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Simultaneous global exact controllability in projection

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Abstract

We consider an infinite number of one dimensional bilinear Schrödinger equations in a segment. We prove the simultaneous global exact controllability in projection of unitarily equivalent sequences of functions.

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Keywords: Schrödinger equation, simultaneous control, global exact controllability, moment problem, perturbation theory, density matrices.

1 Introduction

Let \mathcal{H} be an infinite dimensional Hilbert space. In quantum mechanics, any statistical ensemble can be described by a wave function (pure state) or by a density matrix (mixed state) that is a positive operator of trace 1. For any density matrix ρ , there exists a sequence $\{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{H}$ such that:

$$(1) \quad \rho = \sum_{j \in \mathbb{N}} l_j |\psi_j\rangle\langle\psi_j|, \quad \sum_{j \in \mathbb{N}} l_j = 1, \quad l_j \geq 0 \quad \forall j \in \mathbb{N}.$$

The sequence $\{\psi_j\}_{j \in \mathbb{N}}$ is a set of eigenvectors of ρ and $\{l_j\}_{j \in \mathbb{N}}$ are the corresponding eigenvalues. If $j_0 \in \mathbb{N}$ is such that $l_{j_0} = 1$ then $l_j = 0$ for each

$j \neq j_0$ and the corresponding density matrix represents a pure state up to a phase ambiguity. Hence the density matrices formalism extends the common formulation of the quantum mechanics in terms of wave function.

Let us consider $T > 0$ and a time dependent self-adjoint operator $H(t)$ (called Hamiltonian). The dynamics of a general density matrix ρ is described by the Von Neumann equation:

$$(2) \quad \begin{cases} i \frac{d\rho}{dt}(t) = [H(t), \rho(t)], & t \in (0, T) \\ \rho(0) = \rho^0, \end{cases} \quad ([H, \rho] = H\rho - \rho H),$$

for ρ^0 the initial ensemble. The solution $\rho(t) = U_t \rho(0) U_t^*$, where U_t is the unitary propagator generated by $H(t)$ and:

$$\begin{cases} i \frac{d}{dt} U_t = -iH(t)U_t, & t \in (0, T), \\ U_0 = Id, \end{cases}$$

In the present work, we consider $\mathcal{H} = L^2((0, 1), \mathbb{C})$ and $H(t) = A + u(t)B$, for $A = -\Delta$ the Dirichlet Laplacian, B a bounded symmetric operator and $u \in L^2((0, T), \mathbb{R})$ control function. From now on we call Γ_t^u the unitary propagator U_t . The system (2) is said to be globally exactly controllable if for any couple of density matrices unitarily equivalent ρ^1, ρ^2 there exist $T > 0$ and $u \in L^2((0, T), \mathbb{R})$ such that $\rho^2 = \Gamma_T^u \rho^1 (\Gamma_T^u)^*$. Thanks to the decomposition (1) the controllability of (2) is equivalent (up to ambiguity on the phases) to the simultaneous controllability of the infinite systems:

$$(3) \quad \begin{cases} i \partial_t \psi_j(t, x) = A \psi_j(t, x) + u(t) B \psi_j(t, x), & x \in (0, 1), t \in (0, T), \\ \psi_j(0, x) = \psi_j^0(x), & \forall j \in \mathbb{N}. \end{cases}$$

The state $\psi_j^0(x)$ is the j -th eigenfunction of ρ^0 , $\rho^0 = \sum_{j=1}^{\infty} \lambda_j |\psi_j^0\rangle \langle \psi_j^0|$ and the j -th solution of (3) is $\psi_j(t) = \Gamma_t^u \psi_j^0$. To this purpose we study the simultaneous global exact controllability of infinitely many systems (3) and we rephrase the results in terms of the density matrices.

The controllability of the bilinear Schrödinger equation (3) has been widely studied in the literature and we start by mentioning the work on the bilinear systems of Ball, Mardsen and Slemrod [1] to which the following proposition is related (see also Turinici [15]).

Proposition 1. *The system (3) for $u \in L_{loc}^1((0, T), \mathbb{R})$ admits an unique solution $\psi_j \in C((0, T), \mathcal{H})$, for any initial state in \mathcal{H} . Moreover if S is the unit sphere in \mathcal{H} and:*

$$Z(\psi_0) := \{\psi \in D(A) \mid \exists T > 0, \exists r > 1, \exists u \in L_{loc}^r((0, T), \mathbb{R}) : \psi = \Gamma_u^T \psi_0\},$$

then for every $\psi_0 \in S \cap D(A)$ the attainable set $Z(\psi_0)$ is contained in a countable union of compact sets and it has dense complement in $S \cap D(A)$.

Despite this non controllability result many authors have addressed the problem for weaker notions of controllability.

For instance Beauchard and Laurent [3] prove the local exact controllability of (3) in a neighborhood of the first eigenfunction of A in $H_{(0)}^s := D(|A|^{\frac{s}{2}})$ for $s = 3$ and $B = \mu \in H^3$ improving the work [2].

The global approximate controllability in a Hilbert space has been studied by Chambrion, Mason, Sigalotti, Boscain in [7] and by Boscain, Caponigro, Chambrion, Sigalotti in [4]. In both simultaneous global approximate controllability results are provided.

Let $H_{(V)}^s := D(|A+V|^{\frac{s}{2}})$ for $V \in H^s$, Nersesyan and Morancey in [11] prove the global exact controllability of one dimensional Schrödinger equation with $B = \mu \in H^6$ and a polarizability term in $H_{(V)}^6$.

Morancey in [10] proves the existence of a residual set of functions μ in H^3 such that for $B = \mu$ the simultaneous local exact controllability is verified in $H_{(0)}^3$ for at most three systems (3) and up to phase-shifts.

In [12] Morancey and Nersesyan extend the result and achieve the existence of a residual set of functions μ in H^4 so that for $B = \mu$, one can ensure the simultaneous global exact controllability of any finite number of (3) in $H_{(V)}^4$.

In the present work we use part of the notations of [3], [10], [11], [12] and we carry on the previous works. We provide explicit conditions in B that imply the simultaneous global exact controllability in projection of infinite (3) in $H_{(0)}^3$ by projecting into suitable finite dimensional subspaces of $H_{(0)}^3$. Another goal of this work is to prove the simultaneous local exact controllability in projection for any $T > 0$ up to phase-shifts. In the appendix we develop a perturbation theory technique that we use in order to get rid of an issue appearing in the proof of the local controllability: the “eigenvalues resonances”. The formulation of the controllability for orthonormal basis allows to provide the result in terms of density matrices and unitarily equivalent sets of functions.

1.1 Framework and main results

We denote $\mathcal{H} = L^2((0, 1), \mathbb{C})$, its norm $\|\cdot\|$ and its scalar product $\langle \cdot, \cdot \rangle$. The operator A is the Dirichlet Laplacian ($A = -\frac{d^2}{dx^2}$ and $D(A) = H_0^1((0, 1), \mathbb{C}) \cap H^2((0, 1), \mathbb{C})$), B is a bounded symmetric operator and u is a $L^2((0, T), \mathbb{R})$ control function.

We consider an orthonormal basis $\{\phi_j\}_{j \in \mathbb{N}}$ composed by eigenfunctions of A related to the eigenvalues $\{\lambda_j\}_{j \in \mathbb{N}}$ and:

$$(4) \quad \phi_j(t, x) = e^{-iAt} \phi_j(x) = e^{-i\lambda_j t} \phi_j(x).$$

Let us define the spaces for $s > 0$:

$$\begin{aligned}
H_{(0)}^s &= H_{(0)}^s((0, 1), \mathbb{C}) := D(A^{\frac{s}{2}}), & \|\cdot\|_{H_{(0)}^s} &= \left(\sum_{k=1}^{\infty} |k^s \langle \cdot, \phi_k \rangle|^2 \right)^{\frac{1}{2}}, \\
\ell^\infty(\mathcal{H}) &= \{ \{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{H} \mid \sup_{j \in \mathbb{N}} \|\psi_j\|_{\mathcal{H}} < \infty \}, \\
\ell^2(\mathcal{H}) &= \{ \{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{H} \mid \sum_{j=1}^{\infty} \|\psi_j\|^2 < \infty \}, \\
h^s(\mathcal{H}) &= \{ \{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{H} \mid \sum_{j=1}^{\infty} (j^s \|\psi_j\|)^2 < \infty \}.
\end{aligned}$$

Assumption (I). Let B be a bounded symmetric operator.

1. For any $N \in \mathbb{N}$ there exists $C_N > 0$ such that:

$$\forall j, k \in \mathbb{N}, \quad j \leq N, \quad |\langle \phi_k, B\phi_j \rangle| \geq \frac{C_N}{k^3};$$

2. $\text{Ran}(B|_{H_{(0)}^2}) \subseteq H_{(0)}^2$ and:

$$\text{Ran}(B|_{H_{(0)}^3((0,1),\mathbb{C})}) \subseteq H^3((0, 1), \mathbb{C}) \cap H_0^1((0, 1), \mathbb{C});$$

3. for each $j, k, l, m \in \mathbb{N}$ such that $j^2 - k^2 - l^2 + m^2 = 0$ and $(j, k) \neq (l, m)$, then:

$$\langle \phi_j, B\phi_j \rangle - \langle \phi_k, B\phi_k \rangle - \langle \phi_l, B\phi_l \rangle + \langle \phi_m, B\phi_m \rangle \neq 0.$$

Remark 1. If a bounded operator B satisfies Assumption (I) then $B \in L(H_{(0)}^2, H_{(0)}^2)$. Indeed B is closed in \mathcal{H} , then for every $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ such that $u_n \xrightarrow{\mathcal{H}} u$ and $Bu_n \xrightarrow{\mathcal{H}} v$ we have that $Bu = v$. Now for every $\{u_n\}_{n \in \mathbb{N}} \subset H_{(0)}^2$ such that $u_n \xrightarrow{H_{(0)}^2} u$ and $Bu_n \xrightarrow{H_{(0)}^2} v$, the convergences with respect to the \mathcal{H} -norm are implied and then $Bu = v$. Hence the operator B is closed in $H_{(0)}^2$ and $B \in L(H_{(0)}^2, H_{(0)}^2)$.

Example. Assumption (I) holds for $B : \psi \mapsto x^2\psi$. Indeed the conditions 1) and 2) directly follow and by considering:

$$\begin{cases} |\langle \phi_j, x^2\phi_k \rangle| = \left| \frac{(-1)^{j-k}}{(j-k)^2\pi^2} - \frac{(-1)^{j+k}}{(j+k)^2\pi^2} \right|, & j \neq k, \\ |\langle \phi_k, x^2\phi_k \rangle| \left| \frac{1}{3} - \frac{1}{2k^2\pi^2} \right|, & k \in \mathbb{N}, \end{cases}$$

the condition 3) is verified as:

$$j^2 - k^2 - l^2 + m^2 = 0 \implies j^{-2} - k^{-2} - l^{-2} + m^{-2} \neq 0.$$

Definition 1. Let $\Psi := \{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{H}$ and $\mathcal{H}_N(\Psi) := \text{span}\{\psi_j : j \leq N\}$. The map $\pi_N(\Psi)$ is the orthogonal projector into $\mathcal{H}_N(\Psi)$ and if Ψ is an orthonormal system, then for each $\varphi \in \mathcal{H}$:

$$\pi_N(\Psi)\varphi = \sum_{j=1}^N \psi_j \langle \psi_j, \varphi \rangle.$$

Theorem 2. Let B satisfy Assumption (I). Let $\{\psi_j^1\}_{j \in \mathbb{N}}, \{\psi_j^2\}_{j \in \mathbb{N}} \subset H_{(0)}^3$ unitarily equivalent and $\Psi^3 := \{\psi_j^3\}_{j \in \mathbb{N}} \subset H_{(0)}^3$ a sequence of functions. If $\widehat{\Gamma} \in U(\mathcal{H})$ is a unitary operator so that $\{\widehat{\Gamma}\psi_j^1\}_{j \in \mathbb{N}} = \{\psi_j^2\}_{j \in \mathbb{N}}$ and it satisfies the condition $\widehat{\Gamma}^* \psi_j^3 \in H_{(0)}^3$ for every $j \in \mathbb{N}$, then for any $N \in \mathbb{N}$ there exist $T > 0$ and a control function $u \in L^2((0, T), \mathbb{R})$ such that:

$$\pi_N(\Psi^3) \psi_j^2 = \pi_N(\Psi^3) \Gamma_T^u \psi_j^1, \quad j \in \mathbb{N}.$$

For any couple of elements $\{\psi_j^1\}_{j \in \mathbb{N}}, \{\psi_j^2\}_{j \in \mathbb{N}} \subset H_{(0)}^3$ unitarily equivalent, one can consider a sequence $\Psi^3 := \{\psi_j^3\}_{j \in \mathbb{N}} \subset H_{(0)}^3$ such that $\widehat{\Gamma}^* \psi_j^3 \in H_{(0)}^3$ and so that the assumptions of Theorem 2 are satisfied.

We point out that if we set $\Psi^3 = \Psi^2$, the condition $\widehat{\Gamma}^* \psi_j^3 \in H_{(0)}^3$ is always verified (such as if $\Psi^3 = \Psi^1$). By considering that:

$$(5) \quad \pi_N(\Psi^2) \psi_j^2 = \begin{cases} \psi_j^2, & j \leq N, \\ \pi_N(\Psi^2) \psi_j^2, & j > N, \end{cases}$$

the next corollary straightly follows (if Ψ^2 is composed by linearly independent elements, the second line of (5) is 0).

Corollary 3. Let B satisfy Assumption (I). Let $\Psi^1 := \{\psi_j^1\}_{j \in \mathbb{N}}, \Psi^2 := \{\psi_j^2\}_{j \in \mathbb{N}} \subset H_{(0)}^3$ unitarily equivalent, then for any $N \in \mathbb{N}$ there exist $T > 0$ and a control function $u \in L^2((0, T), \mathbb{R})$ such that:

$$\begin{cases} \Gamma_T^u \psi_j^1 = \psi_j^2, & j \leq N \\ \pi_N(\Psi^2) \Gamma_T^u \psi_j^1 = \pi_N(\Psi^2) \psi_j^2, & j > N. \end{cases}$$

One can notice that Corollary 3 implies the simultaneous global exact controllability without projecting of N systems (3) and provides further informations about the remaining functions. This extends Morancey, Nersesan [12, *Main Theorem*] and it provides conditions in B that imply the controllability.

