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Simultaneous global exact controllability in projection

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Abstract

We consider an infinite number of one-dimensional bilinear Schrödinger equations on a segment. We prove the simultaneous local exact controllability in projection for any positive time and the simultaneous global exact controllability in projection for sufficiently large time.

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Keywords: Schrödinger equation, simultaneous control, global exact controllability, moment problem, perturbation theory, density matrices.

1 Introduction

Let \mathcal{H} be an infinite dimensional Hilbert space. In quantum mechanics, any statistical ensemble can be described by a wave function (pure state) or by a density matrix (mixed state) which is a positive operator of trace 1. For any density matrix ρ , there exists a sequence $\{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{H}$ such that

$$(1) \quad \rho = \sum_{j \in \mathbb{N}} l_j |\psi_j\rangle\langle\psi_j|, \quad \sum_{j \in \mathbb{N}} l_j = 1, \quad l_j \geq 0 \quad \forall j \in \mathbb{N}.$$

The sequence $\{\psi_j\}_{j \in \mathbb{N}}$ is a set of eigenvectors of ρ and $\{l_j\}_{j \in \mathbb{N}}$ are the corresponding eigenvalues. If there exists $j_0 \in \mathbb{N}$ such that $l_{j_0} = 1$ and $l_j = 0$ for each $j \neq j_0$, then the corresponding density matrix represents a pure state up to a phase. For this reason, the density matrices formalism is said to be an extension of the common formulation of the quantum mechanics in terms of wave function.

Let us consider $T > 0$ and a time dependent self-adjoint operator $H(t)$ (called Hamiltonian) for $t \in (0, T)$. The dynamics of a general density matrix ρ is described by the Von Neumann equation

$$(2) \quad \begin{cases} i \frac{d\rho}{dt}(t) = [H(t), \rho(t)], & t \in (0, T), \\ \rho(0) = \rho^0, & ([H, \rho] = H\rho - \rho H), \end{cases}$$

for ρ^0 the initial solution of the problem. The solution is $\rho(t) = U_t \rho(0) U_t^*$, where U_t is the unitary propagator generated by $H(t)$, *i.e.*

$$\begin{cases} i \frac{d}{dt} U_t = -i H(t) U_t, & t \in (0, T), \\ U_0 = Id. \end{cases}$$

In the present work, we consider $\mathcal{H} = L^2((0, 1), \mathbb{C})$ and $H(t) = A + u(t)B$, for $A = -\Delta$ the Dirichlet Laplacian (*i.e.* $D(A) = H^2 \cap H_0^1$), B a bounded symmetric operator and $u \in L^2((0, T), \mathbb{R})$ control function. From now on, we call Γ_t^u the unitary propagator U_t when it is defined. The problem (2) is said to be globally exactly controllable if, for any couple of unitarily equivalent density matrices ρ^1 and ρ^2 , there exist $T > 0$ and $u \in L^2((0, T), \mathbb{R})$ such that $\rho^2 = \Gamma_T^u \rho^1 (\Gamma_T^u)^*$. Thanks to the decomposition (1), the controllability of (2) is equivalent (up to phases) to the simultaneous controllability of the Cauchy problems in \mathcal{H}

$$(3) \quad \begin{cases} i \partial_t \psi_j(t) = A \psi_j(t) + u(t) B \psi_j(t), & t \in (0, T), \\ \psi_j(0) = \psi_j^0, & \forall j \in \mathbb{N}. \end{cases}$$

The state ψ_j^0 is the j -th eigenfunction of ρ^0 corresponding to the eigenvalue λ_j and $\rho^0 = \sum_{j=1}^{\infty} \lambda_j |\psi_j^0\rangle \langle \psi_j^0|$. The j -th solution of (3) is $\psi_j(t) = \Gamma_t^u \psi_j^0$. To this purpose, we study the simultaneous global exact controllability of infinitely many problems (3) and we only rephrase the results in terms of the density matrices.

The controllability of the bilinear Schrödinger equation (3) has been widely studied in the literature and we start by mentioning the work on the bilinear systems of Ball, Mardsen and Slemrod [1]. In the framework of the bilinear Schrödinger equation, the work shows the well-posedness of (3) in \mathcal{H} for controls belonging to $L_{loc}^1((0, T), \mathbb{R})$ and an important non-controllability result. In particular, let S be the unit sphere in \mathcal{H} and

$$Z(\psi_0) := \{\psi \in D(A) \mid \exists T > 0, \exists r > 1, \exists u \in L_{loc}^r((0, T), \mathbb{R}) : \psi = \Gamma_u^T \psi_0\}.$$

For every $\psi_0 \in S \cap D(A)$, the attainable set $Z(\psi_0)$ is contained in a countable union of compact sets and it has dense complement in $S \cap D(A)$.

Despite this non-controllability result, many authors have addressed the problem for weaker notions of controllability. We call M_μ the multiplication operator for a function $\mu \in \mathcal{H}$ and $H_{(0)}^s := D(|A|^{\frac{s}{2}})$ for $s > 0$.

For instance in [3], Beauchard and Laurent improve the work [2] and they prove the local exact controllability of (3) in a neighborhood of the first eigenfunction of A in $S \cap H_{(0)}^3$ when $B = M_\mu$ for a suitable $\mu \in H^3$.

The global approximate controllability in a Hilbert space has been studied by Boscain, Caponigro, Chambrion, Mason and Sigalotti in [4] and [6]. In both, simultaneous global approximate controllability results are provided. Morancey proves in [11] the simultaneous local exact controllability in $S \cap H_{(0)}^3$ for at most three problems (3) and up to phases, when $B = M_\mu$ for suitable $\mu \in H^3$.

In [12], Morancey and Nersesyan extend the result. They provide the existence of a residual set of functions Q in H^4 so that, for $B = M_\mu$ and $\mu \in Q$, the simultaneous global exact controllability is verified for any finite number of (3) in $H_{(V)}^4 := D(|A + V|^2)$ for $V \in H^4$.

In the present work we use part of the notations of [3], [11], [12] and we carry on the previous results. We provide explicit conditions in B that imply the simultaneous global exact controllability in projection of infinitely many problems (3) in $H_{(0)}^3$ by projecting onto suitable finite dimensional subspaces of $H_{(0)}^3$. Another goal of this work is to prove the simultaneous local exact controllability in projection for any $T > 0$ up to phase-shifts. We use different techniques from the Coron's return method usually adopted for those types of results, *e.g.* [11] and [12]. Indeed, in the appendix we develop a perturbation theory technique that we use in order to get rid of an issue appearing in the proof of the local controllability: the "eigenvalues resonances". The formulation of the controllability for orthonormal basis allows to provide the result in terms of density matrices and unitarily equivalent sets of functions.

1.1 Framework and main results

We denote $\mathcal{H} = L^2((0, 1), \mathbb{C})$, its norm $\|\cdot\|$ and its scalar product $\langle \cdot, \cdot \rangle$. The operator A is the Dirichlet Laplacian, *i.e.* $A = -\frac{d^2}{dx^2}$ and $D(A) = H_0^1((0, 1), \mathbb{C}) \cap H^2((0, 1), \mathbb{C})$. The control function u belongs to $L^2((0, T), \mathbb{R})$ and B is a bounded symmetric operator.

We consider an Hilbert basis $\{\phi_j\}_{j \in \mathbb{N}}$ composed by eigenfunctions of A related to the eigenvalues $\{\lambda_j\}_{j \in \mathbb{N}}$ and we have

$$(4) \quad \phi_j(t) = e^{-iAt} \phi_j = e^{-i\lambda_j t} \phi_j.$$

Let us define the spaces for $s > 0$

$$H_{(0)}^s = H_{(0)}^s((0, 1), \mathbb{C}) := D(A^{\frac{s}{2}}), \quad \|\cdot\|_{(s)} = \|\cdot\|_{H_{(0)}^s} = \left(\sum_{k=1}^{\infty} |k^s \langle \cdot, \phi_k \rangle|^2 \right)^{\frac{1}{2}},$$

$$\ell^\infty(\mathcal{H}) = \left\{ \{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{H} \mid \sup_{j \in \mathbb{N}} \|\psi_j\|_{\mathcal{H}} < \infty \right\},$$

$$h^s(\mathcal{H}) = \left\{ \{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{H} \mid \sum_{j=1}^{\infty} (j^s \|\psi_j\|)^2 < \infty \right\}.$$

We call $H^s := H^s((0, 1), \mathbb{C})$, $H_0^s := H_0^s((0, 1), \mathbb{C})$ and, for $N \in \mathbb{N}$

$$(5) \quad I^N := \{(j, k) \in \mathbb{N} \times \{1, \dots, N\} : j \neq k\}.$$

Assumption (I). The bounded symmetric operator B satisfies the following conditions.

1. For any $N \in \mathbb{N}$, there exists $C_N > 0$ so that for every $j \leq N$ and $k \in \mathbb{N}$

$$|\langle \phi_k, B\phi_j \rangle| \geq C_N/k^3.$$

2. $\text{Ran}(B|_{H_{(0)}^2}) \subseteq H_{(0)}^2$ and $\text{Ran}(B|_{H_{(0)}^3}) \subseteq H^3 \cap H_0^1$.

3. For every $N \in \mathbb{N}$ and $(j, k), (l, m) \in I^N$ such that $(j, k) \neq (l, m)$ and $j^2 - k^2 - l^2 + m^2 = 0$, there holds $\langle \phi_j, B\phi_j \rangle - \langle \phi_k, B\phi_k \rangle - \langle \phi_l, B\phi_l \rangle + \langle \phi_m, B\phi_m \rangle \neq 0$.

Remark 1. If a bounded operator B satisfies Assumptions I, then $B \in L(H_{(0)}^2, H_{(0)}^2)$. Indeed, B is closed in \mathcal{H} , so for every $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ such that $u_n \xrightarrow{\mathcal{H}} u$ and $Bu_n \xrightarrow{\mathcal{H}} v$, we have $Bu = v$. Now, for every $\{u_n\}_{n \in \mathbb{N}} \subset H_{(0)}^2$ such that $u_n \xrightarrow{H_{(0)}^2} u$ and $Bu_n \xrightarrow{H_{(0)}^2} v$, the convergences with respect to the \mathcal{H} -norm are implied and $Bu = v$. Hence, the operator B is closed in $H_{(0)}^2$ and $B \in L(H_{(0)}^2, H_{(0)}^2)$. The same argument leads to $B \in L(H_{(0)}^3, H^3 \cap H_0^1)$ since $\text{Ran}(B|_{H_{(0)}^3}) \subseteq H^3 \cap H_0^1$.

Example. Assumptions I are satisfied for $B : \psi \mapsto x^2\psi$. Indeed, the condition 2) is trivially verified, while the first directly follows by considering

$$\begin{cases} |\langle \phi_j, x^2\phi_k \rangle| = \left| \frac{(-1)^{j-k}}{(j-k)^2\pi^2} - \frac{(-1)^{j+k}}{(j+k)^2\pi^2} \right|, & j \neq k, \\ |\langle \phi_k, x^2\phi_k \rangle| = \left| \frac{1}{3} - \frac{1}{2k^2\pi^2} \right|, & k \in \mathbb{N}. \end{cases}$$

The point 3) holds since for $(j, k), (l, m) \in I^N$ so that $(j, k) \neq (l, m)$

$$j^2 - k^2 - l^2 + m^2 = 0 \implies j^{-2} - k^{-2} - l^{-2} + m^{-2} \neq 0.$$

Let $\Psi := \{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{H}$ and $\mathcal{H}_N(\Psi) := \text{span}\{\psi_j : j \leq N\}$. We define $\pi_N(\Psi)$ the orthogonal projector onto $\mathcal{H}_N(\Psi)$.

Definition 1. The problem (3) is said to be simultaneously globally exactly controllable in projection in $H_{(0)}^3$ if, for every $N \in \mathbb{N}$, there exist $T > 0$ and $\Psi := \{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{H}$ so that the following property is verified. For every $\{\psi_j^1\}_{j \in \mathbb{N}}, \{\psi_j^2\}_{j \in \mathbb{N}} \subset H_{(0)}^3$ unitarily equivalent, there exists $u \in L^2((0, T), \mathbb{R})$ such that

$$\pi_N(\Psi)\psi_j^2 = \pi_N(\Psi)\Gamma_T^u\psi_j^1 \quad \forall j \in \mathbb{N}.$$

Definition 2. Let us define

$$O_{\epsilon, T} := \left\{ \{\psi_j\}_{j \in \mathbb{N}} \subset H_{(0)}^3 \mid \langle \psi_j, \psi_k \rangle = \delta_{j, k}; \sup_{j \in \mathbb{N}} \|\psi_j - \phi_j(T)\|_{(3)} < \epsilon \right\}.$$

The problem (3) is said to be simultaneously locally exactly controllable in projection in $H_{(0)}^3$ if, for every $N \in \mathbb{N}$, there exist $\epsilon > 0, T > 0$ and $\Psi := \{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{H}$ so that the following property is verified. For every $\{\psi_j^1\}_{j \in \mathbb{N}} \subset O_{\epsilon, T}$, there exists $u \in L^2((0, T), \mathbb{R})$ such that

$$\pi_N(\Psi)\psi_j^1 = \pi_N(\Psi)\Gamma_T^u\phi_j \quad \forall j \in \mathbb{N}.$$

Let $U(\mathcal{H})$ be the space of the unitary operators on \mathcal{H} . We present the simultaneous local exact controllability in projection for any positive time up to phases.

