > 0$, there exist $\epsilon > 0$ and $\Psi := \{\psi_j\}_{j\in\mathbb{N}} \in O_{\epsilon,T}$ such that the following holds. For any $\{\psi^1_j\}_{j\in\mathbb{N}} \in O_{\epsilon,T}$ and $\widehat{\Gamma} \in U(\mathscr{H})$ such that $\{\widehat{\Gamma}\psi^1_j\}_{j\in\mathbb{N}} = \{\phi_j\}_{j\in\mathbb{N}}$, if $\{\widehat{\Gamma}\phi_j\}_{j\in\mathbb{N}} \subset H^3_{(0)}$, then there exist $\{\theta_j\}_{j\leq N} \subset \mathbb{R}$ and $u \in L^2((0,T),\mathbb{R})$ such that*

$$\begin{cases} \pi_N(\Psi)\psi^1_j = \pi_N(\Psi)e^{i\theta_j}\Gamma^u_T\psi_j & j \leq N, \\ \pi_N(\Psi)\psi^1_j = \pi_N(\Psi)\Gamma^u_T\psi_j, & j > N. \end{cases}$$

Proof. See Proposition 2.1. □

Now, we present the simultaneous global exact controllability in projection up to phases in the components.

Theorem 1.6. *Let B satisfy Assumptions I and $\Psi^3 := \{\psi^3_j\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ be an orthonormal system. Let $\{\psi^1_j\}_{j\in\mathbb{N}}$, $\{\psi^2_j\}_{j\in\mathbb{N}}, \subset H^3_{(0)}$ be complete orthonormal systems so that there exists $\widehat{\Gamma} \in U(\mathscr{H})$ such that $\{\widehat{\Gamma}\psi^2_j\}_{j\in\mathbb{N}} =$*

$\{\psi_j^1\}_{j\in\mathbb{N}}$. *If* $\{\widehat{\Gamma}\psi_j^3\}_{j\in\mathbb{N}} \subset H^3_{(0)}$, *then for any* $N \in \mathbb{N}$, *there exist* $T > 0$, $u \in L^2((0,T),\mathbb{R})$ *and* $\{\theta_k\}_{k\leq N} \subset \mathbb{R}$ *such that*

$$e^{i\theta_k}\langle \psi_k^3, \psi_j^2\rangle = \langle \psi_k^3, \Gamma_T^u \psi_j^1\rangle, \qquad \forall j,k \in \mathbb{N},\ k \leq N. \tag{7}$$

Proof. See Section 3. □

In Theorem 1.6, if $\Psi^3 = \Psi^2$, then $\widehat{\Gamma}\psi_j^3 \in H^3_{(0)}$. As $e^{i\theta_k}\langle \psi_k^2, \psi_j^2\rangle = e^{i\theta_k}\delta_{k,j} = e^{i\theta_j}\langle \psi_k^2, \psi_j^2\rangle$ for every $j,k \in \mathbb{N}$, the relation (7) becomes

$$\begin{aligned} \pi_N(\Psi^2)\ e^{i\theta_j}\psi_j^2 &= \pi_N(\Psi^2)\Gamma_T^u\psi_j^1, && j \leq N,\\ \pi_N(\Psi^2)\ \psi_j^2 &= \pi_N(\Psi^2)\Gamma_T^u\psi_j^1, && j > N. \end{aligned}$$

As Ψ^2 is composed by orthogonal elements, then $\pi_N(\Psi^2)\psi_j^2 = \psi_j^2$ when $j \leq N$, otherwise $\pi_N(\Psi^2)\psi_j^2 = 0$. Then the next corollary follows.

Corollary 1.7. *Let* B *satisfy Assumptions I. Let* $\Psi^1 := \{\psi_j^1\}_{j\in\mathbb{N}}$, $\Psi^2 := \{\psi_j^2\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ *be complete orthonormal systems. For any* $N \in \mathbb{N}$, *there exist* $T > 0$, $u \in L^2((0,T),\mathbb{R})$ *and* $\{\theta_j\}_{j\leq N} \subset \mathbb{R}$ *such that*

$$\begin{cases} \Gamma_T^u\psi_j^1 = e^{i\theta_j}\psi_j^2, & j \leq N,\\ \pi_N(\Psi^2)\ \Gamma_T^u\psi_j^1 = 0, & j > N. \end{cases}$$

Remark. *One can notice that Corollary* 1.7 *implies the simultaneous global exact controllability (without projecting) of* N *bilinear Schrödinger equations. As we have mentioned before, a similar result is proved by Morancey and Nersesyan in* $[MN15, Main\ \ Theorem]$. *They prove the existence of a class of multiplication operators* B *that guarantees the validity of the result. However, Corollary* 1.7 *provides a novelty as we are able to explicit conditions in* B *implying the controllability. Given any bounded operator* B, *one can verify if those assumptions are satisfied, e.g.* $B = x^2$.

Let $P_{\phi_j}^{\perp}$ be the projector onto the orthogonal space of ϕ_j and the operator

$$\widetilde{B}(M,j) = B\big((\lambda_j - A)\big|_{\phi_j^{\perp}}\big)^{-1}\Big(\big((\lambda_j - A)\big|_{\phi_j^{\perp}}\big)^{-1}P_{\phi_j}^{\perp}B\Big)^M P_{\phi_j}^{\perp}B$$

for $M, j \in \mathbb{N}$. When (A,B) satisfies Assumptions I and the following assumptions, the phase ambiguities $\{\theta_j\}_{j\leq N} \subset \mathbb{R}$ appearing in Theorem 1.6 can be removed. Let $\mathbf{0^n}$ be the null vector in $\mathbb{Q}^n$ with $n \in \mathbb{N}$.

Assumptions (A)**.** If for every $N \in \mathbb{N}$ there exists $\{r_j\}_{0\leq j\leq N} \in \mathbb{Q}^{N+1} \setminus \mathbf{0^{N+1}}$ such that $r_0 + \sum_{j=1}^N r_j\lambda_j = 0$, then either we have $\sum_{j=1}^N r_j B_{j,j} \neq 0$, or there exists $M \in \mathbb{N}$ such that $\sum_{j=1}^N r_j\langle \phi_j, \widetilde{B}(M,j)\phi_j\rangle \neq 0$.

Remark. *When the operator B is such that $\{B_{j,j}\}_{j\leq N}$ are rationally independent with $N \in \mathbb{N}$, the Assumptions A are verified as, for any $\{r_j\}_{0\leq j\leq N}\mathbb{Q}^{N+1}\setminus \mathbf{0^{N+1}}$, there holds $\sum_{j=1}^{N} r_j B_{j,j} \neq 0$.*

Theorem 1.8. *Let $N \in \mathbb{N}$. Let B satisfy Assumptions I and Assumptions A. Let $\Psi^3 := \{\psi_j^3\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ and $\{\psi_j^1\}_{j\in\mathbb{N}}$, $\{\psi_j^2\}_{j\in\mathbb{N}}, \subset H^3_{(0)}$ such that there exists $\widehat{\Gamma} \in U(\mathscr{H})$ such that $\{\widehat{\Gamma}\psi_j^2\}_{j\in\mathbb{N}} = \{\psi_j^1\}_{j\in\mathbb{N}}$. If $\{\widehat{\Gamma}\psi_j^3\}_{j\in\mathbb{N}} \subset H^3_{(0)}$, then there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that*

$$\pi_N(\Psi^3)\ \psi_j^2 = \pi_N(\Psi^3)\ \Gamma_T^u \psi_j^1, \qquad j \in \mathbb{N}.$$

Proof. See Paragraph 3. □

Remark. *If $\Psi^3 = \Psi^2$, then the same result of Corollary 1.7 is also provided when B satisfies Assumptions A thanks to Theorem 1.8.*

1.2 Well-posedness

We mention now the crucial result of well-posedness for the problem in $\mathscr{H}$

$$\begin{cases} i\partial_t\psi(t) = A\psi(t) + u(t)\mu\psi(t), \\ \psi(0) = \psi^0, \qquad\qquad t \in (0,T). \end{cases} \tag{8}$$

Proposition 1.9. *[BL10, Proposition 2] Let $\mu \in H^3$, $T > 0$, $\psi^0 \in H^3_{(0)}$ and $u \in L^2((0,T),\mathbb{R})$. There exists a unique mild solution of (8) in $H^3_{(0)}$, i.e. $\psi \in C^0([0,T], H^3_{(0)})$ so that*

$$\psi(t,x) = e^{-iAt}\psi^0(x) - i\int_0^t e^{-iA(t-s)}(u(s)\mu(x)\psi(s,x))ds, \qquad \forall t \in [0,T]. \tag{9}$$

Moreover, for every $R > 0$, there exists $C = C(T,\mu,R) > 0$ such that, if $\|u\|_{L^2((0,T),\mathbb{R})} < R$, then, for every $\psi^0 \in H^3_{(0)}$, the solution satisfies $\|\psi\|_{C^0([0,T],H^3_{(0)})} \leq C\|\psi^0\|_{(3)}$ and $\|\psi(t)\|_{\mathscr{H}} = \|\psi^0\|_{\mathscr{H}}$ for every $t \in [0,T]$.

The result of Proposition 1.9 is also valid if one substitute $\mu \in H^3$ with $B \in L(H^3_{(0)}, H^3 \cap H^1_0)$. When B satisfies Assumptions I, we know that $B \in L(H^3_{(0)}, H^3 \cap H^1_0)$ (see Remark 1.1) and there exists a unique mild solution of (3) in $H^3_{(0)}$ so that

$$\psi_j(t,x) = e^{-iAt}\psi_j^0(x) - i\int_0^t e^{-iA(t-s)}u(s)B\psi_j(s,x)ds.$$

In conclusion, for every $\{\psi_j\}_{j\in\mathbb{N}} \in \ell^\infty(H^3_{(0)})$ (respectively in $h^3(H^3_{(0)})$), it follows that $\{\Gamma_T^u\psi_j\}_{j\in\mathbb{N}} \in \ell^\infty(H^3_{(0)})$ (respectively in $h^3(H^3_{(0)})$).

1.3 Time reversibility

An important feature of the bilinear Schrödinger equation is the time reversibility. If we substitute t with $T-t$ for $T>0$ in the bilinear Schrödinger equation (3), then we obtain

$$\begin{cases} i\partial_t \Gamma^u_{T-t}\psi^0 = -A\Gamma^u_{T-t}\psi^0 - u(T-t)B\Gamma^u_{T-t}\psi^0, & t\in(0,T),\\ \Gamma^u_{T-0}\psi^0 = \Gamma^u_T\psi^0 = \psi^1. \end{cases}$$

We define $\widetilde{\Gamma}^{\widetilde{u}}_t$ such that $\Gamma^u_{T-t}\psi^0 = \widetilde{\Gamma}^{\widetilde{u}}_t\psi^1$ for $\widetilde{u}(t) := u(T-t)$ and

$$\begin{cases} i\partial_t \widetilde{\Gamma}^{\widetilde{u}}_t\psi^1 = (-A-\widetilde{u}(t)B)\widetilde{\Gamma}^{\widetilde{u}}_t\psi^1, & t\in(0,T),\\ \widetilde{\Gamma}^{\widetilde{u}}_0\psi^0 = \psi^1. \end{cases} \tag{10}$$

Thanks to $\psi^0 = \widetilde{\Gamma}^{\widetilde{u}}_T\Gamma^u_T\psi^0$ and $\psi^1 = \Gamma^u_T\widetilde{\Gamma}^{\widetilde{u}}_T\psi^1$, it follows $\widetilde{\Gamma}^{\widetilde{u}}_T = (\Gamma^u_T)^{-1} = (\Gamma^u_T)^*$. The operator $\widetilde{\Gamma}^{\widetilde{u}}_t$ describes the reversed dynamics of Γ^u_t and represents the propagator of (10) generated by the Hamiltonian $(-A-\widetilde{u}(t)B)$.

1.4 Scheme of the work

In Section 2, we provide Proposition 2.1 and its proof. The proposition extends Theorem 1.5 and it ensures the simultaneous local exact controllability in projection for any positive time up to phases. In order to motivate the modification of the problem, we emphasize the obstructions to overcome.
In Section 3, we provide the simultaneous global approximate controllability of N problems (3) in Proposition 3.3, then the simultaneous global exact controllability of N (3) (Proposition 3.4). Those results lead to the proofs of Theorem 1.6 and Theorem 1.8, while in Section 4, we provide the main result in terms of density matrices.
In Appendix 1.3, we explain the time reversibility of the (3), while in Appendix A, we briefly discuss the solvability of the moment problems.
In Appendix B, we develop the perturbation theory technique adopted in the work.

2 Simultaneous locale exact controllability in projection for $T>0$

2.1 Preliminaries

In this section, we discuss the simultaneous local exact controllability in projection. We explain first why we modify the problem.

Let $\Phi = \{\phi_j\}_{j\in\mathbb{N}}$ be an Hilbert basis composed by eigenfunctions of A. We study the local exact controllability in projection in $O_{\epsilon,T}$ with respect to $\pi_N(\Phi)$. Let $\Gamma^u_t\psi_j = \sum_{k=1}^{\infty}\phi_k(T)\langle\phi_k(T),\Gamma^u_t\phi_j\rangle$ be the solution of the j-th

(3). We consider the map $\alpha(u)$, the infinite matrix with elements $\alpha_{k,j}(u) = \langle \phi_k(T), \Gamma_T^u \phi_j \rangle$, for every $k, j \in \mathbb{N}$ and $k \leq N$. Our goal is to prove the existence of $\epsilon > 0$ such that for any $\{\psi_j\}_{j\in\mathbb{N}} \in O_{\epsilon,T}$, there exists $u \in L^2((0,T),\mathbb{R})$ such that

$$\pi_N(\Phi)\Gamma_T^u \phi_j = \pi_N(\Phi)\psi_j, \quad \forall j \in \mathbb{N}.$$

This outcome is equivalent to the local surjectivity of α for $T > 0$. To this end, we want to use the Generalized Inverse Function Theorem ([Lue69, Theorem 1; p. 240]) and we study the surjectivity of $\gamma(v) := (d_u\alpha(0)) \cdot \ v$, the Fréchet derivative of α the infinite matrix that, for $j, k \in \mathbb{N}$ and $k \leq N$,

$$\gamma_{k,j}(v) := \left\langle \phi_k(T), -i \int_0^T e^{-iA(T-s)} v(s) B e^{-iAs} \phi_j ds \right\rangle = -i \int_0^T v(s) e^{-i(\lambda_j - \lambda_k)s} ds B_{k,j},$$

for $B_{k,j} = \langle \phi_k, B\phi_j \rangle = \langle B\phi_k, \phi_j \rangle = \overline{B_{j,k}}$. The surjectivity of γ consists in proving the solvability of the moment problem

$$\frac{x_{k,j}}{B_{k,j}} = -i \int_0^T u(s) e^{-i(\lambda_j - \lambda_k)s} ds, \tag{11}$$

for each infinite matrix x, with elements $x_{k,j}$, belonging to a suitable space. One would use Haraux Theorem as explained in Remark $A.1$ ([KL05, $Theorem$ 4.6]) but the eigenvalues resonances occur: for some $j, k, n, m \in \mathbb{N}$, $(j,k) \neq (n,m)$ and $k, m \leq N$, there holds $\lambda_j - \lambda_k = \lambda_n - \lambda_m$, which implies

$$\frac{x_{k,j}}{B_{k,j}} = -i \int_0^T u(s) e^{-i(\lambda_j - \lambda_k)s} ds = -i \int_0^T u(s) e^{-i(\lambda_n - \lambda_m)s} ds = \frac{x_{n,m}}{B_{n,m}}.$$

An example is $\lambda_7 - \lambda_1 = \lambda_8 - \lambda_4$, but they also appear for all the diagonal terms of γ since $\lambda_j - \lambda_k = 0$ for $j = k$.
We avoid the problem by adopting the following procedure. First, we decompose $A + u(t)B = (A + u_0 B) + u_1(t)B$ for $u_0 \in \mathbb{R}$ and $u_1 \in L^2((0,T),\mathbb{R})$. We consider $A + u_0 B$ instead of A and we modify the eigenvalues gaps by using $u_0 B$ as a perturbating term in order to remove all the non-diagonal resonances. Second, we redefine α in a map $\widehat{\alpha}$ depending on the parameter u_0. We introduce α^{u_0} by acting phase-shifts in order to remove the resonances on the diagonal terms, *i.e.* $\widetilde{\psi}_j(t,x) = \frac{\overline{\widehat{\alpha}_{j,j}(u)}}{|\widehat{\alpha}_{j,j}(u)|}\psi_j(t,x)$, which implies $\alpha^{u_0}_{k,j}(u) = \frac{\overline{\widehat{\alpha}_{j,j}(u)}}{|\widehat{\alpha}_{j,j}(u)|}\widehat{\alpha}_{k,j}(u)$.

