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RIPS INDUCTION: INDEX OF THE DUAL LAMINATION OF AN \mathbb{R} -TREE

THIERRY COULBOIS, ARNAUD HILION

ABSTRACT. Let T be a \mathbb{R} -tree in the boundary of the Outer Space CV_N , with dense orbits. The \mathcal{Q} -index of T is defined by means of the dual lamination of T . It is a generalisation of the Poincaré-Lefschetz index of a foliation on a surface. We prove that the \mathcal{Q} -index of T is bounded above by $2N - 2$, and we study the case of equality. The main tool is to develop the Rips Machine in order to deal with systems of isometries on compact \mathbb{R} -trees.

Combining our results on the \mathcal{Q} -index with results on the classical geometric index of a tree, developed by Gaboriau and Levitt [GL95], we obtain a beginning of classification of trees.

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1. INTRODUCTION

The space of minimal, free and discrete actions by isometries of the free group F_N of finite rank $N \geq 2$ on \mathbb{R} -trees has been introduced by Culler and Vogtmann [CV86]. Its projectivization is called Outer Space, and we denote it by CV_N . It has a Thurston-boundary ∂CV_N , which gives rise to a compactification $\overline{CV}_N = CV_N \cup \partial CV_N$. Elements of this compact space \overline{CV}_N are projective classes $[T]$ of minimal, very small actions by isometries of the free group F_N on \mathbb{R} -trees T (see [CL95] and [BF95]). The reader will find a survey on Outer Space in [Vog02].

In this article, we introduce and study the \mathcal{Q} -index $\text{ind}_{\mathcal{Q}}(T)$ of \mathbb{R} -trees T in ∂CV_N with dense orbits. The \mathcal{Q} -index of an \mathbb{R} -tree (see Section 5.2) naturally extends the Poincaré-Lefschetz index of a foliation on a surface as explained below. The main result of our paper regarding this \mathcal{Q} -index is:

Theorem 5.3. *Let T be an \mathbb{R} -tree with a very small, minimal action of F_N by isometries with dense orbits. Then*

$$\text{ind}_{\mathcal{Q}}(T) \leq 2N - 2.$$

We also characterize the case of equality, see Section 4.5.

This \mathcal{Q} -index characterizes dynamical properties of trees. Using it together with the geometric index introduced by Gaboriau and Levitt [GL95] we obtain a classification of trees.

Theorem 5.3 already has several important consequences.

First, it answers a question of Levitt and Lustig [LL03, Remark 3.6] on the finiteness of the fibres of the map \mathcal{Q} (see below).

In our paper [CH10], we obtain a qualitative classification of fully irreducible outer automorphisms of free groups which extends that of Handel and Mosher [HM07] and of Guirardel [Gui05]. The key point is to interpret the index of an iwip automorphism [GJLL98] as the \mathcal{Q} -index of its repelling tree in ∂CV_N .

In our paper with P. Reynolds [CHR11] we define an induction analogous to Rauzy-Veech for trees in ∂CV_N . As we are working with systems of isometries on compact trees, Theorem 5.3 is crucially used to ensure that there are points where to start the splitting procedure.

1.1. Measured foliations on surfaces. Let Σ be a surface of negative Euler characteristic, $\chi(\Sigma) < 0$, with a measured foliation \mathcal{F} (see [FLP79]). The foliation lifts to a measured foliation $\tilde{\mathcal{F}}$ of the universal cover $\tilde{\Sigma}$ of Σ . The space of leaves of $\tilde{\mathcal{F}}$ is an \mathbb{R} -tree T : the distance in the tree T is given by the transverse measure of the foliation $\tilde{\mathcal{F}}$ (see for instance [Kap01, chapter 11]). This tree comes with a small action of the fundamental group of Σ .

When Σ has non empty boundary, its fundamental group is a free group F_N and T defines an element of \overline{CV}_N . The foliation \mathcal{F} has k -prong singularities which give rise to branch points of valence k in the tree. Locally the picture is as in Figure 1. We say that the foliation and the tree are dual to each other.

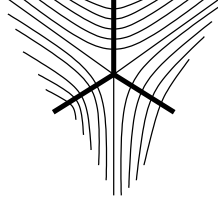


FIGURE 1. 3-prong singularity and the transverse tree (in bold). The local Poincaré-Lefschetz index is $-\frac{1}{2}$ and the local contribution to the \mathcal{Q} -index is 1.

A local index can be defined for each singularity P : the Poincaré-Lefschetz index. In this paper, we rather consider minus two times this index: $\text{ind}(P)$ can be defined as the number of half-leaves reaching the singular point, minus two. Alternatively, $\text{ind}(P)$ is the valence of the corresponding point in the tree, minus two. Adding-up over all singular points in Σ , we obtain the (global) index of the foliation, which turns out to be equal to $-2\chi(\Sigma)$ ($= 2N - 2$ when $\pi_1(\Sigma) = F_N$).

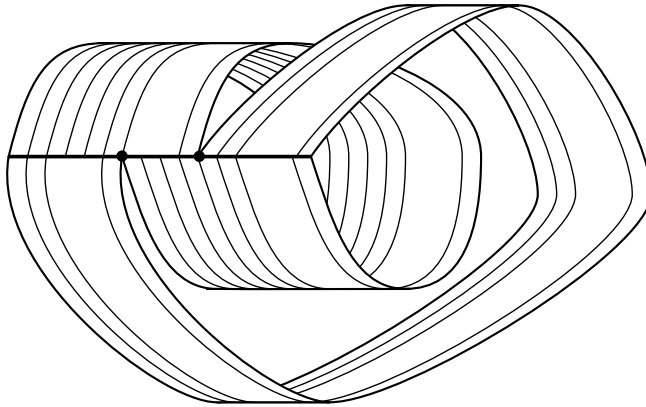


FIGURE 2. Vertical foliation of the mapping torus of an interval exchange transformation. This foliation has two singularities (in bold), each of Poincaré-Lefschetz index $-\frac{1}{2}$ and local \mathcal{Q} -index 1. The \mathcal{Q} -index of the foliation is 2. The surface is a torus with one boundary component: its Euler Characteristic is -1 .

Interval exchange transformations provide examples of such foliated surfaces. Indeed, the mapping torus of an interval exchange transformation is a surface (with boundary), naturally foliated by the vertical direction, as in Figure 2. The transverse measure of the foliation is given by the Lebesgue measure of the interval. We define, in this case, the index of the interval exchange transformation as the index defined above for this foliation and its dual \mathbb{R} -tree.

1.2. Geometric trees. This surface situation has been generalized (see for instance [Bes02]). Let us consider a finite family of partial isometries of an interval (or a finite number of intervals, or even a finite tree). The suspension of these partial isometries gives rise to a 2-complex B (which is not a surface in general), naturally foliated by the vertical direction. As previously, the foliation can be lifted to the universal cover of B , and the space of leaves of this foliation is an \mathbb{R} -tree with an action of $\pi_1(B)$ by isometries. A tree obtained by such a construction is called a geometric tree. In this situation, we can define two local indices: one for the tree, using the valence of branch points, and one for the foliation, using the number of ends of singular leaves. We would like to stress that, contrary to the case of a foliation on a surface, these two indices do not agree, not even locally.

The first index has been introduced by Gaboriau and Levitt [GL95]. In this paper we call it the geometric index, and denote it by $\text{ind}_{\text{geo}}(T)$. It is defined using the valence of the branch points, of the \mathbb{R} -tree T , with an action of the free group by isometries:

$$\text{ind}_{\text{geo}}(T) = \sum_{[P] \in T/F_N} \text{ind}_{\text{geo}}(P).$$

where the local index of a point P in T is

$$\text{ind}_{\text{geo}}(P) = \#(\pi_0(T \setminus \{P\})/\text{Stab}(P)) + 2 \text{rank}(\text{Stab}(P)) - 2.$$

Gaboriau and Levitt [GL95] proved that the geometric index of a geometric tree is equal to $2N - 2$ and that for any tree in the compactification of Outer Space $\overline{\text{CV}}_N$ the geometric index is bounded above by $2N - 2$. Moreover, they proved that the trees in $\overline{\text{CV}}_N$ with geometric index equal to $2N - 2$ are precisely the geometric trees.

The second index is defined from the number of ends of singular leaves. To our knowledge it has never been studied in its own right, although Gaboriau [Gab96] gives a lot of relevant insights. In particular Gaboriau [Gab96, Theorem VI.1] gives partial results to bound this index.

1.3. The \mathcal{Q} -index of an \mathbb{R} -tree. Let T be an \mathbb{R} -tree in the boundary of Outer Space with dense orbits. We denote by \overline{T} its metric completion, ∂T its Gromov-boundary at infinity. The set $\widehat{T} = \overline{T} \cup \partial T$ equipped with the observers' topology (a slight weakening of the metric topology, see [CHL07]) is a compact set denoted \widehat{T}^{obs} .

Let P be a point in T . The map $\mathcal{Q} : \partial F_N \rightarrow \widehat{T}^{\text{obs}}$ is the unique continuous extension (see [CHL07]) of the map

$$\begin{aligned} F_N &\rightarrow T \\ u &\mapsto u \cdot P. \end{aligned}$$

It does not depend on the choice of the point P .

The map \mathcal{Q} can be easily understood in the special case of a tree T dual to a foliation on a 2-complex B . Each leaf of the foliation of \widetilde{B} is a point of the dual tree T . A half-leaf of the foliation of \widetilde{B} defines a point $X \in \partial F_N = \partial \pi_1(B)$ and $\mathcal{Q}(X)$ is the point of T defined by the leaf.

The general definition of the \mathcal{Q} -index of the tree T is given as follows:

$$\text{ind}_{\mathcal{Q}}(T) = \sum_{[P] \in \widehat{T}/F_N} \max(0; \text{ind}_{\mathcal{Q}}(P)).$$

where the local index of a point P in T is:

$$\text{ind}_{\mathcal{Q}}(P) = \#(\mathcal{Q}^{-1}(P)/\text{Stab}(P)) + 2 \text{rank}(\text{Stab}(P)) - 2.$$

Levitt and Lustig [LL03] proved that points in ∂T have exactly one pre-image by \mathcal{Q} (see Proposition 5.2). Thus, only points in \bar{T} contribute to the \mathcal{Q} -index of T .

Our main result states that the \mathcal{Q} -index of an \mathbb{R} -tree in the boundary of Outer Space is bounded above by $2N - 2$. This answers the question of Levitt and Lustig [LL03, Remark 3.6] whether the map $\mathcal{Q} : \partial F_N \rightarrow \widehat{T}$ has finite fibers (in the case where the action is free).

In [CHL08] the dual lamination of T is defined: it is the set of pairs (X, Y) of distinct points in the boundary ∂F_N such that $\mathcal{Q}(X) = \mathcal{Q}(Y)$. More conceptually, the \mathcal{Q} -index of T can indeed be understood in a more general context as that of its dual lamination.

The limit set Ω is the subset of \bar{T} which consists of points with at least two pre-images by the map \mathcal{Q} .

We also describe the trees such that $\text{ind}_{\mathcal{Q}}(T) = 2N - 2$: these are the trees such that all points of T have at least two pre-images by \mathcal{Q} .

Theorem 5.10. *Let T be an \mathbb{R} -tree in the boundary of Outer Space with dense orbits. The \mathcal{Q} -index is maximal: $\text{ind}_{\mathcal{Q}}(T) = 2N - 2$ if and only if T is contained in the limit set Ω .*

An \mathbb{R} -tree dual to a foliation on a surface with boundary of negative Euler characteristic, has maximal \mathcal{Q} -index. We call trees with maximal \mathcal{Q} -index trees of surface type.

1.4. Compact systems of isometries. A traditional strategy to study a tree in the boundary of Outer Space is

- (1) describe any geometric tree by a system of isometries on a finite tree (or even a multi-interval) and then use the Rips Machine;
- (2) approximate any tree by a sequence of geometric trees.

In particular Gaboriau and Levitt [GL95] proved in this way that the geometric index of any tree in Outer Space is bounded above by $2N - 2$.

In [CHL09] a new approach was proposed: to describe an \mathbb{R} -tree by a system of isometries on a compact \mathbb{R} -tree (rather than on a finite tree). The point here is that any tree T in the compactification of Outer Space can be described by a system of isometries on a compact \mathbb{R} -tree: $S_A = (K_A, A)$ (where A is a basis of F_N and K_A is a compact subtree of \bar{T}). This system of isometries encodes all of the original tree T (together with the action of F_N). An index is defined in Section 2.3 for any such system of isometries.

