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A RIGIDITY RESULT FOR OVERDETERMINED ELLIPTIC PROBLEMS IN THE PLANE

ANTONIO ROS, DAVID RUIZ, AND PIERALBERTO SICBALDI

ABSTRACT. Let $f : [0, +\infty) \rightarrow \mathbb{R}$ be a (locally) Lipschitz function and $\Omega \subset \mathbb{R}^2$ a $C^{1,\alpha}$ domain whose boundary is unbounded and connected. If there exists a positive bounded solution to the overdetermined elliptic problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial \vec{\nu}} = 1 & \text{on } \partial\Omega \end{cases}$$

we prove that Ω is a half-plane. In particular, we obtain a partial answer to a question raised by H. Berestycki, L. Caffarelli and L. Nirenberg in 1997.

1. INTRODUCTION

Given a locally Lipschitz function f , a widely open problem is to classify the set of domains $\Omega \subset \mathbb{R}^n$ where there exists a bounded solution u to the overdetermined elliptic problem

$$(1) \quad \begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial \vec{\nu}} = 1 & \text{on } \partial\Omega. \end{cases}$$

Here $\vec{\nu}(x)$ stands for the interior unit normal vector to $\partial\Omega$ at x . In this case we say that Ω is an *f-extremal domain* (see [28] for a motivation of that definition). The case of bounded *f-extremal domains* was completely solved by J. Serrin in [30] (see also [26]): the ball is the unique such domain and any solution is radial. This result has many applications to Physics and Applied Mathematics (see [17, 18, 28, 32]). Instead, the case of unbounded domains Ω is far from being completely understood.

Overdetermined boundary conditions arise naturally in free boundary problems, when the variational structure imposes suitable conditions on the separation interface, see for example [3]. In this context, several methods for studying the regularity of the interface are based on blow-up techniques which lead to the study of an elliptic problem in an

unbounded domain. In this framework, problem (1) in unbounded domains was considered in [7] for $f(u) = u - u^3$ (the Allen-Cahn equation). In that paper, H. Berestycki, L. Caffarelli and L. Nirenberg proposed the following:

Conjecture BCN: If $\mathbb{R}^n \setminus \overline{\Omega}$ is connected, then the existence of a bounded solution to problem (1) implies that Ω is either a ball, a half-space, a generalized cylinder $B^k \times \mathbb{R}^{n-k}$ (B^k is a ball in \mathbb{R}^k), or the complement of one of them.

That question was motivated by the results of the same authors in [7], and some other results concerning exterior domains, i.e. domains that are the complement of a compact region (see [1, 27]).

In [31] the third author gave a counterexample to that conjecture for $n \geq 3$, constructing a periodic perturbation of the straight cylinder $B^{n-1} \times \mathbb{R}$ that supports a periodic solution to problem (1) with $f(t) = \lambda t$. The goal of this paper is to prove that Conjecture BCN is true for $n = 2$ if $\partial\Omega$ is unbounded.

In the last years, a deep parallelism between overdetermined elliptic problems and constant mean curvature (CMC) surfaces has been observed. Serrin's result can be seen as the analogue of the Alexandrov's one ([2]), which asserts that the only embedded compact CMC hypersurfaces in \mathbb{R}^n are round spheres. In [29] F. Schlenk and the third author show that the counterexamples to Conjecture BCN built in [31] belong to a smooth one-parameter family that can be seen as a counterpart of the family of *Delaunay surfaces*. In [34] M. Traizet finds a one-to-one correspondence between 0-extremal domains in dimension 2 and a special class of minimal surfaces (see Section 2 for the exact statement of this result). In [13] M. Del Pino, F. Pacard and J. Wei consider problem (1) for functions f of Allen-Cahn type and they build new solutions in domains in \mathbb{R}^3 whose boundary is close to a dilated Delaunay surface or a dilated minimal catenoid. They also build bounded and monotone solutions to problem (1) for epigraphs in case $n \geq 9$ (this type of solutions do not exist if $n \leq 8$, as has been proved by K. Wang and J. Wei in [36]). The domain in [13] has boundary close to a dilated Bombieri-De Giorgi-Giusti entire minimal graph ([9]).

We point out that almost all those examples of f -extremal domains have boundary with some nontrivial topology. The only exception is the epigraph extremal domain found in [13], which requires $n \geq 9$. Therefore it is natural to consider BCN Conjecture if $\partial\Omega$ has the topology of the Euclidean space and $n \leq 8$. In this paper we solve the case $n = 2$.

Some partial results have been already given in the literature for dimension 2. In [16] A. Farina and E. Valdinoci prove BCN Conjecture if u is monotone along one direction and ∇u is bounded. In [36] the case of f -extremal epigraphs is solved for some nonlinearities f of the Allen-Cahn type. Finally, in [28] the result is proved if either $f(t) \geq \lambda t$, λ being a positive constant, or Ω is contained in a half-plane and ∇u is bounded. It is shown that if $f(t) \geq \lambda t$ (observe that this assumption excludes the prototypical Allen-Cahn nonlinearity) then the unique f -extremal domain with connected boundary is the

disk (see also [14] for a generalization to other geometries). In this paper we prove Conjecture BCN for $n = 2$ under the only assumption that $\partial\Omega$ is unbounded. The exact statement of our result is the following:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ a $C^{1,\alpha}$ domain whose boundary is unbounded and connected. Assume that there exists a bounded solution u to problem (1) for some (locally) Lipschitz function $f : [0, +\infty) \rightarrow \mathbb{R}$. Then Ω is a half-plane and u is parallel, that is, u depends only on one variable.*

We point out that, generally speaking, f -extremal domains always have $C^{2,\alpha}$ regularity, as shown in [35]. Hence, Theorem 1.1 could be stated under less regularity requirements, but for the sake of clarity we have preferred to leave it in that form.

The proof is divided in several steps. First, we show that the curvature of $\partial\Omega$ is bounded. This is done via a blow-up argument, making use of the classification results of [34] for the case $f = 0$. This argument needs some uniform regularity estimates that are given in Section 2, together with other preliminary results. In particular this result implies, via standard regularity for elliptic problems, that ∇u is bounded. This allows us to prove Theorem 1.1 if u is monotone along one direction. This result is basically contained in [16] if $\partial\Omega$ is C^3 ; in Section 4 we relax this regularity assumption by using ideas from the proof of the De Giorgi conjecture in dimension 2. In Section 5 we combine the previous result and the moving plane method (as well as the so-called *tilted moving plane method*) to show that Ω must contain a half-plane. A crucial ingredient in our proof is given in Section 6: we prove the existence of a divergent sequence of points in $p_n \in \partial\Omega$ such that $\partial\Omega$ converges to a straight line near such sequence. In particular, a parallel solution in a half-plane exists, which is given as the limit of $u(\cdot - p_n)$. In Section 7 we use the variational method to construct solutions in large balls converging to the parallel solution as the radius goes to $+\infty$. Section 8 concludes the proof of Theorem 1.1. First we show that the graph of u is above the graphs of those solutions defined in balls: passing to the limit, it is above the parallel solution too. But both solutions are in contact and have the same boundary conditions, so Theorem 1.1 follows from the maximum principle.

2. PRELIMINARY TOOLS

In this section we discuss some preliminary results that will be useful throughout the paper. Throughout the paper, $B_R(p)$ stands for the open ball of center p and radius R .

2.1. $C^{2,\alpha}$ regularity. In this paper we assume that the boundary of our domains is of class $C^{1,\alpha}$. Standard regularity arguments for elliptic equations show that a solution u of (1) is $C^{2,\alpha}$ in Ω and $C^{1,\alpha}$ up to the boundary. However, f -extremal domains always exhibit more regularity, namely $C^{2,\alpha}$. Moreover, the following uniform estimate holds.

Lemma 2.1. *Fix $R > 0$, $\alpha \in (0, 1)$, $p = (p_1, p_2) \in \partial\Omega$ and let $\phi \in C^{1,\alpha}(p_1 - R, p_1 + R)$ be such that $\Gamma_R = \partial\Omega \cap B_R(p) \subset \{(x, \phi(x)); x \in (p_1 - R, p_1 + R)\}$. Define $\Omega_R = \Omega \cap B_R(p)$.*

Let u be a bounded solution of the problem:

$$(2) \quad \begin{cases} \Delta u = h(x) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} = 1 & \text{on } \partial\Omega \end{cases}$$

for some $h \in C^{0,\alpha}$. Take $M = \|h\|_{C^{0,\alpha}(\Omega_R)} + \|u\|_{C^0(\Omega_R)} + \|\phi\|_{C^{1,\alpha}(p_1-R, p_1+R)}$. Then, u, ϕ belong to $C^{2,\alpha}$ and

$$\|u\|_{C^{2,\alpha}(\Omega_{R/2})} + \|\phi\|_{C^{2,\alpha}(p_1-R/2, p_1+R/2)} \leq C,$$

for some $C > 0$ depending only on M, R .

Remark 2.2. The $C^{2,\alpha}$ regularity for overdetermined problems in this fashion was given in [23] (see also [35]). However, the result in [23] needs some additional conditions that do not hold under our assumptions. Moreover, Lemma 2.1 is also concerned with the uniformity of the regularity estimate, which will be crucial later on. The proof we give here is different from [23] and takes advantage of some regularity results for problems with nonlinear oblique boundary conditions (see [25]). It is also worth pointing out that Lemma 2.1 is valid for any dimension n .

Proof. By standard regularity results, we conclude that

$$\|u\|_{C^{1,\alpha}(\Omega_{2R/3})} \leq C,$$

with C depending on M, R (see [20], Theorem 8.33 and the comment that follows, and also Corollaries 8.35, 8.36). Then, we are under the hypotheses of [25][Proposition 11.21]¹. Therefore, there exists $C > 0$ depending on M, R with

$$\|u\|_{C^{2,\alpha}(\Omega_{R/2})} \leq C.$$

Now observe that Γ_R is the 0 level of u , and $|\nabla u| = 1$ there. The implicit function theorem implies that, enlarging C if necessary,

$$\|\phi\|_{C^{2,\alpha}(p_1-R/2, p_1+R/2)} \leq C.$$

This concludes the proof of the lemma.

¹In [25][Proposition 11.21] the estimates are written with respect to a certain weighted Holder norms, and those weights vanish when a point approaches $\partial B_R(p) \cap \Omega$. We avoid the use of those norms by considering estimates in a smaller ball $B_{R/2}(p)$. Moreover, $b(x, u, \nabla u) = |\nabla u|^2 - 1$ and the obliquity condition (11.57b) trivially holds in our setting.

