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# Free group automorphisms with parabolic boundary orbits

ARNAUD HILION

For  $N \geq 4$ , we show that there exist automorphisms of the free group  $F_N$  which have a parabolic orbit in  $\partial F_N$ . In fact, we exhibit a technology for producing infinitely many such examples.

[20E05](#), [20E36](#); [37B05](#)

## 1 Introduction

An automorphism  $\varphi$  of the free group  $F_N$  of rank  $N$  induces a homeomorphism  $\partial\varphi$  of the (Gromov) boundary  $\partial F_N$  of  $F_N$ . The dynamics of the map  $\partial\varphi$  on  $\partial F_N$  has been studied a lot, see [13, 14, 15, 16, 10]. We give a survey of the known results relevant in our context in section 3. In this paper, we focus on the following question:

*Does there exist an automorphism  $\varphi$  of  $F_N$  such that there is a parabolic orbit for the homeomorphism  $\partial\varphi$ ?*

We say that an automorphism  $\varphi$  has a parabolic orbit if there exists two points  $X, Y \in \partial F_N$ ,  $X \neq Y$ , such that:

$$\lim_{k \rightarrow \pm\infty} \partial\varphi^k(Y) = X.$$

We note that this implies that  $X$  is a fixed point of  $\partial\varphi$ . In such a situation, the point  $X \in \partial F_N$  is called a parabolic fixed point for  $\varphi$ , and the set  $\{\partial\varphi^k(Y) \mid k \in \mathbb{Z}\}$  is called a parabolic orbit for  $\varphi$ . We prove:

**Theorem 1.1** *For  $N \geq 4$  there exists an infinite family  $\{\varphi_k \mid k \in \mathbb{N}\}$  of automorphisms of  $F_N$  which have a parabolic orbit, such that for any  $k, k', p, p' \in \mathbb{N}$ ,  $\varphi_k^p$  and  $\varphi_{k'}^{p'}$  are conjugate if and only if  $k = k'$  and  $p = p'$ .*

Discussions with some of the experts of the subject have led the author to feel that the existence of such parabolic orbits come somehow as a surprise. To put Theorem 1.1 in perspective, we would like to mention the following three facts.

First, given a compact set  $K$  and a homeomorphism  $f$  of  $K$ , one says that  $f$  has *North-South dynamics*, if (i)  $f$  has precisely two distinct fixed points  $x^+$  and  $x^-$ , (ii)  $\lim_{k \rightarrow +\infty} f^k(y) = x^+$  and  $\lim_{k \rightarrow +\infty} f^{-k}(y) = x^-$  for all  $y \in K \setminus \{x^-, x^+\}$ , and (iii) the limit of  $f^k$ , when  $k$  tends to infinity, is uniform on compact subsets of  $K \setminus \{x^-\}$  and the limit of  $f^{-k}$  is uniform on compact subsets of  $K \setminus \{x^+\}$ . It is proved in [13] that “most” automorphisms of  $F_N$ , in a precise sense we do not explain here, have North-South dynamics on  $\partial F_N$ . In particular, they cannot have a parabolic orbit.

Second, let  $\delta$  be the automorphism of  $F_2 = \langle a, b \rangle$  defined by  $\delta(a) = a$  and  $\delta(b) = ba$ . The outer automorphism class  $D$  of  $\delta$  is sometimes called a *Dehn twist automorphism*. The reader, who has in mind

the action by isometries of  $SL_2(\mathbb{Z})$  on the hyperbolic plane, should be warned that Dehn twist automorphisms do not give rise to parabolic orbits in  $\partial F_2$ . We give in section 6 a description of all possible dynamics of automorphisms of  $F_2$  in the outer class  $D^n$ , for  $n \in \mathbb{Z}$ .

Third, more generally, it is known that geometric automorphisms of  $F_N$  do not have parabolic orbits in  $\partial F_N$ . We recall that an automorphism  $\varphi$  of  $F_N$  is *geometric* if there exist a surface  $S$  (with non empty boundary) with fundamental group  $\pi_1(S)$  isomorphic to  $F_N$  and a homeomorphism  $f$  of  $S$  which induces  $\varphi$  on  $F_N \cong \pi_1(S)$ . More details are given in section 4.2. As a consequence, since all automorphisms of  $F_2$  are known to be geometric, one obtains:

**Proposition 1.2** *There does not exist an automorphism of  $F_2$  which has a parabolic orbit.*

To our knowledge, the question of the existence of automorphisms with a parabolic orbit is still open for  $F_3$ .

*Acknowledgments.* I would like to express my gratitude to Gilbert Levitt, who has posed the question of the existence of parabolic orbits as part of my thesis project, see[10], and has consistently encouraged me to publish my results since then. I would like to thank Pascal Hubert and Erwan Lanneau for helpful discussions about dilatation coefficients of matrices in  $SL_2(\mathbb{Z})$ . I am grateful to Martin Lustig for his active interest in the present paper.

## 2 A first example

For the impatient reader, we give a first example of an automorphism of  $F_4 = \langle a, b, c, d \rangle$  with a parabolic orbit “inside  $F_4$ ” (using Proposition 3.5, this gives immediately a parabolic orbit in  $\partial F_4$ ).

Let  $\varphi$  be the automorphism defined by:

$$\begin{aligned} \varphi : \quad a &\mapsto a \\ b &\mapsto ba \\ c &\mapsto ca^2 \\ d &\mapsto dc. \end{aligned}$$

The inverse of  $\varphi$  is given by:

$$\begin{aligned} \varphi^{-1} : \quad a &\mapsto a \\ b &\mapsto ba^{-1} \\ c &\mapsto ca^{-2} \\ d &\mapsto da^2c^{-1}. \end{aligned}$$

The common limit point of the forward and backward iteration of  $\varphi$  (called a “parabolic fixed point”) will be the element:  $ba^{-\infty} = ba^{-1}a^{-1}a^{-1}a^{-1} \dots \in \partial F_4$ . The element of  $F_4$  which gives rise to a parabolic orbit with this limit point is  $bd^{-1}$ . We calculate:

$$\begin{aligned} bd^{-1} \xrightarrow{\varphi} bac^{-1} \cdot d^{-1} \xrightarrow{\varphi} bc^{-1} \cdot c^{-1}d^{-1} \xrightarrow{\varphi} ba^{-1}c^{-1} \cdot a^{-2}c^{-1}c^{-1}d^{-1} \xrightarrow{\varphi} ba^{-2}c^{-1} \cdot a^{-4}c^{-1}a^{-2}c^{-1}c^{-1}d^{-1} \xrightarrow{\varphi} \dots \\ b \cdot d^{-1} \xrightarrow{\varphi^{-1}} ba^{-1} \cdot ca^{-2}d^{-1} \xrightarrow{\varphi^{-1}} ba^{-2} \cdot ca^{-4}ca^{-2}d^{-1} \xrightarrow{\varphi^{-1}} ba^{-3} \cdot ca^{-6}ca^{-4}ca^{-2}d^{-1} \xrightarrow{\varphi^{-1}} \dots \end{aligned}$$

In these calculations, we help the reader to follow through the iteration by introducing an extra  $\cdot$  which is “mapped” to the  $\cdot$  in the next iteration step. The crucial feature is that at any of these  $\cdot$  no cancellation does occur. We see that  $\lim_{k \rightarrow +\infty} \varphi^k(bd^{-1}) = \lim_{k \rightarrow +\infty} \varphi^{-k}(bd^{-1}) = ba^{-\infty}$ . A more formal justification is given in section 5.