Theorem 1. Let B satisfy Assumptions I. For every $N \in \mathbb{N}$, there exist $\epsilon > 0$ so that, for every $T > 0$ and $\Psi := \{\psi_j\}_{j \in \mathbb{N}} \subset H_{(0)}^3$, the following holds. For any $\{\tilde{\psi}_j\}_{j \in \mathbb{N}} \in O_{\epsilon, T}$ and $\hat{\Gamma} \in U(\mathcal{H})$ such that $\{\tilde{\psi}_j\}_{j \in \mathbb{N}} = \{\hat{\Gamma}\phi_j\}_{j \in \mathbb{N}}$, if

$$(6) \quad \{\hat{\Gamma}^*\phi_j\}_{j \in \mathbb{N}} \subset H_{(0)}^3,$$

then there exist $\{\theta_j\}_{j \leq N} \subset \mathbb{R}$ and $u \in L^2((0, T), \mathbb{R})$ such that

$$\begin{cases} \pi_N(\Psi)\tilde{\psi}_j = \pi_N(\Psi)e^{i\theta_j}\Gamma_T^u\tilde{\psi}_j & j \leq N, \\ \pi_N(\Psi)\tilde{\psi}_j = \pi_N(\Psi)\Gamma_T^u\tilde{\psi}_j & j > N. \end{cases}$$

Proof. See Proposition 6. □

We present now the simultaneous global exact controllability in projection.

Theorem 2. Let B satisfy Assumptions I and let $\Psi^3 := \{\psi_j^3\}_{j \in \mathbb{N}} \subset H_{(0)}^3$. Let $\{\psi_j^1\}_{j \in \mathbb{N}}, \{\psi_j^2\}_{j \in \mathbb{N}} \subset H_{(0)}^3$ so that there exists $\hat{\Gamma} \in U(\mathcal{H})$ such that $\{\hat{\Gamma}\psi_j^1\}_{j \in \mathbb{N}} = \{\psi_j^2\}_{j \in \mathbb{N}}$. If

$$(7) \quad \{\hat{\Gamma}\psi_j^3\}_{j \in \mathbb{N}} \subset H_{(0)}^3,$$

then for any $N \in \mathbb{N}$, there exist $T > 0$ and $u \in L^2((0, T), \mathbb{R})$ such that

$$(8) \quad \pi_N(\Psi^3) \psi_j^2 = \pi_N(\Psi^3) \Gamma_T^u \psi_j^1, \quad j \in \mathbb{N}.$$

Proof. See Section 3. □

We point out that if $\Psi^3 = \Psi^2$, then $\widehat{\Gamma}^* \psi_j^3 \in H_{(0)}^3$. By considering that

$$(9) \quad \pi_N(\Psi^2) \psi_j^2 = \begin{cases} \psi_j^2, & j \leq N, \\ \pi_N(\Psi^2) \psi_j^2, & j > N, \end{cases}$$

the next corollary straightly follows (if Ψ^2 is composed by orthogonal elements, then the second line of (9) is 0).

Corollary 3. *Let B satisfy Assumption I. Let $\Psi^1 := \{\psi_j^1\}_{j \in \mathbb{N}}$, $\Psi^2 := \{\psi_j^2\}_{j \in \mathbb{N}} \subset H_{(0)}^3$ unitarily equivalent. For any $N \in \mathbb{N}$, there exist $T > 0$ and a control function $u \in L^2((0, T), \mathbb{R})$ such that*

$$\begin{cases} \Gamma_T^u \psi_j^1 = \psi_j^2, & j \leq N \\ \pi_N(\Psi^2) \Gamma_T^u \psi_j^1 = \pi_N(\Psi^2) \psi_j^2, & j > N. \end{cases}$$

Remark. One can notice that Corollary 3 implies the simultaneous global exact controllability (without projecting) of N problems (3). As we have mentioned before, a similar result is proved by Morancey and Nersesyan in [12, *Main Theorem*]. They prove the existence of a class of multiplication operators B that guarantees the simultaneous global exact controllability of a finite number of equations $i\partial_t \psi = (A + V)\psi + u(t)B\psi$ in $D(|A + V|^2)$ for $V \in H^4$. However, Theorem 2 and Corollary 3 provide a novelty as we are able to provide conditions in B implying the validity of the result. Given any bounded operator B , one can verify if those assumptions are satisfied, e.g. $B = x^2$.

Remark 2. Assumptions as (6) and (7) naturally occur when one tries to control simultaneously infinite bilinear Schrödinger equations. Indeed, if $\{\Gamma_T^u \phi_j\}_{j \in \mathbb{N}} = \{\widehat{\Gamma} \phi_j\}_{j \in \mathbb{N}}$ for $T > 0$, $u \in L^2((0, T), \mathbb{R})$ and $\widehat{\Gamma} \in U(\mathcal{H})$, then for every $k \in \mathbb{N}$

$$\sum_{j=1}^{+\infty} j^6 |\langle \phi_k, \Gamma_T^u \phi_j \rangle|^2 = \sum_{j=1}^{+\infty} j^6 |(\Gamma_T^u)^* \phi_k, \phi_j|^2 = \|(\Gamma_T^u)^* \phi_k\|_{(3)}^2.$$

As we present in the next section, $(\Gamma_T^u)^*$ preserves $H_{(0)}^3$ and then, for every $k \in \mathbb{N}$, we have $\|(\Gamma_T^u)^* \phi_k\|_{(3)}^2 < +\infty$. On the other hand, for every $k \in \mathbb{N}$

$$\sum_{j=1}^{+\infty} j^6 |\langle \phi_k, \Gamma_T^u \phi_j \rangle|^2 = \sum_{j=1}^{+\infty} j^6 |\langle \phi_k, \widehat{\Gamma} \phi_j \rangle|^2 = \sum_{j=1}^{+\infty} j^6 |\langle \widehat{\Gamma}^* \phi_k, \phi_j \rangle|^2 = \|\widehat{\Gamma}^* \phi_k\|_{(3)}^2.$$

Now, we rephrase Theorem 2 in terms of density matrices.

Corollary 4. *Let B satisfy Assumptions I and $\rho^1, \rho^2 \in T(\mathcal{H})$ be two density matrices so that $\text{Ran}(\rho^1), \text{Ran}(\rho^2) \subseteq H_{(0)}^3$. We suppose the existence of $\widehat{\Gamma} \in U(\mathcal{H})$ so that $\rho^2 = \widehat{\Gamma}\rho^1\widehat{\Gamma}^*$. Let $\Psi^3 := \{\psi_j^3\}_{j \in \mathbb{N}} \subset H_{(0)}^3$ be such that*

$$\{\widehat{\Gamma}\psi_j^3\}_{j \in \mathbb{N}} \subset H_{(0)}^3,$$

for every $j \in \mathbb{N}$. For any $N \in \mathbb{N}$, there exist $T > 0$ and a control function $u \in L^2((0, T), \mathbb{R})$ such that

$$\pi_N(\Psi^3) \Gamma_T^u \rho^1 (\Gamma_T^u)^* \pi_N(\Psi^3) = \pi_N(\Psi^3) \rho^2 \pi_N(\Psi^3).$$

Proof. See Section 3.3. □

1.2 Well-posedness

We mention now the crucial result of well-posedness for the problem in \mathcal{H}

$$(10) \quad \begin{cases} i\partial_t \psi(t) = A\psi(t) + u(t)\mu\psi(t), \\ \psi(0) = \psi^0, \quad t \in (0, T). \end{cases}$$

Proposition 5. [3, Lemma 1; Proposition 2]

1) *Let $T > 0$ and $\tilde{f} \in L^2((0, T), H_0^1 \cap H^3)$. The function $G : t \mapsto \int_0^t e^{iAs} \tilde{f}(s) ds$ belongs to $C^0([0, T], H_{(0)}^3)$. Moreover*

$$\|G\|_{L^\infty((0, T), H_{(0)}^3)} \leq c_1(T) \|\tilde{f}\|_{L^2((0, T), H^3 \cap H_0^1)},$$

where the constant $c_1(T)$ is uniformly bounded with T in bounded intervals.

2) *Let $\mu \in H^3$, $T > 0$, $\psi^0 \in H_{(0)}^3$ and $u \in L^2((0, T), \mathbb{R})$. There exists a unique mild solution of (10) in $H_{(0)}^3$, i.e. $\psi \in C^0([0, T], H_{(0)}^3)$ so that*

$$(11) \quad \psi(t, x) = e^{-iAt} \psi^0(x) - i \int_0^t e^{-iA(t-s)} (u(s)\mu(x)\psi(s, x)) ds, \quad \forall t \in [0, T].$$

Moreover, for every $R > 0$, there exists $C = C(T, \mu, R) > 0$ such that, if $\|u\|_{L^2((0, T), \mathbb{R})} < R$, then, for every $\psi^0 \in H_{(0)}^3$, the solution satisfies $\|\psi\|_{C^0([0, T], H_{(0)}^3)} \leq C\|\psi^0\|_{(3)}$ and $\|\psi(t)\|_{\mathcal{H}} = \|\psi^0\|_{\mathcal{H}}$ for every $t \in [0, T]$.

The result of Proposition 5 is also valid if one substitute $\mu \in H^3$ with $B \in L(H_{(0)}^3, H^3 \cap H_0^1)$. When B satisfies Assumptions I, we know that $B \in L(H_{(0)}^3, H^3 \cap H_0^1)$ (see Remark 1) and there exists a unique mild solution of (3) in $H_{(0)}^3$ so that

$$\psi_j(t, x) = e^{-iAt} \psi_j^0(x) - i \int_0^t e^{-iA(t-s)} u(s) B \psi_j(s, x) ds.$$

1.3 Time reversibility

We present another feature of the bilinear Schrödinger equation, the time reversibility. First, we substitute t with $T - t$ for $T > 0$ in the bilinear Schrödinger equation (3) and we obtain

$$\begin{cases} i\partial_t \Gamma_{T-t}^u \psi^0 = -A \Gamma_{T-t}^u \psi^0 - u(T-t) B \Gamma_{T-t}^u \psi^0, \\ \Gamma_{T-0}^u \psi^0 = \Gamma_T^u \psi^0 = \psi^1 & t \in (0, T). \end{cases}$$

Then $\Gamma_{T-t}^u \psi^0 = \tilde{\Gamma}_t^{\tilde{u}} \psi^1$ for $\tilde{u}(t) := u(T-t)$ and $\tilde{\Gamma}_t^{\tilde{u}}$ so that

$$(12) \quad \begin{cases} i\partial_t \tilde{\Gamma}_t^{\tilde{u}} \psi^1 = (-A - \tilde{u}(t) B) \tilde{\Gamma}_t^{\tilde{u}} \psi^1, \\ \tilde{\Gamma}_0^{\tilde{u}} \psi^0 = \psi^1, & t \in (0, T). \end{cases}$$

Thanks to $Id = \tilde{\Gamma}_T^{\tilde{u}} \Gamma_T^u$, it follows $\tilde{\Gamma}_T^{\tilde{u}} = (\Gamma_T^u)^{-1} = (\Gamma_T^u)^*$. The operator $\tilde{\Gamma}_t^{\tilde{u}}$ describes the reversed dynamics of Γ_t^u and represents the propagator of (6) generated by the Hamiltonian $(-A - \tilde{u}(t) B)$.

The importance of the time reversibility resides in the fact that all the controllability results that we are going to prove are also verified for the reversed problem. We use this feature in many steps of the next proofs.

1.4 Scheme of the work

In Section 2, we presents Proposition 6 and its proof. The proposition extends Theorem 1 and it provides the simultaneous local exact controllability in projection for any positive time up to phases. In order to motivate the modification of the problem, we emphasize the obstructions to overcome.

In Section 3, at first we prove that the simultaneous local exact controllability in projection of infinite problems (3), under particular conditions, is equivalent to the simultaneous local exact controllability of a finite number of problems. We call this feature “transposition argument”. Second, we prove the simultaneous global approximate controllability for N problems (3) in Proposition 7. Third we gather Proposition 7 with the simultaneous local exact controllability ensured by [12, *Main Theorem*] in order to prove the simultaneous global exact controllability for N problems (3) (Proposition 8). The transposition argument leads to the proof of Theorem 2. In conclusion the proof of Corollary 4 follows from Theorem 2.

In Appendix A, we develop the perturbation theory technique adopted in the proofs of Theorem 2, Corollary 4 and Proposition 6.

2 Simultaneous locale exact controllability in projection for $T > 0$

2.1 Preliminaries

In this section, we discuss the simultaneous local exact controllability in projection and we explain why we modify the problem before proceeding.