2.2. The moving plane method for overdetermined problems. One of the most important tools coming from the maximum principle of elliptic operators is the *moving plane method*, introduced firstly by A. D. Alexandrov [2] for constant mean curvature surfaces and then adapted by J. Serrin [30] to elliptic overdetermined problems (see also [8, 12, 19]).

Let L be a line in \mathbb{R}^2 that intersects Ω , and let L^+ and L^- be the two connected components of $\mathbb{R}^2 \setminus L$. Let us suppose that $\Omega \cap L^-$ has a bounded connected component C (Fig. 1).

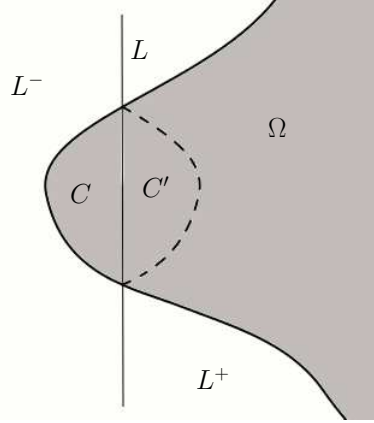


FIGURE 1

The following assertions are well known, and are basically contained in [30]:

- i. the closure of $\partial C \cap L^-$ is a graph over $\partial C \cap L$,
- ii. the closure of $\partial C \cap L^-$ is not orthogonal to L at any point,
- iii. If C' is the reflection of C about L , then the closure of $C \cup C'$ stays within $\overline{\Omega}$,
- iv. If for every $x \in C$ we define $u'(x') = u(x)$ where x' is the symmetric point to x with respect to L , then the graph of the function u' over C' stays under the graph of u , and the two graphs are not tangent in the points of L ,
- v. $\frac{\partial u}{\partial \vec{n}} > 0$ in C where \vec{n} is the normal direction to L pointing towards L^+ .

An immediate consequence of the moving plane method is the following:

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^2$ be an unbounded f -extremal domain such that $\partial\Omega$ is connected. Then for any point $p \in \partial\Omega$, the half-line $N(p)$ given by the half-line starting at p and pointing in the inward normal direction about $\partial\Omega$ with respect to Ω , is contained in Ω . We say that Ω has the property of the inward normal half-line.*

Proof. Observe that if $N(p)$ hits the boundary at a point q , then there is a bounded connected component of $\partial\Omega \setminus \{p, q\}$. By property (ii) the closure of this component cannot be orthogonal to $[p, q]$, but $[p, q] \subset N(p)$ and this gives a contradiction.

The property of the inward normal half-line has an interesting consequence, that will be exploited later on:

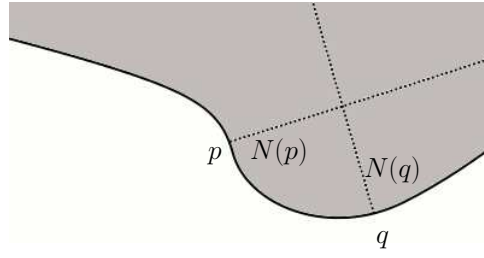


FIGURE 2

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^2$ be an unbounded f -extremal domain such that $\partial\Omega$ is connected, $p \in \partial\Omega$ and $R > 0$. Denote D the connected component of $\Omega \cap B_R(p)$ with p in its boundary. Then $\partial D \cap B_R(p)$ is connected.*

Proof. Otherwise, let us call U the connected component of $\partial D \cap B_R(p)$ containing p . There exists $q \in \partial D \cap B_R(p) \setminus U$ that minimizes the distance from p . Clearly, the segment $[p, q]$ touches q perpendicularly.

Define U' the connected component of $\partial D \cap B_R(p)$ containing q . Clearly, U' separates $B_R(p)$ in two connected components, V and V' , and p belongs to one of them, say, V . Since D is connected, and $p \in \partial D$, it follows that $D \subset V$. As q minimizes the distance of ∂V to p we have that $[p, q]$ lies in V and so the points of $[p, q]$ close to q belong to Ω . By the property of the inward normal half-line, $[p, q] \subset N(q)$ but this is a contradiction because $p \in \partial\Omega$.

2.3. Graph estimates. Let γ be an embedded curve of class $C^{2,\alpha}$ in \mathbb{R}^2 , and let $p \in \Gamma := \text{Im}(\gamma)$. Up to a rigid motion we can assume that p is the origin O of \mathbb{R}^2 and $\vec{\nu}(p) = (0, 1)$. Let κ be the curvature of Γ . As Γ is locally a graph, we have that around the origin Γ can be expressed as

$$(3) \quad \psi(x) = (x, y(x)),$$

with $y(0) = 0$ and $y'(0) = 0$ and then we have the following result.

Lemma 2.5. *If $|\kappa| \leq C$ in Γ then, for any $p \in \Gamma$, Γ contains a graph (3) defined over the interval $(-\varepsilon, \varepsilon)$ with $\psi(0) = p$. Here ε depends only on C , and the functions $y(x)$, $y'(x)$ and $y''(x)$ are uniformly bounded in that interval.*

Proof. As $|\kappa| \leq C$ and $|x| < \frac{1}{2C}$ by integrating the formula

$$(4) \quad \kappa(x) = \frac{d}{dx} \frac{y'}{\sqrt{1+(y')^2}} = \frac{y''}{(1+(y')^2)^{3/2}}$$

we get

$$\frac{|y'|}{\sqrt{1+(y')^2}} \leq C|x| < \frac{1}{2}.$$

Observe now that the tangent vector $\vec{t} = \frac{1}{\sqrt{1+(y')^2}}(1, y')$ satisfies $|\langle \vec{t}, (0, 1) \rangle| < 1/2$. This inequality implies that the graph $\psi(x)$ can be extended to the interval $|x| < \varepsilon$ with $\varepsilon = 1/2C$. Moreover $|y'|$ is bounded in that interval in terms of ε . We estimate the second derivative by using the identity

$$|y''| = |\kappa| (1 + (y')^2)^{3/2}.$$

and that proves the lemma.

2.4. Harmonic overdetermined domains in the plane. When $f \equiv 0$, a classification of the domains of the plane where problem (1) is solvable is given in [34]. Assume that Ω is unbounded and $\partial\Omega$ has a finite number of connected components; then, there exist only three domains Ω where problem (1) is solvable (even for unbounded functions u):

- the half-plane,
- the complement of a ball, and
- the domain

$$(5) \quad \Omega_* = \left\{ (x, y) \in \mathbb{R}^2 : |y| < \frac{\pi}{2} + \cosh(x) \right\}$$

that was first described in [21].

This correspondence gives in particular the following result.

Lemma 2.6. (Corollary of Theorem 5 of [34]). *If Ω is a domain of the plane where problem (1) can be solved for $f \equiv 0$, and the boundary of Ω is unbounded and connected, then Ω is a half-plane and u is linear.*

3. BOUNDEDNESS OF THE CURVATURE

The main result of this section is the following.

Proposition 3.1. *Let Ω be an f -extremal domain with boundary unbounded and connected, and u a bounded solution to (1). Then:*

- i) *The curvature of $\partial\Omega$ is bounded.*
- ii) *The $C^{2,\alpha}$ norm of the function u is bounded in $\bar{\Omega}$.*

Proof. If i) holds, Lemma 2.1 (recall Lemmas 2.4 and 2.5 of Subsection 2.3) implies a uniform estimate of the $C^{2,\alpha}$ norm of u near the boundary. In the interior of Ω , the $C^{2,\alpha}$ norm of u is also bounded due to interior regularity estimates (here we use in an essential way the global boundedness of u). Therefore ii) follows immediately.

We now turn our attention to the proof of i). We recall that the *accumulation set* of a sequence F_n of subsets of \mathbb{R}^2 is the closed set defined by

$$\text{Acc}(F_n) = \{p \in \mathbb{R}^2 : \exists p_n \in F_n \text{ such that } p_n \rightarrow p\}.$$

Let us suppose that $\partial\Omega$ has unbounded curvature, and we will reach a contradiction. The proof uses a blow-up technique.

Step 1: curvature rescaling. Let $\kappa(q)$ denote the curvature of $\partial\Omega$ at the point $q \in \partial\Omega$. If κ is unbounded, then there exists a sequence of points $q_n \in \partial\Omega$ such that $|q_n|$ and $|\kappa(q_n)|$ diverge to $+\infty$ increasingly. Let I_n be the connected component of $\partial\Omega \cap B_1(q_n)$ containing q_n and let $p_n = (x_n, y_n) \in I_n$ be the point where the function

$$p \rightarrow d(p, \partial B_1(q_n)) |\kappa(p)| = (1 - |p - q_n|) |\kappa(p)|, \quad p \in I_n$$

attains its maximum, that clearly exists. We set

$$r_n := d(p_n, \partial B_1(q_n)) = (1 - |p_n - q_n|)$$

and

$$R_n = r_n |\kappa(p_n)|.$$

We have

$$|\kappa(q_n)| \leq (1 - |p_n - q_n|) |\kappa(p_n)| = r_n |\kappa(p_n)| = R_n$$

and then $R_n \rightarrow +\infty$. Since $r_n \leq 1$, we have also that $|\kappa(p_n)|$ and R_n/r_n diverge to $+\infty$. Consider the transformation T_n in \mathbb{R}^2 given by

$$(x, y) \mapsto |\kappa(p_n)| (x - x_n, y - y_n).$$

Define $\Omega_n = T_n(\Omega)$. The image by T_n of the balls $B_{r_n}(p_n) \subset B_1(q_n)$ is given by the balls $B_{R_n}(O)$, where O is the origin of \mathbb{R}^2 . If κ_n is the curvature of $\partial\Omega_n$, we have clearly that

$$\kappa_n = \frac{\kappa}{|\kappa(p_n)|}.$$

Let $J_n = T_n(I_n)$. The function

$$p \mapsto d(p, \partial B_{R_n/r_n}(O)) |\kappa_n(p)| = (R_n/r_n - |p|) |\kappa_n(p)|, \quad p \in J_n$$

attains its maximum at $p = O$ and $|\kappa_n(O)| = 1$ for all n . Let $R > 0$. For n large enough and $p \in J_n \cap B_R(O)$ we have

$$(R_n/r_n - R) |\kappa_n(p)| \leq (R_n/r_n - |p|) |\kappa_n(p)| \leq (R_n/r_n - |O|) |\kappa_n(O)| = R_n/r_n.$$

Then

$$|\kappa_n(p)| \leq \frac{R_n/r_n}{R_n/r_n - R}$$

for all $p \in J_n \cap B_R(O)$, and the curvature of $\partial\Omega_n$ is uniformly bounded on compact sets.