### 3 Basics

This section serves sort of as glossary: We summarize in a sequence of brief subsections the basic definitions and facts which are needed to follow the arguments in the subsequent sections. The expert reader is encouraged to skip the first few subsections (and to go back later to them, if need be). However, the terminology introduced in the last subsections is non-standard and should be read carefully.

#### 3.1 The induced boundary homeomorphism

Let  $F_N$  denote the free group of finite rank  $N \geq 2$ . The boundary  $\partial F_N$  of  $F_N$  is a Cantor set. If  $\mathcal{A} = \{a_1, \dots, a_N\}$  is a basis of  $F_N$ , we denote by  $\mathcal{A}^{\pm 1}$  the set  $\{a_1, \dots, a_N, a_1^{-1}, \dots, a_N^{-1}\}$ . A word  $w = w_1 \dots w_p$  ( $w_i \in \mathcal{A}^{\pm 1}$ ) is *reduced* if  $w_{i+1} \neq w_i^{-1}$ . The free group  $F_N$  can be understood as the set of (finite) reduced words in  $\mathcal{A}^{\pm 1}$ . Then the boundary  $\partial F_N$  is naturally identified to the set of (right) infinite reduced words  $X = x_1 \dots x_p \dots$  with  $x_i \in \mathcal{A}^{\pm 1}$ ,  $x_{i+1} \neq x_i^{-1}$ . The cylinder defined by a reduced word  $w = w_1 \dots w_p$  is the set of right-infinite reduced words  $X = x_1 \dots x_k \dots$  which admit  $w$  as prefix:  $x_i = w_i$  for  $i \in \{1, \dots, p\}$ . A basis of topology of  $\partial F_N$  is given by the set of all such cylinders.

An automorphism  $\varphi$  of a free group  $F_N$  induces a homeomorphism  $\partial\varphi$  of the boundary  $\partial F_N$ . This can easily be checked by considering a standard set of generators of the automorphisms group  $\text{Aut}(F_N)$  of  $F_N$ . Alternatively, this can be seen as a consequence of the fact that a quasi-isometry of a proper Gromov-hyperbolic space induces a homeomorphism on the boundary of this space, see [7]. Indeed,  $F_N$  equipped with the word metric associated to a basis  $\mathcal{A}$ , is a proper Gromov-0-hyperbolic space, and any automorphism of  $F_N$  is a quasi-isometry of  $F_N$  with respect to this metric.

#### 3.2 Compactification of $F_N$

Let  $\bar{F}_N$  denote the union of  $F_N$  and its boundary  $\partial F_N$ , i.e.  $\bar{F}_N = F_N \cup \partial F_N$ . Given a basis of  $F_N$ , if  $w$  is a reduced word, let  $C_w$  be the set of reduced finite or infinite words which have  $w$  as prefix. A basis of topology of  $\bar{F}_N$  is given by the finite subsets of  $F_N$  and the sets  $C_w$  (with  $w$  describing all the reduced words of  $F_N$ ). Then  $\bar{F}_N$  is a compact set, and the inclusions of  $F_N$  and  $\partial F_N$  in  $\bar{F}_N$  are embeddings. If  $\varphi$  is an automorphism of  $F_N$ ,  $\bar{\varphi}$  will denote the map defined by  $\bar{\varphi}(g) = \varphi(g)$  if  $g \in F_N$  and  $\bar{\varphi}(X) = \partial\varphi(X)$  if  $X \in \partial F_N$ . The map  $\bar{\varphi}$  is a homeomorphism of  $\bar{F}_N$ .

#### 3.3 Getting rid of periodicity

Let  $f$  be a homeomorphism of a topological space  $\mathcal{X}$ . We denote by  $\text{Fix}(f) = \{x \in \mathcal{X} \mid f(x) = x\}$  the set of fixed points of  $f$ , and by  $\text{Per}(f) = \bigcup_{k \in \mathbb{N}} \text{Fix}(f^k)$  the set of periodic points of  $f$ .

Levitt and Lustig have proved in [14] that there exists an integer  $p$ , which depends only on the rank  $N$  of  $F_N$ , such that for all  $\varphi \in \text{Aut}(F_N)$ , the periodic points of  $\overline{\varphi}^p$  are fixed points:  $\text{Fix}(\overline{\varphi}^p) = \text{Per}(\overline{\varphi}^p)$ . This result has been refined by Feighn and Handel in [5], where the notion of “forward rotationless” outer automorphism has been introduced. This lead us to say, in this paper, that an automorphism  $\varphi \in \text{Aut}(F_N)$  is *rotationless* if  $\text{Fix}(\overline{\varphi}) = \text{Per}(\overline{\varphi})$ . The previously mentioned result can be rephrased as follows:

**Theorem 3.1** (Levitt-Lustig) *Any automorphism  $\varphi \in \text{Aut}(F_N)$  has a power  $\varphi^p$  ( $p \in \mathbb{N}$ ) which is rotationless.*

### 3.4 Nature of fixed points

Let  $\varphi$  be a rotationless automorphism of  $F_N$ . The set  $\text{Fix}(\varphi)$  is a subgroup of  $F_N$ , which is called the *fixed subgroup* of  $\varphi$ . This fixed subgroup has finite rank, see [4]. More precisely, it is proved in [2] that  $\text{rank}(\text{Fix}(\varphi)) \leq N$ . In particular,  $\text{Fix}(\varphi)$  is a quasiconvex subgroup of  $F_N$ , and thus its boundary  $\partial\text{Fix}(\varphi)$  naturally injects into  $\partial F_N$ . By continuity of  $\overline{\varphi}$ , every point of  $\partial\text{Fix}(\varphi)$  is contained in  $\text{Fix}(\partial\varphi)$ . Following Nielsen, these fixed points of  $\partial\varphi$  are called *singular*; the fixed points of  $\partial\varphi$  which are not singular are called *regular*.

A fixed point  $X$  of  $\partial\varphi$  is *attracting* if there exists a neighbourhood  $U$  of  $X$  in  $\overline{F}_N$  such that the sequence  $\overline{\varphi}^k(x)$  converges to  $X$  for all  $x$  in  $U$ . A fixed point  $X$  of  $\partial\varphi$  is *repulsing* if it is attracting for  $\partial\varphi^{-1}$ . It is proved in [6] that:

**Lemma 3.2** *Let  $\varphi \in \text{Aut}(F_N)$ . A regular fixed point of  $\partial\varphi$  is either attracting or repulsing.* □

However, outside of the regular fixed point set, i.e. for singular fixed points, the dynamics can be quite a bit more complicated. In particular, there may exist *mixed* fixed points, i.e. fixed points which serve as attractor for some orbits, and simultaneously as repeller for others. This phenomenon is rather common; some concrete examples will be spelled out in the subsequent sections.