2.2 The modified problem

Let $N \in \mathbb{N}$ and $u(t) = u_0 + u_1(t)$, for u_0 and $u_1(t)$ real. We introduce the following Cauchy problem

$$\begin{cases} i\partial_t \psi_j(t) = (A + u_0 B)\psi_j(t) + u_1(t) B\psi_j(t), & t \in (0,T),\ j \in \mathbb{N}, \\ \psi_j^0 = \psi_j(0). \end{cases} \tag{12}$$

Its solutions are $\psi_j(t) = \Gamma_t^{u_0+u_1}\psi_j^0$, where $\Gamma_t^{u_0+u_1}$ is the unitary propagator of the dynamics, which is equivalent to the one of the problems (3).
As B is bounded, $A + u_0B$ has pure discrete spectrum. We call $\{\lambda_j^{u_0}\}_{j\in\mathbb{N}}$ the eigenvalues of $A + u_0B$ that correspond to an Hilbert basis composed by eigenfunctions $\Phi^{u_0} := \{\phi_j^{u_0}\}_{j\in\mathbb{N}}$. We set $\phi_j^{u_0}(T) := e^{-i\lambda_j^{u_0}T}\phi_j^{u_0}$ and

$$O_{\epsilon_0,T}^{u_0} := \Big\{\{\psi_j\}_{j\in\mathbb{N}} \subset H^3_{(0)}\big|\ \langle\psi_j,\psi_k\rangle = \delta_{j,k};\ \sup_{j\in\mathbb{N}}\|\psi_j - \phi_j^{u_0}(T)\|_{(3)} < \epsilon_0\Big\}.$$

We choose $|u_0|$ small so that $\lambda_k^{u_0} \neq 0$ for every $k \in \mathbb{N}$ (Lemma $B.4$, Appendix B). The introduction of the new Hilbert basis imposes to define $\widetilde{H}^3_{(0)} := D(|A+u_0B|^{\frac{3}{2}})$ equipped with $\|\cdot\|_{\widetilde{H}^3_{(0)}} = \Big(\sum_{k=1}^{\infty}\big||\lambda_k^{u_0}|^{\frac{3}{2}}\langle\cdot,\phi_k\rangle\big|^2\Big)^{\frac{1}{2}}$. However, from now on, due to Lemma $B.6$ (Appendix B), we have $\widetilde{H}^3_{(0)} \equiv H^3_{(0)}$.

We define $\widehat{\alpha}$, the infinite matrices with elements for $k \leq N$ and $j \in \mathbb{N}$ such that $\widehat{\alpha}_{k,j}(u_1) = \langle\phi_k^{u_0}(T), \Gamma_T^{u_0+u_1}\phi_j^{u_0}\rangle$ and the map α^{u_0} with elements

$$\begin{cases}\alpha_{k,j}^{u_0}(u_1) = \frac{\overline{\widehat{\alpha}_{j,j}(u_1)}}{|\widehat{\alpha}_{j,j}(u_1)|}\widehat{\alpha}_{k,j}(u_1), & j,k \leq N,\\ \alpha_{k,j}^{u_0}(u_1) = \widehat{\alpha}_{k,j}(u_1), & j > N,\ k \leq N.\end{cases} \tag{13}$$

Now, the local surjectivity of the map α^{u_0} in a suitable space is equivalent to the simultaneous local exact controllability in projection up to N phases on $O_{\epsilon_0,T}^{u_0}$ for a suitable $\epsilon_0 > 0$ since for $j \in \mathbb{N}$,

$$\pi_N(\Phi^{u_0})e^{i\theta_j}\Gamma_T^{u_0+u_1}\phi_j^{u_0} = \sum_{k=1}^{N}\phi_k^{u_0}(T)\alpha_{k,j}^{u_0}(u_1), \qquad e^{i\theta_j} := \frac{\overline{\widehat{\alpha}_{j,j}(u_1)}}{|\widehat{\alpha}_{j,j}(u_1)|}. \tag{14}$$

Let $\gamma^{u_0}(v) = ((d_{u_1}\alpha^{u_0})(0))\cdot v$ be the Fréchet derivative of α^{u_0} and $B_{k,j}^{u_0} = \langle\phi_k^{u_0}, B\phi_j^{u_0}\rangle$ for $k \leq N$ and $j \in \mathbb{N}$. Defined $\widehat{\gamma}_{k,j}(v) = ((d_{u_1}\widehat{\alpha})(0))\cdot v$, we compute $\gamma^{u_0}(v)$ such that $\gamma_{k,j}^{u_0} = \big(\overline{\widehat{\gamma}_{j,j}}\delta_{k,j} + \widehat{\gamma}_{k,j} - \delta_{k,j}\Re(\widehat{\gamma}_{j,j})\big)$ when $j,k \leq N$, while $\gamma_{k,j}^{u_0} = \widehat{\gamma}_{k,j}$ when $k \leq N$ and $j > N$. Thus for $k \leq N$ and $j \in \mathbb{N}$,

$$\begin{cases}\gamma_{k,j}^{u_0} = \widehat{\gamma}_{k,j} = -i\int_0^T u_1(s)e^{-i(\lambda_j^{u_0}-\lambda_k^{u_0})s}dsB_{k,j}^{u_0}, & k \neq j,\\ \gamma_{k,k}^{u_0} = \Re(\widehat{\gamma}_{k,k}) = 0, & k = j.\end{cases} \tag{15}$$

The relation $\gamma_{k,k}^{u_0} = 0$ comes from $(i\widehat{\gamma}_{k,k}) \in \mathbb{R}$ since $\widehat{\gamma}_{k,j} = -\overline{\widehat{\gamma}_{j,k}}$ for $j,k \leq N$. Due to the phase-shifts of α^{u_0}, the diagonal elements of γ^{u_0} are all 0.

Remark. *As $O_{\epsilon_0,T}^{u_0}$ is composed by orthonormal elements, we have*

$$T_{\Phi^{u_0}}O_{\epsilon_0,T}^{u_0} = \Big\{\{\psi_j\}_{j\in\mathbb{N}} \subset \ell^\infty(H^3_{(0)})\big|\ \langle\phi_k^{u_0},\psi_j\rangle = -\overline{\langle\phi_j^{u_0},\psi_k\rangle}\Big\}.$$

For every $k \in \mathbb{N}$, from Lemma B.6, there exists $C > 0$ so that

$$\sum_{j=1}^{+\infty} j^6 |\alpha_{k,j}^{u_0}|^2 = \sum_{j=1}^{+\infty} j^6 |\langle \widetilde{\Gamma}_T^{u_0+\widetilde{u}_1} \phi_k^{u_0}, \phi_j^{u_0} \rangle|^2 = \|\widetilde{\Gamma}_T^{u_0+\widetilde{u}_1} \phi_k^{u_0}\|^2_{\widetilde{H}^3_{(0)}} \leq C \|\widetilde{\Gamma}_T^{u_0+\widetilde{u}_1} \phi_k^{u_0}\|^2_{(3)} < \infty$$

as the propagator $\widetilde{\Gamma}_T^{u_0+\widetilde{u}_1}$ (see Appendix 1.3) preserves $H^3_{(0)}$. Hence, $\{\alpha_{k,j}^{u_0}\}_{j\in\mathbb{N}} \in h^3(\mathbb{C})$ for every $k \in \mathbb{N}$, then the maps α^{u_0} and γ^{u_0} take respectively values in $Q^N := \big\{ \{x_{k,j}\}_{\substack{k,j\in\mathbb{N} \\ k\leq N}} \in (h^3(\mathbb{C}))^N \big| \; x_{k,k} \in \mathbb{R}, \quad k \leq N \big\}$ and

$$G^N := \big\{ \{x_{k,j}\}_{\substack{k,j\in\mathbb{N} \\ k\leq N}} \in (h^3(\mathbb{C}))^N \big| \;\; x_{k,j} = -\overline{x_{j,k}}, \; x_{k,k} = 0 \; j,k \leq N \big\}.$$

2.3 Proof of Theorem 1.5

In the next proposition, we ensure the simultaneous local exact controllability in projection for any $T > 0$ up to phases.

Proposition 2.1. *Let $N \in \mathbb{N}$ and B satisfy Assumptions I. For every $T > 0$, there exist $\epsilon > 0$ and $u_0 \in \mathbb{R}$ such that, for any $\{\psi_j\}_{j\in\mathbb{N}} \in O_{\epsilon,T}$ and $\widehat{\Gamma} \in U(\mathscr{H})$ such that $\{\widehat{\Gamma}\psi_j^1\}_{j\in\mathbb{N}} = \{\phi_j\}_{j\in\mathbb{N}}$, if*

$$\big\{\widehat{\Gamma}\phi_j\big\}_{j\in\mathbb{N}} \subset H^3_{(0)}, \tag{16}$$

then there exist a sequence of real numbers $\{\theta_j\}_{j\in\mathbb{N}} = \big\{\big\{\widehat{\theta}_j\big\}_{j\leq N}, 0, ...\big\}$ and $u \in L^2((0,T),\mathbb{R})$ such that

$$\pi_N(\Phi^{u_0})\psi_j = \pi_N(\Phi^{u_0}) e^{i\theta_j} \Gamma_T^u \phi_j^{u_0}, \qquad \forall j \in \mathbb{N}.$$

Proof. **1)** Let u_0 in the neighborhoods defined in Appendix B by Lemma $B.4$, Lemma $B.5$, Lemma $B.6$ and Remark $B.9$. First, the relation (16) is required for the following reason. Let $\{\Gamma_T^u \phi_j^{u_0}\}_{j\in\mathbb{N}} = \{\widehat{\Gamma}\phi_j\}_{j\in\mathbb{N}}$ for $T > 0$, $u \in L^2((0,T),\mathbb{R})$ and $\widehat{\Gamma} \in U(\mathscr{H})$. For $|u_0|$ small enough, thanks to Lemma $B.4$ (Appendix B), there exists $C_1 > 0$ such that $j^6 \leq C_1 |\lambda_j^{u_0}|^3$. From Lemma $B.6$ (Appendix B), there exists $C_2 > 0$ such that, for every $k \in \mathbb{N}$,

$$\begin{cases} \sum_{j=1}^{+\infty} j^6 |\langle \phi_k, \Gamma_T^u \phi_j^{u_0} \rangle|^2 = \sum_{j=1}^{+\infty} j^6 |\langle (\Gamma_T^u)^* \phi_k, \phi_j^{u_0} \rangle|^2 \leq C_1 C_2 \|\widetilde{\Gamma}_T^{\widetilde{u}} \phi_k\|^2_{(3)} < \infty, \\ \sum_{j=1}^{+\infty} j^6 |\langle \phi_k, \Gamma_T^u \phi_j^{u_0} \rangle|^2 = \sum_{j=1}^{+\infty} j^6 |\langle \phi_k, \widehat{\Gamma}\phi_j \rangle|^2 = \sum_{j=1}^{+\infty} j^6 |\langle \widehat{\Gamma}^* \phi_k, \phi_j \rangle|^2 = \|\widehat{\Gamma}^* \phi_k\|^2_{(3)}. \end{cases}$$

Second, thanks to the third point of Remark $B.9$ (Appendix B), the controllability in $O^{u_0}_{\epsilon_0,T}$ implies the controllability in $O_{\epsilon,T}$ for suitable $\epsilon > 0$. Indeed, if $|u_0|$ is small enough, then $\sup_{j\in\mathbb{N}} \|\phi_j - \phi_j^{u_0}\|_{(3)} \leq \epsilon_0$ (Remark $B.9$). For every $\{\psi_j\}_{j\in\mathbb{N}} \in O^{u_0}_{\epsilon_0,T}$, we have $\{\psi_j\}_{j\in\mathbb{N}} \in O_{2\epsilon_0,T}$ since

$$\sup_{j\in\mathbb{N}} \|\psi_j - \phi_j(T)\|_{(3)} \leq \sup_{j\in\mathbb{N}} \|\phi_j^{u_0} - \phi_j(T)\|_{(3)} + \sup_{j\in\mathbb{N}} \|\psi_j - \phi_j^{u_0}(T)\|_{(3)} \leq 2\epsilon_0.$$