Moreover as long as we study the controllability of two unitarily equivalent sequences of functions and we project with respect to the target one, we do not need any further condition as in Theorem 2.

Each time that we try to control in projection with respect a third sequence of functions, the situation is more delicate. Theorem 2 assures that if a “ $H_{(0)}^3$ -compatibility” condition between two elements and a projector is satisfied, one can simultaneously control them in projection.

Now we rephrase the two controllability results in terms of density matrices in the following corollary.

Corollary 4. *Let B satisfy Assumption (I) and $\rho^1, \rho^2 \in T(\mathcal{H})$ be two density matrices unitarily equivalent such that $\text{Ran}(\rho^1), \text{Ran}(\rho^2) \subseteq H_{(0)}^3$.*

1. *Let $\Psi^2 := \{\psi_j^2\}_{j \in \mathbb{N}} \subset H_{(0)}^3$ a complete orthonormal system composed by eigenfunctions of ρ^2 . Then for any $N_1 \in \mathbb{N}$ there exist $T_1 > 0$ and a control function $u_1 \in L^2((0, T_1), \mathbb{R})$ such that:*

$$\pi_{N_1}(\Psi^2) \Gamma_{T_1}^{u_1} \rho^1 (\Gamma_{T_1}^{u_1})^* \pi_{N_1}(\Psi^2) = \pi_{N_1}(\Psi^2) \rho^2 \pi_{N_1}(\Psi^2).$$

2. *Let $\Psi^3 := \{\psi_j^3\}_{j \in \mathbb{N}} \subset H_{(0)}^3$, $\widehat{\Gamma} \in U(\mathcal{H})$ the operator such that $\rho^2 = \widehat{\Gamma} \rho^1 \widehat{\Gamma}^*$. If $\widehat{\Gamma}^* \psi_j^3 \in H_{(0)}^3$ for every $j \in \mathbb{N}$, then for any $N_2 \in \mathbb{N}$ there exist $T_2 > 0$ and a control function $u_2 \in L^2((0, T_2), \mathbb{R})$ such that:*

$$\pi_{N_2}(\Psi^3) \Gamma_{T_2}^{u_2} \rho^1 (\Gamma_{T_2}^{u_2})^* \pi_{N_2}(\Psi^3) = \pi_{N_2}(\Psi^3) \rho^2 \pi_{N_2}(\Psi^3).$$

1.2 Well-posedness of the Cauchy problem

We mention now the crucial result of well-posedness for the system:

$$(6) \quad \begin{cases} i\partial_t \psi(t, x) = -\Delta \psi(t, x) + u(t)\mu(x)\psi(t, x) + f(t, x), \\ \psi(0, x) = \psi^0(x), \quad x \in (0, 1), \quad t \in (0, T). \end{cases}$$

Proposition 5 (Beauchard, Laurent; [3]; Lemma 1 & Proposition 2).

1) *Let $T > 0$ and $\tilde{f} \in L^2((0, T), H_0^1 \cap H^3)$. The function $G : t \mapsto \int_0^t e^{iAs} \tilde{f}(s) ds$ belongs to $C^0([0, T], H_{(0)}^3)$. Moreover:*

$$\|G\|_{L^\infty((0, T), H_{(0)}^3)} \leq c_1(T) \|\tilde{f}\|_{L^2((0, T), H^3 \cap H_{(0)}^1)},$$

where the constant $c_1(T)$ is uniformly bounded with T lying in bounded intervals.

2) *Let $\mu \in H^3((0, 1), \mathbb{R})$, $T > 0$, $\psi^0 \in H_{(0)}^3(0, 1)$, $f \in L^2((0, T), H_0^1 \cap H^3)$ and $u \in L^2((0, T), \mathbb{R})$. Then there exists a unique mild solution of (6) in $H_{(0)}^3(0, 1)$, i.e. a function $\psi \in C_0([0, T], H_{(0)}^3)$ such that for every $t \in [0, T]$:*

$$(7) \quad \psi(t, x) = e^{i\Delta t} \psi^0(x) - i \int_0^t e^{i\Delta(t-s)} (u(s)\mu(x)\psi(s, x) + f(s, x)) ds.$$

Moreover, for every $R > 0$, there exists $C = C(T, \mu, R) > 0$ such that, if $\|u\|_{L^2((0,T),\mathbb{R})} < R$ then the solution satisfies:

$$(8) \quad \|\psi\|_{C^0([0,T],H_{(0)}^3)} \leq C(\|\psi^0\|_{H_{(0)}^3} + \|f\|_{L^2((0,T),H_0^1 \cap H^3)}), \quad \forall \psi_0 \in H_{(0)}^3.$$

If $f \equiv 0$ then:

$$\|\psi\| = \|\psi^0\| \quad \forall t \in [0, T].$$

If B satisfies Assumption (I), then Proposition 5 implies:

$$\psi_j(t, x) = e^{-iAt}\psi_j^0(x) - i \int_0^t e^{-iA(t-s)}u(s)B\psi_j(s, x)ds.$$

The relation (8) becomes $\|\psi_j\|_{C^0([0,T],H_{(0)}^3)} \leq C\|\psi_j^0\|_{H_{(0)}^3}$ and for every $\{\psi_j\}_{j \in \mathbb{N}} \in \ell^\infty(H_{(0)}^3)$ (respectively $h^3(H_{(0)}^3)$) it follows that $\{\Gamma_T^u \psi_j\}_{j \in \mathbb{N}} \in \ell^\infty(H_{(0)}^3)$ (respectively $h^3(H_{(0)}^3)$).

1.3 Time reversibility

Let us now present another feature of the bilinear Schrödinger equation, the time reversibility. First we notice that:

$$\begin{cases} i\partial_t \Gamma_{T-t}^u \psi^0(x) = -A\Gamma_{T-t}^u \psi^0(x) - u(T-t)B\Gamma_{T-t}^u \psi^0(x), \\ \Gamma_{T-0}^u \psi^0(x) = \Gamma_T^u \psi^0(x) = \psi^1(x). \end{cases}$$

Let us define $\tilde{u}(t) := u(T-t)$, $\tilde{\Gamma}_t^{\tilde{u}} := \Gamma_{T-t}^u$. It follows:

$$\begin{cases} i\partial_t \tilde{\Gamma}_t^{\tilde{u}} \psi^1(x) = (-A - \tilde{u}(t)B)\tilde{\Gamma}_t^{\tilde{u}} \psi^1(x), \quad x \in (0, 1), \quad t \in (0, T), \\ \tilde{\Gamma}_0^{\tilde{u}} \psi^0(x) = \psi^1(x) \end{cases}$$

and $\tilde{\Gamma}_t^{\tilde{u}} \Gamma_T^u = \Gamma_{T-t}^u$. Then $\tilde{\Gamma}_T^{\tilde{u}} \Gamma_T^u = Id$ and:

$$(9) \quad \tilde{\Gamma}_T^{\tilde{u}} = (\Gamma_T^u)^{-1} = (\Gamma_T^u)^*.$$

In conclusion for $t > 0$ and $\tilde{u}(\cdot) = u(t - \cdot)$ the operator $\tilde{\Gamma}_t^{\tilde{u}}$ is the propagator related to $(-A - \tilde{u}(t)B)$ and describes the reversed dynamics of Γ_t^u .

The importance of the time reversibility resides in the fact that all the controllability results that we are going to prove are still verified for the reversed system. We will use this feature in many steps of the next proofs.

1.4 Scheme of the work

First, in Section 2 we present the general strategy and the sketch of the proof of the simultaneous local exact controllability in projection. We emphasize the obstructions to overcome in order to motivate the modification of the problem. In Theorem 6 we prove the simultaneous global approximate controllability for N systems (3).

Second, we achieve the simultaneous local exact controllability in projection for any $T > 0$ up to phase-shifts (Section 3, Theorem 8).

Third, we prove that the simultaneous local exact controllability in projection of infinite systems (3), under particular conditions, is equivalent to the simultaneous local exact controllability of a finite number of systems (Section 4, Proposition 9).

Fourth, we use together the simultaneous local exact controllability of a finite number of systems from [12, *Theorem 4.1*] with Theorem 6 in order to obtain the simultaneous global exact controllability for N systems (3) (Section 4, Theorem 10).

The proof of Theorem 2 follows from Theorem 10. We provide the controllability in terms of unitarily equivalent sets of functions and density matrices (Section 4.2).

2 Preliminaries and simultaneous global approximate controllability

In this section we discuss the simultaneous local exact controllability in projection and we explain why we modify the problem before proceeding. Moreover we prove the simultaneous global approximate controllability in $H_{(0)}^3$. In the local result, our purpose is to control different orthogonal systems $\{\psi_j^1\}_{j \in \mathbb{N}}$, $\{\psi_j^2\}_{j \in \mathbb{N}} \in h^3(H_{(0)}^3)$ such that $\|\psi_j^1\| = \|\psi_j^2\|$ for every $j \in \mathbb{N}$, hence we define $\{p_j\}_{j \in \mathbb{N}}$ the $h^3(\mathbb{R})$ sequence such that:

$$p_j = \|\psi_j^1\| = \|\psi_j^2\|, \quad \forall j \in \mathbb{N}.$$

We recall Definition 1 and we set:

$$\mathcal{H}_N := \text{span}\{\phi_j : j \leq N\}, \quad \pi_N := \pi_N(\{\phi_j\}_{j \in \mathbb{N}}) = \sum_{j=1}^N \phi_j \langle \phi_j, \cdot \rangle.$$

We start by studying the local exact controllability in projection with respect to π_N of the systems (3) with initial state $\{\psi_j^0\}_{j \in \mathbb{N}} = \{p_j \phi_j\}_{j \in \mathbb{N}}$ in the neighborhood:

$$(10) \quad O_{\epsilon, T} := \left\{ \{\psi_j\}_{j \in \mathbb{N}} \in h^3(H_{(0)}^3) \mid \langle \psi_j, \psi_k \rangle = p_j p_k \delta_{j,k}; \right. \\ \left. \|\psi_j - p_j \phi_j(T)\|_{h^3(H_{(0)}^3)}^2 < \epsilon \right\}.$$

In other words, let $T > 0$ and $N \in \mathbb{N}$, our goal is to prove the existence of $\epsilon > 0$ such that for any orthogonal system $\{\psi_j^1\}_{j \in \mathbb{N}} \in O_{\epsilon, T}$, there exists a control function $u \in L^2((0, T), \mathbb{R})$ such that:

$$\pi_N \Gamma_T^u p_j \phi_j = \pi_N \psi_j^1, \quad \forall j \in \mathbb{N}.$$

To this purpose, we consider the solution of the j -th system:

$$\psi_j(t) = \Gamma_t^u p_j \phi_j = \sum_{k=1}^{\infty} \phi_k(t) \langle \phi_k(t), p_j \Gamma_t^u \phi_j \rangle = \sum_{k=1}^{\infty} \frac{\phi_k(t)}{p_k} \langle p_k \phi_k(t), p_j \Gamma_t^u \phi_j \rangle$$

and the map $\alpha(u)$, the infinite matrix with elements:

$$\alpha_{k,j}(u) = \langle p_k \phi_k(T), \Gamma_T^u p_j \phi_j \rangle = p_k p_j \langle \phi_k(T), \Gamma_T^u \phi_j \rangle, \quad k, j \in \mathbb{N} \quad k \leq N.$$

The local existence of the control function is equivalent to the local right invertibility of the map α for $T > 0$ (in other words the local surjectivity). To this end, we are going to use the Generalized Inverse Function Theorem (see [9], p. 240) and we study the surjectivity of the Fréchet derivative of α , $\gamma(v) := (d_u \alpha(0)) \cdot v$, the infinite matrix with elements:

$$\begin{aligned} \gamma_{k,j}(v) &:= p_k p_j \left\langle \phi_k(T), -i \int_0^T e^{-iA(T-s)} v(s) B e^{-iAs} \phi_j ds \right\rangle \\ &= -i \int_0^T v(s) e^{-i(\lambda_j - \lambda_k)s} ds B_{k,j}, \quad k \leq N, \quad j \in \mathbb{N}, \end{aligned}$$

for $B_{k,j} = p_k p_j \langle \phi_k, B \phi_j \rangle = p_k p_j \langle B \phi_k, \phi_j \rangle = \overline{B_{j,k}}$. The right invertibility of the map γ consists in proving the solvability of the moment problem:

$$\frac{x_{k,j}}{B_{k,j}} = -i \int_0^T u(s) e^{-i(\lambda_j - \lambda_k)s} ds.$$

for each x , infinite matrix such that $\{x_{k,j} B_{k,j}^{-1}\}_{k,j \in \mathbb{N}} \in (\ell^2)^N$.

One would use Haraux Theorem ([8], p. 67), but there are the ‘‘eigenvalues resonances’’.

1. First, we recall $\lambda_n = \pi^2 n^2$ so that $\lambda_j - \lambda_k = 0$ for all the diagonal terms of γ ($j = k$).
2. Second if the dimension of \mathcal{H}_N is large enough, there happens that $\lambda_j - \lambda_k = \lambda_n - \lambda_m$, for $j, k, n, m \in \mathbb{N}$, $(j, k) \neq (n, m)$ and $k, m \leq N$. An example is $\lambda_7 - \lambda_1 = \lambda_8 - \lambda_4$.

These eigenvalues resonances cause the following constraints:

$$\begin{aligned} \frac{x_{k,j}}{B_{k,j}} &= -i \int_0^T u(s) e^{-i(\lambda_j - \lambda_k)s} ds \\ &= -i \int_0^T u(s) e^{-i(\lambda_n - \lambda_m)s} ds = \frac{x_{n,m}}{B_{n,m}}, \end{aligned}$$

for $k \neq n, m$ and $j \neq n, m$. Hence we adopt the following strategies.