A particular case of a mixed fixed point is the case (defined in the Introduction) of a parabolic fixed point. Thus we obtain as special case the following consequence of Lemma 3.2:

**Remark 3.3** Any parabolic fixed point of  $\varphi$  is singular.

### 3.5 Limit points

Let  $\varphi$  be a rotationless automorphism of  $F_N$ . For any  $x \in \overline{F}_N$ , if the limit  $\lim_{k \rightarrow +\infty} \overline{\varphi}^k(x)$  exists, we denote it by  $\omega_\varphi(x)$ . In [16], Levitt and Lustig have proved:

**Theorem 3.4** (Levitt-Lustig) *Let  $\varphi \in \text{Aut}(F_N)$  be rotationless. Then for any  $x \in \overline{F}_N$  the sequence  $\overline{\varphi}^k(x)$  converges to some element  $\omega_\varphi(x) \in \text{Fix}(\overline{\varphi})$ .* □

A point  $X \in \partial\text{Fix}(\varphi)$  is a  $\omega$ -limit point of  $\varphi$  if there exists  $x \in \overline{F_N}$  such that  $X = \omega_\varphi(x)$ . A point  $X \in \partial\text{Fix}(\varphi)$  is a limit point of  $\varphi$  if it is a  $\omega$ -limit point of  $\varphi$  or  $\varphi^{-1}$ . Let  $L_\varphi^\omega$  denote the set of  $\omega$ -limit points of  $\varphi$  and let  $L_\varphi$  denote the set of limit points of  $\varphi$ .

For any  $g \in F_N$ ,  $g \neq 1$ , the sequence  $g^k$  has a limit in  $\partial F_N$  when  $k \rightarrow +\infty$ : this limit is denoted by  $g^\infty$ .

**Proposition 3.5** *Let  $\varphi \in \text{Aut}(F_N)$  be a rotationless automorphism. If  $g \in F_N \setminus \text{Fix}(\varphi)$ , then:*

$$\omega_\varphi(g) = \omega_\varphi(g^\infty).$$

**Proof** The proof is a simple adaptation of the arguments in the proof of [13, Proposition 2.3]. We fix a basis  $\mathcal{A}$  of  $F_N$ . We note that for all  $g \in F_N \setminus \{1\}$ , the Gromov product  $(g, g^\infty)$  (i.e. the length of longest common prefix) of  $g$  and  $g^\infty$  is bigger than  $\frac{1}{2}(|g| + 1)$  (where  $|g|$  denotes the length of  $g$  in the basis  $\mathcal{A}$ ). If  $g \notin \text{Fix}(\varphi)$ , then the length of  $\varphi^k(g)$ , and thus also the Gromov product  $(\varphi^k(g), (\varphi^k(g))^\infty)$ , tend to infinity. Theorem 3.4 implies that  $\omega_\varphi(g) = \omega_\varphi(g^\infty)$ .  $\square$

Proposition 3.5 shows that  $L_\varphi^\omega = \{\omega_\varphi(X) \mid X \in \partial F_N\}$ . We do not know whether  $L_\varphi^\omega = \{\omega_\varphi(g) \mid g \in F_N\}$  holds.

### 3.6 Isoglossy classes

For any  $\varphi \in \text{Aut}(F_N)$ , two points  $X, Y \in \partial F_N$  are called *isoglossy* (with respect to  $\varphi$ ) if there exists some  $g \in \text{Fix}(\varphi)$  such that  $X = gY$ . It follows directly from this definition that isoglossy is an equivalence relation. The fixed subgroup  $\text{Fix}(\varphi)$  acts naturally on the fixed point set  $\text{Fix}(\partial\varphi)$ , which is thus naturally partitioned into isoglossy classes. If  $X, Y \in \text{Fix}(\partial\varphi)$  are isoglossy, then they are of same ‘‘dynamical type’’: they are simultaneously singular, attracting, repulsing, mixed, parabolic or limit points.

### 3.7 Dynamics graph

Let  $\varphi \in \text{Aut}(F_N)$  be a rotationless automorphism. We associate to  $\varphi$  a graph  $\Gamma_\varphi$ , called the *dynamics graph* of  $\varphi$ . The vertices of  $\Gamma_\varphi$  are the isoglossy classes of points of  $L_\varphi$ . There is an oriented edge from the isoglossy class  $x_1$  to the isoglossy class  $x_2$  if there exists some representatives  $X_i$  of  $x_i$  and  $X \in \partial F_N$  such that  $\omega_{\varphi^{-1}}(X) = X_1$  and  $\omega_\varphi(X) = X_2$ . The main theorem of [10] states that  $\Gamma_\varphi$  is a finite graph. We give in Figure 1 the dynamics graph of an automorphism which has North-South dynamics on  $\partial F_N$ .



Figure 1: North-South dynamics graph

Finally, we note that, for a rotationless automorphism  $\varphi$ , the existence of parabolic orbit is equivalent to the fact that there is an edge of the dynamics graph  $\Gamma_\varphi$  which is a loop.

**Remark 3.6** In [11] G. Levitt introduces a graph in order to code the dynamics of so-called “simple-dynamics homeomorphisms” of the Cantor set  $C$ : a homeomorphism  $f : C \rightarrow C$  has *simple dynamics* if the set  $\text{Fix}(f)$  of its fixed points is finite, and if the sequence  $f^n$  uniformly converges on any compact set disjoint from  $\text{Fix}(f)$ . If  $\varphi \in \text{Aut}(F_N)$  is a rotationless automorphism with trivial fixed subgroup, then  $\partial\varphi$  has simple dynamics, and the graph  $\Gamma_\varphi$  is the same as the one defined in [11]. In this case, the fixed points of  $\partial\varphi$  are either attracting or repulsing. Thus, if one is interested in parabolic orbits, which are the main focus of the present paper, one has to purposefully leave to world of “simple dynamics” homeomorphisms.

## 4 Examples

### 4.1 Inner automorphisms

Let  $i_u \in \text{Aut}(F_N)$  denote the *conjugation*, or *inner automorphism*, by  $u \in F_N$ , i.e.  $i_u(g) = ugu^{-1}$  for all  $g \in F_N$ . The set  $\text{Inn}(F_N)$  of inner automorphisms of  $F_N$  is a normal subgroup of  $\text{Aut}(F_N)$ . The quotient group, denoted by  $\text{Out}(F_N)$ , is the group of *outer automorphisms* of  $F_N$ .