Theorem 5.7. *Let T be an \mathbb{R} -tree with a very small, minimal action of F_N by isometries with dense orbits. The \mathcal{Q} -index of T and the index of the induced system of isometries $S_A = (K_A, A)$, for any basis A , are equal:*

$$\text{ind}_{\mathcal{Q}}(T) = i(S_A).$$

The computation of the index of a tree is thus achieved by computing the index of a system of isometries. We study system of isometries by themselves in Sections 2, 3 and 4.

We improve the classical Rips Machine (see [GLP94, BF95]) to work in the context of systems of isometries on compact \mathbb{R} -trees (or forests). The Rips Machine applied to a system of isometries returns a new system of isometries obtained by erasing parts of the supporting forest. To each system of isometries we associate a finite graph Γ , the index of which is given

by the Euler characteristic. We study the effect of the Rips Machine on this graph: the Rips Machine decreases the index of the graph Γ .

Iterating the Rips Machine infinitely many times, the sequence of associated graphs Γ has a limit $\widehat{\Gamma}$. The index of $\widehat{\Gamma}$ is bounded above by the decreasing sequence of indices. We prove that the index of the limit graph $\widehat{\Gamma}$ is equal to the index of the system of isometries. In fact, in the case of a Levitt system of isometries the graph $\widehat{\Gamma}$ can be viewed as a geometric realization of the dual lamination of the system of isometries. We obtain

Theorem 2.2. *The index of a system of isometries $S = (F, A)$ with independent generators is finite and bounded above by the index of the associated graph Γ .*

The above Theorem 5.10 follows from our characterization of systems of isometries with maximal index:

Theorem 4.8. *Let $S = (F, A)$ be a reduced system of isometries with independent generators, let Γ be its associated graph, and $\widehat{\Gamma}$ be its limit graph. The following are equivalent*

- (1) *The system of isometries S has maximal index,*
- (2) *The graph $\widehat{\Gamma}$ is finite,*
- (3) *The Rips Machine, starting from S , halts after finitely many steps.*

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We are grateful to Vincent Guirardel and Gilbert Levitt for introducing us to mixing properties of trees.

2. SYSTEMS OF ISOMETRIES

2.1. Definitions. We collect in this Section basic facts from [CHL09].

An **\mathbb{R} -tree**, (T, d) is a metric space such that for any two points P, Q in T , there exists a unique arc $[P; Q]$ between them and this arc is isometric to the segment $[0; d(P, Q)]$.

A **compact forest** F is a metric space with finitely many connected components each of which is a compact \mathbb{R} -tree.

A **partial isometry** of a compact forest F is an isometry $a : K \rightarrow K'$, between two compact subtrees K and K' of F . The **domain** of a is K , its **range** is K' . The partial isometry a is **non-empty** if its domain is non-empty. The domain (and the range) of a partial isometry needs not be a whole connected component of F . A **system of isometries** $S = (F, A)$ consists of a compact forest F and of a finite set A of non-empty partial isometries of F .

To such a system of isometries S we associate the oriented graph Γ which has the connected components of F as vertices and the non-empty partial isometries of A as oriented edges. The edge $a \in A$ starts at the connected component of F which contains its domain, and ends at the connected component of F which contains its range.

We regard the reverse edge a^{-1} of the edge $a \in A$ as the inverse partial isometry a^{-1} of a . A reduced path w in the graph Γ , given as a sequence of edges $w = z_1 \cdots z_n$ with $z_i \in A^{\pm 1}$ (such that $z_{i+1} \neq z_i^{-1}$), defines a (possibly empty) partial isometry, also denoted by w : the composition of partial isometries $z_1 \circ z_2 \circ \cdots \circ z_n$. We write this **pseudo-action** on F on the right, i.e.

$$P(u \circ v) = (Pu)v$$

for all point $P \in F$ and for all path w in Γ .

The **pseudo-orbit** of a point P in F is the subset of F which can be reached from P :

$$\{P.w \mid w \text{ reduced path in } \Gamma, P \in \text{dom}(w)\}.$$

A reduced path w in Γ is **admissible** if it is non-empty as a partial isometry of F .

An infinite reduced path X in Γ is **admissible** if all its subpaths are admissible. The domains of the initial subpaths of X are nested compact subtrees, their intersection is the **domain** of X , denoted by $\text{dom}(X)$.

A bi-infinite reduced path Z in Γ is **admissible** if all its subpaths are admissible. A bi-infinite reduced path, $Z = \cdots z_{-2}z_{-1}z_0z_1z_2 \cdots$, has two halves which are infinite reduced paths:

$$Z^+ = z_1z_2 \cdots, \quad Z^- = z_0^{-1}z_{-1}^{-1}z_{-2}^{-1} \cdots.$$

The **domain** of Z is the intersection of the domains of its two halves. Equivalently, a bi-infinite reduced path Z in Γ is admissible if and only if its domain is non-empty.

The **limit set** Ω of a system of isometries $S = (F, A)$ is the set of elements of K which are in the domain of a bi-infinite admissible reduced path in Γ . The limit set is the place where the dynamics of the system of isometries concentrates. Alternatively, Ω is the largest subset of F such that for each $P \in \Omega$ there exists at least two partial isometries $a, b \in A^{\pm 1}$ with $P.a$ and $P.b$ in Ω .

A system of isometries $S = (F, A)$ has **independent generators** (compare Gaboriau [Gab97] and [CHL09]) if the domain of any admissible infinite reduced path X in Γ consists of exactly one point which we denote by $\mathcal{Q}(X)$:

$$\text{dom}(X) = \{\mathcal{Q}(X)\}.$$

In this case, the domain of a bi-infinite admissible reduced path Z in Γ also consists of exactly one point which we also denote by $\mathcal{Q}(Z)$.

If S has independent generators we have:

$$\Omega = \{P \in K \mid \exists Z \text{ bi-infinite admissible, } \mathcal{Q}(Z) = P\}.$$

The **restriction** of a partial isometry a to the compact \mathbb{R} -tree K (at the source) is the (possibly empty) partial isometry $_{K\upharpoonright}a$ which is defined for each $P \in K \cap \text{dom}(a)$. The restriction of a to the compact \mathbb{R} -tree K' (at the destination) is the (possibly empty) partial isometry $a_{\upharpoonright K'}$ which is defined for each $P \in \text{dom}(a)$ such that $Pa \in K'$. The restriction of the partial isometry to the compact \mathbb{R} -trees K and K' is the (possibly empty) partial isometry $_{K\upharpoonright}a_{\upharpoonright K'}$ which is defined for each $P \in K \cap \text{dom}(a)$ such that $P.a$ is in K' .

2.2. Index of a Graph. We denote by $V(\Gamma)$ the set of vertices of a graph Γ and by $E(\Gamma)$ its set of edges.

For a vertex x of a graph Γ the **valence** $v_\Gamma(x)$ of x is the number of edges incident to x . The **index** $i_\Gamma(x) = v_\Gamma(x) - 2$ of x is its valence minus two.

The **index** $i(\Gamma)$ of a finite connected graph Γ is:

$$\begin{aligned} i(\Gamma) &= \max(0; \sum_{x \in V(\Gamma)} i_\Gamma(x)) \\ &= \max(0; 2(\#E(\Gamma) - \#V(\Gamma))) \\ &= \max(0; -2\chi(\Gamma)) \\ &= \max(0; 2 \text{rank}(\pi_1(\Gamma)) - 2) \end{aligned}$$

where $\chi(\Gamma)$ is the Euler characteristic of Γ . The index $i(\Gamma)$ is a homotopy invariant of the graph Γ .

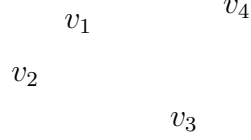


FIGURE 3. A graph of index 2, with vertices of indices $i(v_1) = 2$, $i(v_2) = 0$, $i(v_3) = 1$ and $i(v_4) = -1$.

The **index** of a finite graph Γ is the sum of the indices of its connected components.

The **core** of a graph Γ is the largest subgraph of Γ without vertices of valence 0 and 1 (see [Ger83]), we denote it by $\text{core}(\Gamma)$. The core of a graph is the union of all bi-infinite reduced paths. Alternatively, if Γ is finite, the subgraph $\text{core}(\Gamma)$ is obtained by recursively erasing vertices of valence 0 and the vertices of valence 1 together with their incident edges. The core of a graph may be empty: this is the case, for instance, when the graph is a union finite of trees.

The index of a finite graph can be computed using its core with the easier formula:

$$i(\Gamma) = i(\text{core}(\Gamma)) = \sum_{x \in V(\text{core}(\Gamma))} i_{\text{core}(\Gamma)}(x).$$

We use this formula to define the index of an infinite graph Γ : The core does not have vertices of valence 0 and 1 and thus we can compute the above (possibly infinite) non-negative sum.

For a connected (possibly infinite) graph Γ the index is thus

$$\begin{aligned} i(\Gamma) &= i(\text{core}(\Gamma)) \\ &= \max(0; \#\partial\Gamma + 2\text{rank}(\pi_1(\Gamma)) - 2) \end{aligned}$$

where $\partial\Gamma$ is the set of ends of Γ . If Γ is not connected we sum the above value for each of its connected components.

A **morphism of graphs** $\tau : \Gamma' \rightarrow \Gamma$ maps vertices to vertices, edges to edges and respects incidence.

We will need the following Lemma in our proofs:

Lemma 2.1. *Let $\tau : \Gamma' \rightarrow \Gamma$ be a morphism between two finite graphs. Assume that τ is injective on edges. Then the index of Γ' is smaller or equal to the index of Γ :*

$$i(\Gamma') \leq i(\Gamma).$$

Proof. For each vertex x of Γ , the set of edges incident to the vertices in the fiber $\tau^{-1}(x)$ injects in the set of edges incident to x . Thus

$$\sum_{x' \in \tau^{-1}(x)} i_{\Gamma'}(x') \leq i_{\Gamma}(x).$$

Moreover, τ maps the core of Γ' inside the core of Γ .

We get

$$i(\Gamma') = i(\text{core}(\Gamma')) = \sum_{x \in V(\text{core}(\Gamma'))} i_{\text{core}(\Gamma')}(x) \leq \sum_{x \in V(\text{core}(\Gamma))} i_{\text{core}(\Gamma)}(x) = i(\text{core}(\Gamma)) = i(\Gamma).$$

□

In particular if Γ' is a subgraph of a finite graph Γ

$$i(\Gamma') \leq i(\Gamma).$$

2.3. Index of a system of isometries. Let F be a compact forest and $S = (F, A)$ be a system of isometries. Let Ω be the limit set of S .

For a point P in F , we define its **index** by

$$i_S(P) = \#\{a \in A^{\pm 1} \mid P.a \in \Omega\} - 2.$$

By definition of the limit set, for any point P in Ω , there exists a bi-infinite reduced admissible path $Z = \cdots z_{-1}z_0z_1 \cdots$ in Γ such that $P \in \text{dom}(Z)$. The edges z_1 and z_0^{-1} send P inside Ω , and thus the index of P is greater or equal to 0: $i_S(P) \geq 0$.

Conversely, if the index of a point P in F is non-negative: $i_S(P) \geq 0$, then there exists two elements $a, b \in A^{\pm 1}$, such that $P.a$ and $P.b$ are in the limit set Ω . As $P.a$ is in Ω there exists a bi-infinite reduced admissible path $Z = \cdots z_{-1}z_0z_1 \cdots$ in Γ such that $P.a \in \text{dom}(Z)$. Up to replacing Z by Z^{-1} (the same bi-infinite path with reversed orientation), we assume that $z_1 \neq a^{-1}$. Symmetrically there exists a bi-infinite reduced admissible path $Z' = \cdots z'_{-1}z'_0z'_1 \cdots$ in Γ such that $P.b \in \text{dom}(Z')$ and $z'_0 \neq b^{-1}$. We get that $Z'' = \cdots z'_{-1}z'_0b \cdot az_1z_2 \cdots$ is a bi-infinite reduced path in Γ with $P \in \text{dom}(Z'')$ and thus that P is in the limit set Ω :

$$P \in \Omega \iff i_S(P) \geq 0.$$

The **index** of S is defined by

$$i(S) = \sum_{P \in F} \max(0; i_S(P)) = \sum_{P \in \Omega} i_S(P).$$

As there is a max in the first sum, and by the above equivalence, both sums are non-negative and are well defined possibly as $+\infty$.

The main result of this paper can now be stated

Theorem 2.2. *The index of a system of isometries $S = (F, A)$ with independent generators is finite and bounded above by the index of the associated graph Γ .*

2.4. Cayley graphs. Let $S = (F, A)$ be a system of isometries and let Γ be its associated graph. Let P be a point in F and K be the connected component of F which contains P . Let Γ_0 be the connected component of Γ which contains P and $\tilde{\Gamma}_0$ its universal cover.