Step 2: existence of a limit curve. Given $R > 0$, define $D_n(R)$ the connected component of $\Omega_n \cap B_R(O)$ which has O in its boundary, and $\Gamma_n(R) = \partial D_n(R) \cap B_R(O)$. By Lemma 2.4, $\Gamma_n(R)$ is connected.

Lemma 2.5 implies the existence of $\delta > 0$ such that $\forall p \in \Gamma_n(R/2)$, the connected component of $B_p(\delta) \cap \Gamma_n(R/2)$ passing through p contains a graph Y_n of a function $y_n(x)$ defined on a segment of length δ . Moreover, Lemma 2.1 implies that the functions y_n are of class $C^{2,\alpha}$ for all $\alpha \in (0, 1)$ and satisfy that their $C^{2,\alpha}$ norm is uniformly bounded. Ascoli-Arzelà's Theorem implies that a subsequence of y_n converges to a function $y_\infty \in C^{2,\alpha}(I_R(O))$ in the $C^{2,\alpha}$ -topology, for all $\alpha \in (0, 1)$. A prolongation argument allows to obtain Γ_n and D_n as the union of a subsequence of $\Gamma_n(R_n)$ and $D_n(R_n)$, where R_n is chosen so that $R_n \rightarrow \infty$, and a connected maximal sheet Γ_∞ of class $C^{2,\alpha}$ for all

$\alpha \in (0, 1)$, such that Γ_∞ belongs to the accumulation set of $\{\Gamma_n\}$ and admits an arc-length parametrization $\gamma_\infty(s)$ with $s \in \mathbb{R}$.

Step 3: Γ_∞ is proper. If this was not the case, there exists $p_n = \gamma_\infty(s_n) \in \Gamma_\infty$, where s_n is a divergent sequence and $p_n \rightarrow p \in \mathbb{R}^2$. By Lemma 2.5 and passing to the limit, there is a $\delta > 0$ such that each connected component of $\Gamma_\infty \cap B_\delta(p)$ is a graph. Therefore we can choose n so that $\Gamma_n \cap B_\delta(p)$ has at least three connected component which are curves passing through the consecutive points p_n, p_{n+1} and p_{n+2} . Enlarging n if necessary, we can assume that the distance of those components to p is smaller than $\delta/4$.

Now we can consider a connected component of $D_n \cap B_\delta(p)$ with a boundary formed by two connected components, both of them at a distance to p smaller than $\delta/4$. Take $q \in \partial\Omega_n$ with $|q - p| < \delta/4$. Then, $B_{\delta/2}(q) \cap D_n$ gives a contradiction with Lemma 2.4.

Step 4: Γ_∞ is embedded. By construction, Γ_∞ cannot have transversal self-intersections because this would give rise to transversal self-intersections of Γ_n for n large. But eventually Γ_∞ could have double tangential points, i.e. points p such that there exist $c_1 < c_2$ such that $p = \gamma_\infty(c_1) = \gamma_\infty(c_2)$, and the tangent vectors to γ_∞ satisfy $\vec{t}_\infty(c_1) = -\vec{t}_\infty(c_2)$. Let $\gamma(s), s \in \mathbb{R}$, be a parametrization of $\partial\Omega$, $\vec{\nu}(s)$ be the unit normal vector of the curve $\gamma(s)$ pointing to Ω , and $\vec{\nu}_\infty$ its induced limit unit normal on γ_∞ . Since Γ_∞ belongs to the accumulation set of $\{\Gamma_n\}$, then the geometry of the curves $\gamma(s)$ and $\gamma_\infty(s)$ depend locally on the homotheties T_n although the arc parameters s of these curves are not globally related.

We can suppose that the two values of $\vec{\nu}_\infty$ at p are given by $(-1, 0)$ and $(1, 0)$. Moreover, as it is not possible that all the points $\gamma_\infty(s)$ with $s \in [c_1, c_2]$ to be double points, we can assume that $\gamma_\infty(s)$ is an embedded curve in the open interval $c_1 < s < c_2$ and there exist

$$c_1 < c_3 < c_4 < c_2$$

such that the angle between $\vec{\nu}_\infty(c_3)$ and $\vec{\nu}_\infty(c_4)$, measured in the counterclockwise sense, is strictly less than π .

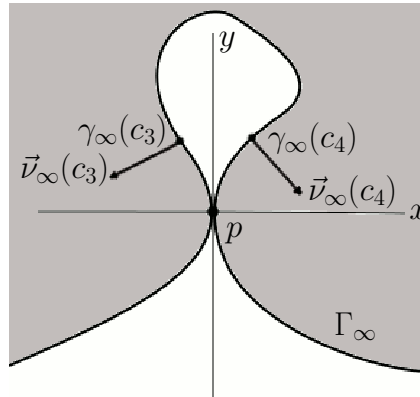


FIGURE 3

By using the arc parameter s of the curve γ we get that there exist four sequences c_1^i, c_2^i, c_3^i and $c_4^i, i = 1, 2, 3, \dots$, such that

- $c_1^1 < c_3^1 < c_4^1 < c_2^1 < c_1^2 < c_3^2 < c_4^2 < c_2^2 < \dots$
- $|c_1^i - c_2^i| \rightarrow 0$,
- $\vec{\nu}(c_1^i)$ converges to $(-1, 0)$ and $\vec{\nu}(c_2^i)$ converges to $(1, 0)$ as $i \rightarrow +\infty$,
- $\vec{\nu}(c_3^i)$ converges to $\vec{\nu}_\infty(c_3)$ and $\vec{\nu}(c_4^i)$ converges to $\vec{\nu}_\infty(c_4)$ as $i \rightarrow +\infty$.

In particular, for i large enough we have that the angle between $\vec{\nu}(c_3^i)$ and $\vec{\nu}(c_4^i)$, measured in the counterclockwise sense, is strictly less than $\pi - \delta$ for some $\delta > 0$. This gives easily a contradiction with the property of the normal inward half-line. In conclusion, Γ_∞ is embedded.

Step 5: one of the connected components of $\mathbb{R}^2 \setminus \Gamma_\infty$ is contained in the accumulation set $Acc(D_n)$. The curve Γ_∞ is properly embedded in \mathbb{R}^2 and hence it separates \mathbb{R}^2 in two connected components. Recall also that D_n and Γ_n are connected (the former by definition, the latter by Lemma 2.4). From these facts the proof of Step 5 is elementary, and we denote by Ω_∞ the domain in \mathbb{R}^2 given by the connected component of $\mathbb{R}^2 \setminus \Gamma_\infty$ contained in the accumulation set $Acc(D_n)$.

Step 6: conclusion. Clearly Ω_∞ is a $C^{2,\alpha}$ domain and $\partial\Omega_\infty = \Gamma_\infty$ with curvature κ_∞ satisfying

$$|\kappa_\infty(s)| \leq 1 = |\kappa_\infty(0)| \text{ for all } s.$$

Define

$$v_n(x, y) = |\kappa(p_n)| u \left(\frac{x}{|\kappa(p_n)|} + x_n, \frac{y}{|\kappa(p_n)|} + y_n \right),$$

$$f_n(x, y) = \frac{1}{|\kappa(p_n)|} f \left(\frac{1}{|\kappa(p_n)|} v_n(x, y) \right).$$

Then v_n solves the problem:

$$(6) \quad \begin{cases} \Delta v_n(x, y) + f_n(x, y) = 0 & \text{in } \Omega_n \\ v_n > 0 & \text{in } \Omega_n \\ v_n = 0 & \text{on } \partial\Omega_n \\ \frac{\partial v_n}{\partial \vec{\nu}} = 1 & \text{on } \partial\Omega_n. \end{cases}$$

Take $p_0 \in \mathbb{R}^2 \setminus \Omega_\infty$, with distance to the boundary greater or equal to a certain positive constant $r_0 > 0$ (this is possible by Step 5). Define ψ in $B_R(p_0)$ as the solution to the problem:

$$(7) \quad \begin{cases} \Delta \psi = 4\pi R^2 \delta_{p_0} - 1 & \text{in } B_R(p_0) \\ \psi = 0 & \text{on } \partial B_R(p_0). \end{cases}$$

Here we denote by δ_{p_0} the Dirac delta measure centered at p_0 . Observe that the function ψ is radially symmetric, has a logarithmic singularity at p_0 and $\frac{\partial \psi}{\partial \vec{\nu}} = 0$ on $\partial B_R(p_0)$.

If n is large enough, then $R_n > R$, $D_n \cap B_R(p_0)$ is connected and the closure of $\Omega_\infty \cap B_R(p_0)$ coincides with $\text{Acc}(D_n \cap B_R(p_0))$.

We claim that $\|v_n\|_{C^{2,\alpha}}$ is bounded in $D_n \cap B_R(p_0)$ for any R fixed. For that we apply Green formula and we obtain:

$$\begin{aligned} \int_{D_n \cap B_R(p_0)} v_n - \psi f_n &= \int_{D_n \cap B_R(p_0)} (\psi \Delta v_n - \Delta \psi v_n) \\ &= \int_{\partial D_n \cap \overline{B_R(p_0)}} \left(\frac{\partial \psi}{\partial \bar{\nu}} v_n - \psi \frac{\partial v_n}{\partial \bar{\nu}} \right) = - \int_{D_n \cap \overline{B_R(p_0)}} \psi. \end{aligned}$$

Observe that the last term is uniformly bounded as it converges to

$$\int_{\partial \Omega_\infty \cap \overline{B_R(p_0)}} \psi.$$

Moreover f_n is bounded in L^∞ . Hence we obtain that $\int_{D_n \cap B_R(p_0)} v_n$ is bounded. Theorem 9.26 of [20] implies that v_n is bounded in L^∞ sense on compact sets. Then we apply [20], Theorem 8.33 and the comment that follows (see also Corollaries 8.35, 8.36 there), to obtain that v_n is bounded in $C^{1,\alpha}$ sense on compact sets. In particular, f_n is bounded in $C^{0,\alpha}$, always on compact sets. Finally, Lemma 2.1 yields the claim.