1. We apply a phase-shift:

$$\widehat{\psi}_j(t, x) = \frac{\overline{\alpha_{j,j}(u)}}{|\alpha_{j,j}(u)|} \psi_j(t, x) \implies \widehat{\alpha}_{k,j}(u) = \frac{\overline{\alpha_{j,j}(u)} \alpha_{k,j}(u)}{|\alpha_{j,j}(u)|}$$

in order to remove the resonances of the diagonal terms.

2. We decompose the operator $A + uB$ as:

$$A + uB = (A + u_0B) + u_1(t)B$$

and consider $A + u_0B$ instead of A , in order to modify the eigenvalues gaps using u_0B as a perturbing term.

The perturbation theory technique is required in order to remove the resonances and solve the moment problem (local controllability), but it is also necessary in the global approximate controllability result that we are going to clarify in the next subsection. In particular one can use it to ensure that the problem admits a non-degenerate chain of connectedness [5, Definition 13].

2.1 Modified problem

Let $N \in \mathbb{N}$ and $u(t) = u_0 + u_1(t)$, for u_0 and $u_1(t)$ real. We consider the systems:

$$(11) \quad \begin{cases} i\partial_t \psi_j(t, x) = (A + u_0B)\psi_j(t, x) + u_1(t)B\psi_j(t, x), \\ \psi_j(0, x) = \psi_j^0(x) \quad \forall j \in \mathbb{N}, \end{cases}$$

with solutions $\psi_j^{u_0}(t, x) = \Gamma_t^{u_0+u_1} \psi_j^0(x)$ where $\Gamma_t^{u_0+u_1}$ is the unitary propagator of the dynamics. We point out that $A + u_0B$ has a compact resolvent thanks to the fact that B is bounded and that A has compact resolvent. Hence the operator $A + u_0B$ has pure discrete spectrum, its eigenvalues $\{\lambda_j^{u_0}\}_{j \in \mathbb{N}}$ correspond to some eigenfunctions $\{\phi_j^{u_0}(x)\}_{j \in \mathbb{N}}$ and we set $\phi_j^{u_0}(T, x) := e^{-i\lambda_j^{u_0}T} \phi_j^{u_0}(x)$.

Now by recalling that $\{p_j\}_{j \in \mathbb{N}} \in h^3(\mathbb{R})$, we introduce:

$$(12) \quad \begin{aligned} O_{\epsilon_0, T}^{u_0} &:= \{ \{ \psi_j \}_{j \in \mathbb{N}} \in h^3(H_{(0)}^3) \mid \langle \psi_j, \psi_k \rangle = p_j p_k \delta_{j,k}; \\ &\quad \| \psi_j - p_j \phi_j^{u_0}(T) \|_{h^3(H_{(0)}^3)}^2 < \epsilon_0 \} \\ \widetilde{O}_{\epsilon, T}^{u_0} &:= \{ \{ f_j \}_{j \leq N} \subset H_{(0)}^3 \mid \langle f_j, f_k \rangle = p_k p_j \delta_{j,k} \quad j, k \leq N; \\ &\quad \sum_{j=1}^N j^6 \| f_j - p_j \phi_j^{u_0}(T) \|_{H_{(0)}^3}^2 < \epsilon \}. \end{aligned}$$

The introduction of the new orthonormal system imposes to define:

$$\tilde{H}_{(0)}^3 := D(|A + u_0 B|^{\frac{3}{2}}), \quad \|\cdot\|_{\tilde{H}_{(0)}^3} = \left(\sum_{k=1}^{\infty} |\lambda_k^{u_0}|^{\frac{3}{2}} |\langle \cdot, \phi_k \rangle|^2 \right)^{\frac{1}{2}}.$$

However we will consider u_0 small enough and B satisfying Assumption (I) so that the hypothesis of Lemma 16 (Appendix A) are verified. Hence $\tilde{H}_{(0)}^3 \equiv H_{(0)}^3$ and there exist two constants $C_1, C_2 > 0$ such that:

$$C_1 \|\cdot\|_{H_{(0)}^s} \leq \|\cdot\|_{\tilde{H}_{(0)}^s} \leq C_2 \|\cdot\|_{H_{(0)}^s},$$

that we abbreviate as $\|\cdot\|_{\tilde{H}_{(0)}^s} \asymp \|\cdot\|_{H_{(0)}^s}$. Afterwards, we define the maps $\hat{\alpha}$ and α^{u_0} , the infinite matrices with elements:

$$(13) \quad \begin{cases} \hat{\alpha}_{k,j}(u_1) = p_k p_j \langle \phi_k^{u_0}(T), \Gamma_T^{u_0+u_1} \phi_j^{u_0} \rangle, & k \leq N, j \in \mathbb{N}, \\ \alpha_{k,j}^{u_0}(u_1) = \frac{\hat{\alpha}_{j,j}(u_1)}{|\hat{\alpha}_{j,j}(u_1)|} \hat{\alpha}_{k,j}(u_1), & j, k \leq N, \\ \alpha_{k,j}^{u_0}(u_1) = \hat{\alpha}_{k,j}(u_1), & j > N, k \leq N, \end{cases}$$

respectively.

Definition 2. Let $\mathcal{H}_N^{u_0} := \text{span}\{\phi_j^{u_0} : j \leq N\}$. The orthogonal projector $\pi_N^{u_0} : \mathcal{H} \rightarrow \mathcal{H}_N^{u_0}$ and the map $\Pi_N^{u_0} : \ell^\infty(\mathcal{H}) \rightarrow \ell^\infty(\mathcal{H}_N^{u_0})$ are respectively such that for each $\psi \in \mathcal{H}$ and $\{\psi_j\}_{j \in \mathbb{N}} \in \ell^\infty(\mathcal{H})$:

$$\pi_N^{u_0} \psi = \sum_{j=1}^N \phi_j^{u_0} \langle \phi_j^{u_0}, \psi \rangle, \quad \Pi_N^{u_0} \{\psi_j\}_{j \in \mathbb{N}} = \{\pi_N^{u_0} \psi_j\}_{j \in \mathbb{N}} \in \ell^\infty(\mathcal{H}_N^{u_0}).$$

2.2 Approximate simultaneous controllability

In this subsection we prove the simultaneous global approximate controllability of the systems (3) by using the theory from [5]. Let us call $U(\mathcal{H})$ the space of the unitary operators on \mathcal{H} .

Definition 3. The problem $(A, B, U, \{\phi_j\}_{j \in \mathbb{N}})$ is said to be finetely simultaneously globally approximately controllable in $H_{(0)}^s$ if for every $r \in \mathbb{N}$, $\psi_1, \dots, \psi_r \in H_{(0)}^s$, $\hat{\Gamma} \in U(\mathcal{H})$ such that $\hat{\Gamma} \psi_1, \dots, \hat{\Gamma} \psi_r \in H_{(0)}^s$ and $\epsilon > 0$, there exist $T > 0$ and $u \in L^2((0, T), \mathbb{R})$ so that for every $1 \leq k \leq r$:

$$\|\hat{\Gamma} \psi_k - \Gamma_T^u \psi_k\|_{H^s} < \epsilon.$$

Theorem 6. *Let us consider the systems (11) and B satisfying Assumption (I). Then $(A, B, U, \{\phi_j\}_{j \in \mathbb{N}})$ is finetely simultaneously globally approximately controllable in $H_{(0)}^3$.*

Proof. Let us consider $(A + u_0B, B, U, \{\phi_j^{u_0}\}_{j \in \mathbb{N}})$ with u_0 contained in the neighborhoods provided by Lemma 16 and Remark 3 (Appendix A). Then the hypothesis of [5, Proposition 65] are satisfied since the operators $-i(A + u_0B)$ and $-i(A + u_0B) - iuB$ are skew-adjoint. Moreover Remark 1 implies the validity of [6, Proposition 6] that ensures that the couple $(A + u_0B, B)$ is 2-weakly coupled. Now $(A + u_0B, B)$ admits a non-degenerate chain of connectedness thanks to Remark 3.

Thus for every $\eta > 0$, $s < 4 - \eta$, $r \in \mathbb{N}$, $\psi_1, \dots, \psi_r \in D(|A + u_0B|^{\frac{s}{2}}) = \tilde{H}_{(0)}^s$, $\hat{\Gamma} \in U(\mathcal{H})$ such that $\hat{\Gamma}\psi_1, \dots, \hat{\Gamma}\psi_r \in \tilde{H}_{(0)}^s$ and for every $\epsilon > 0$, there exists a piecewise constant control $u : [0, T] \rightarrow U$ such that for every $1 \leq k \leq r$:

$$\|\hat{\Gamma}\psi_k - \Gamma_T^u \psi_k\|_{\tilde{H}_{(0)}^s} < \epsilon.$$

Thanks to Lemma 16 the claim is achieved. \square

3 Simultaneous local exact controllability in projection

The aim of the section is to study the surjectivity of the map α^{u_0} , defined in (13) (from now on we omit its dependence in u). If α^{u_0} is locally surjective then the systems (11) are simultaneously locally exactly controllable in projection in $h^3(H_{(0)}^3)$.

Let B^{u_0} be the hermitian infinite matrix with elements $B_{j,k}^{u_0} = p_j p_k \langle \phi_j^{u_0}, B \phi_k^{u_0} \rangle$ for $j \leq N$ and $k \in \mathbb{N}$. We prove the local surjectivity of α^{u_0} by using the Inverse Function Theorem ([9], p. 240) and we focus our efforts on the map $\gamma^{u_0}(v) = ((d_{u_1} \alpha^{u_0})(0)) \cdot v$. One can define $\hat{\gamma}_{k,j}(v) = ((d_{u_1} \hat{\alpha})(0)) \cdot v$ and then $\gamma^{u_0}(v)$ is the infinite matrix with elements:

$$\begin{cases} \gamma_{k,j}^{u_0} = \frac{1}{p_j^2} (\overline{\hat{\gamma}_{j,j}} p_k p_j \delta_{k,j} + p_j^2 \hat{\gamma}_{k,j} - p_k p_j \delta_{k,j} \Re(\hat{\gamma}_{j,j})), & j, k \leq N, \\ \gamma_{k,j}^{u_0} = \hat{\gamma}_{k,j}, & k \leq N, j > N. \end{cases}$$

Thus for $k \leq N$ and $j \in \mathbb{N}$:

$$(14) \quad \begin{cases} \gamma_{k,j}^{u_0} = \hat{\gamma}_{k,j} = -i \int_0^T u_1(s) e^{-i(\lambda_j^{u_0} - \lambda_k^{u_0})s} ds B_{k,j}^{u_0}, & k \neq j, \\ \gamma_{k,k}^{u_0} = \Re(\hat{\gamma}_{k,k}) = 0, & k = j, \end{cases}$$

The relation $\gamma_{k,k}^{u_0} = 0$ comes from the fact that $\gamma_{k,k}^{u_0} = \Im(\gamma_{k,k}^{u_0})$ as one can notice that:

$$(15) \quad \gamma_{k,j}^{u_0} = -\overline{\gamma_{j,k}^{u_0}}, \quad j, k \leq N, k \neq j.$$

3.1 Features of the spaces

Let us study the spaces in which the previous maps take values. We consider $P\Phi^{u_0} = \{p_j\phi_j^{u_0}(T)\}_{j \in \mathbb{N}}$, the space $O_{\epsilon_0, T}^{u_0}$ defined in (12) and:

$$\begin{aligned} \Pi_N^{u_0} O_{\epsilon_0, T}^{u_0} \subseteq & \left\{ \{\psi_j\}_{j \in \mathbb{N}} \in h^3(\mathcal{H}_N^{u_0}) \mid \sum_{j=1}^N j^6 \|\psi_j - p_j\phi_j^{u_0}(T)\|_{H_{(0)}^3}^2 \right. \\ & \left. + \sum_{j=N+1}^{\infty} j^6 \|\psi_j\|_{H_{(0)}^3}^2 < \epsilon_0 \right\}. \end{aligned}$$

Afterwards we introduce the tangent space:

$$\begin{aligned} \Pi_N^{u_0} T_{P\Phi^{u_0}} O_{\epsilon_0, T}^{u_0} = & \left\{ \{f_j\}_{j \in \mathbb{N}} \in h^3(\mathcal{H}_N^{u_0}) \mid \langle \Phi_k^{u_0}, f_j \rangle = -\overline{\langle \Phi_j^{u_0}, f_k \rangle} \right. \\ & \left. j, k \leq N \right\}. \end{aligned}$$

One can notice that:

$$\begin{aligned} (16) \quad & \sum_{k=1}^N \sum_{j=1}^{\infty} |p_j^{-1} j^3 \langle p_k \phi_k^{u_0}(T), \Gamma_T^{u_0+u_1} p_j \phi_j^{u_0} \rangle|^2 \\ & = \sum_{k=1}^N \sum_{j=1}^{\infty} |j^3 \langle \tilde{\Gamma}_T^{u_0+\tilde{u}_1} p_k \phi_k^{u_0}(T), \phi_j^{u_0} \rangle|^2 < \infty, \end{aligned}$$

thanks to the fact that $\tilde{\Gamma}_t^u$ preserves $h^3(H_{(0)}^3)$ (as the propagator Γ_t^u , Section 1.2) and that $\{p_k \phi_k^{u_0}\}_{k \in \mathbb{N}} \in h^3(H_{(0)}^3)$ (Lemma 16). Thus $\{p_j^{-1} j^3 \hat{\alpha}_{k,j}\}_{k,j \in \mathbb{N}} \in (\ell^2(\mathbb{C}))^N$. In conclusion the maps $\hat{\alpha}$, α^{u_0} and γ^{u_0} respectively take value in:

$$\begin{aligned} D^N := & \left\{ \{d_{k,j}\}_{k,j \in \mathbb{N}} \in (h^3(\mathbb{C}))^N \mid \{p_j^{-1} j^3 d_{k,j}\}_{k,j \in \mathbb{N}} \in (\ell^2(\mathbb{C}))^N; \right. \\ & \left. \langle \{d_{k,j}\}_{j \in \mathbb{N}}, \{d_{l,j}\}_{j \in \mathbb{N}} \rangle_{\ell^2} = p_k p_l \|p_j\|_{\ell^2}^2 \delta_{k,l} \quad k, l \leq N \right\}, \\ Q^N := & \left\{ \{x_{k,j}\}_{k,j \in \mathbb{N}} \in (h^3(\mathbb{C}))^N \mid \exists d \in D^N : x_{k,j} = \frac{\overline{d_{j,j}} d_{k,j}}{|d_{j,j}|} \quad j \leq N; \right. \\ & \left. x_{k,j} = d_{k,j} \quad j > N \right\}, \\ G^N := & \left\{ \{x_{k,j}\}_{k,j \in \mathbb{N}} \in (h^3(\mathbb{C}))^N \mid \{p_j^{-1} j^3 x_{k,j}\}_{k,j \in \mathbb{N}} \in (\ell^2(\mathbb{C}))^N; \right. \\ & \left. x_{k,j} = -\overline{x_{j,k}} \quad j, k \leq N; \quad x_{k,k} = 0 \quad k \leq N \right\}. \end{aligned}$$

3.2 Simultaneous local exact controllability in projection for any $T > 0$

Now, we are going to expose the simultaneous local exact controllability in projection of (3) in $O_{\epsilon_0, T}^{u_0}$ defined in (12).