Let $\Phi = \{\phi_j\}_{j \in \mathbb{N}}$ be an Hilbert basis composed by eigenfunctions of A . We start by studying the local exact controllability in projection with respect to $\pi_N(\Phi)$ of functions belonging to $O_{\epsilon, T}$. Let $\Gamma_t^u \psi_j = \sum_{k=1}^{\infty} \phi_k(T) \langle \phi_k(T), \Gamma_t^u \psi_j \rangle$ be the solution of the j -th problem of (3). We consider the map $\alpha(u)$ as the infinite matrix with elements

$$\alpha_{k,j}(u) = \langle \phi_k(T), \Gamma_T^u \phi_j \rangle, \quad k, j \in \mathbb{N} \quad k \leq N.$$

Our goal is to prove the existence of $\epsilon > 0$ such that for any $\{\psi_j\}_{j \in \mathbb{N}} \in O_{\epsilon, T}$, there exists $u \in L^2((0, T), \mathbb{R})$ so that

$$\pi_N(\Phi) \Gamma_T^u \phi_j = \pi_N(\Phi) \psi_j, \quad \forall j \in \mathbb{N}.$$

This outcome is equivalent to the local surjectivity of the map α for $T > 0$. To this end, we want to use the Generalized Inverse Function Theorem (see [10], p. 240) and we study the surjectivity of Fréchet derivative of α , $\gamma(v) := (d_u \alpha(0)) \cdot v$. The map γ is the infinite matrix with elements

$$\begin{aligned} \gamma_{k,j}(v) &:= \left\langle \phi_k(T), -i \int_0^T e^{-iA(T-s)} v(s) B e^{-iAs} \phi_j ds \right\rangle \\ &= -i \int_0^T v(s) e^{-i(\lambda_j - \lambda_k)s} ds B_{k,j}, \quad k \leq N, \quad j \in \mathbb{N}, \end{aligned}$$

for $B_{k,j} = \langle \phi_k, B \phi_j \rangle = \langle B \phi_k, \phi_j \rangle = \overline{B_{j,k}}$. The surjectivity of γ consists in proving the solvability of the moment problem

$$(13) \quad \frac{x_{k,j}}{B_{k,j}} = -i \int_0^T u(s) e^{-i(\lambda_j - \lambda_k)s} ds,$$

for each infinite matrix x , with elements $x_{k,j}$, belonging to a suitable space. One would use the Haraux Theorem ([9], p. 67) but a problem appears. The so-called ‘‘eigenvalues resonances’’ occur when, for $j, k, n, m \in \mathbb{N}$, $(j, k) \neq (n, m)$ and $k, m \leq N$, there holds $\lambda_j - \lambda_k = \lambda_n - \lambda_m$ (we recall $\lambda_l = \pi^2 l^2$). Those relations imply the constraints

$$\frac{x_{k,j}}{B_{k,j}} = -i \int_0^T u(s) e^{-i(\lambda_j - \lambda_k)s} ds = -i \int_0^T u(s) e^{-i(\lambda_n - \lambda_m)s} ds = \frac{x_{n,m}}{B_{n,m}}.$$

An example is $\lambda_7 - \lambda_1 = \lambda_8 - \lambda_4$, but they also appear for all the diagonal terms of γ since $\lambda_j - \lambda_k = 0$ for $j = k$.

We avoid the problem by adopting the following procedure.

- We decompose $A + u(t)B = (A + u_0B) + u_1(t)B$ for $u_0 \in \mathbb{R}$ and $u_1 \in L^2((0, T), \mathbb{R})$. We consider $A + u_0B$ instead of A and we modify the eigenvalues gaps by using u_0B as a perturbing term in order to remove all the non-diagonal resonances.
- The previous point imposes to redefine α in a map $\widehat{\alpha}$ depending on the parameter u_0 . After, we introduce α^{u_0} by acting the following phase-shifts in order to remove the resonances on the diagonal terms

$$\widetilde{\psi}_j(t, x) = \frac{\overline{\widehat{\alpha}_{j,j}(u)}}{|\widehat{\alpha}_{j,j}(u)|} \psi_j(t, x) \implies \alpha_{k,j}^{u_0}(u) = \frac{\overline{\widehat{\alpha}_{j,j}(u)}}{|\widehat{\alpha}_{j,j}(u)|} \widehat{\alpha}_{k,j}(u).$$

2.2 The modified problem

Let $N \in \mathbb{N}$ and $u(t) = u_0 + u_1(t)$, for u_0 and $u_1(t)$ real. We introduce the following Cauchy problem

$$(14) \quad \begin{cases} i\partial_t \psi_j(t) = (A + u_0B)\psi_j(t) + u_1(t)B\psi_j(t), \\ \psi_j(0) = \phi_j^{u_0} \quad t \in (0, T), \quad j \in \mathbb{N}. \end{cases}$$

Its solutions are $\psi_j(t) = \Gamma_t^{u_0+u_1} \phi_j^{u_0}$, where $\Gamma_t^{u_0+u_1}$ is the unitary propagator of the dynamics which is equivalent to the one of the problems (3). The operator $A + u_0B$ has pure discrete spectrum and its eigenvalues $\{\lambda_j^{u_0}\}_{j \in \mathbb{N}}$ correspond to an Hilbert basis composed by eigenfunctions $\Phi^{u_0} := \{\phi_j^{u_0}\}_{j \in \mathbb{N}}$.

We set $\phi_j^{u_0}(T) := e^{-i\lambda_j^{u_0}T} \phi_j^{u_0}$ and we introduce

$$(15) \quad O_{\epsilon_0, T}^{u_0} := \left\{ \{\psi_j\}_{j \in \mathbb{N}} \subset H_{(0)}^3 \mid \langle \psi_j, \psi_k \rangle = \delta_{j,k}; \sup_{j \in \mathbb{N}} \|\psi_j - \phi_j^{u_0}(T)\|_{(3)} < \epsilon_0 \right\}.$$

In addition, we chose $|u_0|$ small enough so that $\lambda_k^{u_0} \neq 0$ for every $k \in \mathbb{N}$ (Lemma 13). The introduction of the new Hilbert basis imposes to define

$$(16) \quad \widetilde{H}_{(0)}^3 := D(|A + u_0B|^{\frac{3}{2}}), \quad \|\cdot\|_{\widetilde{H}_{(0)}^3} = \left(\sum_{k=1}^{\infty} |\lambda_k^{u_0}|^{\frac{3}{2}} |\langle \cdot, \phi_k \rangle|^2 \right)^{\frac{1}{2}}.$$

However, from now on, we consider the hypothesis of Lemma 15 (Appendix A) being verified so that $\widetilde{H}_{(0)}^3 \equiv H_{(0)}^3$.

We define $\widehat{\alpha}$, the infinite matrices with elements for $k \leq N$ and $j \in \mathbb{N}$ so that $\widehat{\alpha}_{k,j}(u_1) = \langle \phi_k^{u_0}(T), \Gamma_T^{u_0+u_1} \phi_j^{u_0} \rangle$ and the map α^{u_0} with elements

$$\begin{cases} \alpha_{k,j}^{u_0}(u_1) = \frac{\overline{\widehat{\alpha}_{j,j}(u_1)}}{|\widehat{\alpha}_{j,j}(u_1)|} \widehat{\alpha}_{k,j}(u_1), & j, k \leq N, \\ \alpha_{k,j}^{u_0}(u_1) = \widehat{\alpha}_{k,j}(u_1), & j > N, \quad k \leq N. \end{cases}$$

Now, we point out that for $k \leq N$

$$(17) \quad \pi_N(\Phi^{u_0})\omega_j \Gamma_T^{u_0+u_1} \phi_j^{u_0} = \sum_{k=1}^N \phi_k^{u_0}(T) \alpha_{k,j}^{u_0}(u_1), \quad \omega_j := \frac{\overline{\widehat{\alpha}_{j,j}(u_1)}}{|\widehat{\alpha}_{j,j}(u_1)|}.$$

Thus, the local surjectivity of the map α^{u_0} in a suitable space (in Q^N defined below in (19)) is equivalent to the simultaneous local exact controllability in projection on $O_{\epsilon_0, T}^{u_0}$ for a suitable $\epsilon_0 > 0$ up to N phases.

Let $\gamma^{u_0}(v) = ((d_{u_1} \alpha^{u_0})(0)) \cdot v$ be the Fréchet derivative of α^{u_0} and let B^{u_0} be the infinite matrix with elements $B_{j,k}^{u_0} = \langle \phi_j^{u_0}, B \phi_k^{u_0} \rangle$ for $j \leq N$ and $k \in \mathbb{N}$. Defined $\widehat{\gamma}_{k,j}(v) = ((d_{u_1} \widehat{\alpha})(0)) \cdot v$, we compute $\gamma^{u_0}(v)$ so that

$$\begin{cases} \gamma_{k,j}^{u_0} = (\widehat{\gamma}_{j,j} \delta_{k,j} + \widehat{\gamma}_{k,j} - \delta_{k,j} \Re(\widehat{\gamma}_{j,j})), & j, k \leq N, \\ \gamma_{k,j}^{u_0} = \widehat{\gamma}_{k,j}, & k \leq N, j > N. \end{cases}$$

Thus for $k \leq N$ and $j \in \mathbb{N}$

$$(18) \quad \begin{cases} \gamma_{k,j}^{u_0} = \widehat{\gamma}_{k,j} = -i \int_0^T u_1(s) e^{-i(\lambda_j^{u_0} - \lambda_k^{u_0})s} ds B_{k,j}^{u_0}, & k \neq j, \\ \gamma_{k,k}^{u_0} = \Re(\widehat{\gamma}_{k,k}) = 0, & k = j, \end{cases}$$

The relation $\gamma_{k,k}^{u_0} = 0$ comes from the fact that $(i\widehat{\gamma}_{k,k}) \in \mathbb{R}$ since $\widehat{\gamma}_{k,j} = -\overline{\widehat{\gamma}_{j,k}}$ for $j, k \leq N$. Thanks to the phase-shifts, the diagonal elements of γ^{u_0} are all 0.

Remark. Thanks to the orthogonality of the elements in $O_{\epsilon_0, T}^{u_0}$ (see (15))

$$T_{\Phi^{u_0}} O_{\epsilon_0, T}^{u_0} = \{ \{f_j\}_{j \in \mathbb{N}} \in \ell^\infty(H_{(0)}^3) \mid \langle \phi_k^{u_0}, f_j \rangle = -\overline{\langle \phi_j^{u_0}, f_k \rangle} \}.$$

We have $T_{\Phi^{u_0}} O_{\epsilon_0, T}^{u_0} \subset \ell^\infty(H_{(0)}^3)$ since $\sup_{j \in \mathbb{N}} \|\psi_j - \phi_j^{u_0}\|_{(3)} \leq \epsilon_0$ for every $\{\psi_j\}_{j \in \mathbb{N}} \in O_{\epsilon_0, T}^{u_0}$. In conclusion, thanks to Remark 2, the maps α^{u_0} and γ^{u_0} take respectively value in

$$(19) \quad \begin{aligned} Q^N &:= \{ \{x_{k,j}\}_{k,j \in \mathbb{N}} \in (h^3(\mathbb{C}))^N \mid x_{k,k} \in \mathbb{R}, \quad k \leq N \}, \\ G^N &:= \{ \{x_{k,j}\}_{k,j \in \mathbb{N}} \in (h^3(\mathbb{C}))^N \mid x_{k,j} = -\overline{x_{j,k}} \ \& \ x_{k,k} = 0 \ j, k \leq N \}. \end{aligned}$$

2.3 Proof of Theorem 1

In the next proposition we ensure the simultaneous local exact controllability in projection for any $T > 0$ up to phases.

Proposition 6. *Let $N \in \mathbb{N}$ and B satisfy Assumptions I. For every $T > 0$, there exist $\epsilon > 0$ and $u_0 \in \mathbb{R}$ such that for any $\{\psi_j\}_{j \in \mathbb{N}} \in O_{\epsilon, T}$ satisfying the relation (6), there exist a sequence of real numbers $\{\theta_j\}_{j \in \mathbb{N}} = \{\{\widehat{\theta}_j\}_{j \leq N}, 0, \dots\}$ and a control function $u \in L^2((0, T), \mathbb{R})$ such that*

$$\pi_N(\Phi^{u_0})\psi_j = \pi_N(\Phi^{u_0})e^{i\theta_j} \Gamma_T^u \phi_j^{u_0}, \quad \forall j \in \mathbb{N}.$$

Proof. 1) Let u_0 belong to the neighborhoods defined in Appendix A by Remark 7, Lemma 13, Lemma 14 and Lemma 15. First, by referring to Remark 2, the relation (6) is required for the following reason. Let $\{\Gamma_T^u \phi_j^{u_0}\}_{j \in \mathbb{N}} = \{\widehat{\Gamma} \phi_j\}_{j \in \mathbb{N}}$ for $T > 0$, $u \in L^2((0, T), \mathbb{R})$ and $\widehat{\Gamma} \in U(\mathcal{H})$. For $|u_0|$ small enough, thanks to Lemma 13, there exists $C_1 > 0$ such that $j^6 \leq C_1 |\lambda_j^{u_0}|^3$ and, thanks to Lemma 15, there exists $C_2 > 0$ such that, for every $k \in \mathbb{N}$

$$\sum_{j=1}^{+\infty} j^6 |\langle \phi_k, \Gamma_T^u \phi_j^{u_0} \rangle|^2 \leq C_1 \|(\Gamma_T^u)^* \phi_k\|_{\widetilde{H}_{(0)}^3}^2 \leq C_1 C_2 \|\widehat{\Gamma}_T^u \phi_k\|_{(3)}^2 < \infty.$$

On the other hand, for every $k \in \mathbb{N}$

$$\sum_{j=1}^{+\infty} j^6 |\langle \phi_k, \Gamma_T^u \phi_j^{u_0} \rangle|^2 = \sum_{j=1}^{+\infty} j^6 |\langle \phi_k, \widehat{\Gamma} \phi_j \rangle|^2 = \sum_{j=1}^{+\infty} j^6 |\langle \widehat{\Gamma}^* \phi_k, \phi_j \rangle|^2 = \|\widehat{\Gamma}^* \phi_k\|_{(3)}^2.$$

Second, thanks to the third point of Remark 7, the controllability in $O_{\epsilon_0, T}^{u_0}$ implies the controllability in $O_{\epsilon, T}$, for suitable ϵ_0 , since

$$\sup_{j \in \mathbb{N}} \|\psi_j - \phi_j(T)\|_{(3)} \leq \sup_{j \in \mathbb{N}} \|\phi_j^{u_0} - \phi_j(T)\|_{(3)} + \sup_{j \in \mathbb{N}} \|\psi_j - \phi_j^{u_0}(T)\|_{(3)}.$$

Third, thanks to the discussion provided for the relation (17), the local surjectivity of the map α^{u_0} guarantees the simultaneous local exact controllability in projection up to phases of (3) with initial state $\{\phi_j^{u_0}\}_{j \in \mathbb{N}}$ on $O_{\epsilon_0, T}^{u_0}$ for ϵ_0 small enough. We consider Generalized Inverse Function Theorem (see [10], p. 240) since Q^N and G^N are Banach spaces. If γ^{u_0} is surjective in G^N , then the local surjectivity of α^{u_0} in Q^N is ensured. Now, γ^{u_0} is surjective when the following moment problem is solvable

$$(20) \quad \frac{x_{k,j}^{u_0}}{B_{k,j}^{u_0}} = -i \int_0^T u(s) e^{-i(\lambda_j^{u_0} - \lambda_k^{u_0})s} ds, \quad j \in \mathbb{N}, k \leq N, k \neq j$$

for $\{x_{k,j}^{u_0}\}_{\substack{j,k \in \mathbb{N} \\ k \leq N}} \in G^N$. We know that $\{x_{k,j}^{u_0}\}_{\substack{j,k \in \mathbb{N} \\ k \leq N}}, \{\gamma_{k,j}^{u_0}\}_{\substack{j,k \in \mathbb{N} \\ k \leq N}} \in (h^3)^N$.