Then, by Ascoli-Arzelà's Theorem v_n converges in $C^{2,\alpha}$ sense (on compact sets) to a solution v of the problem:

$$(8) \quad \begin{cases} \Delta v(x, y) = 0 & \text{in } \Omega_\infty \\ v > 0 & \text{in } \Omega_\infty \\ v = 0 & \text{on } \partial \Omega_\infty \\ \frac{\partial v}{\partial \bar{\nu}} = 1 & \text{on } \partial \Omega_\infty. \end{cases}$$

We apply now Lemma 2.6 and conclude that Ω_∞ is a half-plane. But $\partial \Omega_\infty$ has a point (the origin O) with curvature equal to ± 1 , and this yields the desired contradiction.

4. THE CASE WHEN u IS INCREASING IN ONE VARIABLE

The main result of this section is the following, that represents the answer to our problem if u is increasing in one variable.

Proposition 4.1. *Let Ω be a domain of \mathbb{R}^2 and suppose that $u = u(x, y)$ is a solution of (1) with $\frac{\partial u}{\partial y} > 0$ in $\bar{\Omega}$. Then Ω is a half-plane and u is parallel, that is, u depends only on one variable.*

Remark. The same result is proved in [16], but under the hypothesis that the domain is of class C^3 . Our proof follows the one in [4, 6].

Proof. Let u_x and u_y be the derivatives of u with respect to x and y . Then, we can define:

$$\sigma = \frac{u_x}{u_y}, \quad F = u_y^2 \nabla \sigma.$$

Then σ is a function of class $C^{1,\alpha}$ in $\bar{\Omega}$ and F is just $C^{0,\alpha}$. We claim that

$$(9) \quad \nabla \cdot F = 0 \quad \text{in } \Omega$$

in the distributional sense. To see that, we multiply both sides of $\Delta u + f(u) = 0$ by a test function $\xi \in C_0^\infty(\Omega)$ and integrate by parts, to obtain:

$$(10) \quad \int_{\Omega} [\langle \nabla \xi, \nabla u \rangle - \xi f(u)] = 0.$$

Differentiating such equation with respect to y we obtain

$$\int_{\Omega} [\langle \nabla \xi_y, \nabla u \rangle - \xi_y f(u)] + \int_{\Omega} [\langle \nabla \xi, \nabla u_y \rangle - \xi f'(u) u_y] = 0.$$

Then $v = u_y$ is a weak solution of

$$(11) \quad \Delta v + f'(u)v = 0$$

in Ω and the same holds for $v = u_x$.

Observe that $F = u_y \nabla u_x - u_x \nabla u_y$ and therefore,

$$\int_{\Omega} F \cdot \nabla \xi = \int_{\Omega} u_y \nabla u_x \cdot \nabla \xi - u_x \nabla u_y \cdot \nabla \xi = \int_{\Omega} u_y f'(u) u_x \xi - u_x f'(u) u_y \xi = 0$$

and (9) is proved.

As F is continuous in $\bar{\Omega}$ with 0 divergence in the distributional sense, by [5][Remark 1.8] (see also [10][Theorem 7.2]), we have that the Divergence Gauss theorem is valid in this framework. So, for any $\zeta \in C_0^\infty(\mathbb{R}^2)$ we have

$$\int_{\Omega} \nabla \cdot (\zeta^2 \sigma F) = \int_{\partial\Omega} \zeta^2 \sigma \langle F, \vec{\nu} \rangle,$$

where $\vec{\nu}$ is the inward normal vector about $\partial\Omega$. Recall that in $\partial\Omega$, $\nabla u = \vec{\nu}$; denoting by \vec{e}_1, \vec{e}_2 the vectors $(1, 0)$ and $(0, 1)$, we have

$$\begin{aligned} \langle F, \vec{\nu} \rangle &= \langle \nabla u_x, \vec{\nu} \rangle u_y - \langle \nabla u_y, \vec{\nu} \rangle u_x = (\nabla^2 u)(\vec{\nu}, \vec{e}_1) \langle \nabla u, \vec{e}_2 \rangle - (\nabla^2 u)(\vec{\nu}, \vec{e}_2) \langle \nabla u, \vec{e}_1 \rangle \\ &= (\nabla^2 u)(\vec{\nu}, \vec{\nu}) \langle \vec{\nu}, \vec{e}_1 \rangle \langle \vec{\nu}, \vec{e}_2 \rangle - (\nabla^2 u)(\vec{\nu}, \vec{\nu}) \langle \vec{\nu}, \vec{e}_2 \rangle \langle \vec{\nu}, \vec{e}_1 \rangle = 0 \end{aligned}$$

and then

$$\int_{\Omega} \nabla \cdot (\zeta^2 \sigma u_y^2 \nabla \sigma) = 0.$$

A simple computation gives

$$\nabla \cdot (\zeta^2 \sigma F) = \zeta^2 \sigma \nabla \cdot F + 2 \zeta \sigma u_y^2 \langle \nabla \zeta, \nabla \sigma \rangle + \zeta^2 u_y^2 |\nabla \sigma|^2$$

and using (9) we have

$$\int_{\Omega} \zeta^2 u_y^2 |\nabla \sigma|^2 = -2 \int_{\Omega} \zeta \sigma u_y^2 \langle \nabla \zeta, \nabla \sigma \rangle.$$

From this last formula, using the Hölder inequality we obtain

$$\int_{\Omega} \zeta^2 u_y^2 |\nabla \sigma|^2 \leq 2 \left[\int_{\Omega} \zeta^2 u_y^2 |\nabla \sigma|^2 \right]^{1/2} \left[\int_{\Omega} u_y^2 \sigma^2 |\nabla \zeta|^2 \right]^{1/2}.$$

By Proposition 3.1, the gradient of u is bounded, hence so it is $u_x = u_y \sigma$. Therefore

$$(12) \quad \int_{\Omega} \zeta^2 u_y^2 |\nabla \sigma|^2 \leq C_1 \int_{\mathbb{R}^2} |\nabla \zeta|^2$$

for some constant C_1 . It is well known that in the plane there is a sequence of logarithmic cutoff functions $\{\zeta_n\}_n \subset C_0^\infty(\mathbb{R}^2)$, such that

$$0 \leq \zeta_n \leq 1, \quad \zeta_n = 1 \text{ in } B_n(O) \quad \lim_n \int_{\mathbb{R}^2} |\nabla \zeta_n|^2 = 0.$$

Putting $\zeta = \zeta_n$ in (12) and letting $n \rightarrow \infty$ we obtain

$$\int_{\Omega} u_y^2 |\nabla \sigma|^2 = 0$$

which means that σ is constant, and then

$$u_x(x, y) = C u_y(x, y)$$

for a constant C . Then ∇u is normal to the vector $(1, -C)$, and then u is constant on every line parallel to that vector, i.e. u is parallel.

5. AN f -EXTREMAL DOMAIN CONTAINS A TANGENT HALF-PLANE

In this section we shall prove the following:

Proposition 5.1. *Let Ω be an f -extremal domain in \mathbb{R}^2 whose boundary $\partial\Omega$ is unbounded and connected. Then Ω contains a half-plane H such that $\partial\Omega$ and ∂H are tangent.*

Let Ω be an f -extremal domain in \mathbb{R}^2 whose boundary $\partial\Omega$ is unbounded and connected. Let $\partial\Omega$ be parameterized by $\gamma(t)$, where $t \in \mathbb{R}$ is the arc length parameter. After a rigid motion in the plane we will assume that the orthonormal basis $\{\gamma'(t), \vec{\nu}(t)\}$ given by the tangent vector and the inward pointing unit normal vector along of $\partial\Omega$ is positive. We remark that this implies that when we follow the parameter t from $-\infty$ to $+\infty$ along γ , the domain Ω stays on the left with respect to $\partial\Omega$. We define also the left and right part of $\partial\Omega$, respectively $(\partial\Omega)_l$ and $(\partial\Omega)_r$, as the two components of $\partial\Omega \setminus \{O\}$ given by $\gamma((-\infty, 0))$ and $\gamma((0, +\infty))$ respectively.

We introduce the concept of *limit direction* to the left and to the right for the boundary of a domain.

Definition 5.2. We say that $v_l \in \mathbb{S}^1$ is a *limit direction to the left* if there exists a sequence of points $l_n \in (\partial\Omega)_l$, $l_n = \gamma(t_n)$ with $t_n \rightarrow -\infty$, such that

$$(13) \quad \lim_{n \rightarrow +\infty} \frac{l_n - O}{|l_n - O|} = v_l.$$

We say that $v_r \in \mathbb{S}^1$ is a *limit direction to the right* if there exists a sequence of points $r_n \in (\partial\Omega)_r$, $r_n = \gamma(t_n)$ with $t_n \rightarrow +\infty$, such that

$$(14) \quad \lim_{n \rightarrow +\infty} \frac{r_n - O}{|r_n - O|} = v_r.$$

Obviously, the sets of limit directions to the left and to the right are not empty. To define the angle between a limit direction to the left and a limit direction to the right we consider the connected set $\nu(\Omega) = \{\vec{\nu}(t) / t \in \mathbb{R}\} \subset \mathbb{S}^1$ and we define $\dot{\nu}(\Omega)$ as the interior of $\nu(\Omega)$. We have the following lemma:

Lemma 5.3. *Let Ω be an f -extremal domain with boundary unbounded and connected. Then:*

- (1) $\dot{\nu}(\Omega) = \emptyset$ if and only if Ω is a half-plane.
- (2) $\dot{\nu}(\Omega)$ does not contain any limit direction (to the left or to the right).

Proof. Let us prove the two statements.

- (1) $\dot{\nu}(\Omega) = \emptyset$ if and only if $\nu(\Omega) = \{p\}$, and this can be if and only if $\partial\Omega$ is line.
- (2) Let us suppose that v is a limit direction (to the left or to the right) and $v \in \dot{\nu}(\Omega)$. Let us use the notation $\mathbb{S}^1 = \mathbb{R}/[0, 2\pi[$. Take $v_1, v_2 \in \dot{\nu}(\Omega)$, $v \in [v_1, v_2] \subset \dot{\nu}(\Omega) \subset \mathbb{S}^1$. Let $p_v, p_{v_1}, p_{v_2} \in \partial\Omega$ with $p_v = \gamma(t_v)$, $p_{v_1} = \gamma(t_{v_1})$, $p_{v_2} = \gamma(t_{v_2})$, such that $\vec{\nu}(p_v) = v$, $\vec{\nu}(p_{v_1}) = v_1$, $\vec{\nu}(p_{v_2}) = v_2$. We have that

$$\mathbb{R}^2 \setminus (N(p_{v_1}) \cup \gamma([t_{v_1}, t_{v_2}]) \cup N(p_{v_2}))$$

has two connected components, and $\gamma(\mathbb{R} \setminus [t_{v_1}, t_{v_2}])$ is contained in the component that does not contain $N(p_v)$. This implies that v is not a limit direction, and gives a contradiction.