The homeomorphism  $\partial i_u : \partial F_N \rightarrow \partial F_N$  induced by  $i_u$  is the left translation by  $u$ :  $\partial i_u(X) = uX$ . If  $u \neq 1$ , the map  $\partial i_u$  has precisely 2 fixed points:  $u^\infty$  and  $u^{-\infty}$  (where  $u^\infty$  is the limit of the sequence  $u^k$ , and  $u^{-\infty}$  is the limit of the sequence  $u^{-k}$ , for  $k \rightarrow +\infty$ ). Moreover, for any point  $X \in \partial F_N$  different from  $u^{-\infty}$ , the sequence  $\partial\varphi^k(X)$  converges to  $u^\infty$  when  $k$  tends to infinity. One checks easily that the map  $\partial i_u$  has North-South dynamics, from  $u^{-\infty}$  to  $u^\infty$ , on  $\partial F_N$ , see [13] for instance.

**Remark 4.1** We note that the fixed subgroup of  $i_u$  is cyclic, generated by the root of  $u$  (i.e. the element  $v \in F_N$  such that  $u = v^p$  with  $p \in \mathbb{N}$  maximal). In particular,  $u^\infty$  and  $u^{-\infty}$  are singular fixed points of  $i_u$ . This shows that when defining “ $X$  is an attracting fixed point of  $\varphi$ ” in section 3.4, it makes a crucial difference that we request the neighbourhood  $U$  of  $X$  to be taken in  $\overline{F}_N$  and not just in  $\partial F_N$ .

### 4.2 Geometric automorphisms

Let  $\Sigma$  be a compact surface with fundamental group  $\pi_1(\Sigma)$  isomorphic to  $F_N$  (in particular,  $\Sigma$  has non empty boundary). The surface  $\Sigma$  can be equipped with a hyperbolic metric (i.e. a metric of constant curvature equal to  $-1$ ) in such a way that every boundary component of the boundary of  $\Sigma$  is a geodesic. The universal cover  $\tilde{\Sigma}$  of  $\Sigma$  is then identified with a closed convex subset of the hyperbolic plane  $\mathbb{H}^2$ , and the Gromov boundary  $\partial\tilde{\Sigma}$  of  $\tilde{\Sigma}$ , which is naturally identified with the boundary  $\partial F_N$  of  $F_N$ , injects in the boundary (or circle at infinity)  $S_\infty$  of  $\mathbb{H}^2$ . Since  $S_\infty$  is a circle, it can be equipped with a natural cyclic order. This order on  $S_\infty$  induces a cyclic order on  $\partial F_N$ .

In his fundamental work [19, 20, 21], Nielsen proposed an original and fruitful point of view to study homeomorphisms of surfaces. The basic idea is that the behaviour of a homeomorphism  $f$  of a surface  $\Sigma$  is well reflected by the collection of all the lifts  $\tilde{f}$  of  $f$  to  $\tilde{\Sigma}$  which have each much simpler individual behaviour. This idea is at the origin of what is now called “Nielsen-Thurston classification” of homeomorphisms of surfaces, see [9], and it has much influenced the study of (outer) automorphisms of free groups, see [6, 5, 8].

The key fact is that any lift  $\tilde{f}$  of  $f$  induces a homeomorphism  $\partial\tilde{f}$  of  $\partial\tilde{\Sigma}$ . A basic (but rather fundamental) remark is that  $\partial\tilde{f}$  preserves the cyclic order on  $\partial\tilde{\Sigma} \subseteq S_\infty$ .

An homeomorphism  $f$  of  $\Sigma$  induces an outer automorphism of  $\pi_1(\Sigma)$ , and thus an outer automorphism  $\Phi \in \text{Out}(F_N)$  (in fact, this outer automorphism  $\Phi$  only depends on the mapping class of  $f$ ). Such an outer automorphism  $\Phi$  of  $F_N$  (and also any automorphism  $\varphi \in \Phi$ ) is called *geometric*. Classical Galois theory for covering spaces states that the lifts of  $f$  are in bijective correspondance with the automorphisms in the outer class  $\Phi$ . More precisely, an automorphism  $\varphi \in \Phi$  and a lift  $\tilde{f}$  of  $f$  are in correspondance if, and only if,

$$\varphi(g) \circ \tilde{f} = \tilde{f} \circ g \quad \forall g \in F_N,$$

where the elements of  $F_N$  are considered as deck transformations of  $\tilde{\Sigma}$ . As a consequence, the dynamics of  $\partial\tilde{f}$  on  $\partial\tilde{\Sigma}$  and the dynamics of  $\partial\varphi$  on  $\partial F_N$  are conjugate via the natural identification between  $\partial\tilde{\Sigma}$  and  $\partial F_N$ .

It follows from the previous discussion that, for any geometric automorphism  $\varphi \in \text{Aut}(F_N)$ , the homeomorphism  $\partial\varphi$  of  $\partial F_N$  must preserve a cyclic order on  $\partial F_N$ .

Another fact proved by Nielsen is that  $\partial\tilde{f}$  has at least 2 periodic points on  $\partial F_N$  (for a proof in the context of free groups, see [14]). This means that there exists a positive power of  $\partial\tilde{f}$  which has at least 2 fixed points on  $\partial F_N$ . Both these facts (existence of 2 fixed points and preservation of a cyclic order) yield directly:

**Proposition 4.2** *A geometric automorphism of  $F_N$  cannot have a parabolic orbit in  $\partial F_N$ .*

This fact is particularly meaningful for the free group of rank 2. Indeed, it is well known that any outer automorphism of  $F_2$  can be induced by a homeomorphism of a torus with one boundary component, see [18]. This is precisely how Proposition 1.2 is proved.

### 4.3 Outer automorphisms

Although well known, we believe that at this point it might be wise to alert the less expert reader about a common misunderstanding. It is by no means true that any two automorphisms  $\varphi, \varphi'$  which belong to the same outer automorphism class  $\Phi$ , must have conjugate dynamics. Indeed, their dynamics graphs  $\Gamma_\varphi$  and  $\Gamma_{\varphi'}$  may look quite different. Concrete examples are easy to come by, and some are given in the subsequent sections.

The reader who wants to be more subtle can easily check that indeed some automorphisms in  $\Phi$  have naturally conjugate dynamics. The resulting *isogredience classes* go again all the way back to Nielsen (see also [13]), and one could associate to  $\Phi$  a *total dynamics graph* which is the disjoint union of the  $\Gamma_\varphi$  over a set of representatives for the single isogredience classes. However, this goes beyond the scope of this paper.