Proposition 7. *Let $N \in \mathbb{N}$ and B satisfy Assumption (I). Then for any $T > 0$ there exist $\epsilon_0 > 0$ and $u_0 \in \mathbb{R}$ such that for any $\{\psi_j\}_{j \in \mathbb{N}} \in O_{\epsilon_0, T}^{u_0}$ there exist a sequence of real numbers $\{\theta_j\}_{j \in \mathbb{N}} = \{\{\hat{\theta}_j\}_{j \leq N}, 0, \dots\}$ and a control function $u \in L^2((0, T), \mathbb{R})$ such that:*

$$\{\pi_N^{u_0} \psi_j\}_{j \in \mathbb{N}} = \{\pi_N^{u_0} e^{i\theta_j} \Gamma_T^u p_j \phi_j^{u_0}\}_{j \in \mathbb{N}}.$$

Proof. **1)** First we recall (14) so that $\gamma^{u_0}(u)$ is the matrix with elements:

$$\gamma_{k,j}^{u_0}(u) = -i \int_0^T u(s) e^{-i(\lambda_j^{u_0} - \lambda_k^{u_0})s} B_{k,j}^{u_0} ds, \quad j \in \mathbb{N}, k \leq N, j \neq k.$$

Let u_0 be contained in the neighborhoods provided by the lemmas 14, 15, 16 and Remark 2 (Appendix A). We want to apply Haraux Theorem ([8], p. 67) in order to solve the following moment problem:

$$(17) \quad x_{k,j}^{u_0} = -i \int_0^T u(s) e^{-i(\lambda_j^{u_0} - \lambda_k^{u_0})s} B_{k,j}^{u_0} ds, \quad j \in \mathbb{N}, k \leq N, k \neq j.$$

for every $x^{u_0} \in G^N$. Now the validity of Lemma 15 (Appendix A) implies the existence of \tilde{C}_N such that:

$$|x_{k,j}^{u_0} \langle \phi_k^{u_0}(T), B \phi_j^{u_0}(T) \rangle^{-1}| \leq |\tilde{C}_N^{-1} j^3 x_{k,j}^{u_0}|.$$

Thus:

$$\left\{ x_{k,j} (B_{k,j}^{u_0})^{-1} \right\}_{\substack{k \leq N \\ k, j \in \mathbb{N}}} = \left\{ x_{k,j} p_k^{-1} p_j^{-1} (\langle \phi_k^{u_0}(T), B \phi_j^{u_0}(T) \rangle)^{-1} \right\}_{\substack{k \leq N \\ k, j \in \mathbb{N}}} \in (\ell^2(\mathbb{C}))^N.$$

We point out that one can just solve (17) for:

$$(k, j) \in I := \{(k, j) \in \mathbb{N}^2 \mid k \leq N, k < j \text{ or } k = j = 1\}$$

since the remaining equations are redundant thanks to (15) and to the definition of G^N . Thanks to Lemma 17 (Appendix A) there exist:

$$\mathcal{G}' := \inf_{(k,j), (m,n) \in I} |\lambda_j^{u_0} - \lambda_k^{u_0} - \lambda_n^{u_0} + \lambda_m^{u_0}| > 0,$$

$$\mathcal{G} := \sup_{A \subset I} \inf_{(k,j), (m,n) \in I \setminus A} |\lambda_j^{u_0} - \lambda_k^{u_0} - \lambda_n^{u_0} + \lambda_m^{u_0}| > 0$$

where A runs over the finite subsets of I (we present a precise estimate of \mathcal{G} in the second point of the proof). Hence for $T > \frac{2\pi}{\mathcal{G}}$, one can use Haraux Theorem ([8], p. 67) and solve the moment problem (17). Then γ^{u_0} is surjective up to phase-shifts.

The proof is achieved by using Generalized Inverse Function Theorem ([9], p. 240), which ensures that the map α^{u_0} is locally surjective.

2) Let us estimate the gap:

$$\mathcal{G} := \sup_{A \subset I} \inf_{(k,j),(m,n) \in I \setminus A} |\lambda_j^{u_0} - \lambda_k^{u_0} - \lambda_n^{u_0} + \lambda_m^{u_0}| > 0$$

that defines the minimal time of solvability of the moment problem (17).

Let $M \in \mathbb{N}$, $A^M := \{(k, j) \in I \mid k \leq N, j \leq M\}$ and $I^M := I \setminus A^M$. For M large enough Remark 2 (Appendix A) implies the existence of $C > 0$ such that for every $(k, j), (m, n) \in I^M$:

$$|\lambda_j^{u_0} - \lambda_k^{u_0} - \lambda_n^{u_0} + \lambda_m^{u_0}| \geq C |\lambda_j - \lambda_k - \lambda_n + \lambda_m|$$

Hence we can study the asymptotic gap by considering the unperturbed eigenvalues so that:

$$\mathcal{G} \geq C \sup_{M \in \mathbb{N}} \left(\inf_{(k,j),(m,n) \in I^M} \lambda_j - \lambda_k - \lambda_n + \lambda_m \right).$$

The sequence $\left\{ \inf_{(k,j),(m,n) \in I^M} (\lambda_j - \lambda_k - \lambda_n + \lambda_m) \right\}_{M > N}$ is increasing and positive, thus:

$$\mathcal{G} \geq C \lim_{M \rightarrow \infty} \left(\inf_{(k,j),(m,n) \in I^M} (\lambda_j - \lambda_k - \lambda_n + \lambda_m) \right).$$

For every $(k, j), (m, n) \in I^M$ one knows that $\lambda_k, \lambda_m \leq \pi^2 N^2$, so:

$$\lambda_j - \lambda_k - \lambda_n + \lambda_m > |\lambda_j - \lambda_n| - 2\pi^2 N^2.$$

Now $j, n > M$ and for M large enough $|\lambda_j - \lambda_n| \geq 2\pi^2 N^2$ and $|\lambda_j - \lambda_n| \geq \lambda_{M+2} - \lambda_{M+1} - 2\pi^2 N^2$. Then:

$$\mathcal{G} \geq C \lim_{M \rightarrow \infty} (\lambda_{M+2} - \lambda_{M+1} - 2N^2 \pi^2) = \lim_{M \rightarrow \infty} C \pi^2 (2M + 3 - 2N^2) = \infty.$$

In conclusion the moment problem (17) is solvable for every $T > 0$. \square

Remark. In the proof of Proposition 7, one can get rid of the diagonal resonances by phase-shifting the components instead of the solutions. Indeed we can repeat the proof by substituting the map α^{u_0} with the following one:

$$\begin{cases} \frac{\widehat{\alpha}_{k,k}(u)}{|\widehat{\alpha}_{k,k}(u)|} \widehat{\alpha}_{k,j}(u), & j, k \leq N, \\ \widehat{\alpha}_{k,j}(u), & j > N, k \leq N. \end{cases}$$

In fact, we point out that one can mix the two techniques by phase-shifting a part of the components and a part of the solutions.

In the next Theorem we ensure the simultaneous local exact controllability in projection for any $T > 0$ up to phase-shifts for orthonormal sets.

Theorem 8. *Let $N \in \mathbb{N}$ and B satisfy Assumption (I). Then there exist $T > 0$, $\epsilon > 0$ and $u_0 \in \mathbb{R}$ such that for any:*

$$\begin{aligned} \{\psi_j\}_{j \in \mathbb{N}} \in \{ \{\psi_j\}_{j \in \mathbb{N}} \subset H_{(0)}^3 \mid \langle \psi_j, \psi_k \rangle = \delta_{j,k}; \\ \|\psi_j - \phi_j^{u_0}(T)\|_{\ell^\infty(H_{(0)}^3)} < \epsilon \} \end{aligned}$$

there exist a sequence of real numbers $\{\theta_j\}_{j \in \mathbb{N}} = \{\{\hat{\theta}_j\}_{j \leq N}, 0, \dots\}$ and a control function $u \in L^2((0, T), \mathbb{R})$ such that:

$$(18) \quad \{\pi_N^{u_0} \psi_j\}_{j \in \mathbb{N}} = \{\pi_N^{u_0} e^{i\theta_j} \Gamma_T^u \phi_j^{u_0}\}_{j \in \mathbb{N}}.$$

Proof. For every $\{\psi_j\}_{j \in \mathbb{N}} \in \ell^\infty(H_{(0)}^3)$ and $\{p_j\}_{j \in \mathbb{N}} \in h^3(\mathbb{R})$ it follows that $\{p_j \psi_j\}_{j \in \mathbb{N}} \in h^3(H_{(0)}^3)$. Hence the claim is consequence of Proposition 7. \square

4 Simultaneous global exact controllability in projection

The purpose of the section is to provide the proof of Theorem 2. We would like to start by proving the theorem in $h^3(H_{(0)}^3)$ with an approximate controllability result and the local exact controllability of Proposition 7.

However even if it is possible to prove the simultaneous global approximate controllability for infinite systems in $h^3(H_{(0)}^3)$ thanks to Theorem 6, the flow of the unitary propagator Γ_T^u does not preserve the space $\pi_N(H_{(0)}^3)$ making impossible to reverse and concatenate dynamics.

Thus we have to adopt an alternative strategy and we transpose the problem. In particular we prove that under particular conditions the controllability in $h^3(H_{(0)}^3)$ in projection into a N dimensional space is equivalent to the controllability of N systems in $(H_{(0)}^3)^N$ without projecting. We call this feature ‘‘transposition argument’’. Now the space $(H_{(0)}^3)^N$ is preserved by the propagator, hence we can reverse and concatenate dynamics.

4.1 Finite simultaneous local exact controllability

Let us start by proving that Proposition 7 implies the finite simultaneous local exact controllability of the systems:

$$(19) \quad \begin{cases} i\partial_t f_j^{u_0}(t, x) = -(A + u_0 B) f_j^{u_0}(t, x) - u_1(t) B f_j^{u_0}(t, x), \\ f_j^{u_0}(0, x) = p_j \phi_j^{u_0} \quad j \leq N, \end{cases}$$

in the neighborhood $\tilde{O}_{\epsilon_0, T}^{u_0}$ introduced in (12).

Before proceeding we define the following notation. Let us consider a generic matrix $x := \{x_{j,k}\}_{(j,k) \in J}$ for $J \subseteq \mathbb{N}^2$ set of indices, we call x^T the transposed matrix with elements $x_{j,k}^T = x_{k,j}$.

Proposition 9. *Let $N \in \mathbb{N}$ and B satisfy Assumption (I). Then there exist $T > 0$, $\epsilon_0 > 0$ and $u_0 \in \mathbb{R}$ such that for any $\{\psi_j\}_{j \leq N} \in \tilde{O}_{\epsilon_0, T}^{u_0}$ there exist a sequence of real numbers $\{\theta_k\}_{k \in \mathbb{N}} = \{\{\tilde{\theta}_k\}_{k \leq N}, 0, \dots\}$ and a control function $u \in L^2((0, T), \mathbb{R})$ such that:*

$$\sum_{k=1}^{\infty} \phi_k^{u_0} e^{i\theta_k} \langle \phi_k^{u_0}, \psi_j \rangle = \tilde{\Gamma}_T^u p_j \phi_j^{u_0}, \quad j \leq N.$$

Proof. First, by using the relation (9), one can show that:

$$(20) \quad \begin{aligned} \overline{\langle \phi_k^{u_0}(T), \Gamma_T^u \phi_j^{u_0} \rangle} &= e^{-i\lambda_k^{u_0} T} \langle \Gamma_T^u \phi_j^{u_0}, \phi_k^{u_0} \rangle = e^{-i\lambda_k^{u_0} T} \langle \phi_j^{u_0}, (\Gamma_T^u)^* \phi_k^{u_0} \rangle \\ &= e^{-i(\lambda_k^{u_0} + \lambda_j^{u_0})T} \langle \phi_j^{u_0}(T), \tilde{\Gamma}_T^u \phi_k^{u_0} \rangle. \end{aligned}$$

Now $e^{-i(\lambda_k^{u_0} + \lambda_j^{u_0})T}$ does not depend on u and then the relation (20) implies that the map $\{\langle \phi_k^{u_0}(T), \Gamma_T^u \phi_j^{u_0} \rangle\}_{k, j \in \mathbb{N}, k \leq N}$ is surjective in D^N if and only if $\{\langle \phi_k^{u_0}(T), \tilde{\Gamma}_T^u \phi_j^{u_0} \rangle\}_{k, j \in \mathbb{N}, j \leq N}$ is surjective in the space:

$$\{x := \{x_{k,j}\}_{k, j \in \mathbb{N}, j \leq N} \mid x^T \in D^N\}.$$

Let the map \hat{a} be the infinite matrix so that $\hat{a}_{k,j}(\tilde{u}_1) := \langle p_k \phi_k^{u_0}(T), \tilde{\Gamma}_T^{u_0 + \tilde{u}_1} p_j \phi_j^{u_0} \rangle$. One can ensure that the surjectivity of the map α^{u_0} is equivalent to the surjectivity of the map a^{u_0} , the infinite matrix with elements:

$$\begin{cases} a_{k,j}^{u_0}(\tilde{u}_1) = \frac{\overline{\hat{a}_{k,k}(\tilde{u}_1)}}{|\overline{\hat{a}_{k,k}(\tilde{u}_1)}|} \hat{a}_{k,j}(\tilde{u}_1), & j, k \leq N, \\ a_{k,j}^{u_0}(\tilde{u}_1) = \hat{a}_{k,j}(\tilde{u}_1), & k > N, j \leq N. \end{cases}$$

Thus the simultaneous controllability by projecting into $\mathcal{H}_N^{u_0}$ of infinite systems (11) is equivalent to the simultaneous controllability of N systems without projecting.