This concludes the proof of the lemma.

Now take v_l and v_r two limit directions respectively to the left and to the right. These two vectors separate \mathbb{S}^1 in two connected components, say S_1 and S_2 (in the case where $v_l = v_r$ one of these components is empty). If Ω is a half-plane then $v_l = -v_r$ and then it is natural to define the angle $\theta(v_l, v_r)$ between v_l and v_r as equal to π . If Ω is not a half-plane, then $\dot{\nu}(\Omega)$ is non empty and is contained in only of the components S_1 and S_2 . This fact allows us to define, when Ω is not a half-plane, the angle $\theta(v_l, v_r)$ between v_l and v_r as the length of the component of $\mathbb{S}^1 \setminus \{v_l, v_r\}$ containing $\dot{\nu}(\Omega)$.

Lemma 5.4. *Let Ω be an f -extremal domain with boundary unbounded and connected. Let v_l and v_r be two limit directions, respectively to the left and to the right. Then*

$$\theta(v_l, v_r) \geq \pi.$$

Proof. The proof uses an argument inspired by [15]. Let us suppose that $0 \leq \theta(v_l, v_r) < \pi$; by using a convenient rotation and reflection, we can assume that $v_r \in (0, \pi/2)$, $v_l \in (\pi/2, \pi)$. Moreover, there is $\vec{\nu} = \vec{\nu}(p)$, $p \in \partial\Omega$, an interior normal vector in the interval (v_r, v_l) of \mathbb{S}^1 . Observe that $N(p)$ is contained in Ω by Lemma 2.3.

Given $a > 0$, take the line $L = \{y = a\}$ and the half-plane $L^- = \{y \leq a\}$. By the definition of the limit directions and the connectedness of $\partial\Omega$, the set $L^- \cap \Omega$ is formed

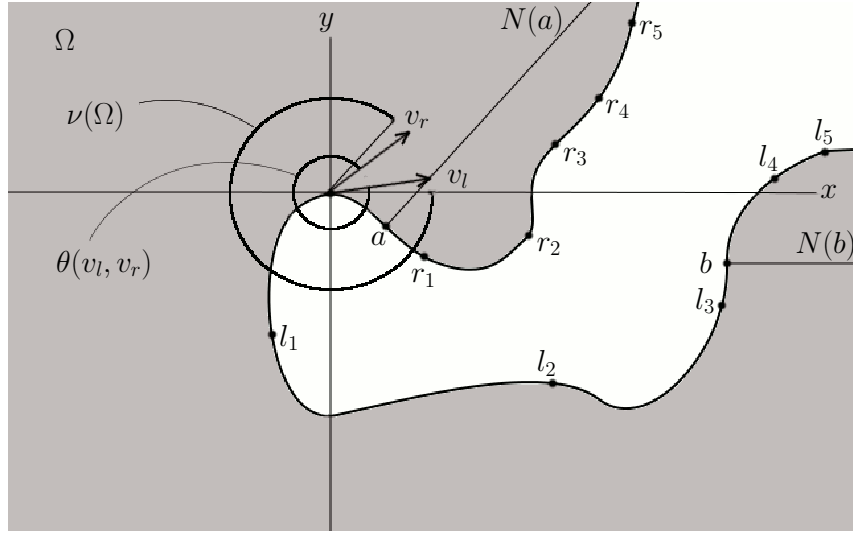


FIGURE 4. In this figure we show the set $\nu(\Omega) \subset \mathbb{S}^1$, a limit direction to the left v_l , limit of the sequence $l_n \in (\partial\Omega)_l$, and a limit direction to the right v_r , limit of the sequence $r_n \in (\partial\Omega)_r$. The angle $\theta(v_l, v_r)$ is the angle of the sector of \mathbb{S}^1 spanned by v_l and v_r and containing $\dot{\nu}(\Omega)$.

by (eventually infinitely many) bounded connected components. Hence we can apply the moving plane method to any of those components. Therefore, $\frac{\partial u}{\partial y} > 0$ in $\Omega \cap L^-$. Since the constant a is arbitrary, Ω is a half-plane by Proposition 4.1, which actually contradicts $\theta(v_l, v_r) < \pi$.

Next lemma excludes from our study the case $\theta(v_l, v_r) = \pi$.

Lemma 5.5. *Let Ω be f -extremal domain with boundary unbounded and connected, and v_l and v_r be two limit directions, respectively to the left and to the right, with $\theta(v_l, v_r) = \pi$. Then Ω is a half-plane and the bounded solution u to problem (1) is parallel.*

Proof. The proof of this lemma is inspired on the *tilted moving plane* introduced in [24] for constant mean curvature surfaces. This procedure has also been applied to elliptic problems in half-planes in [11], and to overdetermined problems in [28].

Up to a suitable rotation \mathcal{R} of \mathbb{R}^2 with center in O we can suppose that $v_l = (-1, 0)$ and $v_r = (1, 0)$. Up to a translation, we can suppose that the origin O of \mathbb{R}^2 belongs to $\partial\Omega$, and $\partial\Omega$ intersect the y -axis transversally. Up to a reflection on the x -axis, we can suppose that there exist $\delta > 0$ such that $\{(0, y) : 0 \leq y \leq \delta\}$ stays in Ω .

Consider $\Omega_1 = \Omega \cap \{x > 0\}$ and $\Omega_2 = \Omega \cap \{x < 0\}$. Given a straight line T , for any $x \in \mathbb{R}^2$ and any subset $X \subset \mathbb{R}^2$ let x' be the reflection of x about T and X' be the reflected image of X about T . Fix $\varepsilon > 0$ small enough and consider the two families of parallel straight lines

$$T_a = \{y = a\} \quad \text{and} \quad T_{\varepsilon, a} = \{y = -\varepsilon x + a\}$$

for $a \in \mathbb{R}$. Let $T = T_{\varepsilon, a}$ be an element of the second family. For $a \geq 0$ the line T cut off from Ω_1 region $\Gamma(T)$. Since $(1, 0)$ is a limit direction of $\partial\Omega$ to the right, $\Gamma(T)$ is made by (eventually infinitely many) bounded connected components. Therefore the moving plane method implies that the reflected image with respect to T of the connected components of $\Gamma(T)$ is contained in Ω , except possibly for the component whose boundary contains O . Let us denote this component by $\Sigma(T)$. The portions of the boundary of $\Sigma(T)$ contained in T , $x = 0$ and $\partial\Omega$ will be denoted respectively by I , J and K . Note that $O \in J \cap K$.

Let $\Sigma'(T)$, K' , J' and O' be respectively the symmetric image of $\Sigma(T)$, K , J and O with respect to T . Define on the closure of $\Sigma'(T)$ the function u'_T given by $u'_T(x) = u(x')$. At the beginning $\Sigma'(T)$ is contained in Ω and $u'_T \leq u$ and we continue the process while this occurs.

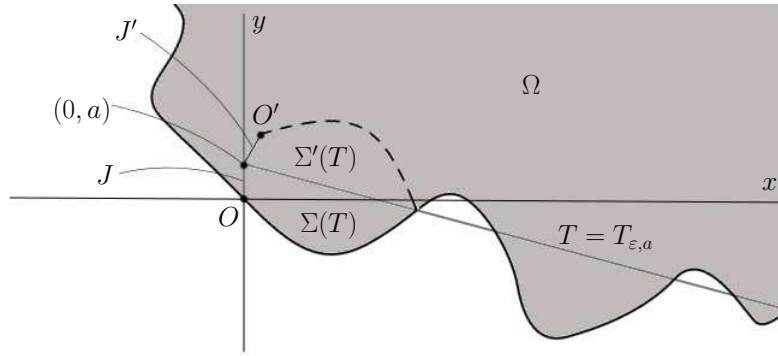


FIGURE 5

The process ends if we meet a first value $a = a(\varepsilon) > 0$ for which one of the following events holds:

- (1) at an interior point, the reflected arc K' touches the boundary of Ω ,
- (2) K meets T orthogonally,
- (3) at a point of $\Sigma'(T) \cup I$, the graph of the resulting function u'_T is tangent to the graph of the function u ,
- (4) O' belongs to $\partial\Omega$,
- (5) when restricted to the segment J' , the graph of the resulting function u'_T is tangent at some interior point to the graph of the function u .

By the moving plane method, we deduce that each one of the first three options implies that $K' \subset \partial\Omega$. Therefore both events (4) and (5) are also true. We conclude that in fact the process can be carried on either for all $a \geq 0$ or until either event (4) or event (5) occurs for a first value $a = a(\varepsilon) > 0$. We can say that the process can be carried on till a reaches the limit value $a(\varepsilon)$, being $a(\varepsilon) = +\infty$ if the process can be carried on for all $a \geq 0$. Since $\partial\Omega$ intersect the y -axis transversally, we have that there exists a constant $C > 0$ such that $a(\varepsilon) > C$.

Now take a sequence of $\varepsilon_i > 0$ going to zero, and repeat all the reasoning with $\varepsilon = \varepsilon_i$. Let $a_1 \in [C, +\infty]$ be the limit of a subsequence of $a(\varepsilon_i)$. Observe that $a_1 = +\infty$ or not depending on the behavior of u on $\Omega \cap \{x = 0\}$.

Now repeat all the process for $\Omega_2 = \Omega \cap \{y < 0\}$ instead of Ω_1 , with lines of positive slope defined by $T_{\varepsilon,a}^* = \{y = \varepsilon x + a\}$, and define the corresponding limit value a_2 . As it happens for a_1 , a_2 depends only on the behavior of the solution u along $\Omega \cap \{x = 0\}$. From this last property, it follows that $a_1 = a_2$.