Third, thanks to the discussion about the relation (14), the local surjectivity of the map α^{u_0} guarantees the simultaneous local exact controllability in projection up to phases (Definition 1.4) of (3) with initial state $\{\phi_j^{u_0}\}_{j\in\mathbb{N}}$ on $O^{u_0}_{\epsilon_0,T}$ for ϵ_0 small enough.
We consider Generalized Inverse function Theorem ([Lue69, Theorem 1; p. 240]) since Q^N and G^N are real Banach spaces. If γ^{u_0} is surjective in G^N, then the local surjectivity of α^{u_0} in Q^N is ensured. The map γ^{u_0} is surjective when the following moment problem is solvable

$$\frac{x^{u_0}_{k,j}}{B^{u_0}_{k,j}} = -i\int_0^T u(s)e^{-i(\lambda_j^{u_0}-\lambda_k^{u_0})s}ds, \qquad j\in\mathbb{N},\ k\leq N,\ k\neq j \tag{17}$$

for every $\left\{x^{u_0}_{k,j}\right\}_{\substack{j,k\in\mathbb{N}\\ k\leq N}} \in G^N$. The equations of (17) for $k=j$ are redundant as $\gamma^{u_0}_{k,k}=0$ and $x^{u_0}_{k,k}=0$ for every $k\leq N$ and $\{x^{u_0}_{k,j}\}_{\substack{k,j\in\mathbb{N}\\ k\leq N}} \in G^N$. Thus, we prove the solvability of the moment problem for $j\neq k$ and $j=k=1$. Now, $\left\{x^{u_0}_{k,j}\right\}_{\substack{j,k\in\mathbb{N}\\ k\leq N}} \in (h^3)^N$ and $\left\{\gamma^{u_0}_{k,j}\right\}_{\substack{j,k\in\mathbb{N}\\ k\leq N}} \in (h^3)^N$. From Lemma B.5 (Appendix B), it follows $\left\{x^{u_0}_{k,j}/B^{u_0}_{k,j}\right\}_{\substack{j,k\in\mathbb{N}\\ k\leq N}} \in (\ell^2(\mathbb{C}))^N$ and $\left\{\gamma^{u_0}_{k,j}/B^{u_0}_{k,j}\right\}_{\substack{j,k\in\mathbb{N}\\ k\leq N}} \in (\ell^2(\mathbb{C}))^N$.
Thanks to Lemma B.8 (Appendix B), for I^N defined in (5), there exist $\mathscr{G}' := \inf_{\substack{(j,k),(n,m)\in I^N\\ (j,k)\neq(n,m)}} |\lambda_j^{u_0}-\lambda_k^{u_0}-\lambda_n^{u_0}+\lambda_m^{u_0}| > 0$ and

$$\mathscr{G} := \sup_{A\subset I^N}\Big(\inf_{\substack{(j,k),(n,m)\in I^N\setminus A\\ (j,k)\neq(n,m)}} |\lambda_j^{u_0}-\lambda_k^{u_0}-\lambda_n^{u_0}+\lambda_m^{u_0}|\Big) \geq \mathscr{G}'$$

where A runs over the finite subsets of I^N. The solvability of the moment problem (17) is guaranteed from Remark A.1 by considering the sequence of numbers $\{\lambda_j^{u_0}-\lambda_k^{u_0}\}_{\substack{j,k\in\mathbb{N},\ k\leq N\\ j\neq k \text{ or } j=k=1}}$. Indeed, $x^{u_0}_{1,1}=0$ and Remark B.9 ensures that $\lambda_j^{u_0}-\lambda_k^{u_0}\neq\lambda_l^{u_0}-\lambda_m^{u_0}$ for every $j,k,l,m\in\mathbb{N}$. The proof is achieved since α^{u_0} is locally surjective for $T>0$ large enough.

2) We show that the first point is valid for every $T>0$ as $\mathscr{G}=+\infty$. Let $A^M := \{(j,n)\in\mathbb{N}^2|\ j,n\geq M;\ j\neq n\}$ for $M\in\mathbb{N}$. Thanks to the relation (30) in the proof of Lemma B.4 (Appendix B), for $|u_0|$ small enough and for every $K\in\mathbb{R}$, there exists $M_K>0$ large enough such that $\inf_{(j,n)\in A^{M_K}}|\lambda_j^{u_0}-\lambda_n^{u_0}|>K$. Indeed, the relation (30) implies that, for $|u_0|$ small enough,

$$|\lambda_j^{u_0}-\lambda_n^{u_0}| \geq |\lambda_j-\lambda_n| - O(|u_0|) \geq 2\pi^2\min\{\lambda_{j+1}-\lambda_j, \lambda_{n+1}-\lambda_n\} - O(|u_0|).$$

Thus $\mathscr{G}\geq\sup_{M\in\mathbb{N}}\big(\inf_{(j,n)\in A^M}|\lambda_j^{u_0}-\lambda_n^{u_0}|-2\lambda_N^{u_0}\big)>0$. Now, for $|u_0|$ small enough, Lemma B.4 (Appendix B) implies the existence of $C>0$ such that

$$\mathscr{G}\geq C\big(\lim_{M\to\infty}\inf_{(j,n)\in A^M}|\lambda_j-\lambda_n|-2\lambda_N\big)\geq C\lim_{M\to\infty}(\lambda_{M+2}-\lambda_{M+1}-2N^2\pi^2)=+\infty.$$

□

3 Simultaneous global exact controllability in projection

The common approach adopted in order to prove the global exact controllability (also simultaneous) consists in gathering the global approximate controllability and the local exact controllability.
However, this strategy can not be used to prove the controllability in projection as the propagator Γ_T^u does not preserve the space $\pi_N(\Psi)H^3_{(0)}$ for any $\Psi := \{\psi_j\}_{j\in\mathbb{N}} \subset H^3_{(0)}$, making impossible to reverse and concatenate dynamics. We adopt an alternative strategy that we call "transposition argument" (see remark below). In particular, under suitable assumptions, we prove that the controllability in projection onto an N dimensional space is equivalent to the controllability of N problems (without projecting).

Remark 3.1. *From time reversibility (Appendix* 1.3*), for every* $j,k\in\mathbb{N}$,

$$\overline{\langle \phi_k^{u_0}(T), \Gamma_T^u \phi_j^{u_0}\rangle} = e^{-i\lambda_k^{u_0}T}\langle \Gamma_T^u\phi_j^{u_0}, \phi_k^{u_0}\rangle = e^{-i(\lambda_k^{u_0}+\lambda_j^{u_0})T}\langle \phi_j^{u_0}(T), \widetilde{\Gamma}_T^{\widetilde{u}}\phi_k^{u_0}\rangle.$$

Now, $e^{-i(\lambda_k^{u_0}+\lambda_j^{u_0})T}$ *does not depend on* u *and the last relation implies that the surjectivity of the two following maps is equivalent*

$$\{\langle \phi_k^{u_0}(T), \Gamma_T^u\phi_j^{u_0}\rangle\}_{\substack{j,k\in\mathbb{N}\\ k\le N}} : L^2((0,T),\mathbb{R}) \longrightarrow \{\{x_{k,j}\}_{\substack{j,k\in\mathbb{N}\\ k\le N}} :\ \{x_{k,j}\}_{j\in\mathbb{N}} \in h^3(\mathbb{C}),\ \forall k\le N\}$$

$$\{\langle \phi_j^{u_0}(T), \widetilde{\Gamma}_T^{\widetilde{u}}\phi_k^{u_0}\rangle\}_{\substack{j,k\in\mathbb{N}\\ k\le N}} : L^2((0,T),\mathbb{R}) \longrightarrow \{\{x_{j,k}\}_{\substack{j,k\in\mathbb{N}\\ k\le N}} :\ \{x_{j,k}\}_{j\in\mathbb{N}} \in h^3(\mathbb{C}),\ \forall k\le N\}.$$

For this reason, the simultaneous global exact controllability in projection onto a suitable N *dimensional space is equivalent to the controllability of* N *problems (without projection).*

The transposition argument is particularly important as it allows to concatenate and reverse dynamics on $(H^3_{(0)})^N$, which is preserved by the propagator when one wants to prove the controllability in projection.
For the simultaneous local exact controllability result, we can use Proposition 2.1 with the transposition argument, but this is not always the most convenient approach. Indeed, when B satisfies Assumptions A, we consider [MN15, *Theorem* 4.1] that requires stronger assumptions on the operator B but provides the result without phase ambiguities (as in Theorem 1.6).

3.1 Approximate simultaneous controllability

In this section, we prove the simultaneous global approximate controllability.

Definition 3.2. The problems (3) are said to be simultaneously globally approximately controllable in $H^s_{(0)}$ if, for every $N\in\mathbb{N}$, $\psi_1,....,\psi_N \in H^s_{(0)}$, $\widehat{\Gamma}\in U(\mathscr{H})$ such that $\widehat{\Gamma}\psi_1,....,\widehat{\Gamma}\psi_N \in H^s_{(0)}$ and $\epsilon>0$, then there exist $T>0$ and $u\in L^2((0,T),\mathbb{R})$ such that $\|\widehat{\Gamma}\psi_k - \Gamma_T^u\psi_k\|_{H^s} < \epsilon$ for every $1\le k\le N$.

Theorem 3.3. *Let B satisfy Assumptions I. The problems* (3) *are simultaneously globally approximately controllable in $H^3_{(0)}$.*

Proof. Let $N \in \mathbb{N}$ and u_0 belong to the neighborhoods provided by Remark $B.7$ and Remark $B.9$ (Appendix B). We define $|||\cdot|||_{(s)} := |||\cdot|||_{L(H^s_{(0)},H^s_{(0)})}$ and $\|f\|_{BV(T)} := \|f\|_{BV((0,T),\mathbb{R})} = \sup_{\{t_j\}_{0\leq j\leq n}\in P}\sum_{j=1}^n |f(t_j)-f(t_{j-1})|$, where $f \in BV((0,T),\mathbb{R})$ and P is the set of the partitions of $(0,T)$ such that $t_0 = 0 < t_1 < ... < t_n = T$. We consider the techniques developed by Chambrion in [Cha12] and we start by choosing $\psi_j = \phi_j$ for every $j \leq N$. Now, $(A + u_0B, B)$ admits a non-degenerate chain of connectedness (see [BdCC13, $Definition$ 3]) thanks to Remark $B.9$ (Appendix B). Up to a reordering of $\{\phi_k\}_{k\in\mathbb{N}}$, we can assume that for every $m \in \mathbb{N}$, the couple $(\pi_m(\Phi)(A + u_0B)\pi_m(\Phi), \pi_m(\Phi)B\pi_m(\Phi))$ admits a non-degenerate chain of connectedness in $\mathscr{H}_m$.

1) Preliminaries:

> **Claim.** For every $\epsilon > 0$, there exist $N_1 \in \mathbb{N}$ and $\widetilde{\Gamma}_{N_1} \in U(\mathscr{H})$ such that $\pi_{N_1}(\Phi)\widetilde{\Gamma}_{N_1}\pi_{N_1}(\Phi) \in SU(\mathscr{H}_{N_1})$ and
>
> $$\|\widetilde{\Gamma}_{N_1}\phi_j - \widehat{\Gamma}\phi_j\|_{(3)} < \epsilon, \qquad \forall j \leq N. \tag{18}$$

Let $N' \in \mathbb{N}$ be such that $N' \geq N$. We apply the orthonormalizing Gram-Schmidt process to $\{\pi_{N'}(\Phi)\widehat{\Gamma}\phi_j\}_{j\leq N}$ and we define the sequence $\{\widetilde{\phi}_j\}_{j\leq N}$ that we complete in $\{\widetilde{\phi}_j\}_{j\leq N'}$, an orthonormal basis of $\mathscr{H}_{N'}$. The operator $\widetilde{\Gamma}_{N'}$ is the unitary map such that $\widetilde{\Gamma}_{N'}\phi_j = \widetilde{\phi}_j$, for every $j \leq N'$. The provided definition implies $\lim_{N'\to\infty}\|\widetilde{\Gamma}_{N'}\phi_j - \widehat{\Gamma}\phi_j\|^2_{(3)} = 0$ for every $j \leq N$. Thus, for every $\epsilon > 0$, there exists $N' \in \mathbb{N}$ large enough such that

$$\|\widetilde{\Gamma}_{N'}\phi_j - \widehat{\Gamma}\phi_j\|_{(3)} < \epsilon, \qquad \forall j \leq N. \tag{19}$$

We denote N_1 the number $N' \geq N$ such that the relation (19) is verified.

2) Finite dimensional controllability: We call T_{ad} the set of the admissible transitions, *i.e.* the couples $(j,k) \in \{1,...,N_1\}^2$ such that $B_{j,k} \neq 0$ and $|\lambda_j - \lambda_k| = |\lambda_m - \lambda_l|$ with $m, l \in \mathbb{N}$ implies $\{j,k\} = \{m,l\}$ or $B_{m,l} = 0$. For every $(j,k) \in \{1,...,N_1\}^2$ and $\theta \in [0,2\pi)$, we define $E^\theta_{j,k}$ the $N_1 \times N_1$ matrix with elements $(E^\theta_{j,k})_{l,m} = 0$, $(E^\theta_{j,k})_{j,k} = e^{i\theta}$ and $(E^\theta_{j,k})_{k,j} = -e^{-i\theta}$, for $(l,m) \in \{1,...,N_1\}^2 \setminus \{(j,k),(k,j)\}$. We call $E_{ad} = \left\{E^\theta_{j,k} \ : \ (j,k) \in T_{ad},\ \theta \in [0,2\pi)\right\}$ and we consider $Lie(E_{ad})$. We introduce the following finite dimensional control system on $SU(\mathscr{H}_{N_1})$

$$\begin{cases} \dot{x}(t) = x(t)v(t), & t \in (0,\tau), \\ x(0) = Id_{SU(\mathscr{H}_{N_1})} \end{cases} \tag{20}$$

where the set of admissible controls v is the set of piecewise constant functions taking value in E_{ad} and $\tau > 0$.

Claim. (20) is controllable, *i.e.* for $R \in SU(\mathscr{H}_{N_1})$, there exist $p \in \mathbb{N}$, $M_1, ..., M_p \in E_{ad}$, $\alpha_1, ..., \alpha_p \in \mathbb{R}^+$ such that $R = e^{\alpha_1 M_1} \circ ... \circ e^{\alpha_p M_p}$.

For every $(j,k) \in \{1, ..., N_1\}^2$, we define the $N_1 \times N_1$ matrices $R_{j,k}$, $C_{j,k}$ and D_j as follow. For $(l,m) \in \{1, ..., N_1\}^2 \setminus \{(j,k),(k,j)\}$,we have $(R_{j,k})_{l,m} = 0$ and $(R_{j,k})_{j,k} = -(R_{j,k})_{k,j} = 1$, while $(C_{j,k})_{l,m} = 0$ and $(C_{j,k})_{j,k} = (C_{j,k})_{k,j} = i$. Moreover, for $(l,m) \in \{1, ..., N_1\}^2 \setminus \{(1,1),(j,j)\}$, $(D_j)_{l,m} = 0$ and $(D_j)_{1,1} = -(D_j)_{j,j} = i$.
Now, $\mathbf{e} := \{R_{j,k}\}_{j,k \leq N_1} \cup \{C_{j,k}\}_{j,k \leq N_1} \cup \{D_j\}_{j \leq N_1}$ is a basis of $su(\mathscr{H}_{N_1})$. Thanks to [Sac00, *Theorem* 6.1], the controllability of (20) is equivalent to prove that $Lie(E_{ad}) \supseteq su(\mathscr{H}_{N_1})$ for $su(\mathscr{H}_{N_1})$ the Lie algebra of $SU(\mathscr{H}_{N_1})$. The claim si valid as it is possible to obtain the matrices $R_{j,k}$, $C_{j,k}$ and D_j for every $j, k \leq N_1$ by iterated Lie brackets of elements in E_{ad}.