The **trajectory tree** of P is the smallest subtree $\tilde{\Gamma}(P) = \tilde{\Gamma}(P, S)$ of $\tilde{\Gamma}_0$ which contains all the admissible paths w based at K such that P is in the domain of w .

Let $\text{Stab}(P)$ be the subgroup of the fundamental group $\pi_1(\Gamma, K)$ of the graph Γ based at K of admissible paths w such that $P.w = P$. The group $\text{Stab}(P)$ is a free group that acts on the tree $\tilde{\Gamma}(P)$.

The **Cayley graph** $\Gamma(P) = \Gamma(P, S)$ of P is the quotient of $\tilde{\Gamma}(P)$ by the action of the stabilizer $\text{Stab}(P)$ (compare Gaboriau [Gab96]). Vertices of the Cayley graph of P are in one-to-one correspondence with the elements of the pseudo-orbit of P in F under the

pseudo-group of isometries. The vertices of the core of the Cayley graph of P are in one-to-one correspondence with the intersection $\omega(P)$ of the pseudo-orbit of P and the limit set Ω .

The index of a vertex P' in $\text{core}(\Gamma(P))$ is equal to the index $i_S(P')$ of the point P' for the system of isometries S . Thus, we get that the index of the core of the Cayley graph $i(\Gamma(P))$ is equal to the contribution of the pseudo-orbit of P to the index of S :

$$i(\Gamma(P)) = i(\text{core}(\Gamma(P))) = \sum_{P' \in \omega(P)} i_S(P').$$

Adding up, for all pseudo-orbits $[P]$ we get

$$i(S) = \sum_{[P]} i(\Gamma(P)).$$

3. RIPS MACHINE

3.1. Elementary step. Let $S = (F, A)$ be a system of isometries on a compact forest F . One step of the Rips Machine produces a new system of isometries $S' = (F', A')$ defined as follows.

The forest F' is the set of all elements of F which are in the domains of at least two distinct partial isometries in $A^{\pm 1}$:

$$F' = \{P \in F \mid \exists a \neq b \in A^{\pm 1}, P \in \text{dom}(a) \cap \text{dom}(b)\}.$$

The set F' has finitely many connected components which are compact \mathbb{R} -trees because it is the finite union of all possible intersections $\text{dom}(a) \cap \text{dom}(b)$ for all pairs of distinct elements a, b of $A^{\pm 1}$.

For each partial isometry $a \in A$ and for each pair of connected components K_0, K_1 of F' , we consider the partial isometry, $a' = a|_{K_0 \cup K_1}$, which is the restriction of a to K_0 and K_1 . The finite set A' consists of all such non-empty partial isometries $a' = a|_{K_0 \cup K_1}$ of F' .

An elementary step of the Rips Machine gives rise to a map τ from the graph Γ' associated to the resulting system of isometries S' , to the original graph Γ . A vertex K' of Γ' is a connected component of F' and it is mapped by τ to the connected component $\tau(K')$ of F which contains K' . Similarly an edge a' of Γ' is a non-empty partial isometry $a' = a|_{K_0 \cup K_1}$ and it is mapped by τ to the original partial isometry a . The map τ is a morphism of oriented graphs.

If w is an admissible reduced path in Γ' , the domain of $\tau(w)$ contains the domain of w and $\tau(w)$ is an admissible path in Γ . Moreover, the image $\tau(w)$ of an admissible reduced path w in Γ' is a reduced path of Γ . Finally, if a bi-infinite reduced path Z in Γ is admissible, then its domain is contained in F' , which leads to the following

Proposition 3.1. *Let S be a system of isometries and S' be the result of the Rips Machine. Then the limit sets and the indices of S and S' are equal:*

$$\Omega = \Omega' \text{ and } i(S) = i(S'). \quad \square$$

3.2. Indices through the Rips Machine. As explained previously, the Rips Machine defines a new system of isometries $S' = (F', A')$ starting from a system of isometries $S = (F, A)$ by erasing the subset E of the forest F which consists of points which belongs to at most one domain of partial isometries of $A^{\pm 1}$.

To better understand the Rips Machine we decompose its elementary step into a finite sequence of elementary moves. Instead of erasing E in one step we successively erase subsets E_i of E . This gives us a finite sequence of system of isometries starting from S and ending at S' . The successive systems of isometries of this sequence differ by an elementary move.

As F is a compact forest, the set E may have infinitely many connected components. We first describe a preliminary move which erases all the “peripheral” ones. Then we are left with a finite forest to erase, which we erase in finitely many elementary moves.

This decomposition of the Rips Machine is used in the next Proposition to prove that the index of the associated graphs is decreasing.

Proposition 3.2. *Let $S = (F, A)$ be a system of isometries and $S' = (F', A')$ be the output of the Rips Machine. Let Γ and Γ' be the associated graphs.*

Then the index $i(\Gamma')$ is smaller or equal than the index $i(\Gamma)$.

Proof. Let E be the part of the forest F which is erased by the Rips Machine:

$$E = F \setminus F' = \{P \in F \mid \#\{a \in A^{\pm 1} \mid P \in \text{dom}(a)\} \leq 1\}.$$

Let E_C be the subset of E which is in the convex hull of F' :

$$E_C = \{P \in E \mid \exists Q, R \in F', P \in [Q, R]\},$$

and let $E_0 = E \setminus E_C$ be the complement of E_C in E .

Let $F_0 = F \setminus E_0 = F' \cup E_C$: F_0 is the convex hull of the connected components of F' in F :

$$F_0 = \{P \in F \mid \exists Q, R \in F', P \in [Q, R]\}.$$

Thus F_0 has finitely many connected components, each of which is a compact \mathbb{R} -tree: F_0 is a compact forest.

Let A_0 be the set of all non-empty restrictions of partial isometries of A to partial isometries of F_0 . Let Γ_0 be the graph associated to the system of isometries $S_0 = (F_0, A_0)$ and let $\tau_0 : \Gamma_0 \rightarrow \Gamma$ be the graph morphism defined as before. As E_0 does not split connected components of F , the map τ_0 is injective and thus by Lemma 2.1,

$$i(\Gamma_0) \leq i(\Gamma)$$

(we note that this inequality can be strict if τ_0 is not onto).

As F' has finitely many connected components, E_C is a finite union of finite open arcs of the form $]P; Q[$ where P and Q are two points of F' .

Lemma 3.3. *The erased part E_C can be decomposed to get a partition*

$$E_C = \alpha_1 \uplus \alpha_2 \uplus \cdots \uplus \alpha_n$$

where each α_i is an open arc such that the number of connected components of F_i is exactly one plus the number connected components of F_{i-1} , where for each $i = 1, \dots, n$, we let $F_i = F_{i-1} \setminus \alpha_i$.

Proof. We recursively define α_i by choosing a connected component C of F_{i-1} which contains at least two connected components of F' . Then we choose a connected component K of F' contained in C and which is not contained in the convex hull C' of $(C \cap F') \setminus K$ in C . We choose the open arc α_i that joins K and C' . Removing α_i from F_{i-1} splits the connected component C of F_{i-1} into two new connected components: K and C' . \square

Let A_i be the set of non-empty restrictions of partial isometries in A to F_i and let Γ_i be the graph associated to the system of isometries $S_i = (F_i, A_i)$. As before we get graph morphisms $\tau_i : \Gamma_i \rightarrow \Gamma_{i-1}$. We observe that the last system of isometries is the output of the Rips Machine: $S_n = S'$. The map τ factors through the graphs Γ_i : $\tau = \tau_0 \circ \tau_1 \circ \cdots \circ \tau_n$.

We now proceed to prove that for each $i = 1, \dots, n$, the index of Γ_i is lower or equal to the index of Γ_{i-1} . This will conclude the proof.

Going from S_{i-1} to S_i corresponds to one of the following **elementary moves** of the Rips Machine. For each i removing the arc α_i from F_{i-1} has one of the following two effects on the graph Γ_{i-1} :

- (1) Split a vertex: α_i joins two connected components, K and K' , of F_i , and no partial isometry in $A_{i-1}^{\pm 1}$ is defined simultaneously on K and K' . Then the map τ_i is injective on edges and, applying Lemma 2.1, the index of Γ_i is smaller or equal to the index of Γ_{i-1} .

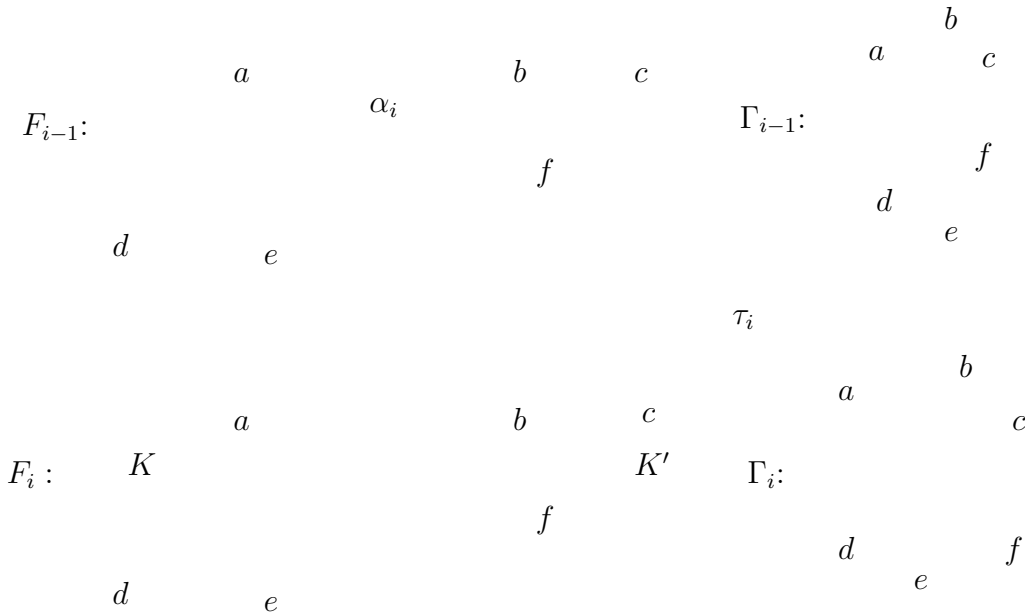


FIGURE 4. Split a vertex move

- (2) Split an edge: α_i joins two connected components K and K' of F_i , and a partial isometry a in $A_{i-1}^{\pm 1}$ is defined on K and K' (and thus its domain $\text{dom}(a)$ contains α_i). By definition of E no other partial isometry is defined on α_i , in particular the range of a is contained in F_i . The graph morphism τ_i maps the two distinct vertices K and K' to the same vertex, $K \cup \alpha_i \cup K'$, of Γ_{i-1} , it maps the two edges $a' = K \upharpoonright a$ and $a'' = K' \upharpoonright a$ which are the restrictions of a to K and K' to the same edge a of Γ_{i-1} . On all other vertices and edges, τ_i is one-to-one. Thus, τ_i is a homotopy equivalence and the index of Γ_i is equal to the index of Γ_{i-1} . Indeed, τ_i is a folding in Stallings' terminology [Sta83].

□

3.3. Iterating the Rips Machine. Let $S_0 = (F_0, A_0)$ be a system of isometries. By repeatedly applying the Rips Machine we get a sequence $S_n = (F_n, A_n)$ of systems of isometries.

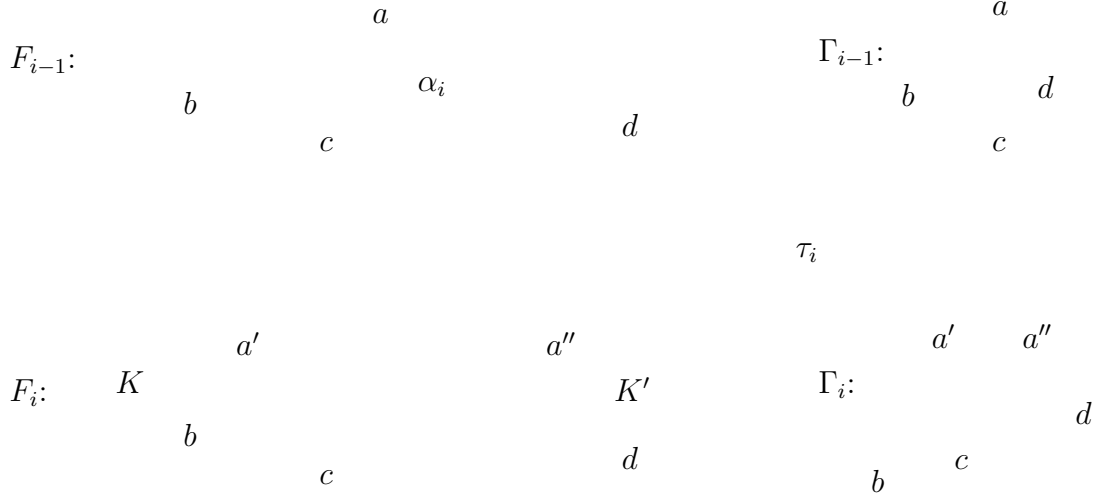


FIGURE 5. Split an edge move

Of course the Rips Machine may halt after some time, that is to say, we do not exclude that $S_{n+1} = S_n$ for n big enough.