Thanks to Lemma 14 and to $|B_{k,j}^{u_0}| = |B_{j,k}^{u_0}|$, it follows

$$\{x_{k,j}^{u_0}/B_{k,j}^{u_0}\}_{\substack{j,k \in \mathbb{N} \\ k \leq N}}, \{\gamma_{k,j}^{u_0}/B_{k,j}^{u_0}\}_{\substack{j,k \in \mathbb{N} \\ k \leq N}} \in (\ell^2)^N.$$

We do not consider the equations of (20) for $k = j$ since $\gamma_{k,k}^{u_0} = 0$ and $x_{k,k}^{u_0} = 0$ for every $k \leq N$ and for every $\{x_{k,j}^{u_0}\}_{\substack{k,j \in \mathbb{N} \\ k \leq N}} \in G^N$. Thanks to Lemma 16 (Appendix A), for I^N defined in (5), there exist

$$\mathcal{G}' := \inf_{\substack{(j,k),(n,m) \in I^N \\ (j,k) \neq (n,m)}} |\lambda_j^{u_0} - \lambda_k^{u_0} - \lambda_n^{u_0} + \lambda_m^{u_0}| > 0,$$

$$\mathcal{G} := \sup_{A \subset I^N} \left(\inf_{\substack{(j,k),(n,m) \in I^N \setminus A \\ (j,k) \neq (n,m)}} |\lambda_j^{u_0} - \lambda_k^{u_0} - \lambda_n^{u_0} + \lambda_m^{u_0}| \right) \geq \mathcal{G}',$$

where A runs over the finite subsets of I . Hence, for $T > \frac{2\pi}{\mathcal{G}}$, Haraux Theorem ([9], p. 67) implies the solvability of the moment problem (20). The proof is achieved since α^{u_0} is locally surjective for $T > 0$ large enough.

2) Now, we show that the first point is valid for every $T > 0$ by proving that $\mathcal{G} = +\infty$. Let $A^M := \{(j, n) \in \mathbb{N}^2 \mid j, n \geq M; j \neq n\}$ for $M \in \mathbb{N}$. Thanks to Lemma 13, for $|u_0|$ small enough and for every $K \in \mathbb{R}$, there exists $M_K > 0$ large enough so that $\inf_{(j,n) \in A^{M_K}} |\lambda_j^{u_0} - \lambda_n^{u_0}| > K$. Thus

$$\mathcal{G} \geq \sup_{M \in \mathbb{N}} \left(\inf_{(j,n) \in A^M} |\lambda_j^{u_0} - \lambda_n^{u_0}| - 2\lambda_N^{u_0} \right) > 0.$$

For $|u_0|$ small enough, Lemma 13 implies the existence of $C > 0$ so that

$$\mathcal{G} \geq C \left(\lim_{M \rightarrow \infty} \inf_{(j,n) \in A^M} |\lambda_j - \lambda_n| - 2\lambda_N \right) \geq C \lim_{M \rightarrow \infty} (\lambda_{M+2} - \lambda_{M+1} - 2N^2\pi^2) = +\infty. \quad \square$$

Remark 3. The proof of Proposition 6 is still valid by phase-shifting the components of the map $u \mapsto \{\Gamma_t^u \phi_j^{u_0}\}$, i.e. by substituting α^{u_0} with

$$\begin{cases} \frac{\widehat{\alpha}_{k,k}(u)}{|\widehat{\alpha}_{k,k}(u)|} \widehat{\alpha}_{k,j}(u), & j, k \leq N, \\ \widehat{\alpha}_{k,j}(u), & j > N, k \leq N. \end{cases}$$

3 Simultaneous global exact controllability in projection

The common approach adopted in order to prove the global exact controllability (also simultaneous) consists in gathering the results of global approximate controllability and local exact controllability.

We would like to use this strategy in order to prove Theorem 2, but it is not possible for the controllability in projection. Indeed, the propagator Γ_T^u does not preserve the space $\pi_N(\Psi)H_{(0)}^3$ for any $\Psi := \{\psi_j\}_{j \in \mathbb{N}} \subset H_{(0)}^3$, making impossible to reverse and concatenate dynamics. In other words, let $\psi^1, \psi^2 \in H_{(0)}^3$. For $T_1, T_2 > 0$, $u_1 \in L^2((0, T_1), \mathbb{R})$ and $u_2 \in L^2((0, T_2), \mathbb{R})$

$$\pi_N(\Psi)\Gamma_{T_1}^{u_1}\psi^1 = \pi_N(\Psi)\widetilde{\Gamma}_{T_2}^{u_2}\psi^2 \not\Rightarrow \pi_N(\Psi)\Gamma_{T_2}^{\widetilde{u}_2}\Gamma_{T_1}^{u_1}\psi^1 = \pi_N(\Psi)\psi^2.$$

Thus, we have to adopt an alternative strategy. We prove that, under particular conditions, the controllability in projection onto an N dimensional space is equivalent to the controllability of N problems (without projecting). We call this feature ‘‘transposition argument’’ (see remark below).

Remark 4. The time reversibility (Section 1.3) implies that for $j, k \in \mathbb{N}$

$$(21) \quad \begin{aligned} \overline{\langle \phi_k^{u_0}(T), \Gamma_T^u \phi_j^{u_0} \rangle} &= e^{-i\lambda_k^{u_0} T} \langle \Gamma_T^u \phi_j^{u_0}, \phi_k^{u_0} \rangle = e^{-i\lambda_k^{u_0} T} \langle \phi_j^{u_0}, (\Gamma_T^u)^* \phi_k^{u_0} \rangle \\ &= e^{-i(\lambda_k^{u_0} + \lambda_j^{u_0}) T} \langle \phi_j^{u_0}(T), \tilde{\Gamma}_T^u \phi_k^{u_0} \rangle. \end{aligned}$$

Now, $e^{-i(\lambda_k^{u_0} + \lambda_j^{u_0}) T}$ does not depend on u and the relation (21) implies that the surjectivity of the two following maps (on suitable spaces) is equivalent

$$\left\{ \langle \phi_k^{u_0}(T), \Gamma_T^u \phi_j^{u_0} \rangle \right\}_{\substack{k, j \in \mathbb{N} \\ k \leq N}}, \quad \left\{ \langle \phi_k^{u_0}(T), \tilde{\Gamma}_T^u \phi_j^{u_0} \rangle \right\}_{\substack{k, j \in \mathbb{N} \\ j \leq N}}.$$

For this reason, under particular conditions, the simultaneous global exact controllability in projection of (3) onto a suitable N dimensional space is equivalent to the controllability of N problems (6) (without projecting).

Remark 5. Keeping in mind our notation, the transposition argument and [12, Proposition 4.4] imply the simultaneous local exact controllability in projection in $\tilde{H}_{(0)}^3$ (defined in (16)) for any $T > 0$ with phases ambiguity in the components. Indeed, $H_{(V)}^3$, defined in [12], corresponds to $\tilde{H}_{(0)}^3$ when $V = u_0 B$ and B is a suitable multiplication operator. In particular, for $N \in \mathbb{N}$, there exists a set of assumptions on the couple (u_0, B) so that, for every $T > 0$, the following result is verified. Let $\mathcal{O}(\Phi^{u_0}) \in \ell^\infty(\tilde{H}_{(0)}^3)$ be a suitable neighborhood of $\Phi^{u_0} = \{\phi_j^{u_0}\}_{j \in \mathbb{N}}$ and $\{\theta_j\}_{j \in \mathbb{N}} \subset \mathbb{R}$. For every $\{\psi_j\}_{j \in \mathbb{N}} \in \mathcal{O}(\Phi_j^{u_0})$ satisfying the relation (6), there exists $u \in L^2((0, T), \mathbb{R})$ so that

$$\sum_{k=1}^N \phi_k^{u_0} e^{i\theta_k} \langle \phi_k^{u_0}, \psi_j \rangle = \pi_N(\Phi^{u_0}) \Gamma_T^u \phi_j^{u_0}, \quad \forall j \in \mathbb{N}.$$

The transposition argument is particularly important as it allows to concatenate and reverse dynamics on $(H_{(0)}^3)^N$ which is preserved by the propagator when one wants to prove the controllability in projection.

For the simultaneous local exact controllability result, we can use Proposition 6 with the transposition argument, but this is not the most convenient approach. Even though Proposition 6 provides the controllability for any $T > 0$, the outcome is true up to phases. For this reason, we consider [12, Theorem 4.1] instead of Proposition 6. The mentioned theorem exhibits the simultaneous local exact controllability of N problems including the control of the phases, even if it is not valid for any $T > 0$ (contrary to Proposition 6). However, we are not able to ensure Theorem 2 for any $T > 0$ since the simultaneous global approximate controllability, adopted in the proof, is not guaranteed for any $T > 0$, as we show in the section below.

3.1 Approximate simultaneous controllability

In this subsection we prove the simultaneous global approximate controllability of the problems (3).

Definition 3. The problem (3) is said to be simultaneously globally approximately controllable in $H_{(0)}^s$ if for every $N \in \mathbb{N}$, $\psi_1, \dots, \psi_N \in H_{(0)}^s$, $\widehat{\Gamma} \in U(\mathcal{H})$ such that $\widehat{\Gamma}\psi_1, \dots, \widehat{\Gamma}\psi_N \in H_{(0)}^s$ and $\epsilon > 0$, then there exist $T > 0$ and $u \in L^2((0, T), \mathbb{R})$ so that for every $1 \leq k \leq N$

$$\|\widehat{\Gamma}\psi_k - \Gamma_T^u \psi_k\|_{H^s} < \epsilon.$$

Proposition 7. Let B satisfy Assumptions I. The problem (3) is simultaneously globally approximately controllable in $H_{(0)}^3$.

Proof. Let $N \in \mathbb{N}$ and u_0 belong to the neighborhoods provided in Appendix A by Remark 6 and Remark 7. We define the norms

$$\|\cdot\|_{(s)} := \|\cdot\|_{L(H_{(0)}^s, H_{(0)}^s)}, \quad \|\cdot\|_{BV((0, T), \mathbb{R})} = \|\cdot\|_{BV(T)},$$

for $\|f\|_{BV((0, T), \mathbb{R})} = \sup_{\{t_j\}_{j \leq n} \in P} \sum_{j=1}^n |f(t_j) - f(t_{j-1})|$ and P the set of the partitions of $(0, T)$ so that $t_0 = 0 < t_1 < \dots < t_n = T$.

1) Let $\lambda > 0$ and $\widehat{H}_{(0)}^4 := D(A(i\lambda - A)) \equiv H_{(0)}^4$. We consider [7, Section 3.10]. For $T > 0$, $u \in L^2((0, T), \mathbb{R})$ and λ large enough, $\|u(t)B(i\lambda_\epsilon - A)^{-1}\|_{(2)} < 1$ and

$$\begin{aligned} M &:= \sup_{t \in [0, T]} \|(i\lambda_\epsilon - A - u(t)B)^{-1}\|_{L(H_{(0)}^2, \widehat{H}_{(0)}^4)} \\ &= \sup_{t \in [0, T]} \|(I - u(t)B(i\lambda_\epsilon - A)^{-1})^{-1}\|_{(2)} \\ &\leq \sup_{t \in [0, T]} \sum_{l=1}^{+\infty} \|(u(t)(i\lambda_\epsilon - A)^{-1}B)^l\|_{(2)} < +\infty. \end{aligned}$$

We know that $B : \widehat{H}_{(0)}^4 \subset H_{(0)}^3 \rightarrow H^3 \cap H_0^1 \subset H_{(0)}^2$ and the techniques of Remark 1 imply that $B \in L(\widehat{H}_{(0)}^4, H_{(0)}^2)$. Now

$$N := \|i\lambda_\epsilon - A - u(\cdot)B\|_{BV([0, T], L(\widehat{H}_{(0)}^4, H_{(0)}^2))} \leq \|u\|_{BV(T)} \|B\|_{L(\widehat{H}_{(0)}^4, H_{(0)}^2)} < +\infty.$$

Thanks to [7, Section 3.10], for every $\psi \in H_{(0)}^4$

$$\|(A + u(T)B - i\lambda)\Gamma_T^u \psi\|_{(2)} \leq Me^{MN} \|(A - i\lambda)\psi\|_{(2)}.$$

Now, $C_1 := \|A(A + u(T)B - i\lambda)^{-1}\|_{(2)} < \infty$. There exists $C_2 > 0$ so that

$$\|\Gamma_T^u \psi\|_{(4)} = \|A\Gamma_T^u \psi\|_{(2)} \leq C_1 Me^{MN} \|(A - i\lambda)\psi\|_{(2)} \leq C_2 Me^{MN} \|\psi\|_{(4)}.$$

In conclusion, for every $T > 0$, $u \in L^2((0, T), \mathbb{R})$ and $\psi^1, \psi^2 \in H_{(0)}^4$, there holds $\|\Gamma_T^u \psi^1 - \psi^2\|_{(4)} < +\infty$.