If $a_1 = a_2 = +\infty$, then u is monotone, and we conclude that Ω is a half-plane by Proposition 4.1. If, instead, $a_1 = a_2 < +\infty$, the line T_{a_1} satisfies that the reflected image of $\Omega \cap \{y < a_1\}$ with respect to T is contained in Ω , $u'_T \leq u$ and one of the assertions (1), (2) or (3) holds (at some point of the y -axis). From the maximum principle we obtain that Ω , and in particular $\partial\Omega$, is symmetric with respect to T . Now, we know that the origin $O \in \partial\Omega$ stays under T . Since $\partial\Omega$ is connected, we have that $(\partial\Omega)$ intersects T and, since $\partial\Omega$ is symmetric with respect to T , the vector $(1, 0)$ would be a limit direction to the left, which is not possible by Lemma 5.4.

From those two lemmas the proof of Proposition 5.1 is immediate.

Proof. (Proposition 5.1). By Lemmas 5.4, 5.5, there are two possibilities: either Ω is a half-plane or $\theta(v_l, v_r) > \pi$ for any limit directions to the left and right. In this last case, assume that $O \in \partial\Omega$, and make a convenient rotation so that:

$$v_r < 0 \text{ and } v_l > \pi,$$

for any limit directions to the right and left, respectively. Then $\partial\Omega \cap \{y \geq -1\}$ is non-empty and compact. Therefore there exists $c \geq 0$ so that $H = \{y \geq c\}$ is the claimed half-plane.

6. BUILDING A PARALLEL SOLUTION STARTING FROM AN f -EXTREMAL DOMAIN

The main result of this section is the following:

Proposition 6.1. *Let Ω be an f -extremal domain with boundary unbounded and connected. There exists a sequence of points $q_n \in \partial\Omega$ satisfying that:*

- (1) $|q_n| \rightarrow +\infty$ and $\frac{q_n}{|q_n|} \rightarrow v \in \mathbb{S}^1$ for some direction to the right v .
- (2) If T_n is the translation in \mathbb{R}^2 that moves q_n to the origin, then $\Omega_n = T_n(\Omega)$ converges to the half-plane

$$\Omega_\infty = \{p \in \mathbb{R}^2 : \langle v^\perp, p \rangle > 0\}.$$

Here v^\perp denotes the vector obtained by rotating v of angle $\pi/2$ measured in the counter-clockwise sense. Moreover, the sequence of functions $u_n(x, y) = u((x, y) - q_n)$ converges to a bounded parallel solution of (1) in Ω_∞ .

Remark 6.2. An analogous statement is true for a certain direction to the left \tilde{v} .

Proof. Observe that the set of the limit directions to the right is closed. Moreover, it is not the whole \mathbb{S}^1 because Ω contains a half-plane. Then, we can choose $v = e^{i\theta}$ a

limit direction to the right such that $e^{i(\theta-\epsilon)}$ is not a limit direction to the right for any $\epsilon \in (0, \epsilon_0)$. Up to a rigid motion, we can assume that $v = (1, 0)$. By the mean value theorem, we can fix the origin $O \in \partial\Omega$ so that $\langle \vec{v}(O), (0, 1) \rangle > 0$. Observe that $\Omega \setminus N(O)$ has two connected components: let Ω_r be the one whose boundary is $N(O) \cup (\partial\Omega)_r$.

Take ϵ small enough. Consider the sector of \mathbb{R}^2 given by

$$C_\epsilon = \{(x, y) \in \mathbb{R}^2 : |y| \leq \epsilon x\}.$$

By the choice of v we know that $(\partial\Omega)_r \cap C_\epsilon$ is unbounded but the part of $\overline{\Omega}_r$ that lies under C_ϵ is compact. If $p = (x^p, y^p) \in (\partial\Omega)_r$ we define also the sector of \mathbb{R}^2 given by

$$G_{p,\epsilon} = \{(x, y) : y \leq y^p - 2\epsilon|x - x^p|\}.$$

Choose $p_\epsilon = (x_\epsilon, y_\epsilon)$ such that

- $p_\epsilon \in C_\epsilon \cap (\partial\Omega)_r$,
- the distance of p_ϵ to the origin is bigger than $1/\epsilon^2$

Observe that $\overline{\Omega}_r \cap G_{p_\epsilon, \epsilon}$ is compact and contained in C_ϵ if ϵ is sufficiently small. In particular there exists a point $q_\epsilon \in \overline{\Omega}_r \cap G_{p_\epsilon, \epsilon} \cap C_\epsilon$ minimizing the function $(x, y) \mapsto y$ (see Figure 6). Such value q_ϵ satisfies that:

- $|q_\epsilon| \rightarrow +\infty$ as $\epsilon \rightarrow 0$.
- $\overline{\Omega}_r \cap G_{q_\epsilon, \epsilon} = \{q_\epsilon\}$.

Now let D_ϵ be the connected component of $B_{1/\sqrt{\epsilon}}(q_\epsilon) \cap \Omega$ containing q_ϵ in its boundary. Observe that D_ϵ is above the sector G_{q_ϵ} . We do a translation T_ϵ in \mathbb{R}^2 , moving q_ϵ to the origin O , and we set $D'_\epsilon = T_\epsilon(D_\epsilon)$.

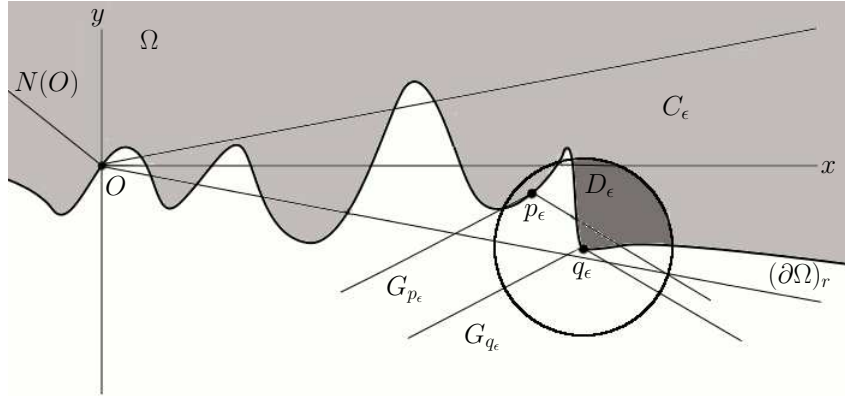


FIGURE 6

We now make ϵ converge to 0. By Proposition 3.1, the curvature of Ω and the $C^{2,\alpha}$ norm of u in $\overline{\Omega}$ are bounded. Following the arguments in the proof of Proposition 3.1 (see in particular Steps 2, 3, 4, 5) we have that the domains D'_ϵ converges to an f -extremal domain with boundary unbounded and connected Ω_∞ . Since $G_\epsilon = T_\epsilon(G_{q_\epsilon})$ converges

to the half-plane $\{y > 0\}$, the domain Ω_∞ is contained in a half-plane. By Lemma 5.5 $\Omega_\infty = \{y > 0\}$, and then the sequence u_n converges to a bounded parallel solution.

7. EXISTENCE OF SOLUTIONS IN BALLS AND ASYMPTOTIC PROPERTIES

The main result of this section is the following:

Proposition 7.1. *Assume that there exists a solution of the problem:*

$$(15) \quad \begin{cases} \varphi''(y) + f(\varphi(y)) = 0 \\ \varphi(0) = 0, \varphi'(0) = 1, \\ \lim_{t \rightarrow +\infty} \varphi(y) = L > 0. \end{cases}$$

Then, there exists $R_0 > 0$ such that for any $R > R_0$ the problem:

$$(16) \quad \begin{cases} \Delta u + f(u) = 0 & x \in B_R(O), \\ u > 0, & x \in B_R(O), \\ u = 0, & x \in \partial B_R(O). \end{cases}$$

admits a radially symmetric solution u_R . Moreover, u_R has the following asymptotic behavior:

- i) $u_R < L$ and for any $\rho \in (0, 1)$, $u_R|_{B_{\rho R}(O)}$ converges uniformly to L as $R \rightarrow +\infty$.
- ii) The functions $v_R(z) = u_R(z - (0, R))$ converges to $u(x, y) = \varphi(y)$ locally in compact sets of $H = \{y > 0\}$.

Remark 7.2. Actually, it will be clear from the proof that if $f(0) \geq 0$ there exist solutions of (16) for any $R > 0$. Instead, if $f(0) < 0$ such existence result is limited to large values of the radius.

In order to prove Proposition 7.1, we need some preliminary work. First, we show that the existence of the ODE (15) is equivalent to some properties on f and its primitive, denoted by:

$$(17) \quad F(u) = \int_0^u f(s) ds.$$

Lemma 7.3. *The following two assertions are equivalent:*

- i) There exists a solution to (15).
- ii) $f(L) = 0$ and $F(L) = 1/2 > F(u)$ for all $u \in [0, L)$.

Moreover, in such case, there exists a sequence $\mu_n < L$, $\mu_n \rightarrow L$ such that $F(\mu_n) > F(u)$ for all $u \in [0, \mu_n)$.

Proof. i) \Rightarrow ii). The limit at infinity of φ in (15) implies that $f(L) = 0$. Moreover, let us recall that the Hamiltonian:

$$H = \frac{1}{2}(\varphi')^2 + F(\varphi),$$

is a constant in y . Observe that $\varphi'(y) \rightarrow 0$ if $y \rightarrow +\infty$, so that such constant is nothing but $F(L)$. Moreover, replacing $y = 0$ we obtain the exact value of $F(L)$:

$$(18) \quad F(L) = \frac{1}{2}\varphi'(0)^2 + F(\varphi(0)) = \frac{1}{2}.$$

Moreover, it is easy to observe that $\varphi'(y) > 0$ for any $y \geq 0$. Then,

$$F(L) = \frac{1}{2}\varphi'(y)^2 + F(\varphi(y)) > F(\varphi(y)) \quad \forall y \in [0, +\infty).$$

ii) \Rightarrow i). In the phase space, let us consider the level set:

$$\mathcal{C} = \{(\varphi, \varphi') \in [0, +\infty)^2 : \frac{1}{2}(\varphi')^2 + F(\varphi) = 1/2\}.$$

This is a smooth curve for any $\varphi' > 0$ as the Implicit function theorem shows. Moreover, for any $\varphi \in [0, L]$, there exists a unique $\varphi' \geq 0$ such that $(\varphi, \varphi') \in \mathcal{C}$. Observe that $\varphi' = 1$ if $\varphi = 0$ and $\varphi' = 0$ if and only if $\varphi = L$. Then, the solution of the Initial Value Problem:

$$(19) \quad \begin{cases} \varphi''(y) + f(\varphi(y)) = 0 \\ \varphi(0) = 0, \quad \varphi'(0) = 1, \end{cases}$$

has image in \mathcal{C} . Since $\varphi' > 0$ for any $\varphi \in (0, L)$, the image of the solution contains all \mathcal{C} except, eventually, the point $(L, 0)$.