In conclusion thanks to Proposition 7, for any $\{\psi_j\}_{j \leq N} \in \tilde{O}_{\epsilon_0, T}^{u_0}$, there exist a sequence of numbers $\{\theta_j\}_{j \in \mathbb{N}} = \{\{\hat{\theta}_j\}_{j \leq N}, 0, \dots\}$ and a control function $u_1 \in L^2((0, T), \mathbb{R})$ such that:

$$\psi_j = \sum_{k=1}^{\infty} \phi_k^{u_0}(T) e^{i\theta_k} \langle \phi_k^{u_0}(T), \tilde{\Gamma}_T^{u_0 + u_1} p_j \phi_j^{u_0} \rangle. \quad \square$$

Remark. We define for $\epsilon > 0$:

$$O(\epsilon) := \left\{ \{\psi_j\}_{j \in \mathbb{N}} \subset H_{(0)}^3 \mid \langle \psi_j, \psi_k \rangle = \delta_{j,k}; \|\psi_j - \phi_j\|_{h^3(H_{(0)}^3)}^2 < \epsilon \right\}.$$

Let us consider the transposition argument and the existing results of simultaneous local exact controllability for N systems from [12, Theorem 4.1]

and [12, *Proposition 4.4*]. One can state the existence of a residual set of functions μ in H^3 such that for $B = \mu$ the following controllability features hold. In particular they imply the simultaneous local exact controllability in projection in $O(\epsilon)$: the first for $T > 0$ large enough and the second for every $T > 0$ up to phase-shifts into the components. In other words $\forall \{\psi_j\}_{j \in \mathbb{N}} \in O(\epsilon)$ there exist $u \in L^2((0, T), \mathbb{R})$ and a sequence $\{\theta_j\}_{j \in \mathbb{N}} = \{\{\widehat{\theta}_j\}_{j \leq N}, 0, \dots\}$ such that:

$$\langle \phi_k, \Gamma_T^u \phi_j \rangle = e^{i\theta_k} \langle \phi_k, \psi_j \rangle, \quad j \in \mathbb{N}, k \leq N.$$

However our purpose is different, we want to explicit conditions in B so that the local controllability in projection is guaranteed for every $T > 0$ and up to phase-shifts as in Theorem 8.

Remark. The transposition argument allows to adopt in the proof of Theorem 2 the result [12, *Theorem 4.1*] instead of Theorem 8. Indeed [12, *Theorem 4.1*] ensures the simultaneous local exact controllability of N systems without the any phase-shifts. The theorem does not hold for any $T > 0$, contrary to Theorem 8, but we have to renounce to the controllability for any positive time when we use Theorem 6.

Now we prove the simultaneous global exact controllability for a finite number of systems (19).

Theorem 10. *Let $N \in \mathbb{N}$ and B satisfy Assumption (I). Then for any $\{\psi_k^1\}_{k \leq N}$, $\{\psi_k^2\}_{k \leq N} \subset H_{(0)}^3$ orthonormal systems, there exist $T > 0$ and a control function $u \in L^2((0, T), \mathbb{R})$ such that:*

$$\psi_k^2 = \widetilde{\Gamma}_T^u \psi_k^1, \quad k \leq N.$$

Proof. Let u_0 be contained in the neighborhoods provided by Remark 3, Lemma 15 and Lemma 16 (Appendix A). Now [12, *Theorem 4.1*] holds if one substitutes V with $u_0 B$ and μ with $-B$. Then there exists $\epsilon > 0$ such that for:

$$N(\epsilon) := \left\{ \{\varphi_k\}_{k \leq N} \subset H_{(0)}^3 \mid \langle \varphi_j, \varphi_k \rangle = \delta_{j,k}; \sum_{j=1}^N \|\varphi_k - \phi_k\|_{H_{(0)}^3} < \epsilon \right\}$$

there exists $T > 0$ so that $\forall \{\varphi_k\}_{k \leq N} \in N(\epsilon)$ there exists a control function $u_1 \in L^2((0, T), \mathbb{R})$ such that $\varphi_k = \Gamma_T^{u_0 + u_1} \phi_k^{u_0}$, for every $k \leq N$.

Afterwards, for each $\{\psi_k\}_{k \leq N} \subset H_{(0)}^3$ such that $\langle \psi_j, \psi_k \rangle = \delta_{j,k}$, Theorem 6 holds if one substitutes A with $-A$ and B with $-B$. Hence there exist $\widetilde{T} > 0$ and a control function $u_2 \in L^2((0, \widetilde{T}), \mathbb{R})$ so that:

$$\sum_{k=1}^N \|\phi_k^{u_0} - \widetilde{\Gamma}_{\widetilde{T}}^{u_0 + u_2} \psi_k\|_{H_{(0)}^3} < \epsilon.$$

Now $\{\tilde{\Gamma}_T^{u_0+u_2}\psi_k\}_{k \leq N} \in N(\epsilon)$ and thanks to the first part of the proof there exists a control function $u_3 \in L^2((0, T), \mathbb{R})$ such that:

$$\tilde{\Gamma}_T^{u_0+u_2}\psi_k = \Gamma_T^{u_0+u_3}\phi_k^{u_0}, \quad \forall k \leq N.$$

By reverting the dynamics and defined:

$$u(t) = u_0 + u_3(t)\chi_{[0, T)}(t) + u_2(T + \tilde{T} - t)\chi_{[T, T+\tilde{T}]}(t)$$

we have that $\psi_k = \Gamma_{T+\tilde{T}}^u \tilde{\phi}_k^{u_0}$. Hence for any $\Psi^2 := \{\psi_j^2\}_{j \leq N} \subset H_{(0)}^3$ orthonormal system there exists a control such that the dynamics of (3) starting from $\{\phi_j^{u_0}\}_{j \leq N}$ reaches Ψ^2 . Thanks to the time reversibility for every $\Psi^1 := \{\psi_j^1\}_{j \leq N} \subset H_{(0)}^3$ orthonormal system there exists a controlled dynamics with initial state Ψ^1 and target $\{\phi_j^{u_0}\}_{j \leq N}$. In conclusion if we concatenate the two, one can ensure the existence of a control such that the associate dynamics starting from Ψ^1 arrives to Ψ^2 by passing through $\{\phi_j^{u_0}\}_{j \leq N}$. \square

4.2 Proofs of the main results

Proof of Theorem 2.

1) Controllability of orthonormal systems:

Let $\Psi^3 := \{\psi_j^3\}_{j \in \mathbb{N}} \in H_{(0)}^3$ be an orthonormal basis. We consider a couple of orthonormal systems $\{\psi_j^1\}_{j \in \mathbb{N}}, \{\psi_j^2\}_{j \in \mathbb{N}} \subset H_{(0)}^3$ and the unitary operator $\hat{\Gamma} \in U(\mathcal{H})$ such that $\hat{\Gamma}\psi_j^1 = \psi_j^2$ and $\hat{\Gamma}^*\psi_j^3 \in H_{(0)}^3$ for every $j \in \mathbb{N}$. The last relation implies that for every $k \leq N$:

$$\tilde{\psi}_k := \sum_{j=1}^{\infty} \psi_j^1 \langle \psi_j^2, \psi_k^3 \rangle = \sum_{j=1}^{\infty} \psi_j^1 \langle \hat{\Gamma}\psi_j^1, \psi_k^3 \rangle = \hat{\Gamma}^*\psi_k^3 \in H_{(0)}^3.$$

Thanks to Theorem 10 there exist $T > 0$ and a control function $u \in L^2((0, T), \mathbb{R})$ such that for every $k \leq N$ we have that $\tilde{\psi}_k = \tilde{\Gamma}_T^u \psi_k^3$. Hence for $j \in \mathbb{N}, k \leq N$:

$$\langle \psi_j^1, \tilde{\Gamma}_T^u \psi_k^3 \rangle = \langle \psi_j^1, \tilde{\psi}_k \rangle = \langle \psi_j^2, \psi_k^3 \rangle,$$

then $\langle \psi_j^2, \psi_k^3 \rangle = \langle \Gamma_T^u \psi_j^1, \psi_k^3 \rangle$ and:

$$(21) \quad \pi_N(\Psi^3)\psi_j^2 = \pi_N(\Psi^3)\Gamma_T^u \psi_j^1, \quad \forall j \in \mathbb{N}.$$

2) Controllability of unitarily equivalent sets of functions:

Let us consider $\{\psi_j^1\}_{j \in \mathbb{N}}, \{\psi_j^2\}_{j \in \mathbb{N}} \subset H_{(0)}^3$ unitarily equivalent. We suppose the existence of a unitary operator $\hat{\Gamma}$ so that $\hat{\Gamma}\psi_j^1 = \psi_j^2$ and $\hat{\Gamma}^*\psi_j^3 \in H_{(0)}^3$ for every $j \in \mathbb{N}$. One knows that $\psi_j^1 = \sum_{k \in \mathbb{N}} a_k^j \psi_k^3$, but $\{\hat{\Gamma}\psi_j^3\}_{j \in \mathbb{N}}$ is an orthonormal basis of \mathcal{H} and:

$$\psi_j^2 = \hat{\Gamma}\psi_j^1 = \sum_{k \in \mathbb{N}} a_k^j \hat{\Gamma}\psi_k^3.$$

Now the first part of the proof implies that there exist $T > 0$ and a control function $u \in L^2((0, T), \mathbb{R})$ such that:

$$\pi_N(\Psi^3) \Gamma_T^u \psi_k^3 = \pi_N(\Psi^3) \widehat{\Gamma} \psi_k^3, \quad \forall k \in \mathbb{N}$$

and then for any $j \in \mathbb{N}$:

$$\pi_N(\Psi^3) \Gamma_T^u \psi_j^1 = \sum_{k \in \mathbb{N}} a_k^j (\pi_N(\Psi^3) \Gamma_T^u \psi_k^3) = \pi_N(\Psi^3) \sum_{k \in \mathbb{N}} a_k^j \widehat{\Gamma} \psi_k^3 = \pi_N(\Psi^3) \psi_j^2.$$

3) Controllability with generic projectors:

Let $\Psi^3 = \{\psi_j^3\}_{j \in \mathbb{N}} \subset H_{(0)}^3$ be a sequence of linearly independent elements. For every $N \in \mathbb{N}$ by considering the Gram-Schmidt orthonormalization process, there exists an orthonormal system $\widetilde{\Psi}^3 := \{\{\widetilde{\psi}_j^3\}_{j \leq N}, 0, \dots\}$ such that:

$$\text{span}\{\psi_j^3 : j \leq N\} = \text{span}\{\widetilde{\psi}_j^3 : j \leq N\}.$$

The claim then follows since $\pi_N(\Psi^3) \equiv \pi_N(\widetilde{\Psi}^3)$. If $\Psi^3 = \{\psi_j^3\}_{j \in \mathbb{N}} \subset H_{(0)}^3$ is a generic sequence of functions, one can extract from Ψ^3 a subsequence of linearly independent elements and repeat as above. \square

Remark. Let us consider the transposition argument and the techniques adopted in the proof of Theorem 2. By using the existing result of simultaneous global exact controllability of N systems of [12, *Main Theorem*], one can prove the simultaneous global exact controllability in projection in $H_{(0)}^4$ for $B = \mu$ and μ in a residual set of H^4 .

Proof of Corollary 4. We start by proving the second point of the corollary. Let $T > 0$ large enough, $\Psi^3 := \{\psi_j^3\}_{j \in \mathbb{N}} \in H_{(0)}^3$ and $\rho^1, \rho^2 \in T(\mathcal{H})$ two density matrices unitarily equivalent such that $\text{Ran}(\rho^1), \text{Ran}(\rho^2) \subseteq H_{(0)}^3$. We suppose that the unitary operator $\widehat{\Gamma} \in U(\mathcal{H})$ such that $\rho^2 = \widehat{\Gamma} \rho^1 \widehat{\Gamma}$ satisfies the condition $\widehat{\Gamma}^* \psi_j^3 \in H_{(0)}^3$ for every $j \in \mathbb{N}$. One can ensure the existence of two complete orthonormal systems $\Psi^1 := \{\psi_j^1\}_{j \in \mathbb{N}}, \Psi^2 := \{\psi_j^2\}_{j \in \mathbb{N}} \in H_{(0)}^3$ respectively composed by eigenfunctions of ρ^1 and of ρ^2 such that:

$$\rho^1 = \sum_{j=1}^{\infty} l_j |\psi_j^1\rangle \langle \psi_j^1|, \quad \rho^2 = \sum_{j=1}^{\infty} l_j |\psi_j^2\rangle \langle \psi_j^2|.$$

The sequence $\{l_j\}_{j \in \mathbb{N}} \subset \mathbb{R}^+$ corresponds to the spectrum of ρ^1 and of ρ^2 . Now Ψ^1, Ψ^2 satisfy the hypothesis of Theorem 2 and there exists a control function $u \in L^2((0, T), \mathbb{R})$ such that $\pi_N(\Psi^3) \Gamma_T^u \psi_j^1 = \pi_N(\Psi^3) \psi_j^2$. Thus:

$$\begin{aligned} \pi_N(\Psi^3) \Gamma_T^u \rho^1 (\Gamma_T^u)^* \pi_N(\Psi^3) &= \sum_{j=1}^{\infty} l_j |\pi_N(\Psi^3) \Gamma_T^u \psi_j^1\rangle \langle \psi_j^1 \Gamma_T^u \pi_N(\Psi^3) | \\ &= \sum_{j=1}^{\infty} l_j \pi_N(\Psi^3) |\psi_j^2\rangle \langle \psi_j^2| \pi_N(\Psi^3) = \pi_N(\Psi^3) \rho^2 \pi_N(\Psi^3). \end{aligned}$$

In conclusion, the first point follows from Corollary 4 such as the second comes from Theorem 2. \square

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A Analytic Perturbation

Let us consider the systems (11) and the eigenvalues $\{\lambda_j^{u_0}\}_{j \in \mathbb{N}}$ of the operator $A + u_0 B$. We recall that A is a self-adjoint operator with discrete spectrum $\sigma(A) = \{\lambda_j\}_{j \in \mathbb{N}}$, B is a bounded symmetric operator and u_0 is real.

Proposition 11. *Let us consider the family of operators $A(u) = A + uB$ u real. Then there exists a neighborhood D of $u = 0$ small enough where the maps $u \mapsto \lambda_j^u$ are analytic $\forall j \in \mathbb{N}$.*

Proof. The claim is consequence of the validity of [5, Proposition 56] that implies [5, Theorem 55]. \square

The next lemma proves the existence of perturbations which do not shrink the eigenvalues gaps. From now on we use the notation $\| \cdot \|$ for the operators norm of bounded operators in \mathcal{H} .

Lemma 12. *Let us consider the Cauchy systems (11) for B a bounded symmetric operator. Then there exists a neighborhood $U(0)$ of $u = 0$ such that for each $u_0 \in U(0)$, there exists $r > 0$ such that $\forall j \in \mathbb{N}$:*

$$\mu_j := \frac{\lambda_j + \lambda_{j+1}}{2} \in \rho(A + u_0 B), \quad \| (A + u_0 B - \mu_j)^{-1} \| \leq r.$$

Proof. Let D be the neighborhood provided by Proposition 11. First, we want to prove that for $u_0 \in D$ small enough the operator $(A + u_0 B - \mu_j)$ is invertible with bounded inverse $\forall j \in \mathbb{N}$. We know that $(A - \mu_j)$ is invertible in a bounded operator and $\mu_j \in \rho(A)$ (resolvent set of A). Let $\delta := \min_{j \in \mathbb{N}} \{|\lambda_{j+1} - \lambda_j|\}$, one has that:

$$\| (A - \mu_j)^{-1} \| \leq \sup_{k \in \mathbb{N}} \frac{1}{|\mu_j - \lambda_k|} = \frac{2}{|\lambda_{j+1} - \lambda_j|} \leq \frac{2}{\delta}.$$

Thus:

$$\| (A - \mu_j)^{-1} u_0 B \| \leq |u_0| \| (A - \mu_j)^{-1} \| \| B \| \leq \frac{2}{\delta} |u_0| \| B \|$$

and, if $|u_0| \leq \frac{\delta(1-\epsilon)}{2\|B\|}$ for $\epsilon \in (0, 1)$, we have that $\|(A - \mu_j)^{-1}u_0B\| \leq 1 - \epsilon$. The operator $(A + u_0B - \mu_j)$ is invertible and one can notice that:

$$\|(A + u_0B - \mu_j)\| \geq \|(A - \mu_j)\| - \|u_0B\| \geq \frac{\delta}{2} - \frac{\delta(1-\epsilon)}{2} = \frac{\delta\epsilon}{2}$$

and in conclusion $\|(A + u_0B - \mu_j)^{-1}\| \leq \frac{2}{\delta\epsilon}$. \square

Lemma 13. *Let us consider the Cauchy systems (11) for B a bounded symmetric operator. Then there exists a neighborhood $U(0)$ of 0 in \mathbb{C} such that $\forall u_0 \in U(0)$:*

$$(A + u_0P_{\phi_k}^\perp B - \lambda_k^{u_0})$$

is invertible with bounded inverse from $D(A) \cap \phi_k^\perp$ to ϕ_k^\perp , $\forall k \in \mathbb{N}$ and $P_{\phi_k}^\perp$ is the projector into the orthogonal space of ϕ_k .

Proof. Let D be the neighborhood provided by Lemma 12. For any $u_0 \in D$, one can consider the decomposition:

$$(A + u_0P_{\phi_k}^\perp B - \lambda_k^{u_0}) = (A - \lambda_k^{u_0}) + u_0P_{\phi_k}^\perp B.$$

The operator $A - \lambda_k^{u_0}$ is invertible with bounded inverse when it acts on the orthogonal space of $\phi_k(t)$ and we want to estimate:

$$\|((A - \lambda_k^{u_0})|_{\phi_k^\perp})^{-1}u_0P_{\phi_k}^\perp B\|.$$

However for every $\psi \in D(A) \cap \text{Ran}(P_{\phi_k}^\perp)$ such that $\|\psi\| = 1$ we have:

$$\|(A - \lambda_k^{u_0})\psi\| \geq \min\{|\lambda_{k+1} - \lambda_k^{u_0}|, |\lambda_k^{u_0} - \lambda_{k-1}|\}\|\psi\|.$$

Let $\delta_k := \min\{|\lambda_{k+1} - \lambda_k^{u_0}|, |\lambda_k^{u_0} - \lambda_{k-1}|\}$. Thanks to Lemma 12 for u_0 small enough, $\lambda_k^{u_0} \in \left(\frac{\lambda_{k-1} + \lambda_k}{2}, \frac{\lambda_k + \lambda_{k+1}}{2}\right)$ and then:

$$\delta_k \geq \min\left\{\left|\lambda_{k+1} - \frac{\lambda_k + \lambda_{k+1}}{2}\right|, \left|\frac{\lambda_{k-1} + \lambda_k}{2} - \lambda_{k-1}\right|\right\} \geq \frac{(2k-1)\pi^2}{2} > k.$$

Afterwards $\|((A - \lambda_k^{u_0})|_{\phi_k^\perp})^{-1}u_0P_{\phi_k}^\perp B\| \leq \frac{1}{\delta_k}|u_0|\|B\|$ and, if $|u_0| \leq (1-r)\frac{\delta_k}{\|B\|}$ for $r \in (0, 1)$, one has that:

$$\|((A - \lambda_k^{u_0})|_{\phi_k^\perp})^{-1}u_0P_{\phi_k}^\perp B\| \leq (1-r) < 1.$$

The operator $A_k := (A - \lambda_k^{u_0} + u_0P_{\phi_k}^\perp B)$ is invertible when it acts into the orthogonal space of $\phi_k(t)$ and:

$$\begin{aligned} \|A_k\| &\geq \|A - \lambda_k^{u_0}\| - \|u_0P_{\phi_k}^\perp B\| \geq \delta_k - \|u_0P_{\phi_k}^\perp B\| \\ &\geq \delta_k - |u_0|\|B\| \geq \delta_k - (1-r)\delta_k = r\delta_k. \end{aligned}$$

In conclusion thanks to the fact that:

$$(22) \quad \left\| \left((A - \lambda_k^{u_0} + u_0 P_{\phi_k}^\perp B) \Big|_{\phi_k^\perp} \right)^{-1} \right\| \leq \frac{1}{r\delta_k} < \frac{1}{rk},$$

the proof is achieved. \square

Lemma 14. *Let us consider the Cauchy systems (11) for B a bounded symmetric operator. Then there exists a neighborhood $U(0)$ of 0 in \mathbb{C} such that, for any $u_0 \in U(0)$ there holds that $\lambda_j^{u_0} \asymp \lambda_j$. In other words there exist two constants $C_1, C_2 > 0$ so that for each $j \in \mathbb{N}$:*

$$(23) \quad C_1 \lambda_j \leq \lambda_j^{u_0} \leq C_2 \lambda_j.$$

Proof. Let $u_0 \in D$ for D the neighborhood provided by Lemma 13. We decompose the eigenfunction $\phi_j^{u_0}(T) = \alpha_j \phi_j(T) + \eta_j$, where α_j is an orthonormalizing constant and η_j is orthogonal to $\phi_j(T)$. Hence:

$$\begin{aligned} \lambda_k^{u_0} \phi_k^{u_0}(T) &= (A + u_0 B)(\alpha_k \phi_k(T) + \eta_k) \\ &= A\alpha_k \phi_k(T) + A\eta_k + u_0 B\alpha_k \phi_k(T) + u_0 B\eta_k, \end{aligned}$$

and:

$$\lambda_k^{u_0} \alpha_k \phi_k(T) + \lambda_k^{u_0} \eta_k = A\alpha_k \phi_k(T) + A\eta_k + u_0 B\alpha_k \phi_k(T) + u_0 B\eta_k.$$

By projecting into the orthogonal space of $\phi_k(T)$:

$$\begin{aligned} \lambda_k^{u_0} \eta_k &= A\eta_k + u_0 P_{\phi_k}^\perp B\alpha_k \phi_k(T) + u_0 P_{\phi_k}^\perp B\eta_k \\ (A + u_0 P_{\phi_k}^\perp B - \lambda_k^{u_0})\eta_k &= -u_0 P_{\phi_k}^\perp B\alpha_k \phi_k(T). \end{aligned}$$

However Lemma 13 ensures that $A + u_0 P_{\phi_k}^\perp B - \lambda_k^{u_0}$ is invertible with bounded inverse when it acts on the orthogonal space of $\phi_k(T)$ and then:

$$(24) \quad \eta_k = -\alpha_k \left((A + u_0 P_{\phi_k}^\perp B - \lambda_k^{u_0}) \Big|_{\phi_k^\perp} \right)^{-1} u_0 P_{\phi_k}^\perp B \phi_k(T).$$

Afterwards:

$$\begin{aligned} \lambda_j^{u_0} &= \langle \alpha_j \phi_j(T) + \eta_j, (A + u_0 B)(\alpha_j \phi_j(T) + \eta_j) \rangle \\ &= |\alpha_j|^2 \lambda_j + u_0 \langle \alpha_j \phi_j(T), B\alpha_j \phi_j(T) \rangle + \langle \alpha_j \phi_j(T), (A + u_0 B)\eta_j \rangle \\ &\quad + \langle \eta_j, (A + u_0 B)\alpha_j \phi_j(T) \rangle + \langle \eta_j, (A + u_0 B)\eta_j \rangle. \end{aligned}$$

By using the relation (24):

$$\begin{aligned} \langle \eta_j, (A + u_0 B)\eta_j \rangle &= \langle \eta_j, (A + u_0 P_{\phi_j}^\perp B - \lambda_j^{u_0})\eta_j \rangle + \lambda_j^{u_0} \|\eta_j\|^2 \\ &= \lambda_j^{u_0} \|\eta_j\|^2 + \left\langle \eta_j, -\alpha_j (A + u_0 P_{\phi_j}^\perp B - \lambda_j^{u_0}) \right. \\ &\quad \left. \cdot \left((A + u_0 P_{\phi_j}^\perp B - \lambda_j^{u_0}) \Big|_{\phi_j^\perp} \right)^{-1} u_0 P_{\phi_j}^\perp B \phi_j(T) \right\rangle. \end{aligned}$$

However $(A + u_0 P_{\phi_j}^\perp B - \lambda_j^{u_0})((A + u_0 P_{\phi_j}^\perp B - \lambda_j^{u_0})|_{\phi_j^\perp})^{-1} = Id$ and then:

$$\langle \eta_j, (A + u_0 B)\eta_j \rangle = \lambda_j^{u_0} \|\eta_j\|^2 - u_0 \alpha_j \langle \eta_j, P_{\phi_j}^\perp B \phi_j(T) \rangle.$$

Moreover we have:

$$\langle \phi_j(T), (A + u_0 B)\eta_j \rangle = u_0 \langle \phi_j(T), B\eta_j \rangle = u_0 \langle P_{\phi_j}^\perp B \phi_j(T), \eta_j \rangle$$

and equivalently $\langle \eta_j, (A + u_0 B)\phi_j(T) \rangle = u_0 \langle \eta_j, P_{\phi_j}^\perp B \phi_j(T) \rangle$. Thus:

$$(25) \quad \lambda_j^{u_0} = |\alpha_j|^2 \lambda_j + u_0 |\alpha_j|^2 B_{j,j} + \lambda_j^{u_0} \|\eta_j\|^2 + u_0 \overline{\alpha_j} \langle P_{\phi_j}^\perp B \phi_j(T), \eta_j \rangle.$$

One can notice that $\alpha_j \in [0, 1]$ and $\|\eta_j\|$ are uniformly bounded in j . We show that the first accumulates at 1 and the second at 0. Indeed from (22) and (24), one has that there exists a constant $C_1 > 0$ such that:

$$(26) \quad \begin{aligned} \|\eta_j\|^2 &\leq |u_0|^2 \left\| \left((A + u_0 P_{\phi_j}^\perp B - \lambda_j^{u_0})|_{\phi_j^\perp} \right)^{-1} \right\|^2 |\alpha_j|^2 \|B\phi_j(T)\|^2 \\ &\leq \frac{|u_0|^2 \|B\phi_j(T)\|^2}{r^2 j^2} \leq \frac{C_1}{j^2}. \end{aligned}$$

for $r \in (0, 1)$, which implies that $\lim_{j \rightarrow \infty} \|\eta_j\| = 0$. Afterwards by contradiction if $\alpha_\infty := \lim_{j \rightarrow \infty} |\alpha_j| \neq 1$, then $\alpha_\infty \in [0, 1)$ and thus:

$$1 = \lim_{j \rightarrow \infty} \|\phi_j^{u_0}(T)\| \leq \lim_{j \rightarrow \infty} |\alpha_j| \|\phi_j(T)\| < 1$$

that is absurd. In conclusion from (25) it follows that $\lambda_j^{u_0} \asymp \lambda_j$. \square

Lemma 15. *Let us consider $T > 0$ and the Cauchy systems (11) for B a bounded symmetric operator satisfying Assumption (I). Then for every $N \in \mathbb{N}$ there exists a neighborhood $U(0)$ of 0 in \mathbb{C} such that, for any $u_0 \in U(0)$ there exists $\tilde{C}_N > 0$ such that for $k \in \mathbb{N}$ and $j \leq N$:*

$$(27) \quad |\langle \phi_k^{u_0}(T), B\phi_j^{u_0}(T) \rangle| \geq \frac{\tilde{C}_N}{k^3}.$$

Proof. We prove (27) for fixed $j \leq N$ then the generalization follows by using the minimum of all the constants defined for every $j \leq N$.