2) Let $\{\psi_j^1\}_{j \leq N}, \{\psi_j^2\}_{j \leq N} \subset H_{(0)}^3$ unitarily equivalent. Thanks to the density of $H_{(0)}^4$ in $H_{(0)}^3$ with respect to the $H_{(0)}^3$ -norm, we know that, for every $\epsilon > 0$, there exist $\{\tilde{\psi}_j^1\}_{j \leq N}, \{\tilde{\psi}_j^2\}_{j \leq N} \subset H_{(0)}^4$ such that for every $j \leq N$

$$\|\psi_j^1 - \tilde{\psi}_j^1\|_{(3)} \leq \epsilon, \quad \|\psi_j^2 - \tilde{\psi}_j^2\|_{(3)} \leq \epsilon.$$

We point out that the same arguments of the first point can be used in order to prove that, for every $T > 0$, $u \in L^2((0, T), \mathbb{R})$ and $\psi \in H_{(0)}^3$, there exists $C_3 > 0$ so that $\|\Gamma_T^u \psi\|_{(3)} \leq C_3 \|\psi\|_{(3)}$. Now

$$\begin{aligned} \|\Gamma_T^u \psi_j^1 - \psi_j^2\|_{(3)} &\leq \epsilon + \|\Gamma_T^u \psi_j^1 - \Gamma_T^u \tilde{\psi}_j^1\|_{(3)} + \|\Gamma_T^u \tilde{\psi}_j^1 - \tilde{\psi}_j^2\|_{(3)} \\ &\leq (C_3 + 1)\epsilon + \|\Gamma_T^u \tilde{\psi}_j^1 - \tilde{\psi}_j^2\|_{(3)}. \end{aligned}$$

Thanks to the Cauchy-Schwarz inequality, for every $\psi \in H_{(0)}^4$, we have

$$\|\psi\|_{(3)}^2 = \langle |A|^{\frac{3}{2}} \psi, |A|^{\frac{3}{2}} \psi \rangle = \langle |A|^2 \psi, |A| \psi \rangle = \|\psi\|_{(2)} \|\psi\|_{(4)}$$

and, thanks to the first point of the proof, there exists $C_4 > 0$ so that

$$\|\Gamma_T^u \psi_j^1 - \psi_j^2\|_{(3)} \leq (C_3 + 1)\epsilon + C_4^{\frac{1}{2}} \|\Gamma_T^u \tilde{\psi}_j^1 - \tilde{\psi}_j^2\|_{(2)}^{\frac{1}{2}}.$$

Afterwards, the hypothesis of [5, Proposition 5] are satisfied since the operators $-i(A+u_0B)$ and $-i(A+u_0B)-iuB$ are skew-adjoint. Moreover, Remark 1 implies the validity of [5, Proposition 6] which ensures that the couple $(A+u_0B, B)$ is 2-weakly coupled (see [5, Definition 1]). Now, $(A+u_0B, B)$ admits a non-degenerate chain of connectedness (see [5, Definition 3]) thanks to Remark 7. Thanks to Remark 6, it follows $\tilde{H}_{(0)}^2 \equiv H_{(0)}^2$ and there exists $C_5 > 0$ so that $\|\cdot\|_{(2)} \leq C_5 \|\cdot\|_{\tilde{H}_{(0)}^2}$. Now [5, Proposition 5] implies the existence of $u : [0, T] \rightarrow U$ so that

$$\|\Gamma_T^u \tilde{\psi}_j^1 - \tilde{\psi}_j^2\|_{(2)} \leq C_5 \|\Gamma_T^u \tilde{\psi}_j^1 - \tilde{\psi}_j^2\|_{\tilde{H}_{(0)}^2} < C_5 \epsilon \quad \forall j \leq N.$$

In conclusion, $\|\Gamma_T^u \psi_j^1 - \psi_j^2\|_{(3)} \leq \tilde{\epsilon}$ for $\tilde{\epsilon} = (C_3 + 1)\epsilon + C_4^{\frac{1}{2}} C_5^{\frac{1}{2}} \epsilon^{\frac{1}{2}}$. \square

3.2 Proofs of Theorem 2 and Corollary 4

In the current section, we provide the proof of Theorem 2 which requires the following proposition.

Proposition 8. *Let $N \in \mathbb{N}$ and B satisfy Assumption I. For any $\{\psi_k^1\}_{k \leq N}, \{\psi_k^2\}_{k \leq N} \subset H_{(0)}^3$ orthonormal systems, there exist $T > 0$ and a control function $u \in L^2((0, T), \mathbb{R})$ such that:*

$$\psi_k^2 = \tilde{\Gamma}_T^u \psi_k^1, \quad k \leq N.$$

Proof. Let $N \in \mathbb{N}$ and let $u_0 \in \mathbb{R}$ belong to the neighborhoods provided in Appendix A by Remark 7, Remark 8, Lemma 14 and Lemma 15.

Keeping in mind the discussion in Remark 5, we consider the assumptions (C_3) , (C_4) and (C_5) introduced in [12, p. 10]. If we substitute V with $u_0 B$ and μ by $-B$, then the statement of [12, Theorem 4.1] is still valid. However, the condition (C_3) is ensured by Lemma 14, while the assumptions (C_4) and (C_5) respectively follow from the first point of Remark 7 and Remark 8. Let $\epsilon, T > 0$ and

$$O_{\epsilon, T}^N := \left\{ \{\psi_j\}_{j \leq N} \subset H_{(0)}^3 \mid \langle \psi_j, \psi_k \rangle = \delta_{j,k}; \sum_{j=1}^N \|\psi_j - \phi_j\|_{(3)} < \epsilon \right\}.$$

Thanks to Lemma 14 and [12, Theorem 4.1], the simultaneous local exact controllability is guaranteed in $O_{\epsilon, T}^N$ for suitable $\epsilon > 0$ and $T > 0$. Lemma 14 allows to obtain the result of [12, Theorem 4.1], not only in a neighborhood of $\tilde{H}_{(0)}^3$, but also in $O_{\epsilon, T}^N \subset H_{(0)}^3$. The controllability is also verified for the problem (6).

Now, Theorem 7 implies the simultaneous global approximate controllability for N problems. For any $\{\psi_j^1\}_{j \leq N} \subset H_{(0)}^3$ composed by orthonormal elements, there exist $T_1 > 0$ and $u_1 \in L^2((0, T_1), \mathbb{R})$ so that

$$\|\Gamma_{T_1}^{u_1} \psi_j^1 - \phi_j^{u_0}\|_{(3)} < \frac{\epsilon}{N}, \quad \forall j \leq N \implies \{\Gamma_{T_1}^{u_1} \psi_j^1\}_{j \leq N} \in O_{\epsilon, T}^N.$$

Thanks to the local controllability, there exists $u \in L^2((0, T), \mathbb{R})$ so that

$$\{\Gamma_{T_1}^{u_1} \psi_j^1\}_{j \leq N} = \{\tilde{\Gamma}_T^u \phi_j^{u_0}\}_{j \leq N} \implies \{\Gamma_T^{\tilde{u}} \Gamma_{T_1}^{u_1} \psi_j^1\}_{j \leq N} = \{\phi_j^{u_0}\}_{j \leq N}.$$

By concatenating and reversing the dynamics, the proof is achieved. \square

Proof of Theorem 2. Let $N \in \mathbb{N}$ and $u_0 \in \mathbb{R}$ in the neighborhoods provided in Appendix A by Remark 7, Remark 8, Lemma 14 and Lemma 15.

1) Controllability in projection of orthonormal systems: Let $\Psi^3 := \{\psi_j^3\}_{j \in \mathbb{N}} \in H_{(0)}^3$ be an orthonormal basis. We consider $\{\psi_j^1\}_{j \in \mathbb{N}}, \{\psi_j^2\}_{j \in \mathbb{N}} \subset H_{(0)}^3$ orthonormal systems and $\hat{\Gamma} \in U(\mathcal{H})$ so that $\hat{\Gamma} \psi_j^1 = \psi_j^2$ and $\hat{\Gamma}^* \psi_j^3 \in H_{(0)}^3$ for every $j \in \mathbb{N}$. The last relation implies that for every $k \leq N$

$$\tilde{\psi}_k := \sum_{j=1}^{\infty} \psi_j^1 \langle \psi_j^2, \psi_k^3 \rangle = \sum_{j=1}^{\infty} \psi_j^1 \langle \hat{\Gamma} \psi_j^1, \psi_k^3 \rangle = \sum_{j=1}^{\infty} \psi_j^1 \langle \psi_j^1, \hat{\Gamma}^* \psi_k^3 \rangle = \hat{\Gamma}^* \psi_k^3 \in H_{(0)}^3.$$

Thanks to Proposition 8, there exist $T > 0$ and $u \in L^2((0, T), \mathbb{R})$ so that $\tilde{\psi}_k = \tilde{\Gamma}_T^u \psi_k^3$, for each $k \leq N$. Hence

$$\langle \psi_j^1, \tilde{\Gamma}_T^u \psi_k^3 \rangle = \langle \psi_j^1, \tilde{\psi}_k \rangle = \langle \psi_j^2, \psi_k^3 \rangle, \quad \forall j, k \in \mathbb{N}, k \leq N.$$

Thanks to Section 1.3, we have $\langle \Gamma_T^{\tilde{u}} \psi_j^1, \psi_k^3 \rangle = \langle \psi_j^1, \tilde{\Gamma}_T^u \psi_k^3 \rangle = \langle \psi_j^2, \psi_k^3 \rangle$ and

$$(22) \quad \pi_N(\Psi^3) \psi_j^2 = \pi_N(\Psi^3) \Gamma_T^{\tilde{u}} \psi_j^1, \quad \forall j \in \mathbb{N}.$$

2) Controllability in projection of unitarily equivalent functions:

Let us consider $\{\psi_j^1\}_{j \in \mathbb{N}}, \{\psi_j^2\}_{j \in \mathbb{N}} \subset H_{(0)}^3$ unitarily equivalent. Let $\Psi^3 := \{\psi_j^3\}_{j \in \mathbb{N}}$ be an Hilbert basis of \mathcal{H} . We suppose the existence of $\hat{\Gamma} \in U(\mathcal{H})$ so that $\hat{\Gamma} \psi_j^1 = \psi_j^2$ and $\hat{\Gamma}^* \psi_j^3 \in H_{(0)}^3$ for every $j \in \mathbb{N}$. One knows that for every $j \in \mathbb{N}$, there exists $\{a_k^j\}_{k \in \mathbb{N}} \in \ell^2(\mathbb{C})$ such that $\psi_j^1 = \sum_{k \in \mathbb{N}} a_k^j \psi_k^3$. However, $\{\hat{\Gamma} \psi_j^3\}_{j \in \mathbb{N}}$ is an Hilbert basis of \mathcal{H} and

$$\psi_j^2 = \hat{\Gamma} \psi_j^1 = \sum_{k \in \mathbb{N}} a_k^j \hat{\Gamma} \psi_k^3.$$

The point 2) implies that there exist $T > 0$ and $u \in L^2((0, T), \mathbb{R})$ so that

$$\pi_N(\Psi^3) \Gamma_T^u \psi_k^3 = \pi_N(\Psi^3) \hat{\Gamma} \psi_k^3$$

for every $k \in \mathbb{N}$, and then for any $j \in \mathbb{N}$

$$\pi_N(\Psi^3) \Gamma_T^u \psi_j^1 = \sum_{k \in \mathbb{N}} a_k^j (\pi_N(\Psi^3) \Gamma_T^u \psi_k^3) = \pi_N(\Psi^3) \sum_{k \in \mathbb{N}} a_k^j \hat{\Gamma} \psi_k^3 = \pi_N(\Psi^3) \psi_j^2.$$

3) Controllability in projection with generic projector: Let $\Psi^3 = \{\psi_j^3\}_{j \in \mathbb{N}} \subset H_{(0)}^3$ be a sequence of linearly independent elements. For every $N \in \mathbb{N}$, by considering the Gram-Schmidt orthonormalization process, there exists an orthonormal system $\tilde{\Psi}^3 := \{\{\tilde{\psi}_j^3\}_{j \leq N}, 0, \dots\}$ such that

$$\text{span}\{\psi_j^3 : j \leq N\} = \text{span}\{\tilde{\psi}_j^3 : j \leq N\}.$$

The claim then follows since $\pi_N(\Psi^3) \equiv \pi_N(\tilde{\Psi}^3)$. If $\Psi^3 = \{\psi_j^3\}_{j \in \mathbb{N}} \subset H_{(0)}^3$ is a generic sequence of functions, then one can extract from Ψ^3 a subsequence of linearly independent elements and repeat as above. \square