3) Finite dimensional estimates: Thanks to the previous claim and to the fact that $\pi_{N_1}(\Phi)\widetilde{\Gamma}_{N_1}\pi_{N_1}(\Phi) \in SU(\mathscr{H}_{N_1})$, there exist $p \in \mathbb{N}$, $M_1, ..., M_p \in E_{ad}$ and $\alpha_1, ..., \alpha_p \in \mathbb{R}^+$ such that

$$\pi_{N_1}(\Phi)\widetilde{\Gamma}_{N_1}\pi_{N_1}(\Phi) = e^{\alpha_1 M_1} \circ ... \circ e^{\alpha_p M_p}. \tag{21}$$

Claim. For every $l \leq p$ and $e^{\alpha_l M_l}$ from (21), there exist $\{T_n^l\}_{l \in \mathbb{N}} \subset \mathbb{R}^+$ and $\{u_n^l\}_{n \in \mathbb{N}}$ such that $u_n^l : (0, T_n^l) \to \mathbb{R}$ for every $n \in \mathbb{N}$ and

$$\lim_{n \to \infty} \|\Gamma_{T_n^l}^{u_n^l}\phi_k - e^{\alpha_l M_l}\phi_k\|_{(3)} = 0, \qquad \forall k \leq N_1, \tag{22}$$

$$\begin{gathered}\sup_{n \in \mathbb{N}} \|u_n^l\|_{BV(T_n)} < \infty, \qquad \sup_{n \in \mathbb{N}} \|u_n^l\|_{L^\infty((0,T_n),\mathbb{R})} < \infty, \\ \sup_{n \in \mathbb{N}} T_n \|u_n^l\|_{L^\infty((0,T_n),\mathbb{R})} < \infty.\end{gathered} \tag{23}$$

We consider the results developed in [Cha12, *Section* 3.1 & *Section* 3.2] by Chambrion and leading to [Cha12, *Proposition* 6] (also adopted in [Duc]). Each $e^{\alpha_l M_l}$ is a rotation in a two dimensional space for every $l \in \{1, ..., p\}$ and the mentioned work allows to explicit $\{T_n^l\}_{l \in \mathbb{N}} \subset \mathbb{R}^+$ and $\{u_n^l\}_{n \in \mathbb{N}}$ satisfying (23) such that $u_n^l : (0, T_n^l) \to \mathbb{R}$ for every $n \in \mathbb{N}$ and

$$\lim_{n \to \infty} \|\pi_{N_1}(\Phi)\Gamma_{T_n^l}^{u_n^l}\phi_k - e^{\alpha_l M_l}\phi_k\| = 0, \qquad \forall k \leq N_1. \tag{24}$$

As $e^{\alpha_l M_l} \in SU(\mathscr{H}_{N_1})$, we have $\lim_{n \to \infty} \|\Gamma_{T_n^l}^{u_n^l}\phi_k - e^{\alpha_l M_l}\phi_k\| = 0$ for $k \leq N_1$.

We consider the propagation of regularity developed by Kato in [Kat53] and adopted in [Duc]. We notice that $i(A + u(t)B - ic)$ is maximal dissipative in $H^2_{(0)}$ for suitable $c := \|u\|_{L^\infty((0,T),\mathbb{R})}$. Let $\lambda > c$ and $\widehat{H}^4_{(0)} := D(A(i\lambda - A)) \equiv H^4_{(0)}$. We know that $B : \widehat{H}^4_{(0)} \subset H^2_{(0)} \to H^2_{(0)}$ and the arguments of

Remark 1.1 imply that $B \in L(\widehat{H}^4_{(0)}, H^2_{(0)})$. For $T > 0$ and $u \in BV((0,T),\mathbb{R})$, we have $|||\, u(t)B(i\lambda - A)^{-1}\,|||_{(2)} < 1$ and

$$M := \sup_{t\in[0,T]} |||\, (i\lambda - A - u(t)B)^{-1}\,|||_{L(H^2_{(0)},\widehat{H}^4_{(0)})} \leq \sup_{t\in[0,T]} \sum_{l=1}^{+\infty} |||\, (u(t)B(i\lambda - A)^{-1})^l\,|||_{(2)} < +\infty.$$

We know that $\|k + f(\cdot)\|_{BV((0,T),\mathbb{R})} = \|f\|_{BV((0,T),\mathbb{R})}$ for every $f \in BV((0,T),\mathbb{R})$ and $k \in \mathbb{R}$. The same idea leads to

$$N := |||\, i\lambda - A - u(\cdot)B\,|||_{BV\left([0,T],L(\widehat{H}^4_{(0)},H^2_{(0)})\right)} = \|u\|_{BV(T)}\, |||\, B\,|||_{L(\widehat{H}^4_{(0)},H^2_{(0)})} < +\infty.$$

We call $C_1 := |||\, A(A + u(T)B - i\lambda)^{-1}\,|||_{(2)} < \infty$ and U^u_t the propagator generated by $A + uB - ic$ such that $U^u_t\psi = e^{-ct}\Gamma^u_t\psi$. Thanks to [Kat53, *Section* 3.10], for every $\psi \in H^4_{(0)}$, it follows $\|(A+u(T)B-i\lambda)U^u_t\psi\|_{(2)} \leq Me^{MN}\|(A - i\lambda)\psi\|_{(2)}$ and

$$\|\Gamma^u_T\psi\|_{(4)} = \|A\Gamma^u_T\psi\|_{(2)} \leq C_1 M e^{MN+cT}\|\psi\|_{(4)}$$

as $|||\, (A - i\lambda)A^{-1}\,|||_{(2)} = |||\, I - i\lambda A^{-1}\,|||_{(2)} \leq 1 + \frac{\lambda}{\pi^2}$. For every $T > 0$, $u \in BV((0,T),\mathbb{R})$ and $\psi \in H^4_{(0)}$, there exists $C(K) > 0$ depending on $K = \left(\|u\|_{BV(T)}, \|u\|_{L^\infty((0,T),\mathbb{R})}, T\|u\|_{L^\infty((0,T),\mathbb{R})}\right)$ such that $\|\Gamma^u_T\psi\|_{(4)} \leq C(K)\|\psi\|_{(4)}$. Then, from (23), there exists $C > 0$ such that

$$|||\, \Gamma^{u^l_n}_{T^l_n}\,|||_{(4)} \leq C. \tag{25}$$

For every $\psi \in H^4_{(0)}$, from the Cauchy-Schwarz inequality, we have $\|A\psi\|^2 \leq \|A^2\psi\|\|\psi\|$ and $\|A^{\frac{3}{2}}\psi\|^4 \leq \left(\langle A^2\psi, A\psi\rangle\right)^2 \leq \|A^2\psi\|^2\|A\psi\|^2$, which imply

$$\|\psi\|^8_{(3)} \leq \|\psi\|^2\|\psi\|^6_{(4)}. \tag{26}$$

In conclusion, the relations (24), (25) and (26) lead to the relation (22).

4) Infinite dimensional estimates:

> **Claim.** There exist $K_1, K_2, K_3 > 0$ such that for every $\epsilon > 0$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that $\|\Gamma^u_T\phi_k - \widehat{\Gamma}\phi_k\|_{(3)} \leq \epsilon$ for every $k \leq N$ and
>
> $$\|u\|_{BV(T)} \leq K_1, \qquad \|u\|_{L^\infty((0,T),\mathbb{R})} \leq K_2, \qquad T\|u\|_{L^\infty((0,T),\mathbb{R})} \leq K_3.$$

Let us assume $p = 2$. The following result is valid for any $p \in \mathbb{N}$. Thanks to (22) and to the propagation of regularity from [Kat53], for every $\epsilon > 0$

and $N_1 \in \mathbb{N}$, there exists $n \in \mathbb{N}$ large enough such that, for every $k \leq N$,

$$\|\Gamma_{T_n^2}^{u_n^2}\Gamma_{T_n^1}^{u_n^1}\phi_k - e^{\alpha_2 M_2}e^{\alpha_1 M_1}\phi_k\|_{(3)} \leq |\!|\!| \Gamma_{T_n^2}^{u_n^2} |\!|\!|_{(3)} \|\Gamma_{T_n^1}^{u_n^1}\phi_k - e^{\alpha_1 M_1}\phi_k\|_{(3)}$$
$$+ \sum_{l=1}^{N_1} \|\big(\Gamma_{T_n^2}^{u_n^2}\phi_l - e^{\alpha_2 M_2}\phi_l\big)\langle \phi_l, e^{\alpha_1 M_1}\phi_k\rangle\|_{(3)} \leq |\!|\!| \Gamma_{T_n^2}^{u_n^2} |\!|\!|_{(3)} \|\Gamma_{T_n^1}^{u_n^1}\phi_k - e^{\alpha_1 M_1}\phi_k\|_{(3)}$$
$$+ \|e^{\alpha_1 M_1}\phi_k\| \Big(\sum_{l=1}^{N_1} \|\big(\Gamma_{T_n^2}^{u_n^2}\phi_l - e^{\alpha_2 M_2}\phi_l\big)\|_{(3)}^2 \Big)^{\frac{1}{2}} \leq \epsilon.$$

In the previous inequality, we considered that $e^{\alpha_1 M_1}\phi_k \in \mathscr{H}_{N_1}$ and that $|\!|\!| \Gamma_{T_n^2}^{u_n^2} |\!|\!|_{(3)}$ is uniformly bounded in $n \in \mathbb{N}$ thanks to the propagation of regularity from [Kat53] and to (23). The identity (21) leads to the existence of $K_1, K_2, K_3 > 0$ such that for every $\epsilon > 0$, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that $\|\Gamma_T^u \phi_k - \widetilde{\Gamma}_{N_1}\phi_k\|_{(3)} < \epsilon$ for every $k \leq N$ and

$$\|u\|_{BV(T)} \leq K_1, \qquad \|u\|_{L^\infty((0,T),\mathbb{R})} \leq K_2, \qquad T\|u\|_{L^\infty((0,T),\mathbb{R})} \leq K_3. \tag{27}$$

The relation (18) and the triangular inequality achieve the claim.

5) Conclusion: For every $\{\psi_j\}_{j\leq N} \subset H^3_{(0)}$, $\widehat{\Gamma} \in U(\mathscr{H})$ such that $\{\widehat{\Gamma}\psi_j\}_{j\leq N} \subset H^3_{(0)}$ and $\epsilon > 0$, there exists a natural number $M \in \mathbb{N}$ such that, for every $l \leq N$, it follows $\|\psi_l\|_{(3)} \leq \big\| \sum_{k=1}^M \phi_k \langle \phi_k, \psi_l\rangle \big\|_{(3)}^2 + \epsilon$ and $\|\widehat{\Gamma}\psi_l\|_{(3)} \leq \big\| \sum_{k=1}^M \widehat{\Gamma}\phi_k \langle \phi_k, \psi_l\rangle \big\|_{(3)}^2 + \epsilon$. The proof is achieved by simultaneously driving $\{\phi_k\}_{k\leq M}$ close enough to $\{\widehat{\Gamma}\phi_k\}_{k\leq M}$ since, for every $l \leq N$, $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ satisfying (27),

$$\|\Gamma_T^u\psi_l - \widehat{\Gamma}\psi_l\|_{(3)} \leq \|\psi_l\| \Big(\sum_{k=1}^M \|\Gamma_T^u\phi_k - \widehat{\Gamma}\phi_k\|_{(3)}^2 \Big)^{\frac{1}{2}} + (\,|\!|\!| \Gamma_T^u |\!|\!|_{(3)} + 1)\epsilon. \quad \square$$

3.2 Proofs of Theorem 1.6 and Theorem 1.8

In the current section, we provide the proofs of Theorem 1.6 and Theorem 1.8, which require the following proposition.

Proposition 3.4. *Let $N \in \mathbb{N}$ and B satisfy Assumptions I.*

1. *For any $\{\psi_k^1\}_{k\leq N}$, $\{\psi_k^2\}_{k\leq N} \subset H^3_{(0)}$ orthonormal systems, there exist $T > 0$, $u \in L^2((0,T),\mathbb{R})$ and $\{\theta_k\}_{k\leq N} \subset \mathbb{R}$ such that $e^{i\theta_k}\psi_k^2 = \widetilde{\Gamma}_T^u\psi_k^1$ for every $k \leq N$.*

2. *If B satisfies Assumptions A, then for any $\{\psi_k^1\}_{k\leq N}$, $\{\psi_k^2\}_{k\leq N} \subset H^3_{(0)}$ orthonormal systems, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ so that $\psi_k^2 = \widetilde{\Gamma}_T^u\psi_k^1$ for every $k \leq N$.*

Proof. Let $N \in \mathbb{N}$ and let $u_0 \in \mathbb{R}$ belong to the neighborhoods provided by Lemma $B.5$, Lemma $B.6$ and Remark $B.9$ (Appendix B).

1) Let $\widetilde{\alpha}^{u_0}$ be the map with elements

$$\begin{cases} \frac{\overline{\widehat{\alpha}_{j,j}(u_1)}}{|\widehat{\alpha}_{j,j}(u_1)|}\widehat{\alpha}_{k,j}(u_1), & j,k \leq N, \\ \widehat{\alpha}_{k,j}(u_1), & k > N,\ j \leq N. \end{cases}$$

The proof of Proposition 2.1 can be repeated in order to prove the local surjectivity of $\widetilde{\alpha}^{u_0}$ for every $T > 0$, instead of α^{u_0} introduced in (13). As explained in Remark 3.1, this result corresponds to the simultaneous local exact controllability up to phases of N problems (3) in a neighborhood

$$O^N_{\epsilon,T} := \left\{ \{\psi_j\}_{j\leq N} \subset H^3_{(0)} \big|\ \langle \psi_j, \psi_k \rangle = \delta_{j,k};\ \sum_{j=1}^N \|\psi_j - \phi_j^{u_0}\|_{(3)} < \epsilon \right\}$$

with $\epsilon > 0$ small. In other words, for any $\{\psi_k\}_{k\leq N} \in O^N_{\epsilon,T}$, there exist $u \in L^2((0,T),\mathbb{R})$ and $\{\theta_j\}_{j\leq N} \subset \mathbb{R}$ so that $\Gamma^u_T \phi_j^{u_0} = e^{i\theta_j}\psi_j$ for any $j \leq N$.