## 5 Parabolic orbits

### 5.1 Structure of a parabolic fixed point

Let  $\varphi \in \text{Aut}(F_N)$  be an automorphism, and  $X \in \text{Fix}(\partial\varphi)$  be a parabolic fixed point for  $\varphi$ . We have seen (cf Remark 3.3) that  $X$  must be singular. A point  $X \in \partial F_N$  is *rational* if it is a fixed point of an inner

automorphism, i.e.  $X = u^\infty$  for some  $u \in F_N \setminus \{1\}$ . It is proved in [10] that singular limit points of  $\varphi$  are rational. We deduce the following:

**Lemma 5.1** *A parabolic fixed point  $X$  of  $\varphi \in \text{Aut}(F_N)$  is a singular rational point:  $X = u^\infty$  with  $u \in \text{Fix}(\varphi)$ .  $\square$*

Moreover, we have:

**Proposition 5.2** *Let  $\varphi$  be an automorphism of  $F_N$ , and  $X \in \text{Fix}(\partial\varphi)$  be a parabolic fixed point for  $\varphi$ . Then any neighborhood of  $X$  in  $\partial F_N$  contains a full orbit  $\{\partial\varphi^k(Y) \mid k \in \mathbb{Z}\} \subset \partial F_N$*

**Proof** We have seen that  $X = u^\infty$ , with  $u \in \text{Fix}(\varphi)$ . We consider a given neighborhood  $\mathcal{V}$  of  $X$ . Let  $\vartheta = \{\partial\varphi^k(Y) \mid k \in \mathbb{Z}\}$  be a parabolic orbit for  $X$ . We note that  $\vartheta \cup \{X\}$  is a compact subset of  $\partial F_N$ . Moreover,  $u^{-\infty} \notin \vartheta \cup \{X\}$  because  $Y \notin \text{Fix}(\partial\varphi)$ . Since the sequence  $(\partial i_u^p)_{p \in \mathbb{N}}$  uniformly converges on compact subsets of  $\partial F_N \setminus \{u^{-\infty}\}$  towards  $u^\infty$  when  $p$  tends to infinity, see section 4.1, the set  $\partial i_u^p(\vartheta)$  is contained in  $\mathcal{V}$ , up to taking  $p$  sufficiently large. We remark that, since  $u \in \text{Fix}(\varphi)$ ,  $\partial i_u^p(\partial\varphi^k(Y)) = \partial\varphi^k(u^p Y)$ , and thus  $\partial i_u^p(\vartheta) = u^p \vartheta$  is a parabolic orbit for  $X$ .  $\square$

## 5.2 Automorphisms of $F_4$ which have parabolic orbits

For any  $k \in \mathbb{N}$ , consider the automorphism  $\varphi_k$  of  $F_4 = \langle a, b, c, d \rangle$  given by:

$$\begin{aligned} \varphi_k : \quad a &\mapsto a \\ b &\mapsto ba \\ c &\mapsto ca^{k+1} \\ d &\mapsto dc \end{aligned}$$

and its inverse:

$$\begin{aligned} \varphi_k^{-1} : \quad a &\mapsto a \\ b &\mapsto ba^{-1} \\ c &\mapsto ca^{-k-1} \\ d &\mapsto da^{k+1}c^{-1} \end{aligned}$$

The rose  $R_4$  is the geometric realization of graph with one vertex and 4 edges. We put an orientation on each edge, and we label them by  $a, b, c$  and  $d$ . We can turn  $R_4$  into a length space by declaring that each edge has length 1. As usual, the automorphisms  $\varphi_k^{\pm 1}$  can be realized as homotopy equivalences  $f_k^{\pm 1}$  of the rose  $R_4$  where each edge is mapped linearly to the edge path with label preassigned by  $\varphi_k^{\pm 1}$ .

In fact, the automorphisms  $\varphi_k^{\pm 1}$  define outer automorphisms which are unipotent polynomially growing in the sense of [1], and the maps  $f_k^+$  satisfy the conclusions of Theorem 5.1.8 of [1]. We do not quote here the statement of this theorem, which would lead us to introduce a lot of technical background, but we freely use in the sequel some consequences of it.

Let  $\mathcal{A}$  be a basis of  $F_N$ . We denote by  $[g]$  the reduced word, in the basis  $\mathcal{A}$ , representing the element  $g \in F_N$ . Let  $\varphi$  be an automorphism of  $F_N$ . A *splitting* of  $g \in F_N$  for  $\varphi$  is a way to write  $g = g_1 \dots g_n$  such that:

- (i)  $n \geq 2$ ,
- (ii) for all  $i \in \{1, \dots, n\}$ ,  $g_i \in F_N \setminus \{1\}$ ,
- (iii) for all  $p \in \mathbb{N}$ , for all  $i \in \{1, \dots, n-1\}$ ,  $[\varphi^p(g_i)][\varphi^p(g_{i+1})] = [\varphi^p(g_i g_{i+1})]$  (this means that no cancellation occurs between  $[\varphi^p(g_i)]$  and  $[\varphi^p(g_{i+1})]$ ).

In that case, we note  $g = g_1 \cdot \dots \cdot g_n$ , and each  $g_i$  is called a *brick* of the splitting.

We now apply that Theorem 5.1.8 of [1] to the given family  $\varphi_k$  and obtain:

**Lemma 5.3** For all  $g \in F_4$ , there exists some  $p_0 \in \mathbb{N}$  such that for all  $p \geq p_0$ ,  $[\varphi_k^p(g)]$  and  $[\varphi_k^{-p}(g)]$  have a splitting, the bricks of which are either edges or paths of the following labels:  $ba^q b^{-1}$ ,  $ca^q c^{-1}$ ,  $ba^q c^{-1}$  or  $ca^q b^{-1}$ , for some  $q \in \mathbb{Z}$ .  $\square$

**Remark 5.4** For the reader who is familiar with the terminology of [1], the edge paths labelled by  $ba^q b^{-1}$ ,  $ca^q c^{-1}$ ,  $ba^q c^{-1}$  or  $ca^q b^{-1}$  are precisely the exceptional paths of the improved train-track map  $f_k$ .

**Remark 5.5** As a consequence of Lemma 5.3, one can easily check that the sequence  $(|[\varphi_k^p(g)]|)_{p \in \mathbb{N}}$  of lengths of  $[\varphi_k^p(g)]$  is bounded above by a polynomial of degree 2 in  $p$ .

It is claimed in [17] that there exists a general algorithm to compute the fixed subgroup of a given automorphism of  $F_N$ . There exist some easier algorithms for special cases: for instance, one could use [3] to compute the fixed subgroup of  $\varphi_k$ . In fact, it is sufficient to determine the so called *indivisible Nielsen paths*, see [2].

Let  $\mathcal{NP}$  denote the set  $\{a^q, ba^q b^{-1}, ca^q c^{-1} \mid q \in \mathbb{Z}\}$ . We notice that  $\mathcal{NP} \subseteq \text{Fix}(\varphi_k)$ . For an element  $g \in F_4$ , we consider the splitting of  $[\varphi_k^p(g)] = g_1 \cdot g_2 \cdot \dots \cdot g_r$  given by Lemma 5.3. If  $g_i \in \mathcal{NP}$  for all  $i \in \{1, \dots, r\}$ , then  $g \in \text{Fix}(\varphi_k)$ . Otherwise, there is some  $i_0$  such that  $g_i \in \mathcal{NP}$  for all  $i \in \{1, \dots, i_0\}$  and  $g_{i_0+1} \notin \mathcal{NP}$ . For simplicity, we write  $\omega_{\varphi_k} = \omega_k$ . Then  $\omega_k(g) = g_1 \dots g_{i_0} \omega_k(g_{i_0+1}) \in \partial F_4$ . This shows that  $\text{Fix}(\varphi_k) = \langle a, bab^{-1}, cac^{-1} \rangle$ .