3.3 Proof of Corollary 4

Let $\psi^1, \psi^2 \in \mathcal{H}$. We define the rank one operator $|\psi^1\rangle\langle\psi^2|$ so that

$$|\psi^1\rangle\langle\psi^2|\psi = \psi^1\langle\psi^2, \psi\rangle, \quad \forall \psi \in \mathcal{H}.$$

We point out that, for any $\hat{\Gamma} \in U(\mathcal{H})$, we have $\hat{\Gamma}|\psi^1\rangle\langle\psi^2| = |\hat{\Gamma}\psi^1\rangle\langle\psi^2|$ and $|\psi^1\rangle\langle\psi^2|\hat{\Gamma}^* = |\psi^1\rangle\langle\hat{\Gamma}\psi^2|$ since for every $\psi \in \mathcal{H}$

$$|\psi^1\rangle\langle\psi^2|\hat{\Gamma}^*\psi = \psi^1\langle\psi^2, \hat{\Gamma}^*\psi\rangle = \psi^1\langle\hat{\Gamma}\psi^2, \psi\rangle = |\psi^1\rangle\langle\hat{\Gamma}\psi^2|\psi.$$

Proof of Corollary 4. Let $T > 0$ large enough and $\Psi^3 := \{\psi_j^3\}_{j \in \mathbb{N}} \in H_{(0)}^3$. Let $\rho^1, \rho^2 \in T(\mathcal{H})$ be two unitarily equivalent density matrices such that $\text{Ran}(\rho^1), \text{Ran}(\rho^2) \subseteq H_{(0)}^3$. We suppose that the unitary operator $\widehat{\Gamma} \in U(\mathcal{H})$ such that $\rho^2 = \widehat{\Gamma} \rho^1 \widehat{\Gamma}$ satisfies the condition $\widehat{\Gamma}^* \psi_j^3 \in H_{(0)}^3$ for every $j \in \mathbb{N}$. One can ensure the existence of two complete orthonormal systems $\Psi^1 := \{\psi_j^1\}_{j \in \mathbb{N}}, \Psi^2 := \{\psi_j^2\}_{j \in \mathbb{N}} \in H_{(0)}^3$ respectively composed by eigenfunctions of ρ^1 and of ρ^2 such that

$$\rho^1 = \sum_{j=1}^{\infty} l_j |\psi_j^1\rangle\langle\psi_j^1|, \quad \rho^2 = \sum_{j=1}^{\infty} l_j |\psi_j^2\rangle\langle\psi_j^2|.$$

The sequence $\{l_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^+$ corresponds to the spectrum of ρ^1 and of ρ^2 . Now, thanks to Theorem 2, there exists a control function $u \in L^2((0, T), \mathbb{R})$ such that $\pi_N(\Psi^3) \Gamma_T^u \psi_j^1 = \pi_N(\Psi^3) \psi_j^2$. Thus

$$\begin{aligned} \pi_N(\Psi^3) \Gamma_T^u \rho^1 (\Gamma_T^u)^* \pi_N(\Psi^3) &= \sum_{j=1}^{\infty} l_j |\pi_N(\Psi^3) \Gamma_T^u \psi_j^1\rangle\langle\psi_j^1 \Gamma_T^u \pi_N(\Psi^3)| \\ &= \sum_{j=1}^{\infty} l_j \pi_N(\Psi^3) |\psi_j^2\rangle\langle\psi_j^2| \pi_N(\Psi^3) = \pi_N(\Psi^3) \rho^2 \pi_N(\Psi^3). \quad \square \end{aligned}$$

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A Analytic Perturbation

Let us consider the problems (14) and the eigenvalues $\{\lambda_j^{u_0}\}_{j \in \mathbb{N}}$ of the operator $A + u_0 B$. Let B be a bounded symmetric operator satisfying Assumptions I. We introduce some classical results by Kato [8].

Definition 4. Let D be a domain of the complex plan. A family $T(z)$ for $z \in D$ of closed operators from a Banach space X to a Banach space Y is said to be a holomorphic family of type (A) if $D(T(z))$ is independent of z and if $T(z)u$ is holomorphic for $z \in D$ and for every $u \in D(T(z))$.

Theorem 9 (Kato; [8]; Theorem VII.3.9). *Let $T(z)$ be a self-adjoint holomorphic family of type (A) defined for z in a neighborhood of an interval $I \subset \mathbb{R}$. Furthermore, let $T(z)$ have a compact resolvent. Then all eigenvalues of $T(z)$ can be represented by functions which are holomorphic in I . More precisely, there is a sequence of scalar-valued functions $z \mapsto \{\lambda_n(z)\}_{n \in \mathbb{N}}$ and operator-valued functions $z \mapsto \{\phi_n(z)\}_{n \in \mathbb{N}}$, all holomorphic on I , such that*

for $z \in I$, the sequence $\{\lambda_n(z)\}_{n \in \mathbb{N}}$ represents all the repeated eigenvalues of $T(z)$ and $\{\phi_n(z)\}_{n \in \mathbb{N}}$ forms a complete orthonormal family of the associated eigenvectors of $T(z)$.

Proposition 10. *Let B be a bounded symmetric operator satisfying Assumptions I. There exists a neighborhood D of $u = 0$ in \mathbb{R} small enough where the maps $u \mapsto \lambda_j^u$ are analytic $\forall j \in \mathbb{N}$.*

Proof. Thanks to [8, Theorem VII.2.6], there exists a neighborhood $D \subset \mathbb{R}$ containing 0 so that the self-adjoint family of operators $A + u_0 B$ is holomorphic of type (A) for $u_0 \in D$. Theorem 9 achieves the proof. \square

The next lemma proves the existence of perturbations which do not shrink the eigenvalues gaps. From now on, we use the notation $\| \cdot \|$ for the operators norm of bounded operators in \mathcal{H} .

Lemma 11. *Let B be a bounded symmetric operator satisfying Assumptions I. There exists a neighborhood $U(0)$ of $u = 0$ in \mathbb{R} such that for each $u_0 \in U(0)$, there exists $r > 0$ such that $\forall j \in \mathbb{N}$*

$$\mu_j := \frac{\lambda_j + \lambda_{j+1}}{2} \in \rho(A + u_0 B), \quad \| (A + u_0 B - \mu_j)^{-1} \| \leq r.$$

Proof. Let D be the neighborhood provided by Proposition 10. First, we prove that for $u_0 \in D$ so that $|u_0|$ is small enough, $(A + u_0 B - \mu_j)$ is invertible with bounded inverse for each $j \in \mathbb{N}$. We know that $(A - \mu_j)$ is invertible in a bounded operator since $\mu_j \in \rho(A)$ (resolvent set of A). For $\delta := \min_{j \in \mathbb{N}} \{|\lambda_{j+1} - \lambda_j|\}$

$$\begin{aligned} \| (A - \mu_j)^{-1} \| &\leq \sup_{k \in \mathbb{N}} \frac{1}{|\mu_j - \lambda_k|} = \frac{2}{|\lambda_{j+1} - \lambda_j|} \leq \frac{2}{\delta}. \\ \implies \| (A - \mu_j)^{-1} u_0 B \| &\leq |u_0| \| (A - \mu_j)^{-1} \| \| B \| \leq \frac{2}{\delta} |u_0| \| B \|. \end{aligned}$$

If $|u_0| \leq \frac{\delta(1-\epsilon)}{2\|B\|}$ for $\epsilon \in (0, 1)$, then $\| (A - \mu_j)^{-1} u_0 B \| \leq 1 - \epsilon$. The operator $(A + u_0 B - \mu_j)$ is invertible and one can notice that

$$\| (A + u_0 B - \mu_j) \| \geq \| (A - \mu_j) \| - \| u_0 B \| \geq \frac{\delta}{2} - \frac{\delta(1-\epsilon)}{2} = \frac{\delta\epsilon}{2}$$

and in conclusion $\| (A + u_0 B - \mu_j)^{-1} \| \leq \frac{2}{\delta\epsilon}$. \square

Lemma 12. *Let B be a bounded symmetric operator satisfying Assumptions I. There exists a neighborhood $U(0)$ of 0 in \mathbb{R} such that $\forall u_0 \in U(0)$*

$$(A + u_0 P_{\phi_k}^\perp B - \lambda_k^{u_0})$$

is invertible with bounded inverse from $D(A) \cap \phi_k^\perp$ to ϕ_k^\perp , for $P_{\phi_k}^\perp$ the projector onto the orthogonal space of ϕ_k and for every $k \in \mathbb{N}$.

Proof. Let D be the neighborhood provided by Lemma 11. For any $u_0 \in D$, one can consider the decomposition

$$(A + u_0 P_{\phi_k}^\perp B - \lambda_k^{u_0}) = (A - \lambda_k^{u_0}) + u_0 P_{\phi_k}^\perp B.$$

The operator $A - \lambda_k^{u_0}$ is invertible with bounded inverse when it acts on the orthogonal space of ϕ_k and we want to estimate

$$\| \| ((A - \lambda_k^{u_0})|_{\phi_k^\perp})^{-1} u_0 P_{\phi_k}^\perp B \| \|.$$

However, for every $\psi \in D(A) \cap \text{Ran}(P_{\phi_k}^\perp)$ such that $\|\psi\| = 1$, we have

$$\| (A - \lambda_k^{u_0})\psi \| \geq \min\{|\lambda_{k+1} - \lambda_k^{u_0}|, |\lambda_k^{u_0} - \lambda_{k-1}|\} \|\psi\|.$$

Let $\delta_k := \min\{|\lambda_{k+1} - \lambda_k^{u_0}|, |\lambda_k^{u_0} - \lambda_{k-1}|\}$. Thanks to Lemma 11, for $|u_0|$ small enough, there holds $\lambda_k^{u_0} \in \left(\frac{\lambda_{k-1} + \lambda_k}{2}, \frac{\lambda_k + \lambda_{k+1}}{2}\right)$ and then

$$\delta_k \geq \min \left\{ \left| \lambda_{k+1} - \frac{\lambda_k + \lambda_{k+1}}{2} \right|, \left| \frac{\lambda_{k-1} + \lambda_k}{2} - \lambda_{k-1} \right| \right\} \geq \frac{(2k-1)\pi^2}{2} > k.$$

Afterwards, $\| \| ((A - \lambda_k^{u_0})|_{\phi_k^\perp})^{-1} u_0 P_{\phi_k}^\perp B \| \| \leq \frac{1}{\delta_k} |u_0| \| \| B \| \|$ and, if $|u_0| \leq (1-r) \frac{\delta_k}{\| \| B \| \|}$ for $r \in (0, 1)$, one has that

$$\| \| ((A - \lambda_k^{u_0})|_{\phi_k^\perp})^{-1} u_0 P_{\phi_k}^\perp B \| \| \leq (1-r) < 1.$$

The operator $A_k := (A - \lambda_k^{u_0} + u_0 P_{\phi_k}^\perp B)$ is invertible when it acts onto the orthogonal space of ϕ_k and

$$\begin{aligned} \| \| A_k |_{\phi_k^\perp} \| \| &\geq \| \| (A - \lambda_k^{u_0}) |_{\phi_k^\perp} \| \| - \| \| u_0 P_{\phi_k}^\perp B |_{\phi_k^\perp} \| \| \geq \delta_k - \| \| u_0 P_{\phi_k}^\perp B \| \| \\ &\geq \delta_k - |u_0| \| \| B \| \| \geq \delta_k - (1-r)\delta_k = r\delta_k. \end{aligned}$$

In conclusion, the proof is achieved since

$$(23) \quad \| \| ((A - \lambda_k^{u_0} + u_0 P_{\phi_k}^\perp B) |_{\phi_k^\perp})^{-1} \| \| \leq \frac{1}{r\delta_k} < \frac{1}{rk}.$$

Lemma 13. *Let B be a bounded symmetric operator satisfying Assumptions I. There exists a neighborhood $U(0)$ of 0 in \mathbb{R} such that, for any $u_0 \in U(0)$, we have $\lambda_j^{u_0} \neq 0$ and $\lambda_j^{u_0} \asymp \lambda_j$ for every $j \in \mathbb{N}$. In other words, there exist two constants $C_1, C_2 > 0$ so that for each $j \in \mathbb{N}$*

$$(24) \quad C_1 \lambda_j \leq \lambda_j^{u_0} \leq C_2 \lambda_j.$$

Proof. Let $u_0 \in D$ for D the neighborhood provided by Lemma 12. We decompose the eigenfunction $\phi_j^{u_0} = a_j \phi_j + \eta_j$, where a_j is an orthonormalizing constant and η_j is orthogonal to ϕ_j . Hence

$$\begin{aligned}\lambda_k^{u_0} \phi_k^{u_0} &= (A + u_0 B)(a_k \phi_k + \eta_k) \\ &= A a_k \phi_k + A \eta_k + u_0 B a_k \phi_k + u_0 B \eta_k, \\ \implies \lambda_k^{u_0} a_k \phi_k + \lambda_k^{u_0} \eta_k &= A a_k \phi_k + A \eta_k + u_0 B a_k \phi_k + u_0 B \eta_k.\end{aligned}$$

By projecting onto the orthogonal space of ϕ_k

$$\begin{aligned}\lambda_k^{u_0} \eta_k &= A \eta_k + u_0 P_{\phi_k}^\perp B a_k \phi_k + u_0 P_{\phi_k}^\perp B \eta_k \\ \implies (A + u_0 P_{\phi_k}^\perp B - \lambda_k^{u_0}) \eta_k &= -u_0 P_{\phi_k}^\perp B a_k \phi_k.\end{aligned}$$