Theorem 3.3 implies the simultaneous global approximate controllability for N problems. For any $\{\psi_j^1\}_{j\leq N} \subset H^3_{(0)}$ composed by orthonormal elements, there exist $T_1 > 0$ and $u_1 \in L^2((0,T_1),\mathbb{R})$ such that

$$\|\Gamma^{u_1}_{T_1}\psi_j^1 - \phi_j^{u_0}\|_{(3)} < \frac{\epsilon}{N}, \quad \forall j \leq N, \qquad \Longrightarrow \quad \{\Gamma^{u_1}_{T_1}\psi_j^1\}_{j\leq N} \in O^N_{\epsilon,T}.$$

The local controllability is also valid for the reversed dynamics of (10), for every $T > 0$, there exist $u \in L^2((0,T),\mathbb{R})$ and $\{\theta_j\}_{j\leq N} \subset \mathbb{R}$ so that

$$\{\Gamma^{u_1}_{T_1}\psi_j^1\}_{j\leq N} = \{e^{i\theta_j}\widetilde{\Gamma}^u_T\phi_j^{u_0}\}_{j\leq N} \qquad \Longrightarrow \quad \{e^{-i\theta_j}\Gamma^{\widetilde{u}}_T\Gamma^{u_1}_{T_1}\psi_j^1\}_{j\leq N} = \{\phi_j^{u_0}\}_{j\leq N}.$$

Then, there exist $T_2 > 0$ and $u_2 \in L^2((0,T_2),\mathbb{R})$ such that $\{e^{-i\theta_j}\Gamma^{u_2}_{T_2}\psi_j^1\}_{j\leq N} = \{\phi_j^{u_0}\}_{j\leq N}$. Now, the same property is valid for the reversed dynamics of (10) and, for every $\{\psi_j^2\}_{j\leq N} \subset H^3_{(0)}$ composed by orthonormal elements, there exist $T_3 > 0$, $u_3 \in L^2((0,T_3),\mathbb{R})$ and $\{\theta'_j\}_{j\leq N} \subset \mathbb{R}$ such that $\{e^{-i\theta'_j}\widetilde{\Gamma}^{u_3}_{T_3}\psi_j^2\}_{j\leq N} = \{\phi_j^{u_0}\}_{j\leq N}$. In conclusion, for $\widetilde{u}_3(\cdot) = u_3(T_3 - \cdot)$, the proof is achieved as

$$\{e^{-i(\theta_j-\theta'_j)}\Gamma^{\widetilde{u}_3}_{T_3}\Gamma^{u_2}_{T_2}\psi_j^1\}_{j\leq N} = \{\psi_j^2\}_{j\leq N}.$$

2) The proof of the second claim follows as in **1)**, with the difference that if B satisfies Assumptions A, then Remark $B.10$ provides the validity of a simultaneous local exact controllability without phase ambiguities.
Indeed, keeping in mind our notation, let $H^3_{(V)}$ be the space defined in [MN15]. We know that $H^3_{(V)}$ corresponds to $\widetilde{H}^3_{(0)}$ when $V = u_0 B$ and B is a suitable multiplication operator. We consider the assumptions $(C3)$,

$(C4)$ and $(C5)$ introduced in [MN15, $p.$ 10]. If we substitute V with $u_0 B$ and μ by $-B$, then the statement of [MN15, $Theorem$ 4.1] is still valid. The condition $(C3)$ is ensured by Lemma $B.5$ (Appendix B), while the assumptions $(C4)$ and $(C5)$ respectively follow from the first point of Remark $B.9$ and Remark $B.10$ (Appendix B). The result of [MN15, $Theorem$ 4.1] is valid in $O^N_{\epsilon,T} \subset H^3_{(0)}$ for suitable $\epsilon > 0$ and $T > 0$ as in the proof of Proposition 2.1. For every $\{\psi_k\}_{k\leq N} \in O^N_{\epsilon,T}$, there exists $u \in L^2((0,T),\mathbb{R})$ such that $\psi_k = \Gamma^u_T \phi^{u_0}_k$ for every $k \leq N$. The remaining part of the proof is achieved as in **1)**. □

Proof of Theorem 1.6. Let $N \in \mathbb{N}$ and $u_0 \in \mathbb{R}$ belong to the neighborhoods provided by Lemma $B.5$, Lemma $B.6$ and Remark $B.9$ (Appendix B). Let $\Psi^3 := \{\psi^3_j\}_{j\in\mathbb{N}} \in H^3_{(0)}$ be an orthonormal systems. We consider $\{\psi^1_j\}_{j\in\mathbb{N}}$, $\{\psi^2_j\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ complete orthonormal systems and $\widehat{\Gamma} \in U(\mathscr{H})$ such that $\widehat{\Gamma}\psi^1_j = \psi^2_j$ and $\widehat{\Gamma}^*\psi^3_j \in H^3_{(0)}$ for every $j \in \mathbb{N}$. Then, for every $k \leq N$,

$$\widetilde{\psi}_k := \sum_{j=1}^{\infty} \psi^1_j \langle \psi^2_j, \psi^3_k\rangle = \sum_{j=1}^{\infty} \psi^1_j \langle \widehat{\Gamma}\psi^1_j, \psi^3_k\rangle = \sum_{j=1}^{\infty} \psi^1_j \langle \psi^1_j, \widehat{\Gamma}^*\psi^3_k\rangle = \widehat{\Gamma}^*\psi^3_k \in H^3_{(0)}.$$

Thanks to the first point of Proposition 3.4, there exist $T > 0$, $u \in L^2((0,T),\mathbb{R})$ and $\{\theta_k\}_{k\leq N} \subset \mathbb{R}$ such that $e^{i\theta_k}\widetilde{\psi}_k = \widetilde{\Gamma}^u_T \psi^3_k$ for each $k \leq N$. Hence

$$\langle \psi^1_j, \widetilde{\Gamma}^u_T \psi^3_k\rangle = \langle e^{i\theta_j}\psi^1_j, e^{i\theta_k}\widetilde{\psi}_k\rangle = \langle \psi^2_j, e^{i\theta_k}\psi^3_k\rangle, \qquad \forall j,k \in \mathbb{N},\ k \leq N.$$

Thanks to the time reversibility (Appendix 1.3), we have

$$\langle \Gamma^{\widetilde{u}}_T \psi^1_j, \psi^3_k\rangle = \langle \psi^1_j, \widetilde{\Gamma}^u_T \psi^3_k\rangle = \langle \psi^2_j, e^{i\theta_k}\psi^3_k\rangle, \qquad \forall j,k \in \mathbb{N},\ k \leq N. \quad \square$$

Proof of Theorem 1.8. Let $N \in \mathbb{N}$ and let $u_0 \in \mathbb{R}$ belong to the neighborhoods provided by Lemma $B.5$, Lemma $B.6$, Remark $B.9$ and Remark $B.10$ (Appendix B).

1) Controllability in projection of orthonormal systems: Let $\Psi^3 := \{\psi^3_j\}_{j\in\mathbb{N}} \in H^3_{(0)}$ be an orthonormal system. Let us consider $\{\psi^1_j\}_{j\in\mathbb{N}}, \{\psi^2_j\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ be complete orthonormal systems and $\widehat{\Gamma} \in U(\mathscr{H})$ be such that $\widehat{\Gamma}\psi^1_j = \psi^2_j$ and $\widehat{\Gamma}^*\psi^3_j \in H^3_{(0)}$ for every $j \in \mathbb{N}$. As in the proof of Theorem 1.6, for every $k \leq N$, we define $\widetilde{\psi}_k := \sum_{j=1}^{\infty} \psi^1_j \langle \psi^2_j, \psi^3_k\rangle$. Thanks to the second point of Proposition 3.4, there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that $\widetilde{\psi}_k = \widetilde{\Gamma}^u_T \psi^3_k$ for each $k \leq N$. Hence

$$\langle \psi^1_j, \widetilde{\Gamma}^u_T \psi^3_k\rangle = \langle \psi^1_j, \widetilde{\psi}_k\rangle = \langle \psi^2_j, \psi^3_k\rangle, \qquad \forall j,k \in \mathbb{N},\ k \leq N.$$

Thanks to Appendix 1.3, we have $\langle \Gamma^{\widetilde{u}}_T \psi^1_j, \psi^3_k\rangle = \langle \psi^1_j, \widetilde{\Gamma}^u_T \psi^3_k\rangle = \langle \psi^2_j, \psi^3_k\rangle$ and then $\pi_N(\Psi^3)\psi^2_j = \pi_N(\Psi^3)\Gamma^{\widetilde{u}}_T \psi^1_j$ for every $j \in \mathbb{N}$.

2) Controllability in projection of unitarily equivalent functions: Let us consider $\{\psi_j^1\}_{j\in\mathbb{N}}, \{\psi_j^2\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ unitarily equivalent. Let $\Psi^3 := \{\psi_j^3\}_{j\in\mathbb{N}}$ be an orthonormal system. We suppose the existence of $\widehat{\Gamma} \in U(\mathscr{H})$ such that $\widehat{\Gamma}\psi_j^1 = \psi_j^2$ and $\widehat{\Gamma}^*\psi_j^3 \in H^3_{(0)}$ for every $j \in \mathbb{N}$. One knows that, for every $j \in \mathbb{N}$, there exists $\{a_k^j\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{C})$ such that $\psi_j^1 = \sum_{k\in\mathbb{N}} a_k^j \psi_k^3$. However, $\{\widehat{\Gamma}\psi_j^3\}_{j\in\mathbb{N}}$ is an Hilbert basis of $\mathscr{H}$ and $\psi_j^2 = \widehat{\Gamma}\psi_j^1 = \sum_{k\in\mathbb{N}} a_k^j \widehat{\Gamma}\psi_k^3$. The point 2) implies that there exist $T > 0$ and $u \in L^2((0,T),\mathbb{R})$ such that $\pi_N(\Psi^3)\ \Gamma_T^u\psi_k^3 = \pi_N(\Psi^3)\ \widehat{\Gamma}\psi_k^3$ for every $k \in \mathbb{N}$, and then for any $j \in \mathbb{N}$,

$$\pi_N(\Psi^3)\ \Gamma_T^u\psi_j^1 = \sum_{k\in\mathbb{N}} a_k^j\big(\pi_N(\Psi^3)\ \Gamma_T^u\psi_k^3\big) = \pi_N(\Psi^3)\ \sum_{k\in\mathbb{N}} a_k^j\widehat{\Gamma}\psi_k^3 = \pi_N(\Psi^3)\ \psi_j^2.$$

3) Controllability in projection with generic projector: Let $\Psi^3 = \{\psi_j^3\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ be a sequence of linearly independent elements. For every $N \in \mathbb{N}$, thanks the Gram-Schmidt orthonormalization process, there exists an orthonormal system $\widetilde{\Psi}^3 := \{\{\widetilde{\psi_j^3}\}_{j\le N}, 0, ...\}$ such that $span\{\psi_j^3 :\ j \le N\} = span\{\widetilde{\psi_j^3} :\ j \le N\}$. The claim follows as $\pi_N(\Psi^3) \equiv \pi_N(\widetilde{\Psi}^3)$. If $\Psi^3 = \{\psi_j^3\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ is a generic sequence of functions, then we extract from Ψ^3 a subsequence of linearly independent elements and repeat as above. □

4 Global exact controllability in projection of density matrices

Let $\psi^1,\ \psi^2 \in \mathscr{H}$. We define the rank one operator $|\psi^1\rangle\langle\psi^2|$ such that $|\psi^1\rangle\langle\psi^2|\psi = \psi^1\langle\psi^2,\psi\rangle$ for every $\psi \in \mathscr{H}$. For any $\widehat{\Gamma} \in U(\mathscr{H})$, we have

$$\widehat{\Gamma}|\psi^1\rangle\langle\psi^2| = |\widehat{\Gamma}\psi^1\rangle\langle\psi^2|, \qquad |\psi^1\rangle\langle\psi^2|\widehat{\Gamma}^* = |\psi^1\rangle\langle\widehat{\Gamma}\psi^2|.$$

Corollary 4.1. *Let B satisfy Assumptions I and Assumptions A. Let $\rho^1, \rho^2 \in T(\mathscr{H})$ be two density matrices so that $Ran(\rho^1)$, $Ran(\rho^2) \subseteq H^3_{(0)}$. We suppose the existence of $\widehat{\Gamma} \in U(\mathscr{H})$ so that $\rho^1 = \widehat{\Gamma}\rho^2\widehat{\Gamma}^*$. Let $\Psi^3 := \{\psi_j^3\}_{j\in\mathbb{N}} \subset H^3_{(0)}$ be such that $\{\widehat{\Gamma}\psi_j^3\}_{j\in\mathbb{N}} \subset H^3_{(0)}$, for every $j \in \mathbb{N}$. For any $N \in \mathbb{N}$, there exist $T > 0$ and a control function $u \in L^2((0,T),\mathbb{R})$ such that*

$$\pi_N(\Psi^3)\ \Gamma_T^u\rho^1(\Gamma_T^u)^*\ \pi_N(\Psi^3) = \pi_N(\Psi^3)\ \rho^2\ \pi_N(\Psi^3).$$

Proof. Let $T > 0$ and $\Psi^3 := \{\psi_j^3\}_{j\in\mathbb{N}} \in H^3_{(0)}$. Let $\rho^1,\ \rho^2 \in T(\mathscr{H})$ be two unitarily equivalent density matrices such that $Ran(\rho^1), Ran(\rho^2) \subseteq H^3_{(0)}$. We suppose that the unitary operator $\widehat{\Gamma} \in U(\mathscr{H})$ such that $\rho^2 = \widehat{\Gamma}\rho^1\widehat{\Gamma}$ satisfies the condition $\widehat{\Gamma}^*\psi_j^3 \in H^3_{(0)}$ for every $j \in \mathbb{N}$. One can ensure the existence of two complete orthonormal systems $\Psi^1 := \{\psi_j^1\}_{j\in\mathbb{N}}$,

$\Psi^2 := \{\psi_j^2\}_{j\in\mathbb{N}} \in H^3_{(0)}$ respectively composed by eigenfunctions of ρ^1 and ρ^2 such that $\rho^1 = \sum_{j=1}^\infty l_j|\psi_j^1\rangle\langle\psi_j^1|$ and $\rho^2 = \sum_{j=1}^\infty l_j|\psi_j^2\rangle\langle\psi_j^2|$. The sequence $\{l_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$ corresponds to the spectrum of ρ^1 and ρ^2. Now, thanks to Theorem 1.8, there exists a control function $u \in L^2((0,T),\mathbb{R})$ such that $\pi_N(\Psi^3)\ \Gamma_T^u\psi_j^1 = \pi_N(\Psi^3)\ \psi_j^2$. Thus

$$\pi_N(\Psi^3)\ \Gamma_T^u\rho^1(\Gamma_T^u)^*\pi_N(\Psi^3) = \sum\nolimits_{j=1}^\infty l_j|\pi_N(\Psi^3)\ \Gamma_T^u\psi_j^1\rangle\langle\psi_j^1\Gamma_T^u\pi_N(\Psi^3)\ |$$
$$= \sum\nolimits_{j=1}^\infty l_j\pi_N(\Psi^3)\ |\psi_j^2\rangle\langle\psi_j^2|\pi_N(\Psi^3) = \pi_N(\Psi^3)\ \rho^2\pi_N(\Psi^3). \qquad \square$$

Acknowledgments. The author thanks Thomas Chambrion for suggesting him the problem and Nabile Boussaïd for the periodic discussions. He is also grateful to Morgan Morancey for the explanation about the works [Mor14] and [MN15].