We now show that $\lim_{t \rightarrow +\infty} \varphi(t) = L$. Otherwise, φ arrives to the value L at a certain time t , and $\varphi'(t) = 0$. However, since $f(L) = 0$, L is an equilibrium of the ODE, and this gives a contradiction with the uniqueness of the solution for the initial value problem.

Observe that the last assertion of Lemma 7.3 would be immediate if f were positive below the value L , and actually we would have a continuum of values satisfying such condition. In general, though, f could change infinitely many times below L . Define

$$m_n = \max \left\{ F(x) : x \in \left[0, L - \frac{1}{n}\right] \right\}, \quad \text{and} \quad \mu_n = \min \left\{ x \in \left[0, L - \frac{1}{n}\right] : F(x) = m_n \right\}.$$

By the definition of μ_n , $F(\mu_n) = m_n > F(x)$ for all $x \in [0, \mu_n)$. We now show that $\mu_n \rightarrow L$. Otherwise, we could pass to a subsequence (still denoted by μ_n) such that $\mu_n \rightarrow \mu < L$. Then, $F(\mu) \leftarrow F(\mu_n) = m_n \rightarrow F(L)$, which implies that $F(\mu) = F(L)$, contradicting ii).

Our intention is now to settle the problem variationally. For that, we need to truncate the function f conveniently for $u < 0$ and $u > L$. Given $\delta > 0$, we define:

$$\tilde{f}(u) = \begin{cases} 0 & \text{if } u \geq L, \\ f(u) & \text{if } u \in [0, L], \\ f(0)(1 + \frac{u}{\delta}) & \text{if } u \in [-\delta, 0], \\ 0 & \text{if } u \leq -\delta. \end{cases}$$

Accordingly, we define $\tilde{F}(u) = \int_0^u \tilde{f}(s) ds$. Observe that for $u \leq -\delta$, $\tilde{F}(u) = -f(0)\delta/2$.

We now fix $\delta > 0$ so that

$$(20) \quad F(u) > |f(0)|\delta/2 \quad \forall u \in [L - 2\delta, L].$$

It is then clear that:

$$(21) \quad 1/2 = \tilde{F}(L) > \tilde{F}(u) \quad \forall u < L, \quad \text{and} \quad \tilde{F}(\mu_n) > \tilde{F}(u) \quad \forall u < \mu_n$$

where μ_n is given by Lemma 7.3, c), and we consider only the terms of the sequence so that $|\mu_n - L| < \delta$. With this truncation, our aim is to find solutions of the problem:

$$(22) \quad \begin{cases} \Delta u + \tilde{f}(u) = 0 & \text{in } B_R(O), \\ u > 0, & \text{in } B_R(O), \\ u = 0, & \text{in } \partial B_R(O). \end{cases}$$

Lemma 7.4. *Let u be a solution of (22). Then*

$$(23) \quad \begin{cases} u(z) \in (0, L) & \text{if } f(0) \geq 0, \\ u(z) \in (-\delta, L) & \text{if } f(0) < 0, \end{cases} \quad \forall z \in B_R(O).$$

Proof. First let us show that $u(z) \leq L$ for any $z \in B_R(O)$. Otherwise, assume that $\max u = u(z_0) > L$. Let $\Omega = \{z \in B_R(O) : u(z) > L\}$. Clearly u is harmonic in Ω and attains a maximum in its interior, which is impossible. In the same way we can prove that $u(z) \geq 0$ (if $f(0) \geq 0$) or $u(z) \geq -\delta$ (if $f(0) < 0$).

We now show the strict inequality. Otherwise, assume that $\max u = L$. Observe also that the constant function L is a solution of $\Delta u + f(u) = 0$. Therefore both solutions are in contact, and this is in contradiction with the maximum principle.

Let us define the energy functional:

$$I_R : H_0^1(B_R(O)) \rightarrow \mathbb{R}, \quad I_R(u) = \int_{B_R(O)} \frac{1}{2} |\nabla u|^2 - \tilde{F}(u).$$

Here $H_0^1(B_R(O))$ denotes the closure of the space $C_0^\infty(B_R(O))$ with the usual Sobolev norm

$$\|u\| = \left(\int_{B_R(O)} |\nabla u|^2 + u^2 \right)^{1/2}.$$

Clearly (22) is the Euler-Lagrange equation of the functional I_R . The following lemma establishes the existence of a minimum for I_R and, therefore, a solution for (22).

Lemma 7.5. *For any fixed $R > 0$, the functional I_R attains its minimum at a radially symmetric function u_R .*

Proof. This is quite standard. Observe that since \tilde{F} is continuous and bounded, the energy functional I_R is coercive and weakly lower semi-continuous. From this we obtain the existence of a minimizer u_R . By making use of the Schwartz rearrangement (see for instance [22]), we can assume that u_R is radially symmetric.

Observe that if $f(0) \geq 0$, by Lemma 7.4 we already have a solution of our problem (16). Instead, if $f(0) < 0$ we still need to show that u_R is positive. But, before, let us give some energy estimates on u_R .

In what follows, we denote by $A(p; R_1, R_2)$ the annulus of radii $R_1 < R_2$.

Lemma 7.6. *There exists $C > 0$ independent of R so that:*

$$(24) \quad -\frac{1}{2}\pi R^2 \leq I_R(u_R) \leq -\frac{1}{2}\pi R^2 + CR,$$

$$(25) \quad \frac{1}{2}\pi R^2 \geq \int_{B_R(O)} \tilde{F}(u_R) \geq \frac{1}{2}\pi R^2 - CR.$$

Proof. Taking into account (21), we have

$$I_R(u_R) = \int_{B_R(O)} \frac{1}{2} |\nabla u_R|^2 - \tilde{F}(u_R) \geq - \int_{B_R(O)} \tilde{F}(u_R) \geq -\frac{1}{2}\pi R^2.$$

From this we obtain the first inequality of (24) and (25).

For the second inequality, let us define $\phi_R \in H_0^1(B_R(O))$,

$$\phi_R(|z|) = \begin{cases} L & |z| \leq R-1, \\ L(R-|z|) & |z| \in [R-1, R]. \end{cases}$$

We now estimate $I_R(\phi_R)$. The gradient term can be estimated as:

$$\int_{B_R(O)} |\nabla \phi_R|^2 = 2\pi \int_{R-1}^R \phi_R'(r)^2 r dr \leq CR.$$

In order to estimate the term $\int_{B_R(O)} \tilde{F}(\phi_R)$, we split it into two terms:

$$\int_{B_{R-1}(O)} \tilde{F}(\phi_R) = \frac{1}{2}\pi(R-1)^2 \geq \frac{1}{2}\pi R^2 - CR,$$

$$\int_{A(0; R-1, R)} \tilde{F}(\phi_R) \geq -CR.$$

In the last estimate we have just used the boundedness of \tilde{F} . The above estimates imply that $I_R(\phi_R) \leq -\frac{1}{2}\pi R^2 + CR$. Since $I_R(u_R) \leq I_R(\phi_R)$, we conclude (24). Finally,

$$-\int_{B_R(O)} \tilde{F}(u_R) \leq I_R(u_R) \leq -\frac{1}{2}\pi R^2 + CR,$$

and (25) follows.

Next lemma is devoted to show the asymptotic behavior of u_R .

Lemma 7.7. *The following assertions hold:*

a) *For any fixed $\rho < L$, there exists $C = C_\rho$ independent of R so that:*

$$\Omega_\rho = \{z \in B_R(O) : u_R(z) < \rho\} \subset A(0; R - C_\rho, R).$$

b) *There exists $R_0 > 0$ such that u_R is positive for $R \geq R_0$.*

Proof. The proof of a) will be made in two steps.

Step 1. For any fixed $\rho < L$, there exists $C = C_\rho$ independent of R so that $|\Omega_\rho| \leq C_\rho R$. Indeed,

$$\begin{aligned} \int_{B_R(O) \setminus \Omega_\rho} \tilde{F}(u_R) &\leq \frac{1}{2}(\pi R^2 - |\Omega_\rho|), \\ \int_{\Omega_\rho} \tilde{F}(u_R) &\leq \max\{\tilde{F}(x) : x < \rho\} |\Omega_\rho| = \left(\frac{1}{2} - \varepsilon\right) |\Omega_\rho|, \end{aligned}$$

where $\varepsilon = \frac{1}{2} - \max\{\tilde{F}(x) : x < \rho\} > 0$ by (21). Adding both terms, we get:

$$\int_{B_R(O)} \tilde{F}(u_R) \leq \frac{1}{2}\pi R^2 - \varepsilon |\Omega_\rho|,$$

and Step 1 follows from (25).

Step 2. Let us fix $R > 0$ and $\mu = \mu_n$ one of the elements of the sequence in Lemma 7.3 satisfying (21). Then $\Omega_\mu = \{z \in B_R(O) : u_R(z) < \mu\}$ is connected.

Observe that Ω_μ always has a connected component touching the boundary $\partial B_R(O)$. Suppose by contradiction that it has an interior connected component too, denoted by U . Then, $u_R(z) < \mu$ for $z \in U$ and $u_R(z) = \mu$ if $z \in \partial U$.

Define:

$$v(z) = \begin{cases} u_R(z) & z \notin U, \\ \mu & z \in U. \end{cases}$$

Clearly, $v \in H_0^1(B_R(O))$ and $\int_U |\nabla u_R|^2 \geq \int_U |\nabla v|^2 = 0$. Moreover, taking into account (21),

$$\int_U \tilde{F}(u_R) \leq \int_U \tilde{F}(\mu) = \int_U \tilde{F}(\mu).$$

Therefore $I_R(v) < I_R(u_R)$, a contradiction that proves Step 2.