However, Lemma 12 ensures that $A + u_0 P_{\phi_k}^\perp B - \lambda_k^{u_0}$ is invertible with bounded inverse when it acts on the orthogonal space of ϕ_k and then

$$(25) \quad \eta_k = -a_k ((A + u_0 P_{\phi_k}^\perp B - \lambda_k^{u_0})|_{\phi_k^\perp})^{-1} u_0 P_{\phi_k}^\perp B \phi_k.$$

Afterwards, we know that

$$\begin{aligned}\lambda_j^{u_0} &= \langle a_j \phi_j + \eta_j, (A + u_0 B)(a_j \phi_j + \eta_j) \rangle \\ &= |a_j|^2 \lambda_j + u_0 \langle a_j \phi_j, B a_j \phi_j \rangle + \langle a_j \phi_j, (A + u_0 B) \eta_j \rangle \\ &\quad + \langle \eta_j, (A + u_0 B) a_j \phi_j \rangle + \langle \eta_j, (A + u_0 B) \eta_j \rangle.\end{aligned}$$

By using the relation (25), there holds

$$\begin{aligned}\langle \eta_j, (A + u_0 B) \eta_j \rangle &= \langle \eta_j, (A + u_0 P_{\phi_j}^\perp B - \lambda_j^{u_0}) \eta_j \rangle + \lambda_j^{u_0} \|\eta_j\|^2 \\ &= \lambda_j^{u_0} \|\eta_j\|^2 + \left\langle \eta_j, -a_j (A + u_0 P_{\phi_j}^\perp B - \lambda_j^{u_0}) \right. \\ &\quad \left. \cdot ((A + u_0 P_{\phi_j}^\perp B - \lambda_j^{u_0})|_{\phi_j^\perp})^{-1} u_0 P_{\phi_j}^\perp B \phi_j \right\rangle.\end{aligned}$$

However, $(A + u_0 P_{\phi_j}^\perp B - \lambda_j^{u_0})((A + u_0 P_{\phi_j}^\perp B - \lambda_j^{u_0})|_{\phi_j^\perp})^{-1} = Id$ and then

$$\langle \eta_j, (A + u_0 B) \eta_j \rangle = \lambda_j^{u_0} \|\eta_j\|^2 - u_0 a_j \langle \eta_j, P_{\phi_j}^\perp B \phi_j \rangle.$$

Moreover, $\langle \phi_j, (A + u_0 B) \eta_j \rangle = u_0 \langle \phi_j, B \eta_j \rangle = u_0 \langle P_{\phi_j}^\perp B \phi_j, \eta_j \rangle$ and equivalently $\langle \eta_j, (A + u_0 B) \phi_j \rangle = u_0 \langle \eta_j, P_{\phi_j}^\perp B \phi_j \rangle$. Thus

$$(26) \quad \lambda_j^{u_0} = |a_j|^2 \lambda_j + u_0 |a_j|^2 B_{j,j} + \lambda_j^{u_0} \|\eta_j\|^2 + u_0 \bar{a}_j \langle P_{\phi_j}^\perp B \phi_j, \eta_j \rangle.$$

One can notice that $|a_j| \in [0, 1]$ and $\|\eta_j\|$ are uniformly bounded in j . We show that the first accumulates at 1 and the second at 0. Indeed, from (23) and (25), one has that there exists a constant $C_1 > 0$ such that

$$(27) \quad \|\eta_j\|^2 \leq |u_0|^2 \left\| ((A + u_0 P_{\phi_j}^\perp B - \lambda_j^{u_0})|_{\phi_j^\perp})^{-1} \right\|^2 |a_j|^2 \|B \phi_j\|^2 \leq \frac{C_1}{j^2}.$$

for $r \in (0, 1)$, which implies that $\lim_{j \rightarrow \infty} \|\eta_j\| = 0$. Afterwards, by contradiction, if $a_\infty := \lim_{j \rightarrow \infty} |a_j| \neq 1$, then $|a_\infty| \in [0, 1)$ and thus

$$1 = \lim_{j \rightarrow \infty} \|\phi_j^{u_0}\| \leq \lim_{j \rightarrow \infty} |a_j| \|\phi_j\| < 1$$

that is absurd. From (26), it follows that $\lambda_j^{u_0} \asymp \lambda_j$ for $|u_0|$ small enough. The relation also implies that $\lambda_j^{u_0} \neq 0$, for every $j \in \mathbb{N}$ and for $|u_0|$ small enough. \square

Lemma 14. *Let B be a bounded symmetric operator satisfying Assumptions I. For every $N \in \mathbb{N}$, there exists a neighborhood $U(0)$ of 0 in \mathbb{R} such that, for any $u_0 \in U(0)$, there exists $\tilde{C}_N > 0$ such that*

$$(28) \quad |\langle \phi_k^{u_0}(T), B\phi_j^{u_0}(T) \rangle| \geq \frac{\tilde{C}_N}{k^3}, \quad \forall j, k \in \mathbb{N}, j \leq N.$$

Proof. We achieve the proof for fixed $j \leq N$. The generalization follows by using the minimum of all the constants defined for every $j \leq N$.

We start by choosing $k \in \mathbb{N}$ such that $k \neq j$ and $u_0 \in D$ for D the neighborhood provided by Lemma 13. Thanks to Assumptions I, we have

$$(29) \quad \begin{aligned} |\langle \phi_k^{u_0}, B\phi_j^{u_0} \rangle| &= |\langle a_k \phi_k + \eta_k, B(a_j \phi_j + \eta_j) \rangle| \\ &\geq C_N \frac{\overline{a_k a_j}}{k^3} - |\overline{a_k} \langle \phi_k, B\eta_j \rangle + a_j \langle \eta_k, B\phi_j \rangle + \langle \eta_k, B\eta_j \rangle|. \end{aligned}$$

1) Expansion of the terms of (29): Thanks to (25), for $|u_0|$ small enough

$$(30) \quad \begin{aligned} \langle \eta_k, B\phi_j \rangle &= \langle \eta_k, P_{\phi_k}^\perp B\phi_j \rangle = \\ &\langle -a_k((A + u_0 P_{\phi_k}^\perp B - \lambda_k^{u_0})|_{\phi_k^\perp})^{-1} u_0 P_{\phi_k}^\perp B\phi_k, P_{\phi_k}^\perp B\phi_j \rangle \\ &= ((A + u_0 P_{\phi_k}^\perp B - \lambda_k^{u_0})|_{\phi_k^\perp})^{-1} u_0 P_{\phi_k}^\perp B\phi_k, P_{\phi_k}^\perp B\phi_j \rangle = \\ &((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} \sum_{n=0}^{\infty} (u_0((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} P_{\phi_k}^\perp B P_{\phi_k}^\perp)^n u_0 P_{\phi_k}^\perp B\phi_k, P_{\phi_k}^\perp B\phi_j \rangle \end{aligned}$$

By defining $M_k := \sum_{n=0}^{\infty} (u_0((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} P_{\phi_k}^\perp B)^n P_{\phi_k}^\perp$, there holds

$$(31) \quad \langle \eta_k, B\phi_j \rangle = -u_0 \langle a_k M_k B\phi_k, ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} P_{\phi_k}^\perp B\phi_j \rangle.$$

Thanks to the fact that $B : D(A) \rightarrow D(A)$

$$\begin{aligned} ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} P_{\phi_k}^\perp B\phi_j &= P_{\phi_k}^\perp B((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} \phi_j \\ - [P_{\phi_k}^\perp B, ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} P_{\phi_k}^\perp] \phi_j &= P_{\phi_k}^\perp B((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} \phi_j \\ - ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} P_{\phi_k}^\perp [B, A]((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} \phi_j. \end{aligned}$$

By calling $\widetilde{B}_k := ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1}P_{\phi_k}^\perp[B, A]$

$$(32) \quad ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1}P_{\phi_k}^\perp B\phi_j = P_{\phi_k}^\perp (B + \widetilde{B}_k)(\lambda_j - \lambda_k^{u_0})^{-1}\phi_j.$$

Let us consider (31) and (32), we have

$$(33) \quad \langle \eta_k, B\phi_j \rangle = -\frac{u_0}{\lambda_j - \lambda_k^{u_0}} \langle a_k M_k B\phi_k, (B + \widetilde{B}_k)\phi_j \rangle.$$

Now, thanks to the same techniques

$$(34) \quad \begin{aligned} |\langle \eta_k, B\eta_j \rangle| &= |\langle B\eta_k, \eta_j \rangle| = |\langle u_0 a_k B((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1}M_k B\phi_k, \\ u_0 a_j ((A - \lambda_j^{u_0})P_{\phi_j}^\perp)^{-1}M_j B\phi_j \rangle| &= \left| \frac{a_j \overline{a_k} u_0^2}{\lambda_k - \lambda_j^{u_0}} \langle \phi_k, L_{k,j}\phi_j \rangle \right| \end{aligned}$$

for $L_{k,j} := (A - \lambda_j^{u_0})B M_k ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1}P_{\phi_k}^\perp B ((A - \lambda_j^{u_0})P_{\phi_j}^\perp)^{-1}M_j B$. We know the existence of $\epsilon > 0$ such that $|a_l| \in (\epsilon, 1)$, for every $l \in \mathbb{N}$. Thanks to (33) and (34), there exists \widehat{C}_N such that from the relation (29), we have

$$(35) \quad \begin{aligned} |\langle \phi_k^{u_0}, B\phi_j^{u_0} \rangle| &\geq \frac{\widehat{C}_N}{k^3} - \left| \frac{u_0}{\lambda_j - \lambda_k^{u_0}} \langle M_k B\phi_k, (B + \widetilde{B}_k)\phi_j \rangle \right| \\ &- \left| \frac{u_0}{\lambda_k - \lambda_j^{u_0}} \langle (B + \widetilde{B}_j)\phi_k, M_j B\phi_j \rangle \right| - \left| \frac{u_0^2}{\lambda_k - \lambda_j^{u_0}} \langle \phi_k, L_{k,j}\phi_j \rangle \right|. \end{aligned}$$

2) Features of the operators $M_k, \widetilde{B}_k, L_{k,j}$: First, the operators M_k are uniformly bounded in $L(H_{(0)}^2, H_{(0)}^2)$, when $|u_0|$ is small enough so that $\|u_0((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1}P_{\phi_k}^\perp B P_{\phi_k}^\perp\|_{L(H_{(0)}^2)} < 1$. Second, (32) implies that

$$\widetilde{B}_k P_{\phi_k}^\perp = ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1}P_{\phi_k}^\perp B (A - \lambda_k^{u_0})P_{\phi_k}^\perp - P_{\phi_k}^\perp B P_{\phi_k}^\perp$$

from which follows that the operators \widetilde{B}_k are uniformly bounded in k in

$$L(H_{(0)}^2 \cap \text{Ran}(P_{\phi_k}^\perp), H_{(0)}^2 \cap \text{Ran}(P_{\phi_k}^\perp)).$$

Third, one can notice that $B((A - \lambda_j^{u_0})P_{\phi_j}^\perp)^{-1}M_j B \in L(H_{(0)}^2, H_{(0)}^2)$ and

$$\begin{aligned} &(A - \lambda_j^{u_0})B M_k ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1}P_{\phi_k}^\perp \\ &= (A - \lambda_j^{u_0})B ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} \sum_{n=0}^{\infty} (u_0 P_{\phi_k}^\perp B ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1})^n P_{\phi_k}^\perp \\ &= (A - \lambda_j^{u_0})((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1}P_{\phi_k}^\perp (\widetilde{B}_k + B) \widetilde{M}_k \end{aligned}$$

for $\widetilde{M}_k := \sum_{n=0}^{\infty} (u_0 P_{\phi_k}^\perp B ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1})^n P_{\phi_k}^\perp$. Now, the operators \widetilde{M}_k and $L_{k,j}$ are uniformly bounded in $L(H_{(0)}^2, H_{(0)}^2)$ as M_k .