A Moment problem

In this appendix, we briefly adapt some results concerning the solvability of the moment problems (as (11) and (17)). Let [BL10, *Proposition* 19; 2)] be satisfied and $\{f_k\}_{k\in\mathbb{Z}}$ be a Riesz basis (see [BL10, *Definition* 2]) in $X = \overline{span\{f_k :\ k\in\mathbb{Z}\}}^{\mathscr{H}} \subseteq \mathscr{H}$, with $\mathscr{H}$ and Hilbert space. For $\{v_k\}_{k\in\mathbb{Z}}$ the unique biorthogonal family to $\{f_k\}_{k\in\mathbb{Z}}$ ([BL10, *Remark* 7]), $\{v_k\}_{k\in\mathbb{Z}}$ is also a Riesz basis of X ([BL10, *Remark* 9]). Thanks to [BL10, *Proposition* 19; 2)], there exist $C_1, C_2 > 0$ such that $C_1\sum_{k\in\mathbb{Z}}|x_k|^2 \leq \|u\|^2_{\mathscr{H}} \leq C_2\sum_{k\in\mathbb{Z}}|x_k|^2$ for every $u(t) = \sum_{k\in\mathbb{Z}} x_k v_k(t)$ with $\{x_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{C})$. Moreover, for every $u\in X$, we know that $u = \sum_{k\in\mathbb{Z}} v_k\langle f_k, u\rangle_{\mathscr{H}}$ since $\{f_k\}_{k\in\mathbb{Z}}$ and $\{v_k\}_{k\in\mathbb{Z}}$ are reciprocally biorthonoromal (see [BL10, *Remark* 9]) and

$$C_1\sum_{k\in\mathbb{Z}}|\langle f_k,u\rangle_{\mathscr{H}}|^2 \leq \|u\|^2_{\mathscr{H}} \leq C_2\sum_{k\in\mathbb{Z}}|\langle f_k,u\rangle_{\mathscr{H}}|^2. \tag{28}$$

When Haraux's Thoerem [KL05, *Theorem* 4.6] is verified, for $T>0$ large enough, $\{e^{i\lambda_k(\cdot)}\}_{k\in\mathbb{Z}}$ is a Riesz basis in $X = \overline{span\{e^{i\lambda_k(\cdot)} :\ k\in\mathbb{Z}\}}^{L^2} \subseteq L^2((0,T),\mathbb{C})$. The relation (28) is satisfied and $F : u\in X \longmapsto \left\{\langle e^{i\lambda_k(\cdot)}, u\rangle_{\mathscr{H}}\right\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{C})$ is invertible. For every sequence $\{x_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{C})$, there exists $u\in X$ such that $x_k = \int_0^T u(s)e^{-i\lambda_k s}ds$ for every $k\in\mathbb{Z}$.

Remark A.1. *Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be an ordered sequence of real numbers such that $\lambda_k \neq -\lambda_l$ for every $k,l\in\mathbb{N}$. Let $G := \inf_{k\neq j}|\lambda_k-\lambda_j| > 0$ and $G' := \sup_{K\subset\mathbb{N}}\inf_{\substack{k\neq j\\ k,j\in\mathbb{N}\setminus K}}|\lambda_k-\lambda_j|$, where K runs over the finite subsets of $\mathbb{Z}$. For $k>0$, we call $\omega_k = -\lambda_k$, while we impose $\omega_k = \lambda_{-k}$ for $k<0$ and $k\neq -l$. We call $\mathbb{Z}^* = \mathbb{Z}\setminus\{0\}$. The sequence $\{\omega_k\}_{k\in\mathbb{Z}^*\setminus\{-l\}}$ satisfies the hypotheses of [KL05, Theorem 4.6] for $\sup_{K\subset\mathbb{Z}^*\setminus\{-l\}}\inf_{\substack{k\neq j\\ k,j\in\mathbb{N}\setminus K}}|\omega_k-\omega_j| = G'$. Given*

$\{x_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{C})$, *we introduce* $\{\widetilde{x}_k\}_{k\in\mathbb{Z}^*\setminus\{-l\}} \in \ell^2(\mathbb{C})$ *such that* $\widetilde{x}_k = x_k$ *for* $k > 0$, *while* $\widetilde{x}_k = \overline{x}_{-k}$ *for* $k < 0$ *and* $k \neq -l$. *For* $T > 2\pi/G$, *there exists* $u \in L^2((0,T),\mathbb{C})$ *such that* $\widetilde{x}_k = \int_0^T u(s)e^{-i\omega_k s}ds$ *for every* $k \in \mathbb{Z}^* \setminus \{-l\}$. *Then*

$$\begin{cases} x_k = \int_0^T u(s)e^{i\lambda_k s}ds = \int_0^T \overline{u}(s)e^{i\lambda_k s}ds, & k \in \mathbb{N}\setminus\{l\}, \\ x_k = \int_0^T u(s)ds, & k = l, \end{cases}$$

which implies that, if $x_l \in \mathbb{R}$, *then* u *is real.*

B Analytic Perturbation

Let us consider the problem (12) and the eigenvalues $\{\lambda_j^{u_0}\}_{j\in\mathbb{N}}$ of the operator $A + u_0B$. When B is a bounded symmetric operator satisfying Assumptions I and $A = -\Delta$ is the Laplacian with Dirichlet type boundary conditions $D(A) = H^2((0,1),\mathbb{C}) \cap H_0^1((0,1),\mathbb{C})$, thanks to [Kat95, *Theorem VII*.2.6] and [Kat95, *Theorem VII*.3.9], the following proposition follows.

Proposition B.1. *Let* B *be a bounded symmetric operator satisfying Assumptions I. There exists a neighborhood* D *of* $u = 0$ *in* $\mathbb{R}$ *small enough where the maps* $u \mapsto \lambda_j^u$ *are analytic for every* $j \in \mathbb{N}$.

The next lemma proves the existence of perturbations, which do not shrink the eigenvalues gaps.

Lemma B.2. *Let* B *be a bounded symmetric operator satisfying Assumptions I. There exists a neighborhood* $U(0)$ *in* $\mathbb{R}$ *of* $u = 0$ *such that, for each* $u_0 \in U(0)$, *there exists* $r > 0$ *such that, for every* $j \in \mathbb{N}$,

$$\mu_j := \frac{\lambda_j + \lambda_{j+1}}{2} \in \rho(A + u_0B), \qquad |||\,(A + u_0B - \mu_j)^{-1}\,||| \leq r.$$

Proof. Let D be the neighborhood provided by Proposition B.1. We know $(A - \mu_j)$ is invertible in a bounded operator and $\mu_j \in \rho(A)$ (resolvent set of A). Let $\delta := \min_{j\in\mathbb{N}}\{|\lambda_{j+1} - \lambda_j|\}$. We know that $|||\,(A-\mu_j)^{-1}\,||| \leq \sup_{k\in\mathbb{N}} \frac{1}{|\mu_j - \lambda_k|} = \frac{2}{|\lambda_{j+1}-\lambda_j|} \leq \frac{2}{\delta}$. Thus

$$|||\,(A-\mu_j)^{-1}u_0B\,||| \leq |u_0|\,|||\,(A-\mu_j)^{-1}\,|||\,|||\,B\,||| \leq \frac{2}{\delta}|u_0|\,|||\,B\,|||$$

and if $|u_0| \leq \frac{\delta(1-\epsilon)}{2\,|||\,B\,|||}$ for $\epsilon \in (0,1)$, then $|||\,(A-\mu_j)^{-1}u_0B\,||| \leq 1-\epsilon$. The operator $(A + u_0B - \mu_j)$ is invertible and $|||\,(A + u_0B - \mu_j)^{-1}\,||| \leq \frac{2}{\delta\epsilon}$ as $\|(A + u_0B - \mu_j)\psi\| \geq \|(A-\mu_j)\psi\| - \|u_0B\psi\| \geq \frac{\delta}{2}\|\psi\| - \frac{\delta(1-\epsilon)}{2}\|\psi\|$ for every $\psi \in D(A)$. □

Lemma B.3. *Let B be a bounded symmetric operator satisfying Assumptions I. There exists a neighborhood $U(0)$ of 0 in $\mathbb{R}$ such that, for every $u_0 \in U(0)$,*

$$(A + u_0 P_{\phi_k}^{\perp} B - \lambda_k^{u_0})$$

is invertible with bounded inverse from $D(A) \cap \phi_k^{\perp}$ to $\phi_k^{\perp}$, for every $k \in \mathbb{N}$ and $P_{\phi_k}^{\perp}$ is the projector onto the orthogonal space of ϕ_k.

Proof. Let D be the neighborhood provided by Lemma $B.2$. For any $u_0 \in D$, one can consider the decomposition $(A+u_0 P_{\phi_k}^{\perp} B-\lambda_k^{u_0}) = (A-\lambda_k^{u_0})+u_0 P_{\phi_k}^{\perp} B$. The operator $A - \lambda_k^{u_0}$ is invertible with bounded inverse when it acts on the orthogonal space of ϕ_k and we estimate $\left|\!\left|\!\left| \left((A - \lambda_k^{u_0})\big|_{\phi_k^{\perp}}\right)^{-1} u_0 P_{\phi_k}^{\perp} B \right|\!\right|\!\right|$. However, for every $\psi \in D(A) \cap Ran(P_{\phi_k}^{\perp})$ such that $\|\psi\| = 1$, we have

$$\|(A - \lambda_k^{u_0})\psi\| \geq \min\{|\lambda_{k+1} - \lambda_k^{u_0}|, |\lambda_k^{u_0} - \lambda_{k-1}|\}\|\psi\|.$$

Let $\delta_k := \min\left\{|\lambda_{k+1} - \lambda_k^{u_0}|, |\lambda_k^{u_0} - \lambda_{k-1}|\right\}$. Thanks to Lemma $B.2$, for $|u_0|$ small enough, $\lambda_k^{u_0} \in \left(\frac{\lambda_{k-1}+\lambda_k}{2}, \frac{\lambda_k+\lambda_{k+1}}{2}\right)$ and then

$$\delta_k \geq \min\left\{\left|\lambda_{k+1} - \frac{\lambda_k + \lambda_{k+1}}{2}\right|, \left|\frac{\lambda_{k-1} + \lambda_k}{2} - \lambda_{k-1}\right|\right\} \geq \frac{(2k-1)\pi^2}{2} > k.$$

Afterwards, $\left|\!\left|\!\left| \left((A - \lambda_k^{u_0})\big|_{\phi_k^{\perp}}\right)^{-1} u_0 P_{\phi_k}^{\perp} B \right|\!\right|\!\right| \leq \frac{1}{\delta_k}|u_0| \left|\!\left|\!\left| B \right|\!\right|\!\right|$ and, if $|u_0| \leq (1 - r)\frac{\delta_k}{\left|\!\left|\!\left| B \right|\!\right|\!\right|}$ for $r \in (0, 1)$, then it follows $\left|\!\left|\!\left| \left((A-\lambda_k^{u_0})\big|_{\phi_k^{\perp}}\right)^{-1} u_0 P_{\phi_k}^{\perp} B \right|\!\right|\!\right| \leq (1-r) < 1$. The operator $A_k := (A - \lambda_k^{u_0} + u_0 P_{\phi_k}^{\perp} B)$ is invertible when it acts on the orthogonal space of ϕ_k and, for every $\psi \in D(A)$,

$$\|A_k\psi\| \geq \|(A - \lambda_k^{u_0})\psi\| - \left|\!\left|\!\left| u_0 P_{\phi_k}^{\perp} B\psi \right|\!\right|\!\right| \geq \delta_k\|\psi\| - \left|\!\left|\!\left| u_0 P_{\phi_k}^{\perp} B \right|\!\right|\!\right| \|\psi\| = r\delta_k\|\psi\|.$$

In conclusion, $\left|\!\left|\!\left| \left((A - \lambda_k^{u_0} + u_0 P_{\phi_k}^{\perp} B)\big|_{\phi_k^{\perp}}\right)^{-1} \right|\!\right|\!\right| \leq \frac{1}{rk}$ for every $k \in \mathbb{N}$. □

Lemma B.4. *Let B be satisfy Assumptions I. There exists a neighborhood $U(0)$ of 0 in $\mathbb{R}$ such that, for any $u_0 \in U(0)$, we have $\lambda_j^{u_0} \neq 0$ and $\lambda_j^{u_0} \asymp \lambda_j$ for every $j \in \mathbb{N}$. In other words, there exist two constants $C_1, C_2 > 0$ such that, for each $j \in \mathbb{N}$, $C_1\lambda_j \leq \lambda_j^{u_0} \leq C_2\lambda_j$.*

Proof. Let $u_0 \in D$ for D the neighborhood provided by Lemma $B.3$. We decompose the eigenfunction $\phi_j^{u_0} = a_j\phi_j + \eta_j$, where a_j is an orthonormalizing constant and η_j is orthogonal to ϕ_j. Hence $\lambda_k^{u_0}\phi_k^{u_0} = (A + u_0 B)(a_k\phi_k + \eta_k)$ and $\lambda_k^{u_0} a_k\phi_k + \lambda_k^{u_0}\eta_k = Aa_k\phi_k + A\eta_k + u_0 Ba_k\phi_k + u_0 B\eta_k$. By projecting onto the orthogonal space of ϕ_k,

$$\lambda_k^{u_0}\eta_k = A\eta_k + u_0 P_{\phi_k}^{\perp} Ba_k\phi_k + u_0 P_{\phi_k}^{\perp} B\eta_k.$$

However, Lemma $B.3$ ensures that $A + u_0 P_{\phi_k}^{\perp} B - \lambda_k^{u_0}$ is invertible with bounded inverse when it acts on the orthogonal space of ϕ_k and then

$$\eta_k = -a_k((A + u_0 P_{\phi_k}^{\perp} B - \lambda_k^{u_0})\big|_{\phi_k^{\perp}})^{-1} u_0 P_{\phi_k}^{\perp} B\phi_k, \tag{29}$$

$$\implies \quad \lambda_j^{u_0} = \langle a_j\phi_j + \eta_j, (A + u_0 B)(a_j\phi_j + \eta_j)\rangle = |a_j|^2\lambda_j + u_0\langle a_j\phi_j, Ba_j\phi_j\rangle$$
$$+ \langle a_j\phi_j, (A + u_0 B)\eta_j\rangle + \langle \eta_j, (A + u_0 B)a_j\phi_j\rangle + \langle \eta_j, (A + u_0 B)\eta_j\rangle.$$

By using the relation (29),

$$\langle \eta_j, (A + u_0 B)\eta_j\rangle = \langle \eta_j, (A + u_0 P_{\phi_k}^{\perp} B - \lambda_j^{u_0})\eta_j\rangle + \lambda_j^{u_0}\|\eta_j\|^2 = \lambda_j^{u_0}\|\eta_j\|^2$$
$$+ \left\langle \eta_j, -a_j(A + u_0 P_{\phi_j}^{\perp} B - \lambda_j^{u_0})((A + u_0 P_{\phi_j}^{\perp} B - \lambda_j^{u_0})\big|_{\phi_j^{\perp}})^{-1} u_0 P_{\phi_j}^{\perp} B\phi_j \right\rangle.$$

However, $(A+u_0 P_{\phi_j}^{\perp} B-\lambda_j^{u_0})((A+u_0 P_{\phi_j}^{\perp})B-\lambda_j^{u_0})\big|_{\phi_j^{\perp}})^{-1} = Id$ and $\langle \eta_j, (A+u_0 B)\eta_j\rangle = \lambda_j^{u_0}\|\eta_j\|^2 - u_0 a_j\langle \eta_j, P_{\phi_j}^{\perp} B\phi_j\rangle$. Moreover, we have $\langle \phi_j, (A+u_0 B)\eta_j\rangle = u_0\langle \phi_j, B\eta_j\rangle = u_0\langle P_{\phi_j}^{\perp} B\phi_j, \eta_j\rangle$ and $\langle \eta_j, (A+u_0 B)\phi_j\rangle = u_0\langle \eta_j, P_{\phi_j}^{\perp} B\phi_j\rangle$. Thus

$$\lambda_j^{u_0} = |a_j|^2\lambda_j + u_0|a_j|^2 B_{j,j} + \lambda_j^{u_0}\|\eta_j\|^2 + u_0\overline{a_j}\langle P_{\phi_j}^{\perp} B\phi_j, \eta_j\rangle. \tag{30}$$

One can notice that $|a_j| \in [0,1]$ and $\|\eta_j\|$ are uniformly bounded in j. We show that the first accumulates at 1 and the second at 0. Indeed, from the proof of Lemma $(B.3)$ and the relation (29), there exists $C_1 > 0$ such that

$$\|\eta_j\|^2 \leq |u_0|^2 \left\| \left((A + u_0 P_{\phi_j}^{\perp} B - \lambda_j^{u_0})\big|_{\phi_j^{\perp}}\right)^{-1} \right\|^2 |a_j|^2 \|B\phi_j\|^2 \leq \frac{C_1}{j^2} \tag{31}$$

for $r \in (0,1)$, which implies that $\lim_{j\to\infty} \|\eta_j\| = 0$. Afterwards, by contradiction, if $|a_j|$ does not converge to 1, then there exists $\{a_{j_k}\}_{k\in\mathbb{N}}$ a subsequence of $\{a_j\}_{j\in\mathbb{N}}$ such that $|a_{j_\infty}| := \lim_{k\to\infty} |a_{j_k}| \in [0,1)$. Now, we have

$$1 = \lim_{k\to\infty} \|\phi_{j_k}^{u_0}\| \leq \lim_{k\to\infty} |a_{j_k}|\|\phi_{j_k}\| + \|\eta_{j_k}\| = \lim_{k\to\infty} |a_{j_k}| + \|\eta_{j_k}\| = |a_{j_\infty}| < 1$$

that is absurd. Then, $\lim_{j\to\infty} |a_j| = 1$. From (30), it follows $\lambda_j^{u_0} \asymp \lambda_j$ for $|u_0|$ small enough. The relation also implies that $\lambda_j^{u_0} \neq 0$ for every $j \in \mathbb{N}$ and $|u_0|$ small enough. □

Lemma B.5. *Let B be a bounded symmetric operator satisfying Assumptions I. For every $N \in \mathbb{N}$, there exists a neighborhood $U(0)$ of 0 in $\mathbb{R}$ such that there exists $\widetilde{C}_N > 0$ such that, for any $u_0 \in U(0)$, we have $|\langle \phi_k^{u_0}, B\phi_j^{u_0}\rangle| \geq \frac{\widetilde{C}_N}{k^3}$ for every $k, j \in \mathbb{N}$ and $j \leq N$.*

Proof. We start by choosing $k \in \mathbb{N}$ such that $k \neq j$ and $u_0 \in D$ for D the neighborhood provided by Lemma $B.4$. Thanks to Assumptions II, we have

$$\begin{aligned} &|\langle \phi_k^{u_0}, B\phi_j^{u_0}\rangle| = |\langle a_k\phi_k + \eta_k, B(a_j\phi_j + \eta_j)\rangle| \\ &\geq C_N \frac{\overline{a_k} a_j}{k^3} - \left|\overline{a_k}\langle \phi_k, B\eta_j\rangle + a_j\langle \eta_k, B\phi_j\rangle + \langle \eta_k, B\eta_j\rangle\right|. \end{aligned} \tag{32}$$

1) Expansion of $\langle \eta_k, B\phi_j\rangle$, $\langle \phi_k, B\eta_j\rangle$, $\langle \eta_k, B\eta_j\rangle$: Thanks to (29), we have $\langle \eta_k, B\phi_j\rangle = \langle -a_k((A+u_0P^\perp_{\phi_k}B-\lambda_k^{u_0})\big|_{\phi_k^\perp})^{-1}u_0P^\perp_{\phi_k}B\phi_k, P^\perp_{\phi_k}B\phi_j\rangle$ for every $k\in\mathbb{N}$ and $j\leq N$, while $\big((A+u_0P^\perp_{\phi_k}B-\lambda_k^{u_0})\big|_{\phi_k^\perp}\big)^{-1}$ corresponds to

$$((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}\sum_{n=0}^{\infty}\big(u_0((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}BP^\perp_{\phi_k}\big)^n$$

for $|u_0|$ small enough. For $M_k := \sum_{n=0}^{\infty}\big(u_0((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}B\big)^nP^\perp_{\phi_k}$,

$$\langle \eta_k, B\phi_j\rangle = -u_0\langle a_kM_kB\phi_k, ((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}B\phi_j\rangle. \tag{33}$$

Thanks to $B: D(A)\to D(A)$, for every $k\in\mathbb{N}$ and $j\leq N$,

$$\begin{aligned}&((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}B\phi_j = P^\perp_{\phi_k}B((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}\phi_j - \big[P^\perp_{\phi_k}B, ((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}\big]\phi_j\\ &= P^\perp_{\phi_k}B((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}\phi_j - ((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}[B,A]((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}\phi_j.\end{aligned}$$

For $\widetilde{B}_k := ((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}[B,A]$, we have $((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}B\phi_j = P^\perp_{\phi_k}(B+\widetilde{B}_k)(\lambda_j-\lambda_k^{u_0})^{-1}\phi_j$. and, for every $k\in\mathbb{N}$ and $j\leq N$,

$$\langle \eta_k, B\phi_j\rangle = -\frac{u_0}{\lambda_j-\lambda_k^{u_0}}\langle a_kM_kB\phi_k, (B+\widetilde{B}_k)\phi_j\rangle. \tag{34}$$

For every $k\in\mathbb{N}$ and $j\leq N$, we obtain

$$\begin{aligned}&|\langle \eta_k, B\eta_j\rangle| = |\langle B\eta_k, \eta_j\rangle| = |\langle u_0a_kB((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}M_kB\phi_k,\\ &u_0a_j((A-\lambda_j^{u_0})P^\perp_{\phi_j})^{-1}M_jB\phi_j\rangle\Big| = \Big|\frac{a_j\overline{a_k}u_0^2}{\lambda_k-\lambda_j^{u_0}}\big\langle\phi_k, L_{k,j}\phi_j\big\rangle\Big|\end{aligned} \tag{35}$$

with $L_{k,j} := (A-\lambda_j^{u_0})BM_k((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}B((A-\lambda_j^{u_0})P^\perp_{\phi_j})^{-1}M_jB$. Now, there exists $\epsilon>0$ such that $|a_l|\in(\epsilon,1)$ for every $l\in\mathbb{N}$. Thanks to (34), (35) and (32), there exists $\widehat{C}_N$ such that

$$\begin{aligned}|\langle \phi_k^{u_0}, B\phi_j^{u_0}\rangle| &\geq \widehat{C}_N\frac{1}{k^3} - \Big|\frac{u_0}{\lambda_j-\lambda_k^{u_0}}\langle M_kB\phi_k, (B+\widetilde{B}_k)\phi_j\rangle\Big|\\ &- \Big|\frac{u_0}{\lambda_k-\lambda_j^{u_0}}\langle (B+\widetilde{B}_j)\phi_k, M_jB\phi_j\rangle\Big| - \Big|\frac{u_0^2}{\lambda_k-\lambda_j^{u_0}}\big\langle\phi_k, L_{k,j}\phi_j\big\rangle\Big|.\end{aligned} \tag{36}$$

2) Features of the operators M_k, $\widetilde{B}_k$, $L_{k,j}$: Each M_k for $k\in\mathbb{N}$ is uniformly bounded in $L(H^2_{(0)}, H^2_{(0)})$ when $|u_0|$ is small enough so that $|||\, u_0((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}BP^\perp_{\phi_k}\,|||_{L(H^2_{(0)})}<1$. The definition of $\widetilde{B}_k$ implies that $\widetilde{B}_kP^\perp_{\phi_k} = ((A-\lambda_k^{u_0})P^\perp_{\phi_k})^{-1}P^\perp_{\phi_k}B(A-\lambda_k^{u_0})P^\perp_{\phi_k} - P^\perp_{\phi_k}BP^\perp_{\phi_k}$. Hence, the operators $\widetilde{B}_k$ are uniformly bounded in k in $L\big(H^2_{(0)}\cap Ran(P^\perp_{\phi_k}), H^2_{(0)}\cap Ran(P^\perp_{\phi_k})\big)$. Third, one

can notice that $B((A-\lambda_j^{u_0})P_{\phi_j}^{\perp})^{-1}M_jB \in L(H^2_{(0)},H^2_{(0)})$ for every $j\in\mathbb{N}$. Then, for every $k\in\mathbb{N}$ and $j\leq N$,

$$(A-\lambda_j^{u_0})BM_k((A-\lambda_k^{u_0})P_{\phi_k}^{\perp})^{-1}P_{\phi_k}^{\perp}=(A-\lambda_j^{u_0})B((A-\lambda_k^{u_0})P_{\phi_k}^{\perp})^{-1}$$
$$\sum_{n=0}^{\infty}\left(u_0P_{\phi_k}^{\perp}B((A-\lambda_k^{u_0})P_{\phi_k}^{\perp})^{-1}\right)^nP_{\phi_k}^{\perp}=(A-\lambda_j^{u_0})((A-\lambda_k^{u_0})P_{\phi_k}^{\perp})^{-1}P_{\phi_k}^{\perp}(\widetilde{B}_k+B)\widetilde{M}_k$$

with $\widetilde{M}_k:=\sum_{n=0}^{\infty}\left(u_0P_{\phi_k}^{\perp}B((A-\lambda_k^{u_0})P_{\phi_k}^{\perp})^{-1}\right)^nP_{\phi_k}^{\perp}$. Now, the operators $\widetilde{M}_k$ are uniformly bounded in $L(H^2_{(0)},H^2_{(0)})$ as M_k. Hence $L_{k,j}$ are uniformly bounded in $L(H^2_{(0)},H^2_{(0)})$.

Let $\{F_l\}_{l\in\mathbb{N}}$ be an infinite uniformly bounded family of operators in $L(H^2_{(0)},H^2_{(0)})$. For every $l,j\in\mathbb{N}$, there exists $c_{l,j}>0$ such that $\sum_{k=1}^{\infty}|k^2\langle\phi_k,F_l\phi_j\rangle|^2<\infty$, which implies $|\langle\phi_k,F_l\phi_j\rangle|\leq\frac{c_{l,j}}{k^2}$ for every $k\in\mathbb{N}$. Now, the constant $c_{l,j}$ can be assumed uniformly bounded in l since, for every $k,j\in\mathbb{N}$,

$$\sup_{l\in\mathbb{N}}|k^2\langle\phi_k,F_l\phi_j\rangle|^2\leq\sup_{l\in\mathbb{N}}\sum_{m\in\mathbb{N}}|m^2\langle\phi_m,F_l\phi_j\rangle|^2\leq\sup_{l\in\mathbb{N}}\|F_l\phi_j\|^2_{(2)}<\infty.$$

Thus, for every infinite uniformly bounded family of operators $\{F_l\}_{l\in\mathbb{N}}$ in $L(H^2_{(0)},H^2_{(0)})$ and for every $j\in\mathbb{N}$, there exists a constant c_j such that

$$|\langle\phi_k,F_l\phi_j\rangle|\leq\frac{c_j}{k^2},\qquad\forall k,l\in\mathbb{N}. \tag{37}$$

3) Conclusion: We know that $|\lambda_j-\lambda_k^{u_0}|^{-1}$ and $|\lambda_k-\lambda_j^{u_0}|^{-1}$ asymptotically behave as k^{-2} thanks to Lemma $B.4$. From the previous point, the families of operators $\{BM_k(B+\widetilde{B}_k)\}_{k\in\mathbb{N}}$, $\{L_{k,j}\}_{k\in\mathbb{N}}$ are uniformly bounded in $L(H^2_{(0)},H^2_{(0)})$ and $BM_j(B+\widetilde{B}_j)\in L(H^2_{(0)},H^2_{(0)})$ for every $1\leq j\leq N$. Hence, we use the relation (37) in (36) and there exist $C_1,C_2,C_3,C_4>0$ depending on $j\in\mathbb{N}$ such that, for $|u_0|$ small enough and $k\in\mathbb{N}$ large enough,

$$|\langle\phi_k^{u_0},B\phi_j^{u_0}\rangle|\geq\widehat{C}_N\frac{1}{k^3}-\frac{C_1|u_0|}{|\lambda_j-\lambda_k^{u_0}|k^2}-\frac{C_2|u_0|}{|\lambda_k-\lambda_j^{u_0}|k^2}-\frac{C_3|u_0|^2}{|\lambda_k-\lambda_j^{u_0}|k^2}\geq C_4\frac{1}{k^3}. \tag{38}$$

Let $K\in\mathbb{N}$ be so that $|\langle\phi_k^{u_0}(T),B\phi_j^{u_0}(T)\rangle|\geq C_4\frac{1}{k^3}$ for every $k>K$. For $j\in\mathbb{N}$, the zeros of the analytic map $u_0\mapsto\{|\langle\phi_k^{u_0}(T),B\phi_j^{u_0}(T)\rangle|\}_{k\leq K}\in\mathbb{R}^K$ are discrete. Then, for $|u_0|$ small enough, $|\langle\phi_k^{u_0}(T),B\phi_j^{u_0}(T)\rangle|\neq 0$ for every $k\leq K$. Thus, for every $j\in\mathbb{N}$ and $|u_0|$ small enough, there exists $C_j>0$ such that $|\langle\phi_k^{u_0}(T),B\phi_j^{u_0}(T)\rangle|\geq\frac{C_j}{k^3}$ for every $k\in\mathbb{N}$. In conclusion, the claim is achieved for every $k\in\mathbb{N}$ and $j\leq N$ with $\widetilde{C}_N=\min\{C_j:\ j\leq N\}$. □

Lemma B.6. *Let B be a bounded symmetric operator satisfying Assumptions I. There exists a neighborhood $U(0)$ of 0 in $\mathbb{R}$ contained in the one introduced in Lemma B.4 such that, for any $u_0 \in U(0)$,*

$$\Big(\sum_{j=1}^{\infty} \big||\lambda_j^{u_0}|^{\frac{3}{2}}\langle \phi_j^{u_0}, \cdot\rangle\big|^2\Big)^{\frac{1}{2}} \asymp \Big(\sum_{j=1}^{\infty} |j^3\langle \phi_j, \cdot\rangle|^2\Big)^{\frac{1}{2}}.$$