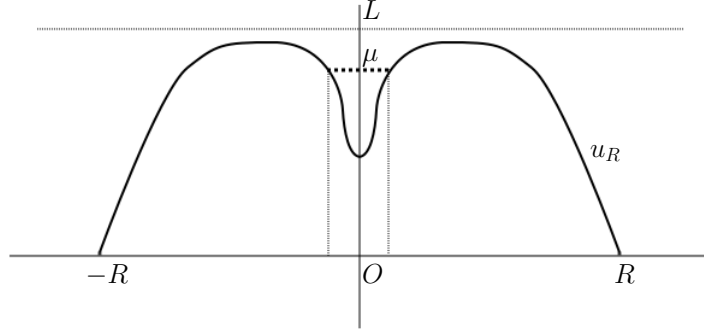


FIGURE 7

Step 1 and 2 readily imply a). Indeed, given $\rho < L$, take $\mu = \mu_n \in (\rho, L)$ one of the elements of the sequence. Since Ω_μ satisfies the statements of Step 1 and 2, $\Omega_\mu \subset A(0, R - C, R)$ for some positive constant C . But $\Omega_\rho \subset \Omega_\mu$, concluding the proof.

We now turn our attention to assertion b). The case $f(0) \geq 0$ is clear from Lemma 7.4, so let us consider the case $f(0) < 0$. Suppose that there exists $r_0 \in [0, R)$ with $u_R(r_0) = -\delta_R \leq 0$, $u'_R(r_0) = 0$. By Lemma 7.4, $\delta_R \in [0, \delta)$. Moreover, by a) we have that $r_0 \in (R - C, R)$ for some positive $C > 0$ independent of R .

Define $v(z) = u_R(z) + \delta_R$, which is a solution of the problem:

$$\begin{cases} \Delta v + g(v) = 0 & \text{in } B_{r_0}(O), \\ v = 0 & \text{in } \partial B_{r_0}(O), \\ \frac{\partial v}{\partial \nu} = 0 & \text{in } \partial B_{r_0}(O) \end{cases}$$

where $g(t) = \tilde{f}(t - \delta_R)$. We now apply the Pohozaev identity (see [33][Chapter III, Lemma 1.4]) to the previous problem, to obtain that

$$(26) \quad \int_{B_{r_0}(O)} G(v) = 0,$$

with $G(t) = \int_0^t g(s) ds = \tilde{F}(v - \delta_R) - \tilde{F}(-\delta_R)$.

We will show now that this is impossible if R is sufficiently large. Indeed, take Ω_μ the set defined in Step 2. Then,

$$\int_{B_{r_0}(O) \setminus \Omega_\mu} G(v) = \int_{B_{r_0}(O) \setminus \Omega_\mu} \tilde{F}(u_R - \delta_R) - \tilde{F}(-\delta_R).$$

Now, $\tilde{F}(-\delta_R) \leq |\tilde{F}(-\delta)| = |f(0)| \frac{\delta}{2}$. Moreover, in Ω_μ , $u_R - \delta_R \geq \mu - \delta_R \geq L - 2\delta$. By (20), we conclude that $\tilde{F}(u_R - \delta_R) - \tilde{F}(-\delta_R) > c > 0$ for any $z \in \Omega_\mu$. Then,

$$\int_{B_{r_0}(O) \setminus \Omega_\mu} G(v) \geq c |B_{r_0}(O) \setminus \Omega_\mu| \geq c' R^2.$$

Moreover,

$$\left| \int_{\Omega_\mu} G(v) \right| \leq \int_{A(0;R-C,R)} |G(v)| = O(R),$$

and hence (26) cannot hold for large R .

We are now able to prove Proposition 7.1.

Proof. (Proposition 7.1) With the previous results, we just need to prove iii). Take $R_n \rightarrow +\infty$, $n \in \mathbb{N}$, $v_n = u_n(z - (0, R_n))$. We first show that in any compact set of $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, v_n is bounded in with respect to the $C^{2,\alpha}$ norm. We use a bootstrap argument in two steps: $f(u_n)$ is a uniformly bounded function, and then u is of class $C^{1,\alpha}$. Then, $f(u_n)$ is a Lipschitz function, and we repeat the argument with $C^{2,\alpha}$ regularity.

As a consequence, v_n converges (up to a subsequence) to a solution of the problem $\Delta v + f(v) = 0$ defined in H . This convergence is $C^{2,\alpha}$ in compact sets of H , with $0 < \alpha < 1$. We now claim that v is parallel.

Take $p = (x, y) \in H$. We denote by ρ_n its distance to the center of the ball $(0, R_n)$, that is, $\rho_n = \sqrt{x^2 + (R_n - y)^2}$. Since u_n is radially symmetric, then $v_n(p) = v_n(0, R_n - \rho_n)$. Observe now that that $R_n - \rho_n \rightarrow y$. Therefore $v_n(p) \rightarrow v(0, y)$, which is independent of x .

We now prove that $v(0) = 0$. With the previous information, we can consider the convergence of sequence $v_n(r) = u_n(r - n)$, which solves:

$$v_n''(r) + \frac{v_n'(r)}{r} + f(v_n(r)) = 0, \quad v_n(0) = 0.$$

If we consider that equation in $r \in [0, 1]$, it is easy to show that it converges in $C^{2,\alpha}$ sense to $v(r)$. In particular, $v(0) = 0$.

Finally, we will show that $v = \varphi$ given in (15) by showing that $\lim_{r \rightarrow +\infty} v(r) = L$. Observe that Lemma 7.4 implies that actually $v(r) \leq L$ for any $r \in (0, +\infty)$. Fix now $\rho > 0$ and take $C > 0$ as given by Lemma 7.7, a). Then, for any $r \in (C, 2R_n - C)$ we have that $v_n(r) \geq \rho$. As a consequence, $v(r) \geq \rho$ for any $r \in (C, +\infty)$, which implies that $\lim_{r \rightarrow +\infty} v(r) = L$. Therefore we have proved the convergence of an adequate subsequence. The uniqueness of the limit implies that actually the whole sequence converges.

8. PROOF OF THE MAIN THEOREM

In order to conclude the proof of our main theorem, we recollect the information from the previous sections.

From Proposition 5.1 we know that Ω contains a half-plane H internally tangent to $\partial\Omega$. Moreover, from Lemmas 5.4, 5.5 we can assume that $\theta(v_l, v_r) > \pi$ for any directions to the left and right v_l, v_r . We can suppose that H is the half-plane $\{y > 0\}$ and the interior tangent point with $\partial\Omega$ is the origin. Moreover, by Proposition 6.1 there exists an unbounded sequence $q_n \in \partial\Omega$ such that, doing translations in \mathbb{R}^2 that move q_n to

the origin we get a sequence of domain Ω_n converging to a limit half-plane Ω_∞ , and a sequence of functions u_n converging to a parallel solution. Recall also that $\frac{q_n}{|q_n|} \rightarrow v$ and $\Omega_\infty = \{p \in \mathbb{R}^2 : \langle p, v^\perp \rangle > 0\}$. By making a rotation, if necessary, we can assume that $v = e^{i\theta}$, $\theta \in (-\pi/2, 0)$.

In section 7 we proved that for every R large enough there exists a radial solution u_R to the problem (16), such that as $R \rightarrow +\infty$

- i) $u_R < L$ and $u_R|_{B_{\rho R}(O)}$ converges uniformly to $L = \lim_{y \rightarrow +\infty} \varphi$ for any $\rho \in (0, 1)$,
- ii) the functions $v_R(z) = u_R(z - (0, R))$ converges to φ locally in compact sets of H , where φ is a solution of (15).

We are now able to prove our main result.

Proof. (Theorem 1.1) Let R large enough and consider the solution u_R . Since the parallel solution u_∞ is obtained as limit of a sequence u_n of translations of the function u in Ω , we get that there exists a point $p \in \Omega$ such that the ball $B_R(p)$ is contained in Ω and the graph of the function u_R defined in $B_R(p)$ stays under the graph of the function u . Moreover, p can be chosen so that $|p - q_n| < 2R$ with q_n an element of the sequence described above.

Now, we claim that we can move the ball $B_R(p)$ inside Ω till it reaches the position of the ball $B_R(q)$ with $q = (0, R)$. Observe that the graph of the function u_R , during the motion, cannot touch the graph of the function u by the maximum principle.

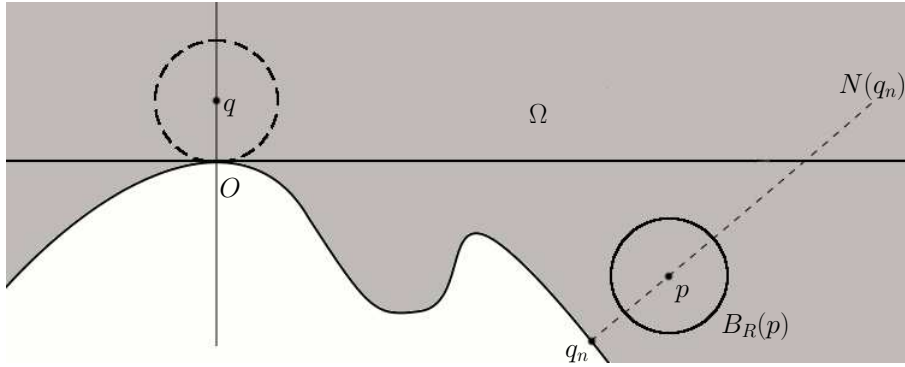


FIGURE 8

Since R is arbitrary, we get that $u_\infty(x, y) := \varphi(y) \leq u(x, y)$ for all $(x, y) \in H$. Moreover, the normal derivative of the functions u and u_∞ is the same at the origin, and by the maximum principle we get

$$u = u_\infty.$$

This shows that $\Omega = H$. Therefore we just need to show the claim.

Proof of the claim: Fix $R > 0$ and fix B a ball of radius $2R$ tangent to a certain point q_n , with sufficiently large n . By Lemma 2.3, inward normal half-line $N(q_n)$ starting at q_n does not intersect $\partial\Omega$. Moreover, at a certain point it reaches the half-plane H .

We move the center of B along $N(q_n)$ till it reaches H . We first show that during that motion, B cannot intersect $\partial\Omega$ at both sides of $N(q_n)$. Indeed, denote by p_1 and p_2 such intersection points, assuming that $|p_1 - q_n| \geq |p_2 - q_n|$. Then, the ball $B_{|p_1 - p_2|+1}(p_1)$ would give a contradiction with Lemma 2.4.

Therefore, when we move the center of B along $N(q_n)$, it eventually intersects $\partial\Omega$ just from one side. Therefore we can move a ball of radius R up to the half-plane H through the other side. From there, we can easily translate it to reach the position of $B_R(q)$.

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