Moreover, we thus obtain all the isoglossy classes of limit points of  $\varphi_k$  by considering all the  $\omega_k(h)$  with  $h \in \{b^{\pm 1}, c^{\pm 1}, d^{\pm 1}, ba^q c^{-1}, ca^q b^{-1} \mid q \in \mathbb{Z}\}$ . A direct computation gives:  $\varphi_k^p(b) = ba^p$ ,  $\varphi_k^p(c) = ca^{p(k+1)}$ ,  $\varphi_k^p(ba^q c^{-1}) = ba^{q-kp} c^{-1}$ ,  $\varphi_k^p(d) = dcca^{k+1} ca^{2(k+1)} \dots ca^{(p-1)(k+1)}$ ,  $\varphi_k^p(d^{-1}) = c^{-1} \cdot d^{-1}$ . We derive that:

- $\omega_k(b) = ba^{+\infty}$ ,
- $\omega_k(c) = \omega_k(ca^q b^{-1}) = ca^{+\infty}$ ,
- $\omega_k(b^{-1}) = \omega_k(c^{-1}) = \omega_k(d^{-1}) = a^{-\infty}$ ,
- $\omega_k(ba^q c^{-1}) = ba^{-\infty}$ ,
- $\omega_k(d) = X_k^+$ ,

where  $X_k^+ = \omega_k(d) = dcca^{k+1} ca^{2k+2} ca^{3k+3} \dots$ . We have thus shown that there are only 5 isoglossy classes in  $L_{\varphi_k}^\omega$ , given by  $X_k^+$ ,  $a^{-\infty}$ ,  $ca^{+\infty}$ ,  $ba^{+\infty}$ ,  $ba^{-\infty}$ .

**Theorem 5.6** The set  $\{\varphi_k \mid k \in \mathbb{N}\}$  is a family of automorphisms of  $F_4$ , such that each  $\varphi_k$  has a parabolic orbit. The dynamics graph of  $\varphi_k$  is given in Figure 2. For any  $k, k', p, p' \in \mathbb{N}$ ,  $\varphi_k^p$  and  $\varphi_{k'}^{p'}$  are conjugate if and only if  $k = k'$  and  $p = p'$ .

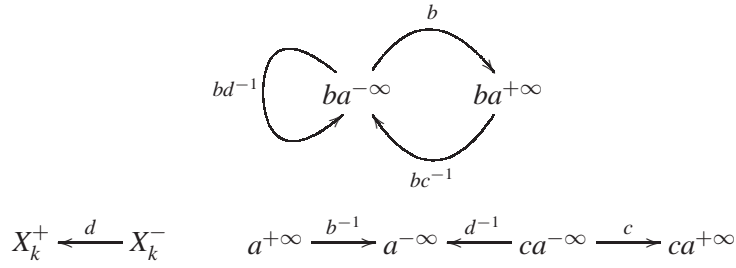


Figure 2: The dynamics graph of  $\varphi_k$  has 3 connected components. A label  $g$  has been added to each edge: it means that  $\omega_k(g)$  is the endpoint of the edge and  $\omega_k^-(g)$  is the origin of the edge.

**Proof** We write  $\omega_{\varphi_k^{-1}} = \omega_k^-$ . Arguing for  $\varphi_k^{-1}$  as we have done for  $\varphi_k$ , we obtain that:

- $\omega_k^-(b^{-1}) = \omega_k^-(c^{-1}) = a^{+\infty}$ ,
- $\omega_k^-(c) = \omega_k^-(d^{-1}) = ca^{-\infty}$ ,
- $\omega_k^-(b) = ba^{-\infty}$ ,
- $\omega_k^-(bc^{-1}) = ba^{+\infty}$ ,
- $\omega_k^-(d) = X_k^-$ ,

where  $X_k^- = \omega_k^-(d) = da^{k+1}c^{-1}a^{2k+2}c^{-1}a^{3k+3}c^{-1} \dots$ . Again, there are only 5 isoglossy classes in  $L_{\varphi_k}^\omega$ , given by  $X_k^-, a^{+\infty}, ca^{-\infty}, ba^{-\infty}, ba^{+\infty}$ .

Note that  $\varphi_k(bd^{-1}) = bac^{-1} \cdot d^{-1}$  is a splitting for  $\varphi_k$ . Hence  $\omega_k(bd^{-1}) = \omega_k(bac^{-1}) = ba^{-\infty}$ . On the other hand,  $b \cdot d^{-1}$  is a splitting for  $\varphi_k^{-1}$ . Hence  $\omega_k^-(bd^{-1}) = \omega_k^-(b) = ba^{-\infty}$ . Thus  $ba^{-\infty}$  is parabolic fixed point for  $\varphi_k$ .

Suppose that  $\varphi_k^p$  and  $\varphi_{k'}^{p'}$  are conjugate ( $k, k', p, p' \in \mathbb{N}$ ): there exists  $\psi \in \text{Aut}(F_4)$  such that  $\varphi_k^p = \psi \varphi_{k'}^{p'} \psi^{-1}$ . Let  $M_k, M_{k'}, P \in GL(4, \mathbb{Z})$  be the matrices obtained by abelianization of respectively  $\varphi_k, \varphi_{k'}$  and  $\psi$ . Then

$$M_k^p = \begin{pmatrix} 1 & p & (k+1)p & \frac{1}{2}(k+1)p(p-1) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & p \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Computing  $M_k^p P = P M_{k'}^{p'}$ , one sees that  $P$  must have the following shape:

$$P = \begin{pmatrix} \lambda_1 & \mu_1 & \mu_2 & \mu_3 \\ 0 & \lambda_2 & 0 & \mu_4 \\ 0 & \mu_5 & \lambda_3 & \mu_6 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$$

with

$$(1) \quad p'(k'+1)\lambda_3 = p(k+1)\lambda_1 \quad \text{and} \quad p'\lambda_3 = p\lambda_4.$$

We deduce that  $\det P = \lambda_1 \lambda_2 \lambda_3 \lambda_4$ , and thus  $\lambda_i \in \{\pm 1\}$ , since  $\det P = \pm 1$ . From (1) we derive  $k = k'$  and  $p = p'$ .  $\square$

### 5.3 Parabolic orbits for $N \geq 5$

For any  $k \in \mathbb{N}$ , consider the automorphism  $\alpha_k$  of  $F_5 = \langle a, b, c, d, e \rangle$  given by:

$$\begin{aligned} \alpha_k : \quad a &\mapsto a \\ b &\mapsto ba \\ c &\mapsto ca^{k+1} \\ d &\mapsto dc \\ e &\mapsto e \end{aligned}$$

Since the restriction of  $\alpha_k$  to  $\langle a, b, c, d \rangle$  is  $\varphi_k$ , it is clear that  $\omega_{\alpha_k}(bd^{-1}) = \omega_{\alpha_k^{-1}}(bac^{-1}) = ba^{-\infty}$  is a parabolic fixed point for  $\alpha_k$ . Considering the abelianization and arguing as previously, we check that if  $k \neq k'$  and  $p \neq p'$ , then  $\alpha_k^p$  and  $\alpha_{k'}^{p'}$  cannot be conjugate.