We also get graph morphisms τ_n from the graph Γ_{n+1} associated to S_{n+1} to the graph Γ_n associated to S_n . Indeed, a partial isometries $a_n \in A_n$ is the restriction of the partial isometry $a = \tau_0 \circ \tau_1 \cdots \tau_{n-1}(a_n)$ in A_0 to connected components of the compact forest F_n .

Lemma 3.4. *The intersection Ω of the nested sequence $(F_n)_{n \in \mathbb{N}}$ of compact subsets of F_0 is equal to the limit set Ω_0 of the system of isometries S_0 .*

Proof. By Proposition 3.1, at each step n the limit sets Ω_{n+1} and Ω_n of the corresponding system of isometries are equal. In particular Ω_0 is contained in F_n at each step n and thus in the nested intersection.

Conversely, let P be a point in the nested intersection $\Omega = \bigcap_{n \in \mathbb{N}} F_n$. For any $n \in \mathbb{N}$, P belongs to F_{n+1} and by definition of the Rips Machine, there exists at least two distinct partial isometries a_n and b_n defined at P in $A_n^{\pm 1}$. Thus $P.a_n$ and $P.b_n$ are in F_n . Up to passing to a subsequence we can assume that a_n and b_n are the restrictions of two fixed and distinct partial isometries a and b in $A_0^{\pm 1}$. We get that for all n , $P.a$ and $P.b$ are in F_n , which proves that $P.a$ and $P.b$ are also in the nested intersection $\Omega = \bigcap_{n \in \mathbb{N}} F_n$. The set Ω is a subset of F_0 such that for any point P in Ω , there exists at least two distinct partial isometries a and b in $A_0^{\pm 1}$ such that $P.a$ and $P.b$ are also in Ω . This proves that Ω is contained in the limit set Ω_0 of the initial system of isometries S_0 . \square

The **limit graph** $\widehat{\Gamma}$ of the system of isometries $S_0 = (F_0, A_0)$ is the (possibly infinite) graph whose vertices are the connected components of the limit set Ω and whose edges are all possible restrictions of partial isometries in A_0 to connected components of Ω . We denote by $\widehat{\tau}_n : \widehat{\Gamma} \rightarrow \Gamma_n$ the graph morphism that maps a connected component C of Ω to the

connected component of F_n that contains C , and which maps an edge e of $\widehat{\Gamma}$ to the partial isometry $a_n \in A_n$ of which it is a restriction.

From the previous Lemma and from the definition of an inverse limit, we deduce:

Lemma 3.5. *The limit graph $\widehat{\Gamma}$ of a system of isometries, $S_0 = (F_0, A_0)$, is the inverse limit of the sequence of graphs $(\Gamma_n)_{n \in \mathbb{N}}$ (together with the sequence of maps $(\tau_n)_{n \in \mathbb{N}}$) associated to the sequence of systems of isometries $(S_n)_{n \in \mathbb{N}}$ obtained from S_0 by iterating the Rips Machine. \square*

By definition of Ω , the graph $\widehat{\Gamma}$ does not have vertices of valence 0 or 1, and we defined its **index** as the non-negative sum

$$i(\widehat{\Gamma}) = \sum_{x \in V(\widehat{\Gamma})} i_{\widehat{\Gamma}}(x).$$

Recall that, by Proposition 3.2, the sequence of indices $(i(\Gamma_n))_{n \in \mathbb{N}}$ is decreasing.

Proposition 3.6. *Let $\widehat{\Gamma}$ be the limit graph of a system of isometries $S_0 = (F_0, A_0)$. Then the index of $\widehat{\Gamma}$ is smaller or equal to the index of Γ_0 :*

$$i(\widehat{\Gamma}) \leq i(\Gamma_0).$$

Proof. For any point P in Ω , by definition, there exist at least two distinct partial isometries a and b in $A_0^{\pm 1}$ defined at P and such that $P.a$ and $P.b$ also lie in Ω . For any $n \in \mathbb{N}$, Ω is contained in F_n : let C_n be the connected component of P in F_n . There are at least two edges going out of the vertex C_n of Γ_n corresponding to the restrictions of a and b to C_n . This proves that the image of $\widehat{\Gamma}$ by $\widehat{\tau}_n$ in Γ_n does not contain vertices of valence 0 or 1: $\widehat{\tau}_n(\widehat{\Gamma})$ is a subgraph of the core of Γ_n .

Let Θ_0 be a finite set of vertices of $\widehat{\Gamma}$. Let Θ be a finite subgraph of $\widehat{\Gamma}$ that contains Θ_0 and all edges incident to elements of Θ_0 . The graph Θ exists because vertices of $\widehat{\Gamma}$ have finite valence bounded above by twice the cardinality of A_0 .

By Lemma 3.5, there exists $n \in \mathbb{N}$ such that Θ is mapped injectively by $\widehat{\tau}_n$ into Γ_n . Arguing as in Lemma 2.1 and using the definition of the index of a graph, the following inequalities hold:

$$\sum_{x \in \Theta_0} i_{\widehat{\Gamma}}(x) = \sum_{x \in \Theta_0} i_{\Theta}(x) \leq \sum_{x \in \Theta_0} i_{\text{core}(\Gamma_n)}(\widehat{\tau}_n(x)) \leq i(\text{core}(\Gamma_n)) = i(\Gamma_n).$$

Using Proposition 3.2 we get that for any finite subset Θ_0 of vertices of $\widehat{\Gamma}$

$$\sum_{x \in \Theta_0} i_{\widehat{\Gamma}}(x) \leq i(\Gamma_0).$$

Taking Θ_0 arbitrarily large we finally get

$$i(\widehat{\Gamma}) \leq i(\Gamma_0). \quad \square$$

A connected component K of Ω is called **regular** if it corresponds to a vertex of valence 2 of the limit graph $\widehat{\Gamma}$. A connected component K of Ω is **singular** if it corresponds to a vertex of valence at least 3 of $\widehat{\Gamma}$.

Corollary 3.7. *All connected components of $\widehat{\Gamma}$ are lines except at most $i(\Gamma_0)$. Moreover, there are at most $i(\Gamma_0)$ singular connected components of Ω . \square*

We are now ready to prove Theorem 2.2 in the special case where the Rips Machine never halts and digs holes everywhere.

Theorem 3.8. *Let $S = (F, A)$ be system of isometries and Γ be its associated graph. Assume that the limit set Ω is totally disconnected. Then the index of S is bounded above by the index of Γ .*

Proof. As connected components of Ω are reduced to single points, the graph $\widehat{\Gamma}$ is the disjoint union of all the cores of the Cayley graphs $\Gamma(P)$:

$$\widehat{\Gamma} = \bigsqcup_{[P]} \text{core}(\Gamma(P)).$$

Thus, the index of $\widehat{\Gamma}$ is equal to the index of S . The Theorem now follows from Proposition 3.6. \square

We turn back to the general case where the limit set has non-trivial (i.e. not reduced to a single point) connected components.

Proposition 3.9. *The limit set Ω of a system of isometries $S_0 = (F_0, A_0)$ has finitely many non-trivial connected components.*

Proof. By Corollary 3.7, there are finitely many singular connected components of Ω .

Let K be a regular connected component of Ω . Then there exists exactly two distinct partial isometries $a \neq b$ in $A_0^{\pm 1}$ with non trivial restrictions $a' = {}_K \lrcorner a|_{\Omega}$ and $b' = {}_K \lrcorner b|_{\Omega}$ to K and Ω . These are the partial isometries which give rise to the two edges of $\widehat{\Gamma}$ going out of the vertex K . By definition of Ω , for each point P of K there exists at least two partial isometries in $A_0^{\pm 1}$ which map P inside Ω . Thus P lies in both the domains of a' and b' and thus $\text{dom}(a') = \text{dom}(b') = K$.

Now, if the range $K.a'$ of a' is also a regular connected component of Ω then a' is an isometry between K and $K.a'$ and in particular they have the same diameter. From the next Lemma we get that there can only be finitely many non-trivial regular connected components.

This proves that Ω only has finitely many non-trivial connected components. \square

Lemma 3.10. *Let K be a compact \mathbb{R} -tree and $(K_i)_{i \in \mathbb{N}}$ be a collection of disjoint subtrees of K . Then*

$$\lim_{i \rightarrow \infty} \text{diam}(K_i) = 0.$$

Proof. By contradiction, assume that there exists $\varepsilon > 0$ such that (extracting a subsequence) $\forall i, \text{diam}(K_i) > \varepsilon$. Let P_i, Q_i be in K_i with $d(P_i, Q_i) \geq \varepsilon$. By compactness of K , up to passing to a subsequence, the sequences P_i and Q_i converge to P and Q respectively in K and $d(P, Q) \geq \varepsilon$. For i big enough $d(P_i, P) < \varepsilon/2$ and $d(Q_i, Q) < \varepsilon/2$, from which we deduce that the segment $[P_i, Q_i]$ contains the midpoint of $[P, Q]$, and therefore the segments $[P_i, Q_i]$ and $[P_j, Q_j]$ are not disjoint for i and j big enough. A contradiction. \square

3.4. Reduced systems of isometries. In this section we introduce an extra hypothesis on the system of isometries. Under this hypothesis using the Rips Machine is considerably easier. For system of isometries on finite trees, such an extra hypothesis was introduced by D. Gaboriau in [Gab96] where it appears in Proposition V.4 as Property (*).

The **set of extremal points** ∂K of a compact \mathbb{R} -tree K is the set of points of K that do not lie in the interior of an interval contained in K . Equivalently a point P is extremal in K if $K \setminus \{P\}$ is connected. The tree K is the convex hull of its extremal points:

$$K = \bigcup_{P, P' \in \partial K} [P; P'].$$

We remark that ∂K may fail to be compact.

Definition 3.11. Let $S_0 = (F_0, A_0)$ be a system of isometries. Let $S_1 = (F_1, A_1)$ be the output of the Rips Machine. The system of isometries S_0 is **reduced** if

- (i) For any point P in F_0 the tree of trajectories $\tilde{\Gamma}(P)$ is infinite.
- (ii) For any partial isometry a in $A_0^{\pm 1}$ the set of extremal points $\partial \text{dom}(a)$ of the domain of a is contained in F_1 .

Lemma 3.12. The graph Γ_0 associated to a reduced system of isometries does not have vertices of negative index, that is to say of valence 0 or 1.

Proof. Let K be a connected component of F and let P be a point in K . From condition (i), the tree of trajectories $\tilde{\Gamma}(P)$ is infinite and thus there exists at least one partial isometry $a \in A_0^{\pm 1}$ defined at P . Let now Q be an extremal point of $\text{dom}(a)$. From condition (ii), Q is in F_1 and thus belongs to at least another domain $\text{dom}(b)$ with $b \in A_0^{\pm 1}$, $a \neq b$. Thus, the vertex K of Γ_0 has at least two outgoing edges a and b . \square

When dealing with the Rips Machine, reduced systems of isometries are easier to handle. The elementary moves “split a vertex” and “split an edge” are described in the proof of Proposition 3.2.

Proposition 3.13. Let $S_0 = (F_0, A_0)$ be a reduced system of isometries. Let $S_1 = (F_1, A_1)$ be the output of the Rips Machine.

Then going from S_0 to S_1 only performs elementary moves of type “split an edge”, and the map $\tau : \Gamma_1 \rightarrow \Gamma_0$ is a homotopy equivalence. In particular

$$i(\Gamma_0) = i(\Gamma_1).$$

Proof. In the proof of Proposition 3.2, starting with a reduced system of isometries, we first get that E_0 is empty. Indeed, let P be an extremal point of F which is not in F_1 . From condition (ii), P does not belong to any domain of a partial isometry in $A^{\pm 1}$ and thus its tree of trajectories consists in a single vertex, which contradicts condition (i).

Then we get that no “split a vertex” move can occur, because the removed points in this move have a tree of trajectories which consists in a single vertex. \square

We can now prove that the reduced condition is inherited while iterating the Rips Machine.

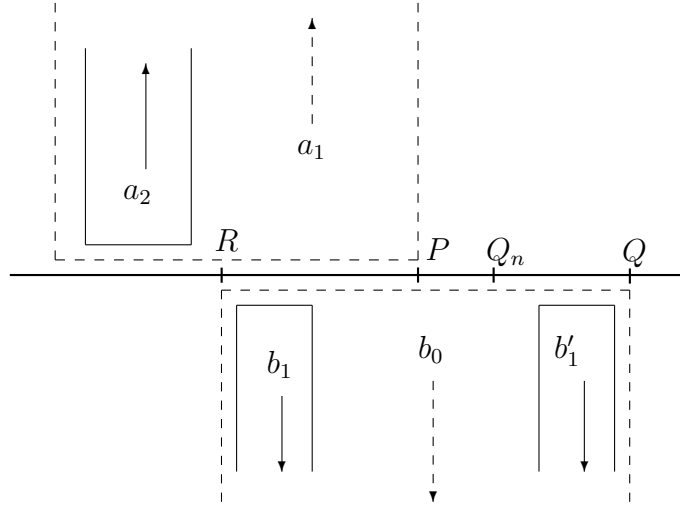
Proposition 3.14. Let $S_0 = (F_0, A_0)$ be a reduced system of isometries. Let $S_1 = (F_1, A_1)$ be the output of the Rips Machine. Then S_1 is reduced.

Proof. For a point P in F_1 , the tree of trajectories of P with respect to S_1 is obtained from the tree of trajectories with respect to S_0 by pruning off the terminal vertices. Thus, if the latter is infinite, so is the former.

By contradiction, let $S_2 = (F_2, A_2)$ be the output of the Rips Machine applied to S_1 , let a_1 be a partial isometry in A_1 and let P be an extremal point in $\partial\text{dom}(a_1)$ which is not in F_2 . Let a_0 be the partial isometry in A_0 of which a_1 is a restriction.

As P is in $\text{dom}(a_1)$, and thus in F_1 , there is at least another partial isometry $b_0 \in A_0$ such that P is in the domain $\text{dom}(b_0)$. There exist two extremal points Q, R in $\partial\text{dom}(b_0)$ such that P lies in the segment $[Q; R]$. By hypothesis, Q, R , as well as $Q.b_0$ and $R.b_0$ lie in F_1 .

As P is an extremal point in $\partial\text{dom}(a_1)$, up to exchanging the two points Q and R , we assume that Q is not in the domain of a_1 and that $[P; Q] \cap \text{dom}(a_1) = \{P\}$. Let (Q_n) be a sequence of points in the open arc $]P; Q[$ that converges to P .



In the original system of isometries S_0 , Q_n has an infinite tree of trajectories, in particular there exists a reduced admissible path $c_n.d_n$ of length 2 in Γ_0 which is defined at Q_n . As Γ_0 is a finite graph, up to passing to a subsequence we assume that this path is constant: for any n the reduced admissible path $c.d$ is defined at Q_n . As domains of partial isometries are closed, the point P is in the domain of $c.d$. Since $c^{-1} \neq d$, by definition of the Rips Machine, the point Pc is in F_1 and there is a restriction c_1 of c in A_1 which is defined at P . As P is not in F_2 and as a_1 is defined at P , the partial isometries c_1 and a_1 are equal. Thus for any integer n the partial isometry a_0 is defined at Q_n and $Q_n a_0$ is in F_1 . There are only finitely many restrictions of a_0 to the connected components of F_1 . Thus a_1 is defined at some Q_n and P is not an extremal point in $\partial\text{dom}(a_1)$. A contradiction. \square

We now state an equivalent characterization of reduced systems of isometries.

Proposition 3.15. *A system of isometries $S_0 = (F_0, A_0)$ is reduced if and only if the two following conditions hold:*

- (i) *For any point P in F the tree of trajectories $\tilde{\Gamma}(P)$ is infinite.*
- (ii') *For any partial isometry a in $A^{\pm 1}$ the set of extremal points $\partial\text{dom}(a)$ of the domain of a is contained in the limit set Ω_0 .*

Proof. As the limit set Ω_0 of S_0 is contained in F_1 we get that condition (ii') implies condition (ii).

Conversely, let $S_0 = (F_0, A_0)$ be a reduced system of isometries and let $S_n = (F_n, A_n)$ be the systems of isometries obtained by iteratively applying the Rips machine. From Proposition 3.14, S_n is reduced. Let P be an extremal point of $\text{dom}(a_0)$ for a partial isometry $a_0 \in A_0^{\pm 1}$. By condition (ii), P is in F_1 as well as $P.a_0$, and thus there exists a restriction $a_1 \in A_1^{\pm 1}$ of a_0 such that P is an extremal point of $\text{dom}(a_1)$. By induction for each n there exists a partial isometry a_n in $A_n^{\pm 1}$ such that P is an extremal point in $\text{dom}(a_n)$ and thus P is in F_n . By Lemma 3.4 we conclude that P is in the limit set Ω_0 . \square

4. COMPUTING THE INDEX OF A SYSTEM OF ISOMETRIES

4.1. Systems of isometries on finite trees. We translate and adapt in this section Proposition 6.1 of [GLP94].

A **finite tree** is an \mathbb{R} -tree which is the convex hull of finitely many of its points. It has finitely many branch points and is the metric realization of a simplicial finite tree. A **finite forest** is a metric space with finitely many connected components each of which is a finite tree.

We remark that removing the branch points, such a finite forest gives a disjoint union of finitely many intervals. The integral of a function defined on F is the integral on this disjoint union of intervals (with respect to Lebesgue measure).

Let $S = (F, A)$ be a system of isometries where F is a finite forest.

The **valence** of a point P in F is

$$v_S(P) = \#\{a \in A^{\pm 1} \mid P \in \text{dom}(a)\}.$$

We emphasize that in Section 2.3 we defined the index $i_S(P)$ by restricting partial isometries to the limit set Ω , and thus for a system of isometries we may have $i_S(P) \neq v_S(P) - 2$.

The function $P \mapsto v_S(P)$ is a finite sum of characteristic functions of finite subtrees. It is Lebesgue integrable.

We translate Proposition 6.1 of [GLP94] to get:

Proposition 4.1. *Let $S = (F, A)$ be a system of isometries with independent generators. Assume that F is a finite forest. Then $\int_{P \in F} (v_S(P) - 2) \leq 0$*

Proof. The sum $\int_{P \in F} v_S(P)$ is twice the sum of the Lebesgue measure of the domains of the partial isometries in A . It is denoted by 2ℓ in Section 6 of [GLP94] while $\int_{P \in F} 2$ is twice the Lebesgue measure of F which is denoted there by $2m$. \square

4.2. Shortening systems of isometries. To use Proposition 4.1 in the broader context of compact forests, we need a standard procedure to shorten a system of isometries to a finite one.

Let F be a compact forest and let $\varepsilon > 0$. We define:

$$F_{\dagger\varepsilon} = \{P \in F \mid \exists P_{-\varepsilon}, P_{+\varepsilon} \in F, P \in [P_{-\varepsilon}, P_{+\varepsilon}], d(P, P_{-\varepsilon}) = d(P, P_{+\varepsilon}) = \varepsilon\}$$

(i.e. P is the midpoint of the segment $[P_{-\varepsilon}, P_{+\varepsilon}]$ which has length 2ε).

The set of extremal points ∂F of a compact forest F is the union of the sets of extremal points of the connected components of F .

Lemma 4.2. *(i) For any $\varepsilon > 0$, for any connected component K of F , $K \cap F_{\dagger\varepsilon} = K_{\dagger\varepsilon}$ is a finite tree (possibly empty);
(ii) For any $\varepsilon > \varepsilon' > 0$, $F_{\dagger\varepsilon} \subset F_{\dagger\varepsilon'}$;*

(iii) $\cup_{\varepsilon>0} F_{\dagger\varepsilon} = F \setminus \partial F$. □

For any partial isometry a of F , we denote by a_ε its restriction to $F_{\dagger\varepsilon}$. We denote by A_ε and S_ε the corresponding finite set of partial isometries and the restricted system of isometries. We remark that if S has independent generators then S_ε also has.

4.3. Pseudo-surface systems of isometries. Let $S = (F, A)$ be a system of isometries, where F is a compact forest and such that each point of F lies inside the domain of at least two different partial isometries in $A^{\pm 1}$. In this case the limit set Ω is equal to F and the Rips Machine does not do anything to S .

If, in addition, the system of isometries S has independent generators, we say that it is **pseudo-surface**.

A key step in our proof of Theorem 2.2 is the following Proposition which is proved using Proposition 4.1 by shortening the system of isometry as in Section 4.2. We note that the following Proposition is obvious if F is a finite tree or a finite forest (cf. for example [Gab96, Processus II.3 5])).

Proposition 4.3. *Let $S = (F, A)$ be a pseudo-surface system of isometries. Then, for any choice of three distinct partial isometries in $A^{\pm 1}$ the intersection of the three domains contains at most one point of F .*

Proof. By definition of pseudo-surface systems of isometries, for any P in F the valence $v_S(P)$ is greater or equal to 2.

By contradiction we assume that there exist three distinct partial isometries a, b and c in $A^{\pm 1}$ such that the intersection of their domains is strictly bigger than a point. As domains are compact subtrees, there exists a non-trivial arc I which is contained in the three domains:

$$\forall P \in I, v_S(P) \geq 3.$$

For any $\varepsilon > 0$ we consider the finite forest $F_{\dagger\varepsilon}$ and the corresponding system of isometries S_ε .

Let ℓ be the length of I . For any $\varepsilon < \ell/3$, the sub-arc J of I which is contained in the domains of the partial isometries $a_\varepsilon, b_\varepsilon$ and c_ε of A_ε contains the middle third of I thus the length of J is bigger than $\ell/3$.

For such an $\varepsilon > 0$, $F_{\dagger\varepsilon}$ is a finite forest by Lemma 4.2 and Proposition 4.1 holds:

$$I_\varepsilon = \int_{P \in F_{\dagger\varepsilon}} (v_{S_\varepsilon}(P) - 2) \leq 0$$

Let \mathcal{P}_ε , resp. \mathcal{N}_ε , be the set of points of $F_{\dagger\varepsilon}$ which contributes positively, resp. negatively, to I_ε :

$$\mathcal{P}_\varepsilon = \{P \in F_{\dagger\varepsilon} \mid v_{S_\varepsilon}(P) \geq 2\} \text{ and } \mathcal{N}_\varepsilon = \{P \in F_{\dagger\varepsilon} \mid v_{S_\varepsilon}(P) \leq 1\}.$$

As the points in J have valence at least 3, we get

$$\int_{P \in \mathcal{P}_\varepsilon} (v_{S_\varepsilon}(P) - 2) \geq \ell/3$$

and thus, we have:

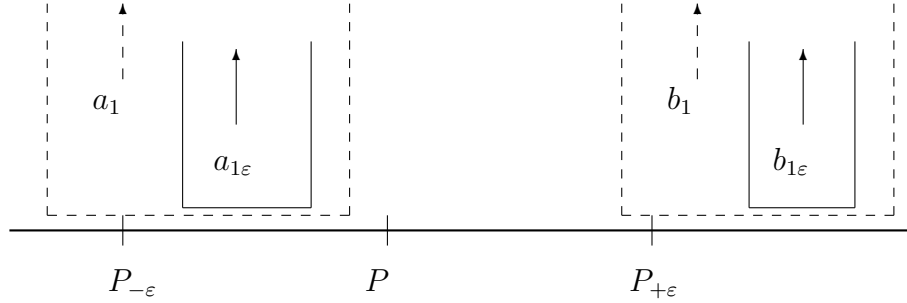
$$0 \geq I_\varepsilon \geq \ell/3 + \int_{P \in \mathcal{N}_\varepsilon} (v_{S_\varepsilon}(P) - 2).$$

Our goal is to prove that the negative part goes to zero to get a contradiction. We only need to prove that the Lebesgue measure of \mathcal{N}_ε goes to zero.

We claim that \mathcal{N}_ε has Lebesgue measure bounded above by $8N^2\varepsilon$ where $N = \#A$ is the cardinality of A .

Let P be in \mathcal{N}_ε , then P is in $F_{\uparrow\varepsilon}$ which means that P is the midpoint of a segment $[P_{-\varepsilon}, P_{+\varepsilon}]$ of length 2ε in F . As S is pseudo-surface, there are at least two elements $a_1, a_2 \in A^{\pm 1}$ which are defined at $P_{-\varepsilon}$ and at least two elements $b_1, b_2 \in A^{\pm 1}$ which are defined at $P_{+\varepsilon}$. As P is in \mathcal{N}_ε at most one of the four partial isometries $a_{1\varepsilon}, a_{2\varepsilon}, b_{1\varepsilon}, b_{2\varepsilon}$ is defined at P . By switching the indices we can assume that $a_{1\varepsilon}$ and $b_{1\varepsilon}$ are not defined at P . Taking ε sufficiently small ensures that the partial isometries $a_{1\varepsilon}$ and $b_{1\varepsilon}$ are not empty.

The domain of $a_{1\varepsilon}$ and the point $P_{-\varepsilon}$ lie in the same connected component of $F \setminus \{P\}$: else P would be located in a segment $[P_{-\varepsilon}, P']$ with P' in the domain of $a_{1\varepsilon}$ and $P_{-\varepsilon}$ in the domain of a_1 , thus P would be in the domain of $a_{1\varepsilon}$. We argue similarly for the domain of $b_{1\varepsilon}$.



We have thus proved that P is in the non-trivial arc joining the disjoint domains of $a_{1\varepsilon}$ and $b_{1\varepsilon}$. The point $P_{-\varepsilon}$ is in the domain of a_1 and thus at distance less than ε of the domain of $a_{1\varepsilon}$. Thus P is at distance less than 2ε from this domain. Hence, the length of the arc joining the disjoint domains of $a_{1\varepsilon}$ and $b_{1\varepsilon}$ is at most 4ε .

If, over all the possible pairs of partial isometries, we sum the lengths of the arcs, we get that the volume of \mathcal{N}_ε is bounded above by $(2N(2N - 1)/2) \times 4\varepsilon$. Which proves the claim and concludes the proof. \square

If the compact forest $F = I$ is an interval and if the system of isometries $S = (F, A)$ is pseudo-surface, Proposition 4.3 states that this is the classical case of an interval exchange transformation and S is usually called surface. This justifies the terminology of pseudo-surface system of isometries.

From Proposition 4.3, it is easy to deduce a rough bound of the index of a pseudo-surface system of isometries.

Corollary 4.4. *Let $S = (F, A)$ be a pseudo-surface system of isometries. Then $i(S)$ is finite and bounded above by a constant depending only on the cardinality of A .*

Proof. We denote, as before, by $N = \#A$ the cardinal of A . From the previous Proposition, there are at most $\binom{2N}{3}$ points in F which belongs to the domains of at least three different partial isometries in $A^{\pm 1}$. Each of these points has valence at most $2N$. Adding up we get that

$$i(S) = \sum_{P \in F} i_S(P) = \sum_{P \in F} (v_S(P) - 2) \leq N(2N - 1)(2N - 2)^2/3. \quad \square$$

We now state a combinatorial Lemma.

Let K be a compact \mathbb{R} -tree and let $\mathcal{K} = (K_a)_{a \in A}$ be a finite collection of compact subtrees of K . For such a collection, and for any point $P \in K$ we denote by $v_{\mathcal{K}}(P)$ the number of elements a of A such that P is in K_a .

Lemma 4.5. *Let $\mathcal{K} = (K_a)_{a \in A}$ be a finite collection of compact subtrees of a compact \mathbb{R} -tree K . Assume that*

- (1) *for any choice of three distinct elements of A the intersection of the corresponding subtrees is at most one point,*
- (2) *any element P of K is in at least two compact subtrees K_a and K_b ($a \neq b \in A$).*

Then

$$\sum_{P \in K} (v_{\mathcal{K}}(P) - 2) = \#A - 2.$$

Proof. Let T be the convex hull in K of all elements $P \in K$ such that $v_{\mathcal{K}}(P) \geq 3$. From our first hypothesis T is a finite tree. For each $a \in A$ the intersection $T_a = K_a \cap T$ is a finite tree. Let $\mathcal{T} = (T_a)_{a \in A}$ be the corresponding collection of finite subtrees of T . We have the equality

$$\sum_{P \in K} (v_{\mathcal{K}}(P) - 2) = \sum_{P \in T} (v_{\mathcal{T}}(P) - 2).$$

Moreover \mathcal{T} satisfies the same hypothesis as \mathcal{K} : the intersection of three of its elements is at most a point and any element of T is in at least two subtrees T_a and T_b , $a \neq b \in A$. We regard T and each T_a as a simplicial tree by considering all the branch points and extremal points as vertices. Each edge of the simplicial tree T belongs to exactly two trees T_a and T_b . Combinatorial computation gives

$$\begin{aligned} \sum_{P \in T} (v_{\mathcal{T}}(P) - 2) &= \sum_{P \in V(T)} (v_{\mathcal{T}}(P) - 2) \\ &= \sum_{P \in V(T)} \sum_{a \in A} \mathbb{1}_{P \in V(T_a)} - 2\#V(T) \\ &= \sum_{a \in A} \sum_{P \in V(T)} \mathbb{1}_{P \in V(T_a)} - 2\#E(T) - 2 \\ &= \sum_{a \in A} \#V(T_a) - \sum_{a \in A} \#E(T_a) - 2 \\ &= \#A - 2. \end{aligned}$$

□

We can now get the correct bound for the index of a pseudo-surface system of isometries

Theorem 4.6. *Let $S = (F, A)$ be a pseudo-surface system of isometries and, let Γ be the associated graph. Then $i(S) = i(\Gamma)$.*

Proof. As S is pseudo-surface, at least two distinct partial isometries are defined at each point of F . Thus, the graph Γ does not have vertices of valence 0 or 1 and its index is given by

$$i(\Gamma) = \sum_{K \in V(\Gamma)} i_{\Gamma}(K).$$

Let K be a connected component of F and let B be the subset of $A^{\pm 1}$ which consists of partial isometries with domains inside K . The set B is also the set of edges going out of the vertex K of the graph Γ and thus

$$i_{\Gamma}(K) = \#B - 2.$$

Let \mathcal{K} be the collection of domains of elements of B . Thus for each point P in K

$$i_S(P) = v_{\mathcal{K}}(P) - 2.$$

By Proposition 4.3, the collection \mathcal{K} satisfies the hypothesis of Lemma 4.5 and we get

$$\sum_{P \in K} (v_{\mathcal{K}}(P) - 2) = \#B - 2.$$

Thus the contribution of the points of K to the index of S is equal to the contribution of the corresponding vertex of Γ :

$$\sum_{P \in K} i_S(P) = i_{\Gamma}(K).$$

Adding up for all connected components K of F , proves the Theorem. \square

4.4. Proof of Theorem 2.2. Using Proposition 3.6, Theorem 2.2 is a consequence of

Proposition 4.7. *Let $S = (F, A)$ be a system of isometries with independent generators. Let $\widehat{\Gamma}$ be its limit graph. Then the index of S is equal to the index of $\widehat{\Gamma}$:*

$$i(S) = i(\widehat{\Gamma}).$$

Proof. Let Ω_0 be the union of all non-trivial connected components of the limit set Ω . By Proposition 3.9, Ω_0 has finitely many connected components, that is to say, Ω_0 is a compact forest. Let $S_0 = (\Omega_0, A_0)$ be the system of isometries which consists of the restrictions of S to Ω_0 . By definition of the limit set, S_0 is a pseudo-surface system of isometries. By Proposition 4.3, the intersection the domains of three distinct partial isometries of $A_0^{\pm 1}$ contains at most one point.

For a vertex of $\widehat{\Gamma}$ corresponding to a connected component K (possibly a single point) of Ω we can apply Lemma 4.5 to the collection given by the domains of the edges going out of K to get:

$$i_{\widehat{\Gamma}}(K) = \sum_{P \in K} (v_S(P) - 2).$$

By Corollary 3.7, $\widehat{\Gamma}$ has finitely many vertices with index strictly positive (and these indices are finite). Adding up for all these singular vertices of $\widehat{\Gamma}$, we get

$$i(S) = i(\widehat{\Gamma}). \quad \square$$

4.5. Systems of isometries with maximal index. From Theorem 2.2, we say that a system of isometries S has **maximal index** if its index is equal to the index of its associated graph Γ : $i(S) = i(\Gamma)$.

The following Proposition characterizes reduced systems of isometries with maximal index.

Theorem 4.8. *Let $S = (F, A)$ be a reduced system of isometries with independent generators, let Γ be its associated graph, and $\widehat{\Gamma}$ be its limit graph. The following are equivalent*

- (1) *The system of isometries S has maximal index,*

- (2) The graph $\widehat{\Gamma}$ is finite,
(3) The Rips Machine, starting from S , halts after finitely many steps.

Proof. As before we denote by $S_n = (F_n, A_n)$ the system of isometries obtained after n steps of the Rips Machine. By Proposition 3.13 and Proposition 3.14 the Rips Machine only performs “split an edge” moves and induces a homotopy equivalence $\tau_n : \Gamma_{n+1} \rightarrow \Gamma_n$ at each step. And thus the index is constant:

$$i(\Gamma_n) = i(\Gamma_0) = i(\Gamma).$$

Moreover at each step the graph Γ_n does not have vertices of valence 0 or 1.

3 \Rightarrow 2: If the Rips Machine halts after step n , then $\Gamma_{n+1} = \Gamma_n = \widehat{\Gamma}$ is a finite graph.

2 \Rightarrow 3: Conversely, at each step n the Rips Machine only performs “split an edge” moves. This move adds one edge to Γ_n to get Γ_{n+1} . If the Rips Machine never halts, then the number of edges of Γ_n goes to infinity. As each of the τ_n is onto, we get that $\widehat{\Gamma}$ is infinite.

3 \Rightarrow 1: If the Rips Machine halts after finitely many steps: for n big enough

$$\widehat{\Gamma} = \Gamma_{n+1} = \Gamma_n,$$

by Proposition 4.7, $i(\widehat{\Gamma}) = i(S)$ and we get that S has maximal index.

1 \Rightarrow 3: Assume that S has maximal index: $i(S) = i(\widehat{\Gamma}) = i(\Gamma)$.

We proceed as in the proof of Proposition 3.6. Let Θ_0 be the finite subset of vertices of $\widehat{\Gamma}$ with valence strictly bigger than 2. Let Θ be the finite subgraph of $\widehat{\Gamma}$ which contains all edges incident to Θ_0 . The graph Θ contains all the index of $\widehat{\Gamma}$:

$$i(\widehat{\Gamma}) = \sum_{x \in \Theta_0} i_{\Theta}(x).$$

For n big enough, $\widehat{\tau}_n$ is injective on Θ and thus for each vertex x in Θ_0

$$i_{\Theta}(x) \leq i_{\Gamma_n}(\widehat{\tau}_n(x)).$$

We assumed that S is reduced and thus that Γ_n does not have vertices of strictly negative index.

By maximality of the index, $i(\widehat{\Gamma}) = i(\Gamma_n)$ and thus we can compute

$$i(\widehat{\Gamma}) = \sum_{x \in \Theta_0} i_{\Theta}(x) = \sum_{x \in \Theta_0} i_{\Gamma_n}(\widehat{\tau}_n(x)) + \sum_{y \in V(\Gamma_n) \setminus \widehat{\tau}_n(\Theta_0)} i_{\Gamma_n}(y) = i(\Gamma_n).$$

We deduce that for each x in Θ_0 and for all $y \in V(\Gamma_n) \setminus \widehat{\tau}_n(\Theta_0)$,

$$i_{\Theta}(x) = i_{\Gamma_n}(\widehat{\tau}_n(x)) \text{ and } i_{\Gamma_n}(y) = 0.$$

The commutative diagram

$$\begin{array}{ccc} & \widehat{\Gamma} & \\ \widehat{\tau}_{n+1} \swarrow & & \searrow \widehat{\tau}_n \\ \Gamma_{n+1} & \xrightarrow{\tau_n} & \Gamma_n \end{array}$$

restricts to graph isomorphisms between Θ and its images. Moreover, $\widehat{\tau}_{n+1}(\Theta_0)$ and $\widehat{\tau}_n(\Theta_0)$ contain all the vertices of strictly positive index of Γ_{n+1} and Γ_n respectively. Thus no “split an edge” move can occur when passing from Γ_n to Γ_{n+1} and thus the Rips Machine does not do anything to the system of isometries S_n . \square

5. TREES

Throughout this Section, T is an \mathbb{R} -tree with a very small, minimal action of the free group F_N of rank N by isometries with dense orbits.

5.1. The map \mathcal{Q} . Let P be a point in T , we consider the equivariant map $\mathcal{Q}_P : F_N \rightarrow T$, $u \mapsto u \cdot P$. This map does not extend continuously to the boundary ∂F_N of F_N . To overcome this difficulty we weaken the topology on T by considering the observers' topology.

Let $\widehat{T} = \overline{T} \cup \partial T$ be the union of the metric completion of T and its (Gromov) boundary. \widehat{T} inherits from the metric on T a well defined topology. However, \widehat{T} is not compact in general.

We consider on \widehat{T} the weaker **observers' topology** and we denote by \widehat{T}^{obs} this topological space. A basis of open sets in \widehat{T}^{obs} is given by the set of connected components of $\widehat{T} \setminus \{P\}$ for all points P . This topology is Hausdorff and \widehat{T}^{obs} is a compact space with the same connected subspaces than \widehat{T} , see [CHL07].

Proposition 5.1 ([CHL07]). *Let T be an \mathbb{R} -tree with a very small, minimal action of F_N by isometries with dense orbits. There exists a unique map \mathcal{Q} that is the continuous extension from ∂F_N to \widehat{T}^{obs} of the map $\mathcal{Q}_P : u \mapsto u \cdot P$. The map \mathcal{Q} does not depend on the choice of a point P . \square*

This map \mathcal{Q} was first introduced by Levitt and Lustig in [LL03, LL08] with a slightly different approach. In particular they proved

Proposition 5.2. *Let T be an \mathbb{R} -tree with a very small, minimal action of F_N by isometries with dense orbits. The map \mathcal{Q} is onto \widehat{T} . The points P in \widehat{T} with strictly more than one pre-image by \mathcal{Q} are in the metric completion \overline{T} of T (and not in the (Gromov) boundary ∂T).*

It has been asked by Levitt and Lustig [LL03, Remark 3.6] whether the map $\mathcal{Q} : \partial F_N \rightarrow \widehat{T}$ has finite fibers (in the case where the action is free). We are going to answer this question and to give a precise bound for the cardinal of the fibers. In this purpose we need to make this question precise by the following definition of the \mathcal{Q} -index.

5.2. The \mathcal{Q} -index. We denote by $\text{Stab}(P)$ the stabilizer in F_N of a point P of \widehat{T} .

It is proved in [GL95] that $\text{Stab}(P)$ is a finitely generated subgroup of F_N . The subgroup $\text{Stab}(P)$ is a free group and its boundary $\partial \text{Stab}(P)$ embeds in the boundary of F_N . For any element $X \in \partial \text{Stab}(P) \subseteq \partial F_N$, Proposition 5.1 proves that $\mathcal{Q}(X) = P$. Elements of $\partial \text{Stab}(P)$ are called **singular**, and the other elements of the fiber $\mathcal{Q}^{-1}(P)$ are **regular**. We denote by $\mathcal{Q}_r^{-1}(P)$ the set of regular points. As \mathcal{Q} is equivariant, $\text{Stab}(P)$ acts on $\mathcal{Q}^{-1}(P)$ and on $\mathcal{Q}_r^{-1}(P)$.

The **\mathcal{Q} -index** $\text{ind}_{\mathcal{Q}}(P)$ of a point P in \widehat{T} is defined by

$$\text{ind}_{\mathcal{Q}}(P) = \#(\mathcal{Q}_r^{-1}(P)/\text{Stab}(P)) + 2 \text{rank}(\text{Stab}(P)) - 2.$$

When $\text{Stab}(P)$ is trivial this definition becomes

$$\text{ind}_{\mathcal{Q}}(P) = \#\mathcal{Q}^{-1}(P) - 2.$$

The \mathcal{Q} -index only depends on the orbit $[P]$ of P under the action of F_N and we can define the **\mathcal{Q} -index** of the tree T by

$$\text{ind}_{\mathcal{Q}}(T) = \sum_{[P] \in \widehat{T}/F_N} \max(0; i_{\mathcal{Q}}([P])).$$

From Proposition 5.2, points in ∂T have exactly one pre-image by \mathcal{Q} . Thus, only points in \overline{T} contribute to the \mathcal{Q} -index of T .

The main goal of this section is to prove the following Theorem:

Theorem 5.3. *Let T be an \mathbb{R} -tree with a very small, minimal action of F_N by isometries with dense orbits. Then*

$$\text{ind}_{\mathcal{Q}}(T) \leq 2N - 2.$$

In the case of a free action of the free group F_N on T this gives the answer to Levitt and Lustig's question:

Corollary 5.4. *Let T be an \mathbb{R} -tree with a free, minimal action of F_N by isometries with dense orbits. Then, there are finitely many orbits of points P in \widehat{T} with strictly more than 2 elements in their \mathcal{Q} -fiber $\mathcal{Q}^{-1}(P)$ and these fibers are finite. \square*

5.3. Dual lamination and compact heart. The **double boundary** of F_N is

$$\partial^2 F_N = (\partial F_N \times \partial F_N) \setminus \Delta$$

where Δ is the diagonal. An element of $\partial^2 F_N$ is a **line**.

Using the map \mathcal{Q} , in [CHL08], the **dual lamination** $L(T)$ to the tree T is defined.

$$L(T) = \{(X, Y) \in \partial^2 F_N \mid \mathcal{Q}(X) = \mathcal{Q}(Y)\}.$$

From this definition, the map \mathcal{Q} naturally induces an equivariant map $\mathcal{Q}^2 : L(T) \rightarrow \widehat{T}$. It is proved in [CHL08] that the map \mathcal{Q}^2 is continuous and its image is a subset Ω of \overline{T} which we call the **limit set**.

We fix a basis A of F_N . Elements of F_N are reduced finite words in the alphabet $A^{\pm 1}$. An element X of ∂F_N is an infinite reduced words in $A^{\pm 1}$, we denote by X_1 its first letter.

The **unit cylinder** $C_A(1)$ of $\partial^2 F_N$ is

$$C_A(1) = \{(X, Y) \in \partial^2 F_N \mid X_1 \neq Y_1\}.$$

Although $\partial^2 F_N$ is not compact, the unit cylinder is compact and indeed a Cantor set.

In [CHL09] the **relative limit set** of T with respect to A is defined:

$$\Omega_A = \mathcal{Q}^2(L(T) \cap C_A(1)).$$

From the continuity of the map \mathcal{Q}^2 , the relative limit set Ω_A is a compact subset of \overline{T} .

The **compact heart** K_A of T is the convex hull of Ω_A .

For any element a of the basis A we consider the partial isometry (which we also denote by a , but which we write on the right) which is the restriction of the action of a^{-1} :

$$\begin{aligned} K_A \cap aK_A &\rightarrow K_A \cap a^{-1}K_A \\ x &\mapsto x.a = a^{-1}x \end{aligned}$$

We get a system of isometries $S_A = (K_A, A)$ as defined in Section 2.

In [CHL09] it is proved that S encodes all the informations given by T and the action of F_N . To be more specific, we summarize results of [CHL09] as follows:

Proposition 5.5 ([CHL09]). *Let T be an \mathbb{R} -tree with a very small, minimal action of F_N by isometries with dense orbits. Let A be a basis of F_N , let K_A be its compact heart and $S_A = (K_A, A)$ be the associated system of isometries. Then*

- (1) S_A has independent generators.
- (2) For any word $u \in F_N$, and for any point $P \in K_A$, $u^{-1}.P \in K_A$ if and only if u is admissible for S_A and $P \in \text{dom}(u)$. In this case $P.u = u^{-1}.P$.
- (3) For any element $X \in \partial F_N$, $\mathcal{Q}(X) = P \in K_A$ if and only if X is admissible and $\{P\} = \text{dom}(X)$.

Proof. Assertion 1 is Lemma 5.1 of [CHL09]. Assertion 2 is proved in Lemma 3.5 (1) and Corollary 5.5 and Assertion 3 is a consequence of Proposition 4.3, Lemma 4.7 and Corollary 5.5. \square

We deduce that for an infinite reduced admissible word X the definition of $\mathcal{Q}(X)$ of Section 2.1 agrees with the definition given by Proposition 5.1. Moreover the relative limit set Ω_A of the \mathbb{R} -tree T is equal to the limit set of the system of isometries S_A .

5.4. The compact heart of a tree is reduced. As explained in Section 3.4, reduced systems of isometries (see Definition 3.11) considerably simplifies the use of the Rips Machine. Fortunately, in the context of \mathbb{R} -trees, which we are studying, and thanks to [CHL09], we can work with reduced systems of isometries.

Proposition 5.6. *Let T be an \mathbb{R} -tree with a minimal very small action of F_N by isometries with dense orbits. Let A be a basis of F_N , let Ω_A be the relative limit set and K_A be the compact heart of T . Let $S_A = (K_A, A)$ the induced system of isometries.*

Then, the system of isometries S_A is reduced.

Proof. By Proposition 5.2, the map \mathcal{Q} is onto \widehat{T} : for any point P in K_A there exists $X \in \partial F_N$ such that $\mathcal{Q}(X) = P$. By Proposition 5.5, X is admissible and P is in the domain of X . Thus, any point P of the compact heart K_A has an infinite tree of trajectories.

Let $S_1 = (F_1, A_1)$ be the output of the Rips Machine. Let a be a partial isometry in A and let P be an extremal point of the domain of a . If P is an extremal point of K_A , then, as K_A is the convex hull of the relative limit set Ω_A , we get that P is in Ω_A and thus in F_1 . If P is not an extremal point of K_A there exists a sequence Q_n of points in F which converges to P and which are not in the domain of a . The points in the compact heart K_A have infinite tree of trajectories, thus for each of these points Q_n there exists a partial isometry b_n in A such that Q_n is in the domain of b_n . As A is finite, up to passing to a subsequence we assume that all the Q_n are in the domain of a partial isometry b in A . The domain of b is close and thus P is in the domain of b . By definition of the Rips Machine we get that P is in F_1 . \square

5.5. Proof of Theorem 5.3. Theorem 5.3 is a consequence of Theorem 2.2 and of the following Theorem which relates the \mathcal{Q} -index of T and the index of the system of isometries S_A defined on its compact heart.

Theorem 5.7. *Let T be an \mathbb{R} -tree with a very small, minimal action of F_N by isometries with dense orbits. Let A be a basis of F_N . The \mathcal{Q} -index of T and the index of the induced system of isometries $S_A = (K_A, A)$ on the heart K_A of T and A are equal:*

$$\text{ind}_{\mathcal{Q}}(T) = i(S_A).$$

Proof. Let P be a point in \widehat{T} and $[P]$ be its orbit under the action of F_N . By Proposition 5.5 assertion 2, the intersection of the orbit $[P]$ and of the compact heart K_A is a pseudo-orbit (possibly empty) of the system of isometries S_A .

Assume that $\text{ind}_{\mathcal{Q}}([P]) \geq 0$. There are at least two distinct elements X, Y in the fiber $\mathcal{Q}^{-1}(P)$. Let u be the common prefix of X and Y , then $X' = u^{-1}X$ and $Y' = u^{-1}Y$ have different first letter and are in the pre-image by \mathcal{Q} of $P' = u^{-1}P$. By definition P' is in the relative limit set Ω_A of the tree T .

This proves that the \mathcal{Q} -index of T can be computed by considering only pseudo-orbits in K_A :

$$\text{ind}_{\mathcal{Q}}(T) = \sum_{[P] \in K_A/F_N} \max(0; \text{ind}_{\mathcal{Q}}([P])).$$

Let P be a point in K_A . By Proposition 5.5, the boundary at infinity of the tree of trajectories $\widetilde{\Gamma}(P)$ is exactly $\mathcal{Q}^{-1}(P)$ and the discussion in Section 2.4 shows that

$$\begin{aligned} \text{ind}_{\mathcal{Q}}([P]) &= \#(\mathcal{Q}_r^{-1}(P)/\text{Stab}(P)) + 2 \text{rank}(\text{Stab}(P)) - 2 \\ &= \#\partial\Gamma(P) + 2 \text{rank}(\text{Stab}(P)) - 2 = \sum_{P' \in [P] \cap K_A} i_S(P'). \end{aligned}$$

Adding up for all points P in K_A , proves the Theorem. □

5.6. Geometric index of a tree. Gaboriau and Levitt in [GL95] introduced the index of T as follows.

Let P a point in T and let $\pi_0(T \setminus \{P\})$ be the set of connected components of T without P . The stabilizer of P acts on this set. The **geometric index** of P is

$$\text{ind}_{\text{geo}}(P) = \#(\pi_0(T \setminus \{P\})/\text{Stab}(P)) + 2 \text{rank}(\text{Stab}(P)) - 2.$$

This index is always non-negative because there are no terminal vertices in a minimal tree. If the action of F_N on the tree T is free the above definition becomes simpler:

$$\text{ind}_{\text{geo}}(P) = \#\pi_0(T \setminus \{P\}) - 2.$$

The geometric index is constant inside an orbit under the action of F_N . The **geometric index** of T is then the sum of the indices over all orbits of points:

$$\text{ind}_{\text{geo}}(T) = \sum_{[P] \in T/F_N} \text{ind}_{\text{geo}}(P).$$

The following Theorem is proved by Gaboriau and Levitt:

Theorem 5.8 ([GL95]). *The geometric index of an \mathbb{R} -tree with a very small minimal action of the free group F_N is bounded above by $2N - 2$.* □

5.7. Botanic of Trees. In this Section we establish a beginning of classification of trees in the boundary of Outer Space. Let T be an \mathbb{R} -tree with a minimal very small action of F_N by isometries with dense orbits. Let A be a basis of F_N , let Ω_A be the relative limit set, let K_A be the compact heart of T and let $S_A = (K_A, A)$ be the associated system of isometries.