We start by choosing $k \in \mathbb{N}$ such that $k \neq j$ and $u_0 \in D$ for D the neighborhood provided by Lemma 14. Thanks to Assumption (I) we have:

$$(28) \quad \begin{aligned} |\langle \phi_k^{u_0}(T), B\phi_j^{u_0}(T) \rangle| &= |\langle \alpha_k \phi_k(T) + \eta_k, B(\alpha_j \phi_j(T) + \eta_j) \rangle| \\ &\geq C_N \frac{\overline{\alpha_k} \alpha_j}{k^3} - |\overline{\alpha_k} \langle \phi_k(T), B\eta_j \rangle + \alpha_j \langle \eta_k, B\phi_j(T) \rangle + \langle \eta_k, B\eta_j \rangle|. \end{aligned}$$

1) Expansion of $\langle \eta_k, B\phi_j(T) \rangle$, $\langle \phi_k(T), B\eta_j \rangle$, $\langle \eta_k, B\eta_j \rangle$:

Thanks to (24):

$$(29) \quad \begin{aligned} \langle \eta_k, B\phi_j(T) \rangle &= \langle \eta_k, P_{\phi_k}^\perp B\phi_j(T) \rangle = \\ &\langle -\alpha_k((A + u_0 P_{\phi_k}^\perp B - \lambda_k^{u_0})|_{\phi_k^\perp})^{-1} u_0 P_{\phi_k}^\perp B\phi_k(T), P_{\phi_k}^\perp B\phi_j(T) \rangle. \end{aligned}$$

We point out that for u_0 small enough:

$$\begin{aligned} ((A + u_0 P_{\phi_k}^\perp B - \lambda_k^{u_0})|_{\phi_k^\perp})^{-1} &= \\ ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} \sum_{n=0}^{\infty} (u_0((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} P_{\phi_k}^\perp B P_{\phi_k}^\perp)^n & \end{aligned}$$

and by defining $M_k := \sum_{n=0}^{\infty} (u_0((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} P_{\phi_k}^\perp B)^n P_{\phi_k}^\perp$ the relation (29) becomes:

$$(30) \quad \langle \eta_k, B\phi_j(T) \rangle = -u_0 \langle \alpha_k M_k B\phi_k(T), ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} P_{\phi_k}^\perp B\phi_j(T) \rangle.$$

Thanks to the fact that $B : D(A) \rightarrow D(A)$:

$$\begin{aligned} ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} P_{\phi_k}^\perp B\phi_j(T) &= P_{\phi_k}^\perp B((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} \phi_j(T) \\ - [P_{\phi_k}^\perp B, ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} P_{\phi_k}^\perp] \phi_j(T) &= P_{\phi_k}^\perp B((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} \phi_j(T) \\ - ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} P_{\phi_k}^\perp [B, A] ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} \phi_j(T) & \end{aligned}$$

and by calling $\tilde{B}_k := ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} P_{\phi_k}^\perp [B, A]$:

$$(31) \quad \begin{aligned} ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} P_{\phi_k}^\perp B\phi_j(T) &= P_{\phi_k}^\perp (B + \tilde{B}_k) ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} \phi_j(T) \\ &= P_{\phi_k}^\perp (B + \tilde{B}_k) (\lambda_j - \lambda_k^{u_0})^{-1} \phi_j(T). \end{aligned}$$

Let us consider (30), by using (31) we have:

$$(32) \quad \langle \eta_k, B\phi_j(T) \rangle = -\frac{u_0}{\lambda_j - \lambda_k^{u_0}} \langle \alpha_k M_k B\phi_k(T), (B + \tilde{B}_k) \phi_j(T) \rangle.$$

Now one can use the same techniques and obtain:

$$(33) \quad \begin{aligned} |\langle \eta_k, B\eta_j \rangle| &= |\langle B\eta_k, \eta_j \rangle| = | \langle u_0 \alpha_k B((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} M_k B\phi_k(T), \\ u_0 \alpha_j ((A - \lambda_j^{u_0})P_{\phi_j}^\perp)^{-1} M_j B\phi_j(T) \rangle | &= \left| \frac{\alpha_j \bar{\alpha}_k u_0^2}{\lambda_k - \lambda_j^{u_0}} \langle \phi_k(T), L_{k,j} \phi_j(T) \rangle \right| \end{aligned}$$

for $L_{k,j} := (A - \lambda_j^{u_0})B M_k ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} P_{\phi_k}^\perp B ((A - \lambda_j^{u_0})P_{\phi_j}^\perp)^{-1} M_j B$. Now there exists $\epsilon > 0$ such that $\alpha_l \in (\epsilon, 1)$ for every $l \in \mathbb{N}$. Thanks to (32), (33) there exists \hat{C}_N such that from the relation (28) we have:

$$(34) \quad \begin{aligned} |\langle \phi_k^{u_0}(T), B\phi_j^{u_0}(T) \rangle| &\geq \hat{C}_N \frac{1}{k^3} - \left| \frac{u_0}{\lambda_j - \lambda_k^{u_0}} \langle M_k B\phi_k(T), (B + \tilde{B}_k) \phi_j(T) \rangle \right| \\ - \left| \frac{u_0}{\lambda_k - \lambda_j^{u_0}} \langle (B + \tilde{B}_j) \phi_k(T), M_j B\phi_j(T) \rangle \right| &- \left| \frac{u_0^2}{\lambda_k - \lambda_j^{u_0}} \langle \phi_k(T), L_{k,j} \phi_j(T) \rangle \right|. \end{aligned}$$

2) Features of the operators $M_k, \tilde{B}_k, L_{k,j}$:

First, the operators M_k are uniformly bounded in $L(H_{(0)}^2, H_{(0)}^2)$ if u_0 is small enough so that:

$$\|u_0((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1}P_{\phi_k}^\perp B P_{\phi_k}^\perp\|_{H_{(0)}^2} < 1.$$

Second, the relation (31) implies that:

$$\tilde{B}_k P_{\phi_k}^\perp = ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1}P_{\phi_k}^\perp B(A - \lambda_k^{u_0})P_{\phi_k}^\perp - P_{\phi_k}^\perp B P_{\phi_k}^\perp$$

from which follows that the operators \tilde{B}_k are uniformly bounded in k in:

$$L(H_{(0)}^2 \cap \text{Ran}(P_{\phi_k}^\perp), H_{(0)}^2 \cap \text{Ran}(P_{\phi_k}^\perp)).$$

Third, one can notice that $B((A - \lambda_j^{u_0})P_{\phi_j}^\perp)^{-1}M_j B \in L(H_{(0)}^2, H_{(0)}^2)$ and:

$$\begin{aligned} & (A - \lambda_j^{u_0})B M_k ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1}P_{\phi_k}^\perp \\ &= (A - \lambda_j^{u_0})B ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1} \sum_{n=0}^{\infty} (u_0 P_{\phi_k}^\perp B ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1})^n P_{\phi_k}^\perp \\ &= (A - \lambda_j^{u_0})((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1}P_{\phi_k}^\perp (\tilde{B}_k + B) \tilde{M}_k \end{aligned}$$

for $\tilde{M}_k := \sum_{n=0}^{\infty} (u_0 P_{\phi_k}^\perp B ((A - \lambda_k^{u_0})P_{\phi_k}^\perp)^{-1})^n P_{\phi_k}^\perp$. Now the operators \tilde{M}_k are uniformly bounded in $L(H_{(0)}^2, H_{(0)}^2)$ such as M_k . Hence $L_{k,j}$ are uniformly bounded in $L(H_{(0)}^2, H_{(0)}^2)$.

Let now $\{F_l\}_{l \in \mathbb{N}}$ be an infinite family of uniformly bounded operators in $L(H_{(0)}^2, H_{(0)}^2)$. We know that for every $l \in \mathbb{N}$ there exists $c_l > 0$ such that:

$$\sum_{k=1}^{\infty} |k^2 \langle \phi_k, F_l \phi_j \rangle|^2 < \infty, \implies |\langle \phi_k, F_l \phi_j \rangle| \leq \frac{c_l}{k^2}.$$

Now one can choose constants c_l uniformly bounded in l since:

$$\begin{aligned} \sup_{l \in \mathbb{N}} |k^2 \langle \phi_k, F_l \phi_j \rangle|^2 &= \sup_{l \in \mathbb{N}} \sum_{m \in \mathbb{N}} |m^2 \langle \phi_m, F_l \phi_j \rangle|^2 - \sup_{l \in \mathbb{N}} \sum_{m \neq k} |m^2 \langle \phi_m, F_l \phi_j \rangle|^2 \\ &\leq 2 \sup_{l \in \mathbb{N}} \|F_l \phi_j\|_{H_{(0)}^2}^2 < \infty. \end{aligned}$$

Thus for every infinite family of uniformly bounded operators F_l in $L(H_{(0)}^2, H_{(0)}^2)$ there exists a constant c such that:

$$(35) \quad |\langle \phi_k, F_l \phi_j \rangle| \leq \frac{c}{k^2}.$$

3) Conclusion:

Thanks to the previous point, one knows that the families of operators $\{BM_k(B + \tilde{B}_k)\}_{k \in \mathbb{N}}$, $\{L_{k,j}\}_{k \in \mathbb{N}}$ are uniformly bounded in $L(H_{(0)}^2, H_{(0)}^2)$ and $BM_j(B + \tilde{B}_j) \in L(H_{(0)}^2, H_{(0)}^2)$ for every $1 \leq j \leq N$. Hence in (34) we use the relation (35). We point out that $|\lambda_j - \lambda_k^{u_0}|^{-1}$ and $|\lambda_k - \lambda_j^{u_0}|^{-1}$ asymptotically behave as k^{-2} thanks to Lemma 14. Thus there exist $C_1, C_2, C_3, C_4 > 0$ such that for u_0 small enough:

$$\begin{aligned} |\langle \phi_k^{u_0}(T), B\phi_j^{u_0}(T) \rangle| &\geq \widehat{C}_N \frac{1}{k^3} - \frac{C_1 |u_0|}{|\lambda_j - \lambda_k^{u_0}| k^2} - \frac{C_2 |u_0|}{|\lambda_k - \lambda_j^{u_0}| k^2} - \frac{C_3 |u_0|^2}{|\lambda_k - \lambda_j^{u_0}| k^2} \\ &\geq C_4 \frac{1}{k^3}. \end{aligned}$$

Now the relation (27) is verified for $k = j$ thanks to the analyticity and by choosing u_0 such that $|\langle \phi_j^{u_0}(T), B\phi_j^{u_0}(T) \rangle| = c \neq 0$. In conclusion (27) is valid for every $k \in \mathbb{N}$ and for a fixed $j \leq N$ for $\tilde{C}_N = \min\{C_4, c \cdot j^3\}$. \square

Lemma 16. *Let us consider $T > 0$ and the Cauchy systems (11). Then there exists a neighborhood $U(0)$ of 0 in \mathbb{C} such that, for any $u_0 \in U(0)$:*

$$\left(\sum_{j=1}^{\infty} |\lambda_j^{u_0}|^{\frac{3}{2}} |\langle \phi_j^{u_0}(T), \cdot \rangle|^2 \right)^{\frac{1}{2}} \asymp \left(\sum_{j=1}^{\infty} |j^3 \langle \phi_j(T), \cdot \rangle|^2 \right)^{\frac{1}{2}}.$$

Proof. Let D be the neighborhood provided by Lemma 15. The validity of Lemma 14 implies that $\lambda_k^{u_0} \asymp \lambda_k$ and then:

$$\sum_{j=1}^{\infty} |\lambda_j^{u_0}|^{\frac{3}{2}} |\langle \phi_j^{u_0}(T), \cdot \rangle|^2 \asymp \sum_{j=1}^{\infty} |j^3 \langle \phi_j^{u_0}(T), \cdot \rangle|^2.$$

By using the decomposition $\phi_j^{u_0}(T) = \alpha_j \phi_j(T) + \eta_j$ where α_j is an orthonormalizing constant and η_j is orthogonal to $\phi_k(T)$, we have:

$$\left(\sum_{j \in \mathbb{N}} |j^3 \langle \phi_j^{u_0}(T), \psi \rangle|^2 \right)^{\frac{1}{2}} \leq \left(\sum_{j \in \mathbb{N}} |j^3 \langle \phi_j(T), \psi \rangle|^2 \right)^{\frac{1}{2}} + \left(\sum_{j \in \mathbb{N}} |j^3 \langle \eta_j, \psi \rangle|^2 \right)^{\frac{1}{2}}.$$

We want to prove the existence a constant $C_1 > 0$ such that $\sum_{j=1}^{\infty} |j^3 \langle \eta_j, \psi \rangle|^2 \leq C_1 \|\psi\|_{H_{(0)}^3}^2$. Now thanks to the Minkowski inequality:

$$\begin{aligned} \left(\sum_{j \in \mathbb{N}} |j^3 \langle \eta_j, \psi \rangle|^2 \right)^{\frac{1}{2}} &= \left(\sum_{j \in \mathbb{N}} \left| \sum_{\substack{k \in \mathbb{N} \\ k \neq j}} j^3 \langle \eta_j, \phi_k(T) \rangle \langle \phi_k(T), \psi \rangle \right|^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{k \in \mathbb{N}} |\langle \phi_k(T), \psi \rangle| \left(\sum_{\substack{j \in \mathbb{N} \\ j \neq k}} |j^3 \langle \eta_j, \phi_k(T) \rangle|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

The argument used in order to obtain the relation (30) is still valid and one can prove that for $j \neq k$:

$$\langle \eta_j, \phi_k \rangle = -u_0 \bar{\alpha}_j \langle M_j B \phi_j(T), (A - \lambda_j^{u_0})^{-1} \phi_k(T) \rangle.$$

Thanks to (23), there exists a constant $C_1 > 0$ such that for $j, k \in \mathbb{N}$, $j \neq k$:

$$(36) \quad \begin{aligned} \sum_{\substack{j \neq k \\ j \in \mathbb{N}}} |j^3 \langle \eta_j, \phi_k \rangle|^2 &\leq \sum_{j=1}^{\infty} j^6 |u_0|^2 |(\lambda_k - \lambda_j^{u_0})^{-1}|^2 |\langle M_j B \phi_j(T), \phi_k \rangle|^2 \\ &\leq |u_0|^2 \sum_{\substack{j \neq k \\ j \in \mathbb{N}}} \left| \frac{j^3 \langle \phi_j(T), M_j B \phi_k(T) \rangle}{(k^2 - \lambda_j^{u_0})} \right|^2 \leq C_1 |u_0|^2 \frac{\|M_j B \phi_k\|_{H^1}^2}{k^2}. \end{aligned}$$

However the operators $M_j B$ are uniformly bounded in $L(H_{(0)}^2, H_{(0)}^2)$ as we have showed in the proof of Lemma 15, hence there exists a constant $C_2 > 0$ such that:

$$\sum_{\substack{j \neq k \\ j \in \mathbb{N}}} |j^3 \langle \eta_j, \phi_k(T) \rangle|^2 \leq C_2^2 k^{-2}.$$

Thus there exists a constant $C_3 > 0$ such that:

$$\begin{aligned} \left(\sum_{j \in \mathbb{N}} |j^3 \langle \eta_j, \psi \rangle|^2 \right)^{\frac{1}{2}} &\leq C_2 \sum_{k \in \mathbb{N}} |\langle \phi_k(T), \psi \rangle| k^{-1} \leq C_2 \|\langle \phi_k(T), \psi \rangle\|_{\ell^2} \|k^{-1}\|_{\ell^2} \\ &\leq C_3 \|\langle \phi_k(T), \psi \rangle\|_{h^3} \end{aligned}$$

and:

$$\left(\sum_{j=1}^{\infty} |j^3 \langle \phi_j^{u_0}(T), \psi \rangle|^2 \right)^{\frac{1}{2}} \leq \|\psi\|_{H_{(0)}^3} + C_3 \|\psi\|_{H_{(0)}^3}.$$

Now, the lower bound is due to the same argument by adopting the decomposition $\phi_j(T) = \tilde{\alpha}_j \phi_j^{u_0}(T) + \tilde{\eta}_j$ for $\tilde{\alpha}_j = \alpha_j^{-1}$ and $\tilde{\eta}_j = -\tilde{\alpha}_j \eta_j$. Indeed for u_0 small enough thanks to (26) there exists $\delta > 0$ such that $\alpha_j \in (\delta, 1]$ for every $j \in \mathbb{N}$. It follows that $|\tilde{\alpha}_j| \leq \delta^{-1}$ for every $j \in \mathbb{N}$ and one can repeat as before. \square

Lemma 17. *Let us consider the Cauchy systems (11) for B a bounded symmetric operator. For $\epsilon > 0$ small enough, there exists U_ϵ , a non zero measure subset of \mathbb{C} non containing 0, such that for each $u_0 \in U_\epsilon$:*

$$\inf_{\substack{k, j, m, n \in \mathbb{N} \\ k \neq j \neq m \neq n \\ j, n \leq N}} |\lambda_k^{u_0} - \lambda_j^{u_0} - \lambda_m^{u_0} + \lambda_n^{u_0}| > \epsilon.$$

Moreover, for every $\delta > 0$ small there exists $\epsilon > 0$ such that $\text{dist}(U_\epsilon, 0) < \delta$.

Proof. Let us consider the neighborhood D provided by Lemma 13. Here the maps $\lambda_k^u - \lambda_j^u - \lambda_m^u + \lambda_n^u$ are analytic for each $k, j, n, m \in \mathbb{N}$. One can notice that the number of elements such that:

$$(37) \quad \lambda_j - \lambda_k - \lambda_n + \lambda_m = 0, \quad j, n \in \mathbb{N}, \quad k, m \leq N$$

is finite. Indeed $\lambda_k = k^2\pi^2$ and the relation (37) corresponds to $j^2 - k^2 = n^2 - m^2$. We have that $|j^2 - n^2| = |k^2 - m^2| \leq N^2 - 1$ which is satisfied for a finite number of elements. Thus:

$$R := \{(k, j, m, n) \in (\mathbb{N} \times \{1, \dots, N\})^2 : k \neq j \neq m \neq n; \lambda_k - \lambda_j - \lambda_m + \lambda_n = 0\},$$

is a finite set.

1) First, let $(k, j, m, n) \in R$, the set:

$$V_{(k,j,m,n)} = \{u \in D \mid \lambda_k^u - \lambda_j^u - \lambda_m^u + \lambda_n^u = 0\}$$

is a discrete subset of D or is equal to D . However by using the relation (25):

$$\begin{aligned} & \lambda_k^u - \lambda_j^u - \lambda_m^u + \lambda_n^u = \\ & |\alpha_k|^2 \lambda_k + u_0 |\alpha_k|^2 B_{k,k} + \lambda_k^{u_0} \|\eta_k\|^2 + u_0 \overline{\alpha_k} \langle P_{\phi_k}^\perp B \phi_k(T), \eta_k \rangle \\ & - |\alpha_j|^2 \lambda_j - u_0 |\alpha_j|^2 B_{j,j} - \lambda_j^{u_0} \|\eta_j\|^2 - u_0 \overline{\alpha_j} \langle P_{\phi_j}^\perp B \phi_j(T), \eta_j \rangle \\ & - |\alpha_m|^2 \lambda_m - u_0 |\alpha_m|^2 B_{m,m} - \lambda_m^{u_0} \|\eta_m\|^2 - u_0 \overline{\alpha_m} \langle P_{\phi_m}^\perp B \phi_m(T), \eta_m \rangle \\ & + |\alpha_n|^2 \lambda_n + u_0 |\alpha_n|^2 B_{n,n} + \lambda_n^{u_0} \|\eta_n\|^2 + u_0 \overline{\alpha_n} \langle P_{\phi_n}^\perp B \phi_n(T), \eta_n \rangle. \end{aligned}$$

Hence it is true that:

$$(38) \quad \begin{aligned} \lambda_k^u - \lambda_j^u - \lambda_m^u + \lambda_n^u &= |\alpha_k|^2 \lambda_k - |\alpha_j|^2 \lambda_j - |\alpha_m|^2 \lambda_m + |\alpha_n|^2 \lambda_n \\ &+ (|\alpha_k|^2 B_{k,k} - |\alpha_j|^2 B_{j,j} - |\alpha_m|^2 B_{m,m} + |\alpha_n|^2 B_{n,n}) u_0 \\ &+ o(u_0). \end{aligned}$$

For u small enough thanks to the fact that $\lim_{|u_0| \rightarrow 0} |\alpha_j|^2 = 1$ and to the third point of Assumption (I) the quantity $\lambda_k^u - \lambda_j^u - \lambda_m^u + \lambda_n^u$ can not be constantly equal to 0. Then $V_{(k,j,m,n)}$ is discrete and:

$$V = \{u \in D \mid \exists (k, j, m, n) \in R : \lambda_k^u - \lambda_j^u - \lambda_m^u + \lambda_n^u = 0\}$$

is a discrete subset of D . Thanks to the fact that R is a finite set and to the analyticity, the subset of $D \setminus V$:

$$\tilde{U}_\epsilon := \{u \in D : \forall (k, j, n, m) \in R \mid |\lambda_k^u - \lambda_j^u - \lambda_m^u + \lambda_n^u| \geq \epsilon\}$$

has positive measure for $\epsilon > 0$ small enough. Moreover, for any $\delta > 0$ small, there exists $\epsilon_0 > 0$ such that $\text{dist}(0, \tilde{U}_{\epsilon_0}) < \delta$.

2) Let $(k, j, m, n) \notin R$ such that $k \neq j \neq m \neq n$, we know that $\lambda_l^0 = \pi^2 l^2$ and:

$$|\lambda_k^0 - \lambda_j^0 - \lambda_m^0 + \lambda_n^0| = \pi^2 |k^2 - j^2 - m^2 + n^2| > \pi^2.$$

First, from the relation (25) we obtain that $\lambda_j^{u_0} \leq |\alpha_j|^2 \lambda_j + |u_0| C_1$ and $\lambda_j^{u_0} \geq |\alpha_j|^2 \lambda_j - |u_0| C_2$ for suitable constants $C_1, C_2 > 0$ non depending on the index j . Thus:

$$(39) \quad |\lambda_k^{u_0} - \lambda_j^{u_0} - \lambda_m^{u_0} + \lambda_n^{u_0}| \geq ||\alpha_k|^2 \lambda_k - |\alpha_j|^2 \lambda_j - |\alpha_m|^2 \lambda_m + |\alpha_n|^2 \lambda_n| - |u_0|(2C_1 + 2C_2).$$

Now, thanks to the relation (24) one has that $\lim_{k \rightarrow \infty} |\alpha_k|^2 = 1$ and for any u in D , ϵ small enough there exists $M_\epsilon \in \mathbb{N}$ such that:

$$\begin{aligned} & ||\alpha_k|^2 \lambda_k - |\alpha_j|^2 \lambda_j - |\alpha_m|^2 \lambda_m + |\alpha_n|^2 \lambda_n| \geq \pi^2 - \epsilon, \\ & \forall k, j, m, n \in R^C, \quad k \neq j \neq m \neq n, \quad k, j, m, n \geq M_\epsilon. \end{aligned}$$

However $\lim_{|u_0| \rightarrow 0} |\alpha_k|^2 = 1$ uniformly in k thanks to (26) and then there exists a neighborhood $W_\epsilon \subseteq D$ such that for each $u \in W_\epsilon$:

$$\begin{aligned} & ||\alpha_k|^2 \lambda_k - |\alpha_j|^2 \lambda_j - |\alpha_m|^2 \lambda_m + |\alpha_n|^2 \lambda_n| \geq \pi^2 - \epsilon, \\ & \forall k, j, m, n \in R^C, \quad k \neq j \neq m \neq n, \quad 1 \leq k, j, m, n < M_\epsilon. \end{aligned}$$

Thus for each $u \in W_\epsilon$ and $k, j, m, n \in R^C$ such that $k \neq j \neq m \neq n$:

$$|\lambda_k^u - \lambda_j^u - \lambda_m^u + \lambda_n^u| \geq \pi^2 - \epsilon.$$

3) Now for every $\epsilon \in (0, \pi^2)$ it follows that $0 \in W_\epsilon$. The proof is achieved thanks to the first point by considering that for $\epsilon_1 > 0$ small enough $\tilde{U}_{\epsilon_1} \cap W_\epsilon$ is non zero measure subset of D . For every $u_0 \in \tilde{U}_{\epsilon_1} \cap W_\epsilon$ and for each $(k, j, m, n) \in (\mathbb{N} \times \{1, \dots, N\})^2$ such that $k \neq j \neq m \neq n$:

$$|\lambda_k^{u_0} - \lambda_j^{u_0} - \lambda_m^{u_0} + \lambda_n^{u_0}| \geq \min\{\pi^2 - \epsilon, \epsilon_1\}. \quad \square$$

Remark 2. Let the decomposition of the pertubated eigenfunction:

$$\phi_k^{u_0}(T) = \alpha_k \phi_k(T) + \eta_k,$$

where α_j is an orthonormalizing constant and η_j is orthogonal to $\phi_j(T)$.

First the finiteness of the eigenvalue resonances proved in the proof of Lemma 17 implies that for M large enough, for every $j, n > M$ and for every $k, m \leq N$:

$$|\lambda_j - \lambda_k - \lambda_n + \lambda_m| \geq \pi^2.$$

Now thanks to (26) for $|u_0|$ small enough there exists $\delta > 0$ such that $\alpha_k \in (\delta, 1]$ for every $k \in \mathbb{N}$. Thus relation (25) implies that for a neighborhood $U(0)$ of $u = 0$ in \mathbb{C} small enough there exists $0 < C < \delta$ such that for every $u_0 \in U(0)$ and for every $j, n > M$ and for every $k, m \leq N$:

$$|\lambda_j^{u_0} - \lambda_k^{u_0} - \lambda_n^{u_0} + \lambda_m^{u_0}| \geq C |\lambda_j - \lambda_k - \lambda_n + \lambda_m|.$$

Remark 3. By using the same techniques of the proofs of Lemma 15 and of Lemma 17, one can ensure the existence of a neighborhood $U(0)$ of 0 in \mathbb{C} such that, for any $u_0 \in U(0)$ we have:

1. $\lambda_k^{u_0} - \lambda_j^{u_0} - \lambda_m^{u_0} + \lambda_n^{u_0} \neq 0$ for all $j, k, n, m \in \mathbb{N}$ and $k \neq j \neq m \neq n$.
2. $B_{j,k}^{u_0} = \langle \psi_j^{u_0}(T), B\phi_k^{u_0}(T) \rangle \neq 0$ for every $j, k \in \mathbb{N}$.
3. The numbers $\{1, \lambda_j^{u_0}\}_{j \in \mathbb{N}}$ are rationally independent, i.e. for any $M \in \mathbb{N}$ and $\{r_j\}_{0 \leq j \leq M} \subset \mathbb{Q} \setminus \{0\}$ it holds $r_0 + \sum_{j=1}^M r_j \lambda_j^{u_0} \neq 0$.

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