If  $N \geq 6$ , we split  $F_N = F_4 * F_2 * F_{N-6}$ . We first recall some facts about  $\text{Out}(F_2)$ . It is well known, since Nielsen [18], that the abelianisation morphism from  $\text{Out}(F_2)$  to  $GL_2(\mathbb{Z})$  is an isomorphism. If  $M \in SL_2(\mathbb{Z})$  has a trace bigger than 2, then  $M$  has an eigenvalue  $\lambda > 1$  which is an algebraic unity of a quadratic extension of  $\mathbb{Q}$ : we call  $\lambda$  the dilatation of  $M$ . For all  $k \in \mathbb{N}$  prime, there exists  $M_k \in SL_2(\mathbb{Z})$  such that the dilatation  $\lambda_k$  of  $M_k$  belongs to  $\mathbb{Q}(\sqrt{k}) \setminus \mathbb{Q}$ . This implies in particular that for all  $p \in \mathbb{N}$ ,  $\lambda_k^p \in \mathbb{Q}(\sqrt{k}) \setminus \mathbb{Q}$ . We choose  $\theta_k \in \text{Aut}(F_2)$  in the outer class represented by  $M_k$ . Then the automorphism  $\theta_k^p$  has growth rate equal to  $\lambda_k^p$ .

We define  $\beta_k \in \text{Aut}(F_N)$  by  $\beta_k = \varphi_1 * \theta_k * id$ , where  $id$  is the identity on  $F_{N-6}$ . Again,  $\omega_{\beta_k}(bd^{-1}) = \omega_{\beta_k^{-1}}(bac^{-1}) = ba^{-\infty}$  is a parabolic fixed point for  $\beta_k$ . Since  $\varphi_1$  is polynomially growing, it follows that the growth rate of  $\beta_k^p$  is  $\lambda_k^p$  (see for instance [12]). This proves that  $\beta_k^p$  is not conjugate to  $\beta_{k'}^{p'}$  if  $k \neq k'$  or  $p \neq p'$ , because the growth rate is a conjugacy invariant and because  $\mathbb{Q}(\sqrt{k}) \cap \mathbb{Q}(\sqrt{k'}) = \mathbb{Q}$  (if  $k$  and  $k'$  are prime integers).

This finishes the proof Theorem 1.1. In view of Proposition 1.2, it remains to ask the following question, the answer of which we do not know:

**Question 5.7** *Does there exist an automorphism of  $F_3$  which has a parabolic orbit?*

## 6 Dehn twist automorphisms of $F_2$

In this last section, we calculate the dynamics graphs of all the automorphisms in the outer class of  $\delta^n$  ( $n \in \mathbb{Z}$ ,  $n \neq 0$ ), where  $\delta$  is the automorphism of  $F_2 = \langle a, b \rangle$  defined by  $\delta(a) = a$  and  $\delta(b) = ba$ .

Let  $D \in \text{Out}(F_2)$  be the outer class of  $\delta$ . As explained in section 4.2, the automorphisms in the outer class  $D^n$  ( $n \in \mathbb{Z}$ ) cannot have parabolic orbits. We are going to describe more precisely the dynamics induced on  $\partial F_N$  by the automorphisms in the outer class  $D^n$  ( $n \in \mathbb{Z}$ ,  $n \neq 0$ ). For that, we pursue the strategy of [6, 13], where the interested reader will be able to find details of the following constructions.

The rose  $R_2$  is the geometric realization of the graph with one vertex and 2 edges. We put an orientation on each edge, and we label them by  $a$  and  $b$ . We can turn  $R_2$  in a length space by declaring that each edge

has length 1. We represent  $D^n$  by an homotopy equivalence  $f$  of  $R_2$  defined in the following way:  $f$  is the identity on the edge  $a$  and linearly sends the edge  $b$  to the edge path labelled  $ba^n$ .

The universal cover  $\tilde{R}_2$  of  $R_2$  is a tree, equipped by the action of  $F_2$  by deck transformations. We lift the labels of the edges of  $R_2$  to the edges of  $\tilde{R}_2$ . Equivalently,  $\tilde{R}_2$  can be considered as the Cayley graph of  $F_2$  relative to the generating set  $\{a, b\}$ . Let  $T$  be the tree obtained by contracting in  $\tilde{R}_2$  all the edges labelled by  $a$ : the action of  $F_N$  on  $\tilde{R}_2$  induces an action of  $F_2$  on  $T$  by isometries. We note that the stabilizer of a vertex of  $T$  is conjugate to the subgroup  $\langle a \rangle \subset F_N$  generated by  $a$ .

As in the geometric case (see section 4.2) the automorphisms in the outer class  $D^n$  are in 1:1 correspondance with the lifts of  $f$  to  $\tilde{R}_2$ . Moreover, these lifts of  $f$  induce isometries of  $T$ . More precisely, the isometry  $H$  of  $T$  associated to the automorphism  $\delta^n \in D^n$  satisfies

$$\delta^n(g) \circ H = H \circ g \quad \forall g \in F_N,$$

where the elements of  $F_N$  are considered as isometries of  $T$ . Then, for  $u \in F_N$ , the map  $H_u = u \circ H$  is the isometry of  $T$  associated to the automorphism  $i_u \circ \delta^n \in D^n$ , since  $(i_u \circ \delta^n)(g) \circ H_u = H_u \circ g$  holds for all  $g \in F_N$ .

If  $H_u$  is a hyperbolic isometry of  $T$ , then  $i_u \circ \delta^n$  has North-South dynamics and the fixed points of  $i_u \circ \delta^n$  are determined by the ends of the axis of  $H_u$  in  $T$ , see [13].

If  $H_u$  is an elliptic isometry, let  $P \in T$  be a fixed point of  $H_u$ . There exists some  $w \in F_N$  such that the stabilizer of  $P$  in  $F_N$  is  $w \langle a \rangle w^{-1}$ . The fact that  $P$  is a fixed point of  $H_u$  then results in the existence of an integer  $k \in \mathbb{Z}$  such that  $u\delta^n(w) = wa^k$ . Or equivalently, such that  $i_u \circ \delta^n = i_w \circ (i_{a^k} \circ \delta^n) \circ i_w^{-1}$ . Indeed,

$$\begin{aligned} i_u \circ \delta^n &= i_{wa^k(\delta^n(w))^{-1}} \circ \delta^n \\ &= i_{wa^k\delta^n(w^{-1})} \circ \delta^n \\ &= i_w \circ i_{a^k} \circ i_{\delta^n(w^{-1})} \circ \delta^n \\ &= i_w \circ i_{a^k} \circ \delta^n \circ i_{w^{-1}}. \end{aligned}$$

The dynamics of  $\partial(i_u \circ \delta^n)$  is thus conjugate to the dynamics of  $\partial(i_{a^k} \circ \delta^n)$  for some  $k \in \mathbb{Z}$ . We are now going to study in more detail the automorphisms  $i_{a^k} \circ \delta^n$  for  $k \in \mathbb{Z}$ , and in particular, to give their dynamics graphs.

The inverse of  $i_{a^k} \circ \delta^n$  is  $i_{a^{-k}} \circ \delta^{-n}$ . We note that:

$$\begin{array}{ll} i_{a^k} \circ \delta^n : & a \mapsto a \\ & b \mapsto a^k b a^{n-k} \\ & b^{-1} \mapsto a^{k-n} b^{-1} a^{-k} \end{array} \quad \begin{array}{ll} i_{a^{-k}} \circ \delta^{-n} : & a \mapsto a \\ & b \mapsto a^{-k} b a^{k-n} \\ & b^{-1} \mapsto a^{n-k} b^{-1} a^k. \end{array}$$

Thus the dynamics of  $\partial(i_{a^k} \circ \delta^n)$  depends on the sign of  $k$  and of  $n - k$ .

**Remark 6.1** Let  $\sigma \in \text{Aut}(F_N)$  defined by  $\sigma(a) = a^{-1}$  and  $\sigma(b) = b^{-1}$ . We note that  $i_{a^k} \circ \delta^n$  and  $i_{a^{n-k}} \circ \delta^n$  are conjugate by the involution  $\sigma$ .

**First case:** Assume  $k(n - k) = 0$ . Since  $\delta^n$  and  $i_{a^n} \circ \delta^n$  are conjugate by  $\sigma$  (see Remark 6.1), we focus on  $\delta^n$ . One can check that  $\text{Fix}(\delta^n) = \langle a, bab^{-1} \rangle$ . Let  $X$  be a point in  $\partial F_2 \setminus \partial \langle a, bab^{-1} \rangle$ , and let  $x$  be the longest prefix of  $X$  in  $\langle a, bab^{-1} \rangle$ . Then  $X = xY$ , with no cancellation between  $x$  and  $Y$ , and the first letter of  $Y$  is equal to  $b$  or to  $b^{-1}$ . If  $Y$  begins by  $b$ , then  $\omega_{\delta^n}(Y) = ba^\infty$  and  $\omega_{\delta^{-n}}(Y) = ba^{-\infty}$ . If  $Y$  begins

by  $b^{-1}$ , then  $\omega_{\delta^n}(Y) = a^{-\infty}$  and  $\omega_{\delta^{-n}}(Y) = a^\infty$ . Hence  $\delta^n$  has 2 isoglossy classes of  $\omega$ -limit points (with representatives  $ba^\infty$  and  $a^{-\infty}$ ), and  $\delta^{-n}$  has 2 isoglossy classes of  $\omega$ -limit points (with representatives  $ba^{-\infty}$  and  $a^\infty$ ). The dynamics graph of  $\delta^n$  is given in Figure 3.

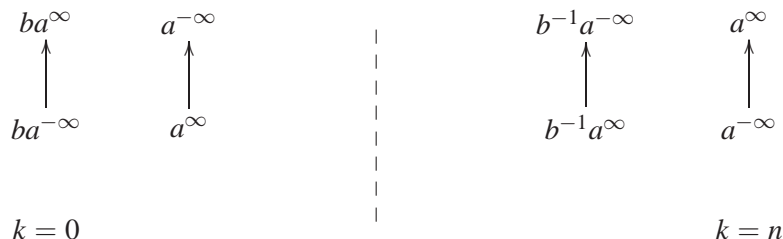


Figure 3: Dynamics graph of  $i_{a^k} \circ \delta^n$  for  $k(n-k) = 0$ .

**Second case:**  $k(n-k) < 0$ . We suppose that  $k > n$  (from which one deduces the case  $k < 0$  by using Remark 6.1). The fixed subgroup is  $\text{Fix}(i_{a^k} \circ \delta^n) = \langle a \rangle$ . We note that  $\omega_{i_{a^k} \circ \delta^n}(b) = \omega_{i_{a^k} \circ \delta^n}(b^{-1}) = a^\infty$  and  $\omega_{(i_{a^k} \circ \delta^n)^{-1}}(b) = \omega_{(i_{a^k} \circ \delta^n)^{-1}}(b^{-1}) = a^{-\infty}$ . It follows that  $\partial(i_{a^k} \circ \delta^n)$  has North-South dynamics on  $\partial F_2$ , see Figure 4.

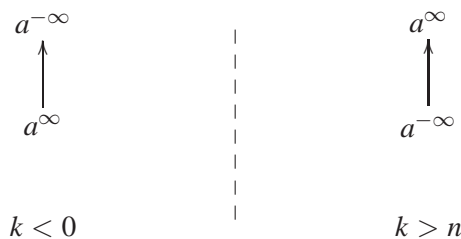


Figure 4: Dynamics graph of  $i_{a^k} \circ \delta^n$  for  $k(n-k) < 0$ .

**Third case:**  $k(n-k) > 0$ , i.e.  $0 < k < n$ . We check that the fixed subgroup is equal to  $\text{Fix}(i_{a^k} \circ \delta^n) = \langle a \rangle$ . We note that  $\omega_{i_{a^k} \circ \delta^n}(b) = a^\infty$ ,  $\omega_{i_{a^k} \circ \delta^n}(b^{-1}) = a^{-\infty}$ ,  $\omega_{(i_{a^k} \circ \delta^n)^{-1}}(b) = a^{-\infty}$  and  $\omega_{(i_{a^k} \circ \delta^n)^{-1}}(b^{-1}) = a^\infty$ . For  $x \in \partial F_2$ , it follows that  $\omega_{i_{a^k} \circ \delta^n}(X)$  and  $\omega_{(i_{a^k} \circ \delta^n)^{-1}}(X)$  depend only on the first occurrence of the letter  $b$  or  $b^{-1}$  in  $X$ : if it is  $b$ , then  $\omega_{i_{a^k} \circ \delta^n}(X) = a^\infty$  and  $\omega_{(i_{a^k} \circ \delta^n)^{-1}}(X) = a^{-\infty}$ ; if it is  $b^{-1}$ , then  $\omega_{i_{a^k} \circ \delta^n}(X) = a^{-\infty}$  and  $\omega_{(i_{a^k} \circ \delta^n)^{-1}}(X) = a^\infty$ . We say that  $\partial(i_{a^k} \circ \delta^n)$  has semi-North-South dynamics on  $\partial F_2$ , see Figure 5.



Figure 5: Dynamics graph of  $i_{a^k} \circ \delta^n$  for  $k(n-k) > 0$ .

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