Let $\{F_l\}_{l \in \mathbb{N}}$ be an infinite family of uniformly bounded operators in $L(H_{(0)}^2, H_{(0)}^2)$

$$\forall l \in \mathbb{N}, \exists c_l > 0 : \sum_{k=1}^{\infty} |k^2 \langle \phi_k, F_l \phi_j \rangle|^2 < \infty, \implies |\langle \phi_k, F_l \phi_j \rangle| \leq \frac{c_l}{k^2}.$$

Now, one can choose constants c_l uniformly bounded in l since

$$\begin{aligned} \sup_{l \in \mathbb{N}} |k^2 \langle \phi_k, F_l \phi_j \rangle|^2 &= \sup_{l \in \mathbb{N}} \sum_{m \in \mathbb{N}} |m^2 \langle \phi_m, F_l \phi_j \rangle|^2 - \sup_{l \in \mathbb{N}} \sum_{m \neq k} |m^2 \langle \phi_m, F_l \phi_j \rangle|^2 \\ &\leq 2 \sup_{l \in \mathbb{N}} \|F_l \phi_j\|_{(2)}^2 < \infty. \end{aligned}$$

Thus, for every family of uniformly bounded operators F_l in $L(H_{(0)}^2, H_{(0)}^2)$

$$(36) \quad \exists c > 0 : |\langle \phi_k, F_l \phi_j \rangle| \leq \frac{c}{k^2}.$$

3) Conclusion: The operators $\{BM_k(B + \tilde{B}_k)\}_{k \in \mathbb{N}}$, $\{L_{k,j}\}_{k \in \mathbb{N}}$ are uniformly bounded in $L(H_{(0)}^2, H_{(0)}^2)$ and $BM_j(B + \tilde{B}_j) \in L(H_{(0)}^2, H_{(0)}^2)$ for every $1 \leq j \leq N$. Hence, we use the relation (36) in (35). Now, $|\lambda_j - \lambda_k^{u_0}|^{-1} \sim k^{-2}$ and $|\lambda_k - \lambda_j^{u_0}|^{-1} \sim k^{-2}$, thanks to Lemma 13. Thus, there exist $C_1, C_2, C_3, C_4 > 0$ so that for $|u_0|$ small enough

$$\begin{aligned} |\langle \phi_k^{u_0}(T), B\phi_j^{u_0}(T) \rangle| &= |\langle \phi_k^{u_0}(T), B\phi_j^{u_0}(T) \rangle| \geq \hat{C}_N \frac{1}{k^3} - \frac{C_1 |u_0|}{|\lambda_j - \lambda_k^{u_0}| k^2} \\ &\quad - \frac{C_2 |u_0|}{|\lambda_k - \lambda_j^{u_0}| k^2} - \frac{C_3 |u_0|^2}{|\lambda_k - \lambda_j^{u_0}| k^2} \geq C_4 \frac{1}{k^3}. \end{aligned}$$

Now, the relation (28) is verified for $k = j$ thanks to the analyticity and by choosing u_0 such that $|\langle \phi_j^{u_0}(T), B\phi_j^{u_0}(T) \rangle| = c \neq 0$. For fixed $j \leq N$, the relation (28) is valid for a constant $C_j = \min\{C_4, c \cdot j^3\}$. In conclusion, (28) is valid for every $j \leq N$ by imposing $\tilde{C}_N = \min_{j \leq N} \{C_j\}$. \square

Lemma 15. *Let B be a bounded symmetric operator satisfying Assumptions I. There exists a neighborhood $U(0)$ of 0 in \mathbb{R} contained in the one introduced in Lemma 13 so that, for any $u_0 \in U(0)$*

$$\left(\sum_{j=1}^{\infty} \|\lambda_j^{u_0}\|^{\frac{3}{2}} |\langle \phi_j^{u_0}, \cdot \rangle|^2 \right)^{\frac{1}{2}} \asymp \left(\sum_{j=1}^{\infty} |j^3 \langle \phi_j, \cdot \rangle|^2 \right)^{\frac{1}{2}}.$$

Proof. For $|u_0|$ small enough, we prove that there exist $C_1 > 0$ such that $\| |A + u_0 B|^{\frac{s}{2}} \psi \| \leq C_1 \| |A|^{\frac{s}{2}} \psi \|$ for $s = 3$. We start by assuming $s = 4$. For any $\psi \in H_{(0)}^4$, there exist $C_2, C_3 > 0$ such that

$$\begin{aligned} \| |A + u_0 B|^2 \psi \| &= \|(A + u_0 B)^2 \psi\| \leq \|A^2 \psi\| + |u_0|^2 \|B^2 \psi\| + |u_0| \|AB \psi\| \\ &\quad + |u_0| \|BA \psi\| \leq \|A^2 \psi\| + |u_0|^2 \|B^2 \psi\| + |u_0| \| \|B\|_{L(H_{(0)}^2)} \|A \psi\| \\ &\quad + |u_0| \| \|B\| \|A \psi\| \leq C_2 \|A^2 \psi\| + C_3 \|\psi\| \leq (C_2 + C_3) \| |A|^2 \psi \|. \end{aligned}$$

Now, the proof of [5, Lemma 1] implies the validity of the relation also for $s = 3$. There exists $C > 0$ so that for every $\psi \in H_{(0)}^3$

$$\|\psi\|_{\tilde{H}_{(0)}^3} = \| |A + u_0 B|^{\frac{3}{2}} \psi \| \leq C \| |A|^{\frac{3}{2}} \psi \| = C \|\psi\|_{H_{(0)}^3}.$$

The opposite inequality follows by the same techniques thanks to the decomposition $A = (A + u_0 B) - u_0 B$. \square

Remark 6. Let B be a bounded symmetric operator satisfying Assumptions I. The techniques of the proof of Lemma 15 also allow to prove the existence of a neighborhood $U(0)$ of 0 in \mathbb{R} so that, for any $u_0 \in U(0)$

$$\|\cdot\|_{\tilde{H}_{(0)}^2} = \left(\sum_{j=1}^{\infty} \|\lambda_j^{u_0} \langle \phi_j^{u_0}, \cdot \rangle\|^2 \right)^{\frac{1}{2}} \asymp \left(\sum_{j=1}^{\infty} |j^2 \langle \phi_j, \cdot \rangle|^2 \right)^{\frac{1}{2}} = \|\cdot\|_{(2)}.$$

Lemma 16. *Let B be a bounded symmetric operator satisfying Assumptions I and $N \in \mathbb{N}$. Let $\epsilon > 0$ small enough and I^N be the set defined in (5). There exists a $U_\epsilon \subset \mathbb{R} \setminus \{0\}$ such that for each $u_0 \in U_\epsilon$*

$$\inf_{\substack{(j,k),(n,m) \in I^N \\ (j,k) \neq (n,m)}} |\lambda_j^{u_0} - \lambda_k^{u_0} - \lambda_n^{u_0} + \lambda_m^{u_0}| > \epsilon$$

Moreover, for every $\delta > 0$ small there exists $\epsilon > 0$ such that $\text{dist}(U_\epsilon, 0) < \delta$.

Proof. Let us consider the neighborhood D provided by Lemma 12. The maps $\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u$ are analytic for each $j, k, n, m \in \mathbb{N}$ and $u \in D$. One can notice that the number of elements such that

$$(37) \quad \lambda_j - \lambda_k - \lambda_n + \lambda_m = 0, \quad j, n \in \mathbb{N}, \quad k, m \leq N$$

is finite. Indeed $\lambda_k = k^2 \pi^2$ and (37) corresponds to $j^2 - k^2 = n^2 - m^2$. We have $|j^2 - n^2| = |k^2 - m^2| \leq N^2 - 1$ which is satisfied for a finite number of elements. Thus, for I the set defined in (5), the following set is finite

$$R := \{((j, k), (n, m)) \in (I^N)^2 : (j, k) \neq (n, m); \lambda_j - \lambda_k - \lambda_n + \lambda_m = 0\}.$$

1) Let $((j, k), (n, m)) \in R$, the set $V_{(j,k,n,m)} = \{u \in D \mid \lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u = 0\}$ is either a discrete subset of D or equal to D . Thanks to the relation (26)

$$\begin{aligned} & \lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u = \\ & |a_j|^2 \lambda_j + u_0 |a_j|^2 B_{j,j} + \lambda_j^{u_0} \|\eta_j\|^2 + u_0 \bar{a}_j \langle P_{\phi_j}^\perp B \phi_j, \eta_j \rangle \\ & - |a_k|^2 \lambda_k - u_0 |a_k|^2 B_{k,k} - \lambda_k^{u_0} \|\eta_k\|^2 - u_0 \bar{a}_k \langle P_{\phi_k}^\perp B \phi_k, \eta_k \rangle \\ & - |a_n|^2 \lambda_n - u_0 |a_n|^2 B_{n,n} - \lambda_n^{u_0} \|\eta_n\|^2 - u_0 \bar{a}_n \langle P_{\phi_n}^\perp B \phi_n, \eta_n \rangle \\ & + |a_m|^2 \lambda_m + u_0 |a_m|^2 B_{m,m} + \lambda_m^{u_0} \|\eta_m\|^2 + u_0 \bar{a}_m \langle P_{\phi_m}^\perp B \phi_m, \eta_m \rangle, \end{aligned}$$

$$(38) \implies \lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u = |a_j|^2 \lambda_j - |a_k|^2 \lambda_k - |a_n|^2 \lambda_n + |a_m|^2 \lambda_m \\ + (|a_j|^2 B_{j,j} - |a_k|^2 B_{k,k} - |a_n|^2 B_{n,n} + |a_m|^2 B_{m,m}) u_0 + o(u_0).$$

For $|u|$ small enough, thanks to the fact that $\lim_{|u_0| \rightarrow 0} |a_j|^2 = 1$ and to the third point of Assumptions I, $\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u$ can not be constantly equal to 0. Then, $V_{(j,k,n,m)}$ is discrete and

$$V = \{u \in D \mid \exists (j, k, n, m) \in R : \lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u = 0\}$$

is a discrete subset of D . Thanks to the fact that R is a finite set

$$\tilde{U}_\epsilon := \{u \in D : \forall (j, k, n, m) \in R \mid |\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u| \geq \epsilon\}$$

has positive measure for $\epsilon > 0$ small enough. Moreover, for any $\delta > 0$ small, there exists $\epsilon_0 > 0$ such that $\text{dist}(0, \tilde{U}_{\epsilon_0}) < \delta$.

2) Let $((j, k), (n, m)) \in (I^N)^2 \setminus R$ be different numbers, we know that

$$|\lambda_j^0 - \lambda_k^0 - \lambda_n^0 + \lambda_m^0| = \pi^2 |j^2 - k^2 - n^2 + m^2| > \pi^2.$$

First, thanks (26), $\lambda_j^{u_0} \leq |a_j|^2 \lambda_j + |u_0| C_1$ and $\lambda_j^{u_0} \geq |a_j|^2 \lambda_j - |u_0| C_2$ for suitable constants $C_1, C_2 > 0$ non depending on the index j . Thus

$$(39) \quad |\lambda_j^{u_0} - \lambda_k^{u_0} - \lambda_n^{u_0} + \lambda_m^{u_0}| \geq ||a_j|^2 \lambda_j - |a_k|^2 \lambda_k - |a_n|^2 \lambda_n + |a_m|^2 \lambda_m| \\ - |u_0| (2C_1 + 2C_2).$$

Now, thanks to the relation (25), $\lim_{k \rightarrow \infty} |a_k|^2 = 1$. For any u in D and ϵ small enough, there exists $M_\epsilon \in \mathbb{N}$ so that for $R^C := (I^N)^2 \setminus R$

$$||a_j|^2 \lambda_j - |a_k|^2 \lambda_k - |a_n|^2 \lambda_n + |a_m|^2 \lambda_m| \geq \pi^2 - \epsilon, \\ \forall ((j, k), (n, m)) \in R^C, \quad j, k, n, m \geq M_\epsilon.$$

However $\lim_{|u_0| \rightarrow 0} |a_k|^2 = 1$ uniformly in k thanks to (27) and then there exists a neighborhood $W_\epsilon \subseteq D$ such that for each $u \in W_\epsilon$

$$||a_j|^2 \lambda_j - |a_k|^2 \lambda_k - |a_n|^2 \lambda_n + |a_m|^2 \lambda_m| \geq \pi^2 - \epsilon, \\ \forall ((j, k), (n, m)) \in R^C, \quad 1 \leq j, k, n, m < M_\epsilon.$$

Thus for each $u \in W_\epsilon$ and $((j, k), (n, m)) \in R^C$ such that $(j, k) \neq (n, m)$

$$|\lambda_j^u - \lambda_k^u - \lambda_n^u + \lambda_m^u| \geq \pi^2 - \epsilon.$$

3) The proof is achieved since for $\epsilon_1 > 0$ small enough, $\tilde{U}_{\epsilon_1} \cap W_\epsilon$ is non zero measure subset of D . For any $u_0 \in \tilde{U}_{\epsilon_1} \cap W_\epsilon$ and for any $((j, k), (n, m)) \in (I^N)^2$ so that $(j, k) \neq (n, m)$, we have

$$|\lambda_j^{u_0} - \lambda_k^{u_0} - \lambda_n^{u_0} + \lambda_m^{u_0}| \geq \min\{\pi^2 - \epsilon, \epsilon_1\}. \quad \square$$

Remark 7. Let B be a bounded symmetric operator satisfying Assumptions I. By using the techniques of the proofs of Lemma 14 and Lemma 16, one can ensure the existence of a neighborhood U_1 of u_0 in \mathbb{R} and U_2 , a countable subset of \mathbb{R} so that, for any $u_0 \in U(0) := (U_1 \setminus U_2) \setminus \{0\}$ we have

1. For every $N \in \mathbb{N}$, $\lambda_j^{u_0} - \lambda_k^{u_0} - \lambda_n^{u_0} + \lambda_m^{u_0} \neq 0$ for all $(j, k), (n, m) \in I^N$ (see (5)) so that $(j, k) \neq (n, m)$.
2. $B_{j,k}^{u_0} = \langle \psi_j^{u_0}(T), B\phi_k^{u_0}(T) \rangle \neq 0$ for every $j, k \in \mathbb{N}$.
3. For $\epsilon > 0$, if $|u_0|$ is small enough, then $\sup_{j \in \mathbb{N}} \|\phi_j - \phi_j^{u_0}\|_{(3)} \leq \epsilon$.

Remark 8. Let B be a bounded symmetric operator satisfying Assumptions I. As Remark 7, there exists a neighborhood U_1 of u_0 in \mathbb{R} and U_2 , a countable subset of \mathbb{R} containing $u = 0$ so that, for any $u_0 \in U(0) := (U_1 \setminus U_2) \setminus \{0\}$, the numbers $\{1, \lambda_j^{u_0}\}_{j \in \mathbb{N}}$ are rationally independent, i.e. for any $M \in \mathbb{N}$ and $\{r_j\}_{0 \leq j \leq M} \subset \mathbb{Q} \setminus \{0\}$ it holds $r_0 + \sum_{j=1}^M r_j \lambda_j^{u_0} \neq 0$.

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