We first recall the existing terminology of geometric trees. The tree T is **geometric** if it can be obtained from a system of isometries on a finite tree, as explained in the Introduction (see for instance [Gab97, Bes02]). Geometric trees can be alternatively characterized thanks to the following:

Theorem 5.9 ([GL95], see also [CHL09, Corollary 6.1]). *Let T be an \mathbb{R} -tree with a minimal, very small action by isometries of F_N with dense orbits. The following are equivalent:*

- (1) T is geometric;
- (2) the geometric index is maximal: $\text{ind}_{\text{geo}}(T) = 2N - 2$;
- (3) K_A is a finite tree. □

We now introduce more terminology. The tree T is of **surface type** if the Rips Machine, starting with the system of isometries $S_A = (K_A, A)$, halts after finitely many steps. More precisely, a tree of surface type is:

- a **surface** tree if it is geometric (this terminology is justified by the fact that a tree dual to measured foliation on a surface with boundary is a surface tree.),
- a **pseudo-surface** tree if it is not geometric.

(Note that the fact that a tree is pseudo-surface does not exactly correspond to the fact that the system of isometries $S_A = (K_A, A)$ is pseudo-surface, according to the definition given in Section 4.3).

By Theorem 4.8 and Theorem 5.7 we get the following characterization of trees of surface type:

Theorem 5.10. *Let T be an \mathbb{R} -tree with a minimal, very small action by isometries of F_N with dense orbits. The tree T is of surface type if and only if its \mathcal{Q} -index is maximal: $\text{ind}_{\mathcal{Q}}(T) = 2N - 2$. □*

This proves in particular that being of surface type is a property of T and does not depend on the choice of a basis A of the free group F_N .

The tree T is of **Levitt type** if its relative limit set Ω_A is totally disconnected (i.e. the connected components of Ω_A are points). More precisely, a tree of Levitt type is

- **Levitt** if it is geometric (these trees were discovered by Levitt [Lev93] and are also termed thin or exotic).
- **pseudo-Levitt** if it is not geometric.

We now prove that being of Levitt type is a property of T and does not depend on the choice of a basis A of F_N .

Let $L(T)$ be the dual lamination of the tree T . The **limit set** Ω of T is the image in the metric completion \bar{T} of $L(T)$ by the map \mathcal{Q}^2 :

$$\Omega = \mathcal{Q}^2(L(T)).$$

Contrary to the relative limit set Ω_A , the limit set Ω is in general not closed.

Theorem 5.11. *The tree T is of Levitt type if and only if the limit set Ω is totally disconnected.*

Proof. By definition, the relative limit set $\Omega_A = \mathcal{Q}^2(L(T) \cap C_A(1))$ is a subset of Ω . Thus, if Ω does not contain a non-trivial arc, Ω_A neither.

Conversely, the double boundary of F_N is the union of the translates of the unit cylinder and

$$L(T) = \bigcup_{u \in F_N} u(L(T) \cap C_A(1)) \quad \text{and} \quad \Omega = \bigcup_{u \in F_N} u\Omega_A$$

In particular if I is a non-trivial arc in the limit set Ω , it is the countable union of its intersections with translates of the relative limit set Ω_A . Using Baire's Property for I , we get that Ω_A contains a non-trivial arc. \square

We remark that there are trees in the boundary of Outer Space which are neither of surface or Levitt type. These are trees of **mixed type**, that is to say their relative limit set Ω_A contains non-trivial arcs but have infinitely many connected components.

Remark 5.12. *A general classification of systems of isometries, in particular a Theorem à-la Imanishi to decompose trees of mixed type would be of interest. More generally, the question of understanding the relationships between mixing properties of trees, indecomposability of systems of isometries and minimality of laminations seems to be natural. Together with Reynolds [CHR11] we prove that indecomposable trees and minimal (up to diagonal leaves) laminations are dual to each other. In this spirit, see also the work of Reynolds [Rey10].*

5.8. Mixing trees. In this section we give sufficient hypothesis on a tree to enforce that it is either of surface type or of Levitt type.

We first describe the limit set of trees of surface type.

Proposition 5.13. *Let T be an \mathbb{R} -tree with a minimal, very small action by isometries of F_N with dense orbits. If the tree T is of surface type, then the limit set Ω is connected and contains T .*

Proof. As T is minimal, and as Ω is F_N -invariant, we get that Ω is connected if and only if Ω contains T :

$$T \subseteq \Omega \subseteq \bar{T}.$$

If T is of surface type, the Rips Machine starting with the system of isometries $S_A = (K_A, A)$, halts after finitely many steps, and thus $\Omega_A = F_n$ for some n , where F_n is the forest remaining after n steps of the Rips Machine. The system of isometries is reduced, hence the pseudo-orbit of each point in K_A is infinite and thus meets F_n , which proves that $K_A \subseteq F_N \cdot \Omega_A$. Moreover the orbit of each point in T meets K_A , thus $T \subseteq F_N K_A$, which concludes the proof. \square

A converse of this Proposition that requires stronger hypothesis on T is proved in Proposition 5.14

A **segment** of an \mathbb{R} -tree is a subset isometric to a compact real interval which is not reduced to a point. The action of F_N on an \mathbb{R} -tree T by isometries is **arc-dense** if every segment of T meets every orbits. Following [Mor88], the action of F_N on an \mathbb{R} -tree T by isometries is **mixing** if for every segments I and J in T , the segment J is covered by finitely many translates of I : there exists finitely many elements u_1, \dots, u_r of F_N such that

$$J \subseteq u_1 I \cup \dots \cup u_r I.$$

It is obvious that a mixing action is arc-dense. An arc-dense action has dense orbits and is minimal.

Proposition 5.14. *Let T be an \mathbb{R} -tree with a mixing action of F_N by isometries. Then T is either of surface type or of Levitt type.*

Proof. Let A be a basis of F_N , let Ω_A be the relative limit set, K_A the compact heart of T and $S_A = (K_A, A)$ be the associated system of isometries.

By contradiction assume that T is neither of surface type or of Levitt type. Then by definition of Levitt type, Ω_A contains a non-trivial connected component and thus a segment I .

Let $S_0 = S_A$ and let $S_n = (F_n, A_n)$ be the sequence of systems of isometries obtained from S_A by applying the Rips Machine. By definition of surface type, the Rips Machine runs forever.

Let E_0 be the set of points of $F_0 = K_A$ erased at the first step of the Rips Machine: $E_0 = F_0 \setminus F_1$. As S_0 is reduced, E_0 is contained in the convex hull in F_0 of F_1 and is a finite union of finite trees and an open subset of F_0 .

Let $E_n = F_n \setminus F_{n+1}$ be the subset of F_n erased at the $n + 1$ step of the Rips Machine. Let D_n be the subset of E_0 defined by

$$D_n = \{P \in E_0 \mid \exists u \in F_N, |u| = n \text{ and } P.u_i \in E_i, \text{ for } i = 1, \dots, n\}$$

where u_i is the prefix of u of length i . By definition, for each n , $D_{n+1} \subset D_n$. As the Rips Machine runs forever, D_n is a non-empty subset of F_0 . We distinguish two cases.

First assume that the nested intersection of the open non-empty subsets D_n is non-empty and let $P_0 \in \bigcap_{n \in \mathbb{N}} D_n$. As P_0 is in the open subset E_0 it is not an extremal point of F and as T is arc-dense, there exists $u \in F_N$ such that $uP_0 \in I$. From Proposition 5.5, the partial isometry u^{-1} is defined at P_0 and $P_0u^{-1} = uP_0$ is in the relative limit set Ω_A . By definition of the Rips Machine, for n bigger than $|u|$, P_0 is not in D_n . A contradiction.

Assume now that the nested intersection $\bigcap_{n \in \mathbb{N}} D_n$ is empty and let P_0 be in the nested intersection of compact subsets

$$P_0 \in \bigcap_{n \in \mathbb{N}} \overline{D}_n.$$

Then, there exists n_0 , such that for n bigger than n_0 , P_0 is in $\overline{D}_n \setminus D_n$. Recall that \overline{D}_{n_0} is a finite tree and let Q be a point of D_{n_0} . Then Q is not an extremal point of K_A and let $J = [P_0, Q]$. The segment J intersects all the D_n for $n \in \mathbb{N}$. If P_0 is in T (and not in $\overline{T} \setminus T$) then, as T is mixing, there there exist $u_1, \dots, u_r \in F_N$ such that

$$J \subseteq u_1I \cup \dots \cup u_rI.$$

The partial isometries u_1, \dots, u_r are not empty and for each k , $u_kI \cap K_A = Iu_k^{-1}$ thus, using Proposition 5.5,

$$J \subseteq Iu_1^{-1} \cup \dots \cup Iu_r^{-1}.$$

Hence, for n bigger than all the lengths of the u_i , $J \cap D_n$ is empty. A contradiction.

Thus, P_0 is in $\overline{T} \setminus T$. We get that P_0 is an extremal point of \overline{E}_0 and, as E_0 is open in K_A , P_0 is in $\overline{E}_0 \setminus E_0$. As P_0 is not in E_0 there are at least two partial isometries $a, b \in A^{\pm 1}$ defined at P_0 . One at least of a and b is not defined in E_0 , say a , and thus is defined only at P_0 . Thus, in T , there are at least two directions going out from P_0 (one containing K_A and the other containing $a^{-1}K_A$). This contradicts the fact that P_0 is in $\overline{T} \setminus T$. \square

Corollary 5.15. *Let T be an \mathbb{R} -tree with a mixing action of F_N by isometries. Let Ω be the limit set of T . The following are equivalent:*

- (1) T is of surface type;
- (2) Ω_A has finitely many components (i.e. Ω_A is a finite forest);
- (3) Ω is connected;
- (4) Ω contains T , that is to say $T \subseteq \Omega \subseteq \overline{T}$.

Proof. The equivalence of conditions 2 and 1 follows from the definition of surface type. Conditions 3 and 4 are equivalent because T is minimal. We proved in Proposition 5.13 that condition 1 implies condition 3. From the dichotomy of Proposition 5.14 and from Theorem 5.11 we get that condition 3 implies condition 1 \square

6. BOTANIC MEMO

In this Section we give a glossary of our classification of trees for the working mathematician.

Let T be an \mathbb{R} -tree with a minimal, very small action of F_N by isometries with dense orbits. We assume that the action is indecomposable (see the definition in Guirardel [Gui08]) or at least that T is not of mixed type.

For a basis A of F_N , Ω_A is the relative limit set and $K_A = \text{conv}(\Omega_A)$ is the compact heart. The compact heart K_A is either a finite tree (and in the good cases an interval) or not. This dichotomy is a property of T and does not depend of the choice of a particular basis A of F_N .

As we assumed that T is indecomposable, Ω_A is either a compact forest, that is to say it has finitely many connected components (and in the good cases $\Omega_A = K_A$ is a tree) or totally disconnected (and in the good cases a Cantor set). This dichotomy is a property of T and does not depend on the choice of a particular basis A of F_N .

The limit set of T is $\Omega = \mathcal{Q}^2(L(T)) = F_N \cdot \Omega_A$. It is either totally disconnected (if Ω_A is), or it is connected (if Ω_A is a forest): in the later case, Ω is a tree, $T \subseteq \Omega \subseteq \bar{T}$.

For such a tree T we considered two indices: the geometric index $\text{ind}_{\text{geo}}(T)$ and the \mathcal{Q} -index $\text{ind}_{\mathcal{Q}}(T)$. Both indices are bounded above by $2N - 2$. We sum up the terminology for T and the results of Section 5.7 in the following table.

		geometric	not geometric
		K_A is a finite tree	K_A is not a finite tree
		\Updownarrow	\Updownarrow
		$\text{ind}_{\text{geo}}(T) = 2N - 2$	$\text{ind}_{\text{geo}}(T) < 2N - 2$
Surface type	Ω is a tree ($T \subset \Omega$) \Updownarrow Ω_A is a finite forest \Updownarrow $\text{ind}_{\mathcal{Q}}(T) = 2N - 2$	surface	pseudo-surface
Levitt type	Ω is totally disconnected \Updownarrow Ω_A is totally disconnected \Updownarrow $\text{ind}_{\mathcal{Q}}(T) < 2N - 2$	Levitt	pseudo-Levitt

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