Proof. Let D be the neighborhood provided by Lemma $B.4$. For $|u_0|$ small enough, we prove that there exist $C_1 > 0$ such that $\||A + u_0B|^{\frac{s}{2}}\psi\| \leq C_1\||A|^{\frac{s}{2}}\psi\|$ for $s = 3$. We start with $s = 4$ and we recall that $B \in L(H^2_{(0)})$ thanks to Remark 1.1. For any $\psi \in H^4_{(0)}$, there exists $C_2 > 0$ such that

$$\|(A + u_0B)^2\psi\| \leq \|A^2\psi\| + |u_0|^2\|B^2\psi\| + |u_0|\|A\psi\|(\,|||\,B\,|||_{(2)} + |||\,B\,|||\,) \leq C_2\||A|^2\psi\|.$$

The proof of [BdCC13, *Lemma* 1] implies the validity of the relation also for $s = 3$. There exists $C > 0$ such that $\|\psi\|_{\widetilde{H}^3_{(0)}} = \||A + u_0B|^{\frac{3}{2}}\psi\| \leq C\||A|^{\frac{3}{2}}\psi\| = C\|\psi\|_{H^3_{(0)}}$ for every $\psi \in H^3_{(0)}$. Now, $H^2_{(0)} = D(|A|) = D(|A + u_0B|) = \widetilde{H}^2_{(0)}$ and B preserves $\widetilde{H}^2_{(0)}$ since $B : H^2_{(0)} \longrightarrow H^2_{(0)}$. The arguments of Remark 1.1 imply that $B \in L(\widetilde{H}^2_{(0)})$ and the opposite inequality follows as above thanks to the identity $A = (A + u_0B) - u_0B$. □

Remark B.7. *Let B be a bounded symmetric operator satisfying Assumptions I. The techniques of the proof of Lemma B.6 also allow to prove that, for $s \in (0, 3)$, there exists a neighborhood $U(0)$ of 0 in $\mathbb{R}$ such that, for any $u_0 \in U(0)$, it follows $\Big(\sum_{j=1}^{\infty}\big|(\lambda_j^{u_0})^{\frac{s}{2}}\langle \phi_j^{u_0}, \cdot\rangle\big|^2\Big)^{\frac{1}{2}} \asymp \Big(\sum_{j=1}^{\infty}|j^s\langle \phi_j, \cdot\rangle|^2\Big)^{\frac{1}{2}}$.*

Lemma B.8. *Let B be a bounded symmetric operator satisfying Assumptions I and $N \in \mathbb{N}$. Let $\epsilon > 0$ small enough and I^N be the set defined in (5). There exists a $U_\epsilon \subset \mathbb{R} \setminus \{0\}$ such that, for each $u_0 \in U_\epsilon$,*

$$\inf_{\substack{(j,k),(n,m)\in I^N \\ (j,k)\neq(n,m)}} |\lambda_j^{u_0} - \lambda_k^{u_0} - \lambda_n^{u_0} + \lambda_m^{u_0}| > \epsilon.$$

Moreover, for every $\delta > 0$ small there exists $\epsilon > 0$ such that $dist(U_\epsilon, 0) < \delta$.

Proof. Let us consider the neighborhood D provided by Lemma $B.3$. The maps $\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u$ are analytic for each $j, k, n, m \in \mathbb{N}$ and $u \in D$. One can notice that the number of elements such that

$$\lambda_j - \lambda_k - \lambda_n + \lambda_m = 0, \qquad j, n \in \mathbb{N}, \quad k, m \leq N \tag{39}$$

is finite. Indeed $\lambda_k = k^2\pi^2$ and (39) corresponds to $j^2 - k^2 = n^2 - m^2$. We have $|j^2 - n^2| = |k^2 - m^2| \leq N^2 - 1$, which is satisfied for a finite number of elements. Thus, for I^N (defined in (5), the following set is finite

$$R := \{((j,k),(n,m)) \in (I^N)^2 \ : \ (j,k) \neq (n,m);\ \lambda_j - \lambda_k - \lambda_n + \lambda_m = 0\}.$$

1) Let $((j,k),(n,m)) \in R$, the set $V_{(j,k,n,m)} = \{u \in D \big| \ \lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u = 0\}$ is a discrete subset of D or equal to D. Thanks to the relation (30),

$$\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u = |a_j|^2\lambda_j + u|a_j|^2 B_{j,j} + \lambda_j^u\|\eta_j\|^2 + u\overline{a_j}\langle P_{\phi_j}^{\perp} B\phi_j, \eta_j\rangle - |a_k|^2\lambda_k$$
$$- u|a_k|^2 B_{k,k} - \lambda_k^u\|\eta_k\|^2 - u\overline{a_k}\langle P_{\phi_k}^{\perp} B\phi_k, \eta_k\rangle - |a_n|^2\lambda_n - u|a_n|^2 B_{n,n} - \lambda_n^u\|\eta_n\|^2$$
$$- u\overline{a_n}\langle P_{\phi_n}^{\perp} B\phi_n, \eta_n\rangle + |a_m|^2\lambda_m + u|a_m|^2 B_{m,m} + \lambda_m^u\|\eta_m\|^2 + u\overline{a_m}\langle P_{\phi_m}^{\perp} B\phi_m, \eta_m\rangle$$

$$\implies \quad \lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u = |a_j|^2\lambda_j - |a_k|^2\lambda_k - |a_n|^2\lambda_n + |a_m|^2\lambda_m \tag{40}$$
$$+ (|a_j|^2 B_{j,j} - |a_k|^2 B_{k,k} - |a_n|^2 B_{n,n} + |a_m|^2 B_{m,m})u + o(u).$$

For $|u|$ small enough, thanks to $lim_{|u|\to 0}|a_j|^2 = 1$ and to the third point of Assumptions I, $\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u$ can not be constantly equal to 0. Then, $V_{(j,k,n,m)}$ is discrete and $V = \{u \in D \big| \ \exists (j,k,n,m) \in R : \lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u = 0\}$ is a discrete subset of D. As R is a finite set $\widetilde{U}_\epsilon := \{u \in D : \forall (j,k,n,m) \in R \big| \ |\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u| \geq \epsilon\}$ has positive measure for $\epsilon > 0$ small enough. Moreover, for any $\delta > 0$ small, there exists $\epsilon_0 > 0$ such that $dist(0, \widetilde{U}_{\epsilon_0}) < \delta$.

2) Let $((j,k),(n,m)) \in (I^N)^2 \setminus R$ be different numbers. We know that $|\lambda_j^0 - \lambda_k^0 - \lambda_n^0 + \lambda_m^0| = \pi^2|j^2 - k^2 - n^2 + m^2| > \pi^2$. First, thanks to (30), we have $\lambda_j^u \leq |a_j|^2\lambda_j + |u|C_1$ and $\lambda_j^u \geq |a_j|^2\lambda_j - |u|C_2$ for suitable constants $C_1, C_2 > 0$ non depending on the index j. Thus

$$|\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u| \geq ||a_j|^2\lambda_j - |a_k|^2\lambda_k - |a_n|^2\lambda_n + |a_m|^2\lambda_m| - |u|(2C_1 + 2C_2).$$

Now, $\lim_{k\to\infty}|a_k|^2 = 1$. For any u in D and ϵ small enough, there exists $M_\epsilon \in \mathbb{N}$ such that $||a_j|^2\lambda_j - |a_k|^2\lambda_k - |a_n|^2\lambda_n + |a_m|^2\lambda_m| \geq \pi^2 - \epsilon$ for every $((j,k),(n,m)) \in R^C := (I^N)^2 \setminus R$ and $j,k,n,m \geq M_\epsilon$. However $\lim_{|u|\to 0}|a_k|^2 = 1$ uniformly in k thanks to (31) and then there exists a neighborhood $W_\epsilon \subseteq D$ such that, for each $u \in W_\epsilon$, it follows $||a_j|^2\lambda_j - |a_k|^2\lambda_k - |a_n|^2\lambda_n + |a_m|^2\lambda_m| \geq \pi^2 - \epsilon$ for every $((j,k),(n,m)) \in R^C$ and $1 \leq j,k,n,m < M_\epsilon$. Thus, for each $u \in W_\epsilon$ and $((j,k),(n,m)) \in R^C$ such that $(j,k) \neq (n,m)$, we have $|\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u| \geq \pi^2 - \epsilon$.

3) The proof is achieved since, for $\epsilon_1 > 0$ small enough, $\widetilde{U}_{\epsilon_1} \cap W_\epsilon$ is a non-zero measure subset of D. For any $u \in \widetilde{U}_{\epsilon_1} \cap W_\epsilon$ and for any $((j,k),(n,m)) \in (I^N)^2$ such that $(j,k) \neq (n,m)$, we have $|\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u| \geq \min\{\pi^2 - \epsilon, \epsilon_1\}$. $\square$

Remark B.9. *Let B be a bounded symmetric operator satisfying Assumptions I. By using the techniques of the proofs of Lemma B.5 and Lemma B.8, one can ensure the existence of a neighborhood U_1 of u_0 in $\mathbb{R}$ and U_2, a countable subset of $\mathbb{R}$ such that, for any $u_0 \in U(0) := (U_1 \setminus U_2) \setminus \{0\}$, we have:*

1. *For every $N \in \mathbb{N}$, $(j,k),(n,m) \in I^N$ (see (5)) such that $(j,k) \neq (n,m)$, there holds $\lambda_j^{u_0} - \lambda_k^{u_0} - \lambda_n^{u_0} + \lambda_m^{u_0} \neq 0$.*

2. $B_{j,k}^{u_0} = \langle \psi_j^{u_0}(T), B\phi_k^{u_0}(T)\rangle \neq 0$ *for every* $j,k \in \mathbb{N}$.

3. *For* $\epsilon > 0$, *if* $|u_0|$ *is small enough, then* $\sup_{j\in\mathbb{N}} \|\phi_j - \phi_j^{u_0}\|_{(3)} \leq \epsilon$.

Remark B.10. *Let* B *be a bounded symmetric operator satisfying Assumptions II and Assumptions A. As Remark B.9, there exists a neighborhood* U_1 *of* u_0 *in* $\mathbb{R}$ *and* U_2, *a countable subset of* $\mathbb{R}$ *containing* $u = 0$ *such that, for any* $u_0 \in U(0) := (U_1 \setminus U_2) \setminus \{0\}$ *and* $N \in \mathbb{N}$, *the numbers* $\{1\} \cup \{\lambda_j^{u_0}\}_{j\leq N}$ *are rationally independent. Indeed, we denote*

$$x_{j,M}^{u_0} := B\big((\lambda_j^{u_0} - A)\big|_{\phi_j^\perp}\big)^{-1}\Big(\big((\lambda_j^{u_0} - A)\big|_{\phi_j^\perp}\big)^{-1} P_{\phi_j}^\perp B\Big)^M P_{\phi_j}^\perp B, \qquad \forall j, M \in \mathbb{N}.$$

As $(1 - \|\eta_j\|^2) = |\alpha_j|^2$ *for every* $j \in \mathbb{N}$, *by using* (29) *in* (30), *for* $|u_0|$ *small,*

$$\lambda_j^{u_0} = \frac{|\alpha_j|^2\lambda_j}{1-\|\eta_j\|^2} + \frac{u_0|\alpha_j|^2 B_{j,j}}{1-\|\eta_j\|^2} - \frac{u_0|\alpha_j|^2}{1-\|\eta_j\|^2}\Big\langle P_{\phi_j}^\perp B\phi_j, ((A + u_0 P_{\phi_j}^\perp B - \lambda_j^{u_0})\big|_{\phi_j^\perp})^{-1}$$
$$u_0 P_{\phi_j}^\perp B\phi_j\Big\rangle = \lambda_j + u_0 B_{j,j} + u_0^2\Big\langle \phi_j, \sum_{M=0}^{+\infty}\big(u_0^M x_{j,M}^{u_0}\big)\,\phi_j\Big\rangle.$$

Let $x_{j,M} = \langle \phi_j, \widetilde{B}(M,j)\phi_j\rangle$ *with* $\widetilde{B}(M,j)$ *defined in Assumptions A and* $j, M \in \mathbb{N}$. *We have* $\lim_{|u_0|\to 0} x_{j,M}^{u_0} = x_{j,M}$. *Let* $M \leq N$ *and* $\mathbf{r} := \{r_j\}_{j\leq M} \in \mathbb{Q}^M \setminus \mathbf{0^M}$. *Thanks to Assumptions A, the map* $u \mapsto r_1 + \sum_{j=2}^M r_j\lambda_j^u$ *is non-constant and analytic. The set* $V_{\mathbf{r}}$ *of its positive zeros is discrete. The property is valid for* $U_2 := \cup_{M\leq N}\cup_{\mathbf{r}\in\mathbb{Q}^M\setminus\mathbf{0^M}} V_{\mathbf{r}}$ *that is discrete.*

References

[BCCS12] U. Boscain, M. Caponigro, T. Chambrion, and M. Sigalotti. A weak spectral condition for the controllability of the bilinear Schrödinger equation with application to the control of a rotating planar molecule. *Comm. Math. Phys.*, 311(2):423–455, 2012.

[BdCC13] N. Boussaï d, M. Caponigro, and T. Chambrion. Weakly coupled systems in quantum control. *IEEE Trans. Automat. Control*, 58(9):2205–2216, 2013.

[Bea05] K. Beauchard. Local controllability of a 1-D Schrödinger equation. *J. Math. Pures Appl. (9)*, 84(7):851–956, 2005.

[BL10] K. Beauchard and C. Laurent. Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control. *J. Math. Pures Appl. (9)*, 94(5):520–554, 2010.

[BMS82] J. M. Ball, J. E. Marsden, and M. Slemrod. Controllability for distributed bilinear systems. *SIAM J. Control Optim.*, 20(4):575–597, 1982.

[Cha12] T. Chambrion. Periodic excitations of bilinear quantum systems. *Automatica J. IFAC*, 48(9):2040–2046, 2012.

[CMSB09] T. Chambrion, P. Mason, M. Sigalotti, and U. Boscain. Controllability of the discrete-spectrum Schrödinger equation driven by an external field. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(1):329–349, 2009.

[Duc] A. Duca. Construction of the control function for the global exact controllability and further estimates. *submitted: https://hal.archives- ouvertes.fr/hal-0152017*.

[Kat53] T. Kato. Integration of the equation of evolution in a Banach space. *J. Math. Soc. Japan*, 5:208–234, 1953.

[Kat95] T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[KL05] V. Komornik and P. Loreti. *Fourier series in control theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2005.

[Lue69] D. G. Luenberger. *Optimization by vector space methods*. John Wiley & Sons, Inc., New York-London-Sydney, 1969.

[MN15] M. Morancey and V. Nersesyan. Simultaneous global exact controllability of an arbitrary number of 1D bilinear Schrödinger equations. *J. Math. Pures Appl. (9)*, 103(1):228–254, 2015.

[Mor14] M. Morancey. Simultaneous local exact controllability of 1D bilinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 31(3):501–529, 2014.

[Sac00] Yu. L. Sachkov. Controllability of invariant systems on Lie groups and homogeneous spaces. *J. Math. Sci. (New York)*, 100(4):2355–2427, 2000. Dynamical systems, 8.