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► **To cite this version:**

Romuald Elie, Ludovic Moreau, Dylan Possamaï. On a class of path-dependent singular stochastic control problems. *SIAM Journal on Control and Optimization*, 2018, 56 (5), pp.3260-3295. 10.1137/17M1114235 . hal-01481393

HAL Id: hal-01481393

<https://hal.science/hal-01481393>

Submitted on 2 Mar 2017

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On a class of path-dependent singular stochastic control problems

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February 1, 2017

Abstract

This paper studies a class of non-Markovian singular stochastic control problems, for which we provide a novel probabilistic representation. The solution of such control problem is proved to identify with the solution of a Z -constrained BSDE, with dynamics associated to a non singular underlying forward process. Due to the non-Markovian environment, our main argumentation relies on the use of comparison arguments for path dependent PDEs. Our representation allows in particular to quantify the regularity of the solution to the singular stochastic control problem in terms of the space and time initial data. Our framework also extends to the consideration of degenerate diffusions, leading to the representation of the solution as the infimum of solutions to Z -constrained BSDEs. As an application, we study the utility maximization problem with transaction costs for non-Markovian dynamics.

Keywords: singular control, constrained BSDEs, path-dependent PDEs, viscosity solutions, transaction costs, regularity.

1 Introduction

The study of singular stochastic problems initiated by Chernoff [11] and Bather and Chernoff [2, 1] gave rise to a large literature, mostly motivated by its large scope of applications in economics or mathematics. This includes in particular the well-known monotone follower problem, see for instance Karatzas [27], real options decision modeling related to optimal investment issues, see Davis, Dempster, Sethi and Vermes [14], Dynkin games, see Karatzas and Wang [30] or Boetius [4], optimal stopping problems, see Karatzas and Shreve [28, 29], Boetius and Kohlmann [5], or Benth and Reikvam [3], as well as optimal switching problems, see Guo and Tomecek [22]. For all these questions, the problem of interest is naturally modeled under the form of a singular control of finite variation problem. In this abundant literature, it is quite striking that very few studies take into consideration this type of questions in a non-Markovian framework. One of the purpose of his paper is to try to fill this gap¹.

In a Markovian environment, the solution to nice singular stochastic control problems characterizes typically as the unique weak solution to a variational inequality, where the linear part of the dynamics is combined with a constraint on the gradient of the solution. For example, when modeling the optimization of dividend flow for a firm, as initiated in continuous time by Jeanblanc-Picqué and Shiryaev [25], the optimal singular actions identify to paying dividends as soon as the underlying wealth process hits a free boundary, where the gradient of the value function reaches its upper-bound value 1. This example

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¹We learned while finishing this paper that Bouchard, Cheridito and Hu [6] were working on similar questions. Their work in preparation will deal with more general controlled SDEs, where the f term in (2.1) below will be allowed to depend on the state process. However, their method of proof will be purely probabilistic and thus of a completely different nature compared to ours. Furthermore, their approach will not, *a priori*, allow for degenerate diffusions

naturally suggests a connection between singular stochastic control problems and stochastic processes with gradient constraints, and more precisely backward stochastic differential equations (BSDEs) with constraints on the gain process, as introduced by Cvitanović, Karatzas and Soner [13]. The main initial motivation for the introduction of such type of equation was the super-hedging of claims with portfolio constraint. We establish in this paper that the solution to such BSDEs provides a nice probabilistic representation for solution to singular stochastic control problems.

More precisely, the class of non-Markovian stochastic control problem of interest is of the form

$$v^{\text{sing}} : (t, \mathbf{x}) \mapsto \sup_{K \in \mathcal{U}_{\text{sing}}^t} \mathbb{E}^{\mathbb{P}_0^t} [U(\mathbf{x} \otimes_t X^{t,\mathbf{x},K})], \quad 0 \leq t \leq T,$$

where $\mathcal{U}_{\text{sing}}^t$ denotes the set of multidimensional càdlàg non decreasing \mathbb{F}^t -adapted process starting from 0 and in L^p , for $p \geq 1$. The controlled underlying process X has the following non-Markovian singular dynamics

$$X^{t,\mathbf{x},K} = \mathbf{x}(t) + \int_t^\cdot \mu_s^{t,\mathbf{x}}(X^{t,\mathbf{x},K}) ds + \int_t^\cdot f_s dK_s + \int_t^\cdot \sigma_s^{t,\mathbf{x}}(X^{t,\mathbf{x},K}) dB_s^t,$$

where μ and σ are functional maps satisfying usual conditions, as detailed in Assumption 2.1 below. By density argument, it is worth noticing that we may also reduce to taking the supremum over the subset of absolutely continuous controls. We can also consider a weak version of such control problem.

After a well suited Girsanov type probability transform, we rewrite $v^{\text{sing}}(t, \mathbf{x})$ in a form that looks similar to a a face-lift type transformation of the terminal reward. Hereby, this provides the intuition behind the representation of $v^{\text{sing}}(t, \mathbf{x})$ as the solution to a BSDE with Z -constraint. The convex constraints imposed on the integrand process Z are induced by the directions f and represented via the convex set

$$K_t := \left\{ q \in \mathbb{R}^d \text{ s.t. } (f_t^\top q) \cdot \mathbf{e}_i \leq 0 \text{ for all } i \in \{1, \dots, d\} \right\}.$$

For any time t , we verify that $v^{\text{sing}}(t, \mathbf{x})$ identifies to $\mathcal{Y}^{t,\mathbf{x}}$ where $(\mathcal{Y}^{t,\mathbf{x}}, \mathcal{Z}^{t,\mathbf{x}})$ is the minimal solution to the constrained BSDE

$$\mathcal{Y}^{t,\mathbf{x}} \geq U^{t,\mathbf{x}}(\mathbf{X}^{t,\mathbf{x}}) - \int_t^T \mathcal{Z}_s^{t,\mathbf{x}} \cdot dB_s^t, \quad ((\sigma_s^{t,\mathbf{x}})^\top)^{-1}(\mathbf{X}^{t,\mathbf{x}}) \mathcal{Z}_s^{t,\mathbf{x}} \in K_s.$$

The line of proof relies on the observation that penalized versions of both the singular problem and BSDE are solutions of the same path-dependent partial differential equation (PPDE for short), for which we are able to provide a comparison theorem. As far as we know, it is the first time that the newly introduced theory of viscosity solutions of path-dependent PDEs (see the works of Ekren, Keller, Ren, Touzi and Zhang [16, 17, 18, 34, 35, 36]) is used to prove such a representation. Even though the ideas are reminiscent of the approach that one could have used in the Markovian case (see for instance Peng and Xu [33] for probabilistic interpretation of classical variational inequalities through BSDEs with constraints), we believe that using it successfully in a non-Markovian setting will open the door to many new potential applications of the PPDE theory.

Our main motivation for investigating such singular stochastic control problem, was the problem of utility maximization faced by an investor in a market presenting both transaction costs and non-Markovian dynamics. From the point of view of applications, having non-Markovian dynamics can be seen as a very desirable effect, as this case encompasses stochastic volatility models for instance. However, in such a framework our previous representation does not apply directly, since it requires non-degeneracy of the diffusion matrix of X , and due to transaction costs, the dynamics of wealth needs to be viewed as a bi-dimensional stochastic process driven by a one dimensional noise. We therefore extend our representation in order to include degenerate volatility coefficients. Our line of proof relies on compactness properties together with convex order ordering arguments as in Pagès [31]. In such

degenerate context, the solution to the singular control problem identifies with the infimum of a family of constrained BSDEs.

As a by-product, the probabilistic representation of v^{sing} in terms of a BSDE solution allows to derive insightful properties on the singular stochastic control problem. First, it automatically provides a dynamic programming principle for such problem. Second, this representation allows us to quantify the regularity of v^{sing} in terms of the initial data points. We observe that v^{sing} is Lipschitz in space as well as 1/2-Holder in time. Obtaining such results for singular control problems is in general a very hard task (see the discussion in [10, Section 4.2] for instance), and our approach could be one potential and promising solution.

The paper is organized as follows: Section 2 presents the class of singular control problems of interest and derives alternative mathematical representations. Section 3 presents the corresponding constrained BSDE representation. The connection is derived in Section 4 via Path-dependent PDE arguments. The consideration of degenerate volatility together with the example on transaction costs example are discussed in Section 5. Finally Section 6 provides the applications of such representation in terms of dynamic programming and regularity of the solution.

Notations: For any $d \in \mathbb{N} \setminus \{0\}$, and for every vector $x \in \mathbb{R}^d$, we will denote its entries by x^i , $1 \leq i \leq d$. For any $p \geq l$ and $(x_l, x_{l+1}, \dots, x_p) \in (\mathbb{R}^d)^{p-l+1}$, we will also sometimes use the notation $x_{l:p} := (x_l, x_{l+1}, \dots, x_p)$.

2 The singular control problem

2.1 Preliminaries

We fix throughout the paper a time horizon $T > 0$. For any $(t, x) \in [0, T] \times \mathbb{R}^d$, we denote by $\mathbf{\Lambda}^{t,x}$ the space of continuous functions \mathbf{x} on $[0, T]$, satisfying $\mathbf{x}_t = x$, $B^{t,x}$ the corresponding canonical process and $\mathbb{F}^{t,x,o} := (\mathcal{F}_s^{t,x,o})_{t \leq s \leq T}$ the (raw) natural filtration of $B^{t,x}$. It is classical result that the σ -algebra $\mathcal{F}_T^{t,x,o}$ coincides with the Borel σ -algebra on $\mathbf{\Lambda}^{t,x}$, for the topology of uniform convergence. We will simplify notations when $x = 0$ by setting $\Omega^t := \mathbf{\Lambda}^{t,0}$, $\Omega := \Omega^0$, $B^t := B^{t,0}$, $B := B^0$, $\mathbb{F}^{t,o} := \mathbb{F}^{t,0,o}$ and $\mathbb{F}^o := \mathbb{F}^{0,o}$. Besides, we will denote generically by \mathcal{C}^t the space of continuous functions on $[t, T]$, without any reference to their values at time t . Moreover, \mathbb{P}_x^t will denote the Wiener measure on $(\mathbf{\Lambda}^{t,x}, \mathcal{F}_T^{t,x,o})$, that is the unique measure on this space which makes the canonical process $B^{t,x}$ a Brownian motion on $[t, T]$, starting from x at time t . We will often make use of the completed natural filtration of $\mathbb{F}^{t,x,o}$ under the measure \mathbb{P}_x^t , which we denote $\mathbb{F}^{t,x} := (\mathcal{F}_s^{t,x})_{t \leq s \leq T}$. Again we simplify notations by setting $\mathbb{F}^t := \mathbb{F}^{t,0}$ and $\mathbb{F} := \mathbb{F}^0$, and we emphasize that all these filtrations satisfy the usual assumptions of completeness and right-continuity. For any $t \in [0, T]$, any $s \in [t, T]$ and any $\mathbf{x} \in \mathcal{C}^t$, we will abuse notations and denote

$$\|\mathbf{x}\|_{\infty,s} := \sup_{t \leq u \leq s} \|\mathbf{x}_u\|,$$

where $\|\cdot\|$ is the usual Euclidean norm on \mathbb{R}^d , which we denote simply by $|\cdot|$ when $d = 1$. Furthermore, the usual inner product on \mathbb{R}^d is denoted by $x \cdot y$, for any $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$.

For any $(t, s) \in [0, T] \times [t, T]$ and any $\mathbf{x} \in \mathcal{C}^t$, we define $\mathbf{x}^s \in \mathcal{C}^s$ by

$$\mathbf{x}^s(r) := \mathbf{x}(r), \quad r \in [s, T].$$

We also define the following concatenation operation on continuous paths. For any $0 \leq t < t' \leq s \leq T$, for any $(x, x') \in \mathbb{R}^d \times \mathbb{R}^d$ and any $(\mathbf{x}, \mathbf{x}') \in \mathbf{\Lambda}^{t,x} \times \mathbf{\Lambda}^{t',x'}$, we let $\mathbf{x} \otimes_s \mathbf{x}' \in \mathbf{\Lambda}^{t,x}$ be defined as

$$(\mathbf{x} \otimes_s \mathbf{x}')(r) := \mathbf{x}(r) \mathbf{1}_{t \leq r \leq s} + (\mathbf{x}'(r) + \mathbf{x}(s) - \mathbf{x}'(s)) \mathbf{1}_{s < r \leq T}.$$

Let us consider some $(t, x, s, \mathbf{x}) \in [0, T] \times \mathbb{R}^d \times [t, T] \times \mathcal{C}^t$. We will also denote, for simplicity, by $x \otimes_s \mathbf{x}$ the concatenation between the constant path equal to x on $[0, T]$ and \mathbf{x} . That being said, for any map $g : [0, T] \times \mathcal{C}^0$ and for any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$, we will denote by $g^{t, \mathbf{x}}$ the map from $[t, T] \times \mathcal{C}^t$ defined by

$$g^{t, \mathbf{x}}(s, \mathbf{x}') := g(s, \mathbf{x} \otimes_t \mathbf{x}').$$

Furthermore, we also use the following (pseudo)distance, defined for any $(t, t', s, s') \in [0, T]^4$, any $(x, x') \in \mathbb{R}^d \times \mathbb{R}^d$ and any $(\mathbf{x}, \mathbf{x}') \in \Lambda^{s, x} \times \Lambda^{s', x'}$ by

$$d_\infty((t, \mathbf{x}), (t', \mathbf{x}')) := \sqrt{|t' - t|} + \sup_{0 \leq r \leq T} \|(x \otimes_s \mathbf{x})(r \wedge t) - (x' \otimes_{s'} \mathbf{x}')(r \wedge t')\|.$$

2.2 A first version of the control problem

The first set of control processes that we will consider will be typical of singular stochastic control. More precisely, we define

$$\mathcal{U}_{\text{sing}}^t := \left\{ (K_s)_{t \leq s \leq T}, \text{ which are càdlàg, } \mathbb{R}^d\text{-valued, } \mathbb{F}^t\text{-adapted, null at } t, \right. \\ \left. \text{with non-decreasing entries and s.t. for any } p \geq 1, \mathbb{E} [K_T^p] < +\infty \right\}.$$

We next consider the following maps

$$\mu : [0, T] \times \mathcal{C}^0 \mapsto \mathbb{R}^d \text{ and } \sigma : [0, T] \times \mathcal{C}^0 \mapsto \mathbb{S}^d,$$

where \mathbb{S}^d is the set of $d \times d$ matrices (which we endow with the operator norm associated to $\|\cdot\|$, which we still denote $\|\cdot\|$ for simplicity) as well as a bounded map $f : [0, T] \mapsto \mathbb{S}^d$.

The following assumption will be in force throughout the paper

Assumption 2.1. (i) *The maps μ and σ are progressively measurable, in the sense that for any $(\mathbf{x}, \mathbf{x}') \in \mathcal{C}^0 \times \mathcal{C}^0$ and any $t \in [0, T]$, we have for $\varphi = \mu, \sigma$*

$$\mathbf{x}(s) = \mathbf{x}'(s), \text{ for all } s \in [0, t] \Rightarrow \varphi(s, \mathbf{x}) = \varphi(s, \mathbf{x}'), \text{ for all } s \in [0, t].$$

(ii) *μ and σ have linear growth in \mathbf{x} , uniformly in t , that is there exists a constant $C > 0$ such that for every $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$*

$$\|\mu_t(\mathbf{x})\| + \|\sigma_t(\mathbf{x})\| \leq C \left(1 + \|\mathbf{x}\|_{\infty, t}\right).$$

(iii) *μ and σ are uniformly Lipschitz continuous in \mathbf{x} , that is there exists a constant $C > 0$ such that for any $(t, \mathbf{x}, \mathbf{x}') \in [0, T] \times \mathcal{C}^0 \times \mathcal{C}^0$ we have*

$$\|\mu_t(\mathbf{x}) - \mu_t(\mathbf{x}')\| + \|\sigma_t(\mathbf{x}) - \sigma_t(\mathbf{x}')\| \leq C \|\mathbf{x} - \mathbf{x}'\|_{\infty, t}.$$

(iv) *For any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}$, $\sigma_t(\mathbf{x})$ is an invertible matrix and the matrix $\sigma_t^{-1}(\mathbf{x})f$ is uniformly bounded in (t, \mathbf{x}) .*

For any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$ and $K \in \mathcal{U}_{\text{sing}}^t$, we denote respectively by $\mathbf{X}^{t, \mathbf{x}}$ and $X^{t, \mathbf{x}, K}$ the unique strong solutions on $(\Omega^t, \mathcal{F}_T^{t, \circ}, \mathbb{P}_0^t)$ of the following SDEs (existence and uniqueness under Assumption 2.1 are classical results which can be found for instance in [24], see Theorems 14.18 and 14.21)

$$\begin{aligned} \mathbf{X}^{t, \mathbf{x}} &= \mathbf{x}(t) + \int_t^\cdot \mu_s^{t, \mathbf{x}}(\mathbf{X}^{t, \mathbf{x}}) ds + \int_t^\cdot \sigma_s^{t, \mathbf{x}}(\mathbf{X}^{t, \mathbf{x}}) dB_s^t, \mathbb{P}_0^t\text{-a.s.}, \\ X^{t, \mathbf{x}, K} &= \mathbf{x}(t) + \int_t^\cdot \mu_s^{t, \mathbf{x}}(X^{t, \mathbf{x}, K}) ds + \int_t^\cdot f_s dK_s + \int_t^\cdot \sigma_s^{t, \mathbf{x}}(X^{t, \mathbf{x}, K}) dB_s^t, \mathbb{P}_0^t\text{-a.s.} \end{aligned} \tag{2.1}$$

By Assumption 2.1, and standard estimates on non-Markovian SDEs, we have the following lemma

Lemma 2.2. For all $p \geq 2$, there is $C_p > 0$, depending only on p, T and the constant C in Assumption 2.1, such that for all $(t, t'; \mathbf{x}, \mathbf{x}'; K, K') \in [0, T] \times [t, T] \times \Lambda^2 \times (\mathcal{U}_{\text{sing}}^t)^2$

$$\mathbb{E}^{\mathbb{P}_0^t} \left[\sup_{t \leq s \leq t'} \|X_s^{t, \mathbf{x}, K} - \mathbf{x}(t)\|^p \right] \leq C_p \left((t' - t)^{\frac{1}{2}} (1 + \|\mathbf{x}\|_{\infty, t}^p) + (1 + (t' - t)^{\frac{1}{2}}) \mathbb{E}^{\mathbb{P}_0^t} [\|K_{t'} - K_t\|^p] \right), \quad (2.2)$$

$$\mathbb{E}^{\mathbb{P}_0^t} \left[\sup_{t \leq s \leq T} \|X_s^{t, \mathbf{x}, K}\|^p \right] \leq C_p \left(1 + \|\mathbf{x}\|_{\infty, t}^p + \mathbb{E}^{\mathbb{P}_0^t} [\|K_T - K_t\|^p] \right), \quad (2.3)$$

$$\mathbb{E}^{\mathbb{P}_0^t} \left[\sup_{t \leq s \leq T} \|X_s^{t, \mathbf{x}, K} - X_s^{t, \mathbf{x}', K'}\|^p \right] \leq C_p \left(\|\mathbf{x} - \mathbf{x}'\|_{\infty, t}^p + \mathbb{E}^{\mathbb{P}_0^t} \left[\left\| \int_t^T d(K_s - K'_s) \right\|^p \right] \right). \quad (2.4)$$

The stochastic control problem we are interested in is then

$$v^{\text{sing}}(t, \mathbf{x}) := \sup_{K \in \mathcal{U}_{\text{sing}}^t} \mathbb{E}^{\mathbb{P}_0^t} [U(\mathbf{x} \otimes_t X^{t, \mathbf{x}, K})], \quad (2.5)$$

where the reward function $U : \mathcal{C}^0 \rightarrow \mathbb{R}$ is assumed to satisfy

Assumption 2.3. For any $(\mathbf{x}, \mathbf{x}') \in \mathcal{C}^0 \times \mathcal{C}^0$, we have for some $C > 0$ and some $r \geq 0$

$$|U(\mathbf{x}) - U(\mathbf{x}')| \leq C \|\mathbf{x} - \mathbf{x}'\|_{\infty, T} \left(1 + \|\mathbf{x}\|_{\infty, T}^r + \|\mathbf{x}'\|_{\infty, T}^r \right).$$

Notice that it is clear from (2.3) that under Assumption 2.3, we have

$$|v^{\text{sing}}(t, \mathbf{x})| < +\infty, \text{ for any } (t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0.$$

2.3 A first simplification

Let us consider for any $t \in [0, T]$ the following subset \mathcal{U}^t of $\mathcal{U}_{\text{sing}}^t$ consisting of controls which are absolutely continuous with respect to the Lebesgue measure on $[t, T]$

$$\bar{\mathcal{U}}^t := \left\{ K \in \mathcal{U}_{\text{sing}}^t, K_s = \int_t^s \nu_r dr, \mathbb{P}_0^t - a.s., \text{ with } (\nu_s)_{t \leq s \leq T}, \mathbb{F}^t\text{-predictable and } (\mathbb{R}_+)^d\text{-valued} \right\}.$$

For any $K \in \bar{\mathcal{U}}^t$, it will be simpler for us to consider the corresponding process ν , so that we define

$$\mathcal{U}^t := \left\{ (\nu_s)_{t \leq s \leq T}, (\mathbb{R}_+)^d\text{-valued, } \mathbb{F}^t\text{-predictable, and s.t. } \mathbb{E}^{\mathbb{P}_0^t} \left[\left\| \int_t^T \nu_s ds \right\|^p \right] < +\infty, \forall p \geq 1 \right\}.$$

Then, for any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$ and $\nu \in \mathcal{U}^t$, we denote by $X^{t, \mathbf{x}, \nu}$ the unique strong solution on $(\Omega^t, \mathcal{F}_T^{t, \circ}, \mathbb{P}_0^t)$ of the following SDE

$$X^{t, \mathbf{x}, \nu} = \mathbf{x}(t) + \int_t^\cdot \mu_s^{t, \mathbf{x}}(X^{t, \mathbf{x}, \nu}) ds + \int_t^\cdot f_s \nu_s ds + \int_t^\cdot \sigma_s^{t, \mathbf{x}}(X^{t, \mathbf{x}, \nu}) dB_s^t, \mathbb{P}_0^t\text{-a.s.} \quad (2.6)$$

We can then define

$$v(t, \mathbf{x}) := \sup_{\nu \in \mathcal{U}^t} \mathbb{E}^{\mathbb{P}_0^t} [U(\mathbf{x} \otimes_t X^{t, \mathbf{x}, \nu})]. \quad (2.7)$$

Our first result is that the maximization under $\mathcal{U}_{\text{sing}}^t$ and \mathcal{U}^t actually lead to the same value function. Notice that for such a result to hold, the continuity assumptions that we made on the functions intervening in our problem are crucial. Indeed, as shown by Heinricher and Mizel [23], such an approximation result does not always hold.

Proposition 2.4. Under Assumptions 2.1 and 2.3, we have for any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$

$$v^{\text{sing}}(t, \mathbf{x}) = v(t, \mathbf{x}).$$

Proof. First of all, it is a classical result that \mathcal{U}^t is dense in $\mathcal{U}_{\text{sing}}^t$ in the sense that for any $K \in \mathcal{U}_{\text{sing}}^t$, there is some sequence $(\nu^n)_{n \geq 0} \subset \mathcal{U}^t$ such that

$$\mathbb{E}^{\mathbb{P}_0^t} \left[\sup_{t \leq s \leq T} \left\| K_s - \int_t^s \nu_r^n dr \right\|^2 \right] \xrightarrow{n \rightarrow +\infty} 0. \quad (2.8)$$

It is also clear from Assumption 2.3, (2.3) and (2.4) with $\mathbf{x} = \mathbf{x}'$ that for any $(t, \mathbf{x}; K, K') \in [0, T] \times \mathcal{C}^0 \times (\mathcal{U}_{\text{sing}}^t)^2$, we have for some constant $C_0 > 0$ which may vary from line to line

$$\begin{aligned} & \left| \mathbb{E}^{\mathbb{P}_0^t} [U(\mathbf{x} \otimes_t X^{t, \mathbf{x}, K})] - \mathbb{E}^{\mathbb{P}_0^t} [U(\mathbf{x} \otimes_t X^{t, \mathbf{x}, K'})] \right| \\ & \leq C_0 \mathbb{E}^{\mathbb{P}_0^t} \left[\left\| \mathbf{x} \otimes_t (X^{t, \mathbf{x}, K} - X^{t, \mathbf{x}, K'}) \right\|_{\infty, T}^2 \right]^{\frac{1}{2}} \left(1 + \mathbb{E}^{\mathbb{P}_0^t} \left[\left\| \mathbf{x} \otimes_t X^{t, \mathbf{x}, K} \right\|_{\infty, T}^{2r} \right]^{\frac{1}{2}} + \mathbb{E}^{\mathbb{P}_0^t} \left[\left\| \mathbf{x} \otimes_t X^{t, \mathbf{x}, K'} \right\|_{\infty, T}^{2r} \right]^{\frac{1}{2}} \right) \\ & \leq C_0 \mathbb{E}^{\mathbb{P}_0^t} \left[\left\| \int_t^T d(K_s - K'_s) \right\|^2 \right]^{\frac{1}{2}} \left(1 + \|\mathbf{x}\|_{\infty, t}^r + \mathbb{E}^{\mathbb{P}_0^t} \left[\left\| \int_t^T dK_s \right\|^{2r} \right]^{\frac{1}{2}} + \mathbb{E}^{\mathbb{P}_0^t} \left[\left\| \int_t^T dK'_s \right\|^{2r} \right]^{\frac{1}{2}} \right). \end{aligned}$$

We thus deduce immediately that the map $K \mapsto \mathbb{E}^{\mathbb{P}_0^t} [U(\mathbf{x} \otimes_t X^{t, \mathbf{x}, K})]$ is continuous with respect to the convergence in (2.8). Hence the result. \square

In the rest of the paper, we will therefore focus on the value function v instead of v^{sing} .

2.4 Weak formulation for v

For any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$ and $\nu \in \mathcal{U}^t$, we now define the following \mathbb{P}_0^t -equivalent measure

$$\frac{d\mathbb{P}^{t, \mathbf{x}, \nu}}{d\mathbb{P}_0^t} = \mathcal{E} \left(\int_t^\cdot (\sigma_s^{t, \mathbf{x}})^{-1} (\mathbf{X}^{t, \mathbf{x}}) f_s \nu_s \cdot dB_s^t \right).$$

The weak formulation of the control problem (2.7) is defined as

$$v^{\text{weak}}(t, \mathbf{x}) := \sup_{\nu \in \mathcal{U}^t} \mathbb{E}^{\mathbb{P}^{t, \mathbf{x}, \nu}} [U^{t, \mathbf{x}}(\mathbf{X}^{t, \mathbf{x}})], \text{ for any } (t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0. \quad (2.9)$$

The following result gives the equivalence between the two formulations of the control problem. It is a classical result and we refer the reader to Proposition 4.1 in [7] or Theorem 4.5 of [20].

Proposition 2.5. *Under Assumptions 2.1 and 2.3, we have for any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$*

$$v^{\text{weak}}(t, \mathbf{x}) = v(t, \mathbf{x}).$$

2.5 A canonical weak formulation

In this section we introduce yet another interpretation of the value function v , which will be particularly well suited when we will use the theory of viscosity solutions for path-dependent PDEs.

For any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$, let us define the following probability measure on $(\Lambda^{t, \mathbf{x}_t}, \mathcal{F}_T^{t, \mathbf{x}_t, o})$

$$\mathbb{P}_0^{t, \mathbf{x}} := \mathbb{P}_0^t \circ (\mathbf{X}^{t, \mathbf{x}})^{-1}.$$

Since σ is assumed to be invertible, it is a classical result that we have

$$\mathbb{F}^{t, o} = \mathbb{F}^{\mathbf{X}^{t, \mathbf{x}}}, \text{ and } \mathbb{F}^t = \overline{\mathbb{F}^{\mathbf{X}^{t, \mathbf{x}}}}^{\mathbb{P}_0^t}, \quad (2.10)$$

where $\mathbb{F}^{\mathbf{X}^{t, \mathbf{x}}}$ denotes the raw natural filtration of $\mathbf{X}^{t, \mathbf{x}}$ and $\overline{\mathbb{F}^{\mathbf{X}^{t, \mathbf{x}}}}^{\mathbb{P}_0^t}$ its completion under \mathbb{P}_0^t . As an immediate consequence, all these filtrations satisfy the Blumenthal 0 – 1 law as well as the predictable martingale representation property.

This implies that the process below is a Brownian motion on $(\Lambda^{t,\mathbf{x}_t}, \mathcal{F}_T^{t,\mathbf{x}_t,o}, \mathbb{F}^{t,\mathbf{x}_t}, \mathbb{P}_0^{t,\mathbf{x}})$

$$W^{t,\mathbf{x}} := \int_t^\cdot (\sigma_s^{t,\mathbf{x}})^{-1}(B^{t,\mathbf{x}_t})(dB_s^{t,\mathbf{x}_t} - \mu_s^{t,\mathbf{x}}(B^{t,\mathbf{x}_t})ds), \mathbb{P}_0^{t,\mathbf{x}}\text{-a.s.}$$

Indeed, by definition of $\mathbb{P}_0^{t,\mathbf{x}}$, we know that

$$\text{the law of } B^{t,\mathbf{x}_t} \text{ under } \mathbb{P}_0^{t,\mathbf{x}} = \text{the law of } \mathbf{X}^{t,\mathbf{x}} \text{ under } \mathbb{P}_0^t. \quad (2.11)$$

Hence, since we do have

$$B^t = \int_t^\cdot (\sigma_s^{t,\mathbf{x}})^{-1}(\mathbf{X}^{t,\mathbf{x}})(d\mathbf{X}_s^{t,\mathbf{x}} - \mu_s^{t,\mathbf{x}}(\mathbf{X}^{t,\mathbf{x}})ds), \mathbb{P}_0^t\text{-a.s.},$$

the result follows immediately by the Lévy characterization of Brownian motion and (2.10). Moreover, we clearly also have that

$$\text{the law of } W^{t,\mathbf{x}} \text{ under } \mathbb{P}_0^{t,\mathbf{x}} = \text{the law of } B^t \text{ under } \mathbb{P}_0^t. \quad (2.12)$$

Next, for any $\nu \in \mathcal{U}^t$, we define the following probability measure $\mathbb{P}_\nu^{t,\mathbf{x}}$ on $(\Lambda^{t,\mathbf{x}_t}, \mathcal{F}_T^{t,\mathbf{x}_t,o})$

$$\frac{d\mathbb{P}_\nu^{t,\mathbf{x}}}{d\mathbb{P}_0^{t,\mathbf{x}}} := \mathcal{E} \left(\int_t^\cdot (\sigma_s^{t,\mathbf{x}})^{-1}(B^{t,\mathbf{x}_t}) f_s \nu_s(W^{t,\mathbf{x}}) dW_s^{t,\mathbf{x}} \right)_T,$$

where it is understood that we interpret ν as a (Borel) map from \mathcal{C}^t to \mathbb{R}^d . We have the following simple result

Lemma 2.6. *We have for all $t \in [0, T]$*

$$\mathbb{E}^{\mathbb{P}^{t,\mathbf{x},\nu}} [U^{t,\mathbf{x}}(\mathbf{X}^{t,\mathbf{x}})] = \mathbb{E}^{\mathbb{P}_\nu^{t,\mathbf{x}}} [U^{t,\mathbf{x}}(B^{t,\mathbf{x}})], \text{ for every } \nu \in \mathcal{U}^t.$$

Proof. Fix some $t \in [0, T]$ and some $\nu \in \mathcal{U}^t$. We have, using successively the definition of $\mathbb{P}_\nu^{t,\mathbf{x}}$, (2.11), (2.12) and the definition of $\mathbb{P}^{t,\mathbf{x},\nu}$

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_\nu^{t,\mathbf{x}}} [U^{t,\mathbf{x}}(B^{t,\mathbf{x}_t})] &= \mathbb{E}^{\mathbb{P}_0^{t,\mathbf{x}}} \left[\mathcal{E} \left(\int_t^\cdot (\sigma_s^{t,\mathbf{x}})^{-1}(B^{t,\mathbf{x}_t}) f_s \nu_s(W^{t,\mathbf{x}}) dW_s^{t,\mathbf{x}} \right)_T U^{t,\mathbf{x}}(B^{t,\mathbf{x}_t}) \right] \\ &= \mathbb{E}^{\mathbb{P}_0^t} \left[\mathcal{E} \left(\int_t^\cdot (\sigma_s^{t,\mathbf{x}})^{-1}(\mathbf{X}^{t,\mathbf{x}}) f_s \nu_s(B^t) dB_s^t \right)_T U^{t,\mathbf{x}}(\mathbf{X}^{t,\mathbf{x}}) \right] \\ &= \mathbb{E}^{\mathbb{P}^{t,\mathbf{x},\nu}} [U^{t,\mathbf{x}}(\mathbf{X}^{t,\mathbf{x}})]. \end{aligned}$$

□

As a consequence of Lemma 2.6, we deduce immediately that for any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$

$$v(t, \mathbf{x}) = \sup_{\nu \in \mathcal{U}^t} \mathbb{E}^{\mathbb{P}_\nu^{t,\mathbf{x}}} [U^{t,\mathbf{x}}(B^{t,\mathbf{x}_t})]. \quad (2.13)$$

2.6 Approximating the value function

To obtain our main probabilistic representation result for the value function v (and thus for v^{sing}), we will use, as mentioned before, the theory of viscosity solutions of path-dependent PDEs. However, in order to do so we will have to make a small detour, and first approximate v .

For any integer $n > 0$ and any $t \in [0, T]$, we let $\mathcal{U}^{t,n}$ denote the subset of \mathcal{U}^t consisting of processes ν such that $0 \leq \nu_s^i \leq n$, for $i = 1, \dots, d$, for Lebesgue almost every $s \in [t, T]$. We then define the approximating value function for all $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$

$$v^n(t, \mathbf{x}) := \sup_{\nu \in \mathcal{U}^{t,n}} \mathbb{E}^{\mathbb{P}_0^t} [U^{t,\mathbf{x}}(X^{t,\mathbf{x},\nu})] = \sup_{\nu \in \mathcal{U}^{t,n}} \mathbb{E}^{\mathbb{P}_\nu^{t,\mathbf{x}}} [U^{t,\mathbf{x}}(B^{t,\mathbf{x}})], \quad (2.14)$$

where the second equality can be proved exactly as for v in Lemma 2.6.

We have the following simple result.

Lemma 2.7. *Under Assumptions 2.1 and 2.3, for every $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$, we have that the sequence $(v^n(t, \mathbf{x}))_{n \geq 1}$ is non-decreasing and*

$$v^n(t, \mathbf{x}) \xrightarrow{n \rightarrow +\infty} v(t, \mathbf{x}).$$

Proof. It is clear that the sequence is non-decreasing, as the sequence of sets $\mathcal{U}^{\cdot n}$ is. Moreover, since the elements of \mathcal{U}^t have by definition moments of any order, it is clear that $\cup_{n \geq 1} \mathcal{U}^{t, n}$ is dense in \mathcal{U}^t , in the sense that for any $\nu \in \mathcal{U}^t$, there exists a sequence $(\nu^m)_{m \geq 1}$ such that for any $m \geq 1$, $\nu^m \in \mathcal{U}^{t, m}$ and

$$\mathbb{E}^{\mathbb{P}_0^t} \left[\int_t^T \|\nu_r - \nu_r^m\|^2 dr \right] \xrightarrow{m \rightarrow +\infty} 0. \quad (2.15)$$

By the same arguments as in the proof of Proposition 2.4, we deduce that

$$v(t, \mathbf{x}) = \sup_{\nu \in \cup_{n \geq 1} \mathcal{U}^{t, n}} \mathbb{E}^{\mathbb{P}_0^t} [U^{t, \mathbf{x}}(X^{t, \mathbf{x}, \nu})] = \lim_{n \rightarrow +\infty} v^n(t, \mathbf{x}),$$

since the sets $\mathcal{U}^{t, n}$ are non-decreasing with respect to n . \square

3 The corresponding constrained BSDEs

3.1 Spaces and norms

We now define the following family of convex sets, for any $t \in [0, T]$:

$$K_t := \left\{ q \in \mathbb{R}^d \text{ s.t. } (f_t^\top q) \cdot \mathbf{e}_i \leq 0 \text{ for all } i \in \{1, \dots, d\} \right\}$$

where $(\mathbf{e}_i)_{1 \leq i \leq d}$ denotes the usual canonical basis of \mathbb{R}^d , and where for any $M \in \mathbb{S}^d$, M^\top denotes its usual transposition.

Remark 3.1. This form of constraint that one wishes intuitively to impose on the gradient of the value function v is quite natural according to representation (2.13). Recall that f describes the direction in which the underlying forward process is pushed in case of singular action.

We next introduce for any $p \geq 1$ the following spaces

$$\mathbb{S}_t^p := \left\{ (Y_s)_{t \leq s \leq T}, \mathbb{R}\text{-valued, } \mathbb{F}^t\text{-progressively measurable, càdlàg, } \mathbb{P}_0^t\text{-a.s., and s.t. } \|Y\|_{\mathbb{S}_t^p} < +\infty \right\},$$

where

$$\|Y\|_{\mathbb{S}_t^p}^p := \mathbb{E}^{\mathbb{P}_0^t} \left[\sup_{t \leq s \leq T} |Y_s|^p \right].$$

$$\mathbb{G}_{t, \mathbf{x}}^p := \left\{ (Y_s)_{t \leq s \leq T}, \mathbb{R}\text{-valued, } \mathbb{F}^{t, \mathbf{x}}\text{-progressively measurable, càdlàg, } \mathbb{P}_0^{t, \mathbf{x}}\text{-a.s., } \|Y\|_{\mathbb{G}_t^p} < +\infty \right\},$$

where

$$\|Y\|_{\mathbb{G}_{t, \mathbf{x}}^p}^p := \mathbb{E}^{\mathbb{P}_0^{t, \mathbf{x}}} \left[\sup_{t \leq s \leq T} |Y_s|^p \right].$$

$$\mathbb{H}_t^p := \left\{ (Z_s)_{t \leq s \leq T}, \mathbb{R}^d\text{-valued, } \mathbb{F}^t\text{-predictable, } \|Z\|_{\mathbb{H}_t^p} < +\infty \right\},$$

where

$$\|Z\|_{\mathbb{H}_t^p}^p := \mathbb{E}^{\mathbb{P}_0^t} \left[\left(\int_t^T \|Z_s\|^2 ds \right)^{\frac{p}{2}} \right].$$

$$\mathfrak{H}_{t, \mathbf{x}}^p := \left\{ (Z_s)_{t \leq s \leq T}, \mathbb{R}^d\text{-valued, } \mathbb{F}^{t, \mathbf{x}}\text{-predictable and s.t. } \|Z\|_{\mathfrak{H}_t^p} < +\infty \right\},$$

where

$$\|Z\|_{\mathfrak{H}_{t, \mathbf{x}}^p}^p := \mathbb{E}^{\mathbb{P}_0^{t, \mathbf{x}}} \left[\left(\int_t^T \|Z_s\|^2 ds \right)^{\frac{p}{2}} \right].$$

3.2 Strong formulation for the BSDE

For any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$, we would like to solve the K -constrained BSDE with generator 0 and terminal condition $U^{t, \mathbf{x}}(\mathbf{X}^{t, \mathbf{x}})$, that is to say we want to find a pair $(\mathcal{Y}^{t, \mathbf{x}}, \mathcal{Z}^{t, \mathbf{x}}) \in \mathbb{S}_t^2 \times \mathbb{H}_t^2$ such

$$\mathcal{Y}^{t, \mathbf{x}} \geq U^{t, \mathbf{x}}(\mathbf{X}^{t, \mathbf{x}}) - \int_t^T \mathcal{Z}_s^{t, \mathbf{x}} \cdot dB_s^t, \quad \mathbb{P}_0^t\text{-a.s.} \quad (3.1)$$

$$((\sigma_s^{t, \mathbf{x}})^\top)^{-1}(\mathbf{X}^{t, \mathbf{x}}) \mathcal{Z}_s^{t, \mathbf{x}} \in K_s, \quad ds \otimes d\mathbb{P}_0^t\text{-a.e.}, \quad (3.2)$$

and such that if there is another pair $(\tilde{\mathcal{Y}}^{t, \mathbf{x}}, \tilde{\mathcal{Z}}^{t, \mathbf{x}}) \in \mathbb{S}_t^2 \times \mathbb{H}_t^2$ satisfying (3.1) and (3.2), then we have $\mathcal{Y}^{t, \mathbf{x}} \leq \tilde{\mathcal{Y}}^{t, \mathbf{x}}$, \mathbb{P}_0^t -a.s. When it exists, the pair $(\mathcal{Y}^{t, \mathbf{x}}, \mathcal{Z}^{t, \mathbf{x}})$ is called the minimal solution of the K -constrained BSDE.

Such constrained BSDEs have been studied in the literature, first in [13] and then by Peng in [32]. However, all these existence results rely on the assumption that there is at least one solution (which does not have to be the minimal one) to the problem. This forces us to adopt the following assumption,

Assumption 3.2. *For every $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$, there exists a pair $(\bar{\mathcal{Y}}^{t, \mathbf{x}}, \bar{\mathcal{Z}}^{t, \mathbf{x}}) \in \mathbb{S}_t^2 \times \mathbb{H}_t^2$ such that*

$$\bar{\mathcal{Y}}^{t, \mathbf{x}} \geq U^{t, \mathbf{x}}(\mathbf{X}^{t, \mathbf{x}}) - \int_t^T \bar{\mathcal{Z}}_s^{t, \mathbf{x}} \cdot dB_s^t, \quad \mathbb{P}_0^t\text{-a.s.}$$

$$((\sigma_s^{t, \mathbf{x}})^\top)^{-1}(\mathbf{X}^{t, \mathbf{x}}) \bar{\mathcal{Z}}_s^{t, \mathbf{x}} \in K_s, \quad ds \otimes d\mathbb{P}_0^t\text{-a.e.}$$

Remark 3.3. This assumption simply indicates that it is indeed possible to satisfy the Z -constraint as well as solve the BSDE. Such constrained BSDEs have been first introduced in order to find the super-hedging price of a claim under portfolio constraints, and such condition in this framework simply indicates that one can find an admissible portfolio strategy that indeed super-hedges the claim of interest.

We then have immediately from [13] the following

Proposition 3.4. *Let Assumption 3.2, Assumption 2.1 and Assumption 2.3 hold. Then, the minimal solution of the K -constrained BSDE (3.1) exists.*

Proof. Since it is clear by Assumption 2.3 and (2.3) that $\mathbb{E}^{\mathbb{P}_0^t}[|U^{t, \mathbf{x}}(\mathbf{X}^{t, \mathbf{x}})|^2] < +\infty$, the result is an immediate consequence of the main result in [13]. \square

Since Assumption 3.2 is rather implicit, let us discuss some sufficient conditions under which it holds. We refer the reader to Assumption 7.1 in [13] for the proof of the sufficiency.

Lemma 3.5. *Fix some $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$. If there exist a constant $C \in \mathbb{R}$ and a process $\varphi \in \mathbb{H}_t^2$ such that $((\sigma_s^{t, \mathbf{x}})^\top)^{-1}(\mathbf{X}^{t, \mathbf{x}}) \varphi_s \in K_s$, $ds \otimes d\mathbb{P}_0^t$ -a.e. and such that*

$$U^{t, \mathbf{x}}(\mathbf{X}^{t, \mathbf{x}}) \leq C + \int_t^T \varphi_s \cdot dB_s^t, \quad \mathbb{P}_0^t\text{-a.e.}, \quad (3.3)$$

then Assumption 3.2 is satisfied. Moreover, (3.3) holds if for instance U is bounded.

Proof. We only have to prove that (3.3) holds when U is bounded. It suffices to notice that in this case we can take C to be a bound for U and $\varphi = 0$ since $0 \in K_s$ for every $s \in [0, T]$. \square

3.3 Weak formulation for the BSDE

It will also be important for us to look at the weak version of the constrained BSDE, where for any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$, we now look for a pair $(\mathfrak{Y}^{t, \mathbf{x}}, \mathfrak{Z}^{t, \mathbf{x}}) \in \mathfrak{G}_{t, \mathbf{x}}^2 \times \mathfrak{H}_{t, \mathbf{x}}^2$ such that

$$\mathfrak{Y}^{t, \mathbf{x}} \geq U^{t, \mathbf{x}}(B^{t, \mathbf{x}t}) - \int_{\cdot}^T \mathfrak{Z}_s^{t, \mathbf{x}} \cdot dW_s^{t, \mathbf{x}}, \mathbb{P}_0^{t, \mathbf{x}}\text{-a.s.} \quad (3.4)$$

$$((\sigma_s^{t, \mathbf{x}})^\top)^{-1}(B^{t, \mathbf{x}t}) \mathfrak{Z}_s^{t, \mathbf{x}} \in K_s, ds \otimes d\mathbb{P}_0^{t, \mathbf{x}}\text{-a.e.}, \quad (3.5)$$

and such that if there is another pair $(\tilde{\mathfrak{Y}}^{t, \mathbf{x}}, \tilde{\mathfrak{Z}}^{t, \mathbf{x}}) \in \mathfrak{G}_{t, \mathbf{x}}^2 \times \mathfrak{H}_{t, \mathbf{x}}^2$ satisfying (3.4) and (3.5), then we have $\mathfrak{Y}^{t, \mathbf{x}} \leq \tilde{\mathfrak{Y}}^{t, \mathbf{x}}, \mathbb{P}_0^{t, \mathbf{x}}\text{-a.s.}$

Again, we need an assumption in order to ensure the existence of the minimal solution.

Assumption 3.6. *For every $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$, there exists a pair $(\bar{\mathfrak{Y}}^{t, \mathbf{x}}, \bar{\mathfrak{Z}}^{t, \mathbf{x}}) \in \mathfrak{G}_{t, \mathbf{x}}^2 \times \mathfrak{H}_{t, \mathbf{x}}^2$ such that*

$$\bar{\mathfrak{Y}}^{t, \mathbf{x}} \geq U^{t, \mathbf{x}}(B^{t, \mathbf{x}t}) - \int_{\cdot}^T \bar{\mathfrak{Z}}_s^{t, \mathbf{x}} \cdot dW_s^{t, \mathbf{x}}, \mathbb{P}_0^{t, \mathbf{x}}\text{-a.s.}$$

$$((\sigma_s^{t, \mathbf{x}})^\top)^{-1}(B^{t, \mathbf{x}t}) \bar{\mathfrak{Z}}_s^{t, \mathbf{x}} \in K_s, ds \otimes d\mathbb{P}_0^{t, \mathbf{x}}\text{-a.e.}$$

We then deduce immediately

Proposition 3.7. *Let Assumption 3.6, Assumption 2.1 and Assumption 2.3 hold. Then, the minimal solution $(\mathcal{Y}^{t, \mathbf{x}}, \mathcal{Z}^{t, \mathbf{x}})$ of the K -constrained BSDE (3.2) exists. Moreover, if $\mathcal{Y}^{t, \mathbf{x}}$ and $\mathfrak{Y}^{t, \mathbf{x}}$ both exist, we have that*

$$\text{the law of } \mathcal{Y}^{t, \mathbf{x}} \text{ under } \mathbb{P}_0^t = \text{the law of } \mathfrak{Y}^{t, \mathbf{x}} \text{ under } \mathbb{P}_0^{t, \mathbf{x}}.$$

Remark 3.8. Of course the sufficient conditions of Lemma 3.5 can readily be adapted in this context. In particular, both Assumptions 3.2 and 3.6 hold if U is bounded.

As an immediate consequence of the Blumenthal 0 – 1 law and Proposition 3.7, we have the following equality for every $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$

$$\mathcal{Y}_t^{t, \mathbf{x}} = \mathfrak{Y}_t^{t, \mathbf{x}}.$$

The aim of this paper is to show that for all $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$, we have

$$v(t, \mathbf{x}) = \mathcal{Y}_t^{t, \mathbf{x}} = \mathfrak{Y}_t^{t, \mathbf{x}}.$$

3.4 The penalized BSDEs

Exactly as we have approximated the value function v by v^n defined in (2.14), it will be useful for us to consider approximations of the K -constrained BSDEs introduced in the previous section. It is actually a very well-known problem, which already appeared in [13], and which can be solved by considering the so-called penalized BSDEs associated to the K -constrained BSDE. Before doing so, we need to introduce, for any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$, the map

$$\rho : q \in \mathbb{R}^d \mapsto q^+ \cdot \mathbf{1}_d \quad (3.6)$$

where for each $q := (q_1, \dots, q_d)^\top \in \mathbb{R}^d$ we have used the notation: $q^+ := (q_1^+, \dots, q_d^+)^\top$.

Under Assumptions 2.1 and 2.3, we can then define for any $(t, \mathbf{x}, n) \in [0, T] \times \mathcal{C}^0 \times \mathbb{N}^*$, $(\mathcal{Y}^{t, \mathbf{x}, n}, \mathcal{Z}^{t, \mathbf{x}, n}) \in \mathfrak{S}_t^2 \times \mathfrak{H}_t^2$ as the unique solution of the following BSDE

$$\mathcal{Y}^{t, \mathbf{x}, n} = U^{t, \mathbf{x}}(\mathbf{X}^{t, \mathbf{x}}) + \int_{\cdot}^T n \rho(f_s^\top ((\sigma_s^{t, \mathbf{x}})^\top)^{-1}(\mathbf{X}^{t, \mathbf{x}}) \mathcal{Z}_s^{t, \mathbf{x}, n}) ds - \int_{\cdot}^T \mathcal{Z}_s^{t, \mathbf{x}, n} dB_s^t, \mathbb{P}_0^t\text{-a.s.} \quad (3.7)$$

Notice that existence and uniqueness hold using for instance the results in [19], since under Assumptions 2.1 and 2.3, the terminal condition is obviously square-integrable, the generator $z \mapsto \rho(f_s^\top((\sigma^{t,\mathbf{x}})^\top)^{-1}(\mathbf{X}^{t,\mathbf{x}})z)$ is null at 0 and uniformly Lipschitz continuous in z (we remind the reader that $\sigma^{-1}f$ is bounded, so its transpose is bounded as well).

It is then a classical result (see [13] or [32]) that under Assumption 3.2, for any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$

$$\mathcal{Y}_s^{t,\mathbf{x},n} \xrightarrow[n \rightarrow +\infty]{\uparrow} \mathcal{Y}_s^{t,\mathbf{x}}, \text{ for any } s \in [t, T], \mathbb{P}_0^t - a.s., \text{ and } \|\mathcal{Y}^{t,\mathbf{x}} - \mathcal{Y}^{t,\mathbf{x},n}\|_{\mathbb{H}_t^2} \xrightarrow[n \rightarrow +\infty]{} 0. \quad (3.8)$$

Alternatively, we may consider the penalized BSDEs in weak formulation

$$\mathfrak{Y}^{t,\mathbf{x},n} = U^{t,\mathbf{x}}(B^{t,\mathbf{x}}) + \int_t^T n\rho(f_s^\top((\sigma_s^{t,\mathbf{x}})^\top)^{-1}(B^{t,\mathbf{x}})\mathfrak{Z}_s^{t,\mathbf{x},n})ds - \int_t^T \mathfrak{Z}_s^{t,\mathbf{x},n}dW_s^{t,\mathbf{x}}, \mathbb{P}_0^{t,\mathbf{x}}\text{-a.s.},$$

which also admit a unique solution under Assumptions 2.1 and 2.3, and which are such that under Assumption 3.6

$$\mathfrak{Y}_s^{t,\mathbf{x},n} \xrightarrow[n \rightarrow +\infty]{\uparrow} \mathfrak{Y}_s^{t,\mathbf{x}}, \text{ for any } s \in [t, T], \mathbb{P}_0^{t,\mathbf{x}} - a.s., \text{ and } \|\mathfrak{Y}^{t,\mathbf{x}} - \mathfrak{Y}^{t,\mathbf{x},n}\|_{\mathfrak{H}_{t,\mathbf{x}}^2} \xrightarrow[n \rightarrow +\infty]{} 0. \quad (3.9)$$

4 The representation formula

The main goal of this section is to prove the following representation

$$v(t, \mathbf{x}) = \mathcal{Y}_t^{t,\mathbf{x}} = \mathfrak{Y}_t^{t,\mathbf{x}}, \text{ for every } (t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0.$$

In order to prove this result, we will show that both $v^n(t, \mathbf{x})$ defined in (2.14) and $u^n(t, \mathbf{x}) := \mathfrak{Y}_t^{t,\mathbf{x},n}$ are viscosity solutions of a semi-linear path-dependent PDE, for which a comparison result holds. It will then imply that $v^n(t, \mathbf{x}) = u^n(t, \mathbf{x})$, and the desired result will be obtained by passing to the limit when n goes to $+\infty$, see Lemma 2.7 and (3.9).

4.1 A crash course on PPDEs

In this section, we follow closely [34] to introduce all the notions needed for the definition of viscosity solutions of path-dependent PDEs. Let us start with the notions of regularity we will consider.

Definition 4.1. (i) For any $(t, \mathbf{x}, \tilde{\mathbf{x}}) \in [0, T] \times \mathcal{C}^0 \times \mathcal{C}^t$, any $s \in [t, T]$ and any $d \geq 1$, we say that a \mathbb{R}^d -valued process on $(\mathbf{\Lambda}^{t,\mathbf{x}_t}, \mathcal{F}_T^{t,\mathbf{x}_t,o})$ is in $C^0([s, T] \times \mathbf{\Lambda}^{s,(\mathbf{x} \otimes_t \tilde{\mathbf{x}})^s}, \mathbb{R}^d)$ when it is continuous with respect to the distance d_∞ , that is, for any $\varepsilon > 0$, for any $(r_1, r_2, \mathbf{x}_1, \mathbf{x}_2) \in [s, T]^2 \times \mathcal{C}^s \times \mathcal{C}^s$, there exists $\delta > 0$ such that

$$\text{if } d_\infty((r_1, \mathbf{x}_1), (r_2, \mathbf{x}_2)) \leq \delta, \text{ then } \left\| u^{s,\tilde{\mathbf{x}}}(r_1, \mathbf{x}_1) - u^{s,\tilde{\mathbf{x}}}(r_2, \mathbf{x}_2) \right\| \leq \varepsilon.$$

(ii) For any $(t, \mathbf{x}, \tilde{\mathbf{x}}) \in [0, T] \times \mathcal{C}^0 \times \mathcal{C}^t$ and any $s \in [t, T]$, we say that a \mathbb{R} -valued process u on $(\mathbf{\Lambda}^{t,\mathbf{x}_t}, \mathcal{F}_T^{t,\mathbf{x}_t,o})$ belongs to $C^{1,2}([s, T] \times \mathbf{\Lambda}^{s,(\mathbf{x} \otimes_t \tilde{\mathbf{x}})^s})$ if $u \in C^0([s, T] \times \mathbf{\Lambda}^{s,(\mathbf{x} \otimes_t \tilde{\mathbf{x}})^s}, \mathbb{R})$ and if there exists $(Z, \Gamma) \in C^0([t, T] \times \mathbf{\Lambda}^{t,\mathbf{x}_t}, \mathbb{R}^d) \times C^0([t, T] \times \mathbf{\Lambda}^{t,\mathbf{x}_t}, \mathbb{R})$ such that

$$u_z^{s,\tilde{\mathbf{x}}} - u_s(\tilde{\mathbf{x}}) = \int_s^z \Gamma_r^{s,\tilde{\mathbf{x}}} dr + \int_s^z Z_r^{s,\tilde{\mathbf{x}}} \cdot dB_r^{s,(\mathbf{x} \otimes_t \tilde{\mathbf{x}})^s}, \quad z \in [s, T], \mathbb{P}_0^{s,(\mathbf{x} \otimes_t \tilde{\mathbf{x}})} - a.s.$$

We then denote for any $\tilde{\mathbf{x}} \in \mathcal{C}^t$ and any $s \in [t, T]$,

$$\begin{aligned} \mathcal{L}^{t,\mathbf{x}}u(s, \tilde{\mathbf{x}}) &:= \partial_t u_s(\tilde{\mathbf{x}}) + \mu_s^{t,\mathbf{x}}(\tilde{\mathbf{x}}) \cdot Du(s, \tilde{\mathbf{x}}) + \frac{1}{2} \text{Tr}[\sigma_s^{t,\mathbf{x}}(\sigma_s^{t,\mathbf{x}})^\top(\tilde{\mathbf{x}})D^2u(s, \tilde{\mathbf{x}})] := \Gamma_s(\tilde{\mathbf{x}}), \\ Du(s, \tilde{\mathbf{x}}) &:= ((\sigma_s^{t,\mathbf{x}}(\tilde{\mathbf{x}}))^\top)^{-1}Z_s(\tilde{\mathbf{x}}). \end{aligned}$$

Let us then denote, for any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$, by $\mathcal{T}^{t, \mathbf{x}}$ the set of $\mathbb{F}^{t, \mathbf{x}, 0}$ -stopping times taking values in $[t, T]$, by $\mathcal{T}_+^{t, \mathbf{x}}$ the subset of $\mathcal{T}^{t, \mathbf{x}}$ consisting of the stopping times taking values in $(t, T]$, and for any $H \in \mathcal{T}^{t, \mathbf{x}}$, by $\mathcal{T}_H^{t, \mathbf{x}}$ and $\mathcal{T}_{H,+}^{t, \mathbf{x}}$, the subsets of $\mathcal{T}^{t, \mathbf{x}}$ consisting of stopping times taking values respectively in $[t, H]$ and $(t, H]$.

Next, we define for any $N \geq 1$

$$\mathcal{P}^{t, \mathbf{x}, N} := \{ \mathbb{P}_\nu^{t, \mathbf{x}}, \nu \in \mathcal{U}^{t, N} \},$$

$$\mathcal{M}^{t, \mathbf{x}, N} := \left\{ \mathbb{Q} \text{ s.t. } \frac{d\mathbb{Q}}{d\mathbb{P}_0^{t, \mathbf{x}}} = \mathcal{E} \left(\int_t^T b_s dW_s^{t, \mathbf{x}} \right), b, \mathbb{F}^{t, \mathbf{x}, t}\text{-predictable s.t. } \|b\|_\infty \leq N \right\}.$$

For any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$, and for any $w \in C^0([0, T] \times \Lambda^{0, \mathbf{x}_0}, \mathbb{R})$, we now define the sets of test functions for w as

$$\overline{\mathcal{A}}^n w(t, \mathbf{x}) := \left\{ \varphi \in C^{1,2}([t, T] \times \Lambda^{t, \mathbf{x}^t}), 0 = (\varphi - w^{t, \mathbf{x}})(t, \mathbf{x}^t) > \overline{\mathcal{E}}_t^n [(\varphi - w^{t, \mathbf{x}})(\cdot, B^{t, \mathbf{x}^t})_{\tau \wedge H}] \right. \\ \left. \text{for some } H \in \mathcal{T}^{t, \mathbf{x}} \text{ and for all } \tau \in \mathcal{T}_{H,+}^{t, \mathbf{x}} \right\},$$

$$\underline{\mathcal{A}}^n w(t, \mathbf{x}) := \left\{ \varphi \in C^{1,2}([t, T] \times \Lambda^{t, \mathbf{x}^t}), 0 = (\varphi - w^{t, \mathbf{x}})(t, \mathbf{x}^t) < \underline{\mathcal{E}}_t^n [(\varphi - w^{t, \mathbf{x}})(\cdot, B^{t, \mathbf{x}^t})_{\tau \wedge H}] \right. \\ \left. \text{for some } H \in \mathcal{T}^{t, \mathbf{x}} \text{ and for all } \tau \in \mathcal{T}_{H,+}^{t, \mathbf{x}} \right\},$$

where for all $\mathcal{F}_T^{t, \mathbf{x}^t}$ -measurable ξ such that the quantities below are finite

$$\overline{\mathcal{E}}_t^n[\xi] = \sup_{\mathbb{Q} \in \mathcal{M}^{t, \mathbf{x}, \bar{n}}} \mathbb{E}^{\mathbb{Q}}[\xi], \underline{\mathcal{E}}_t^n[\xi] = \inf_{\mathbb{Q} \in \mathcal{M}^{t, \mathbf{x}, \bar{n}}} \mathbb{E}^{\mathbb{Q}}[\xi], \text{ with } \bar{n} := n\sqrt{d} \max(1, \|\sigma^{-1} f\|).$$

Finally, we define for every $(t, \mathbf{x}, \varphi) \in [0, T] \times \mathcal{C}^0 \times C^{1,2}([t, T] \times \Lambda^{t, \mathbf{x}^t})$ the following PPDE

$$-\mathcal{L}^{t, \mathbf{x}} \varphi(t, \mathbf{x}^t) - n\rho \left(f_t^\top D\varphi(t, \mathbf{x}^t) \right) = 0. \quad (4.1)$$

Definition 4.2. Fix some $x \in \mathbb{R}^d$ and let $u \in C^0([0, T] \times \Lambda^{0, x}, \mathbb{R})$. We say that

(i) u is a viscosity subsolution of PPDE (4.1) if for any $(t, \mathbf{x}, \varphi) \in [0, T] \times \mathcal{C}^0 \times \underline{\mathcal{A}}^n u(t, \mathbf{x})$

$$-\mathcal{L}^{t, \mathbf{x}} \varphi(t, \mathbf{x}^t) - n\rho \left(f_t^\top D\varphi(t, \mathbf{x}^t) \right) \leq 0.$$

(ii) u is a viscosity supersolution of PPDE (4.1) if for any $(t, \mathbf{x}, \varphi) \in [0, T] \times \mathcal{C}^0 \times \overline{\mathcal{A}}^n u(t, \mathbf{x})$

$$-\mathcal{L}^{t, \mathbf{x}} \varphi(t, \mathbf{x}^t) - n\rho \left(f_t^\top D\varphi(t, \mathbf{x}^t) \right) \geq 0.$$

(iii) u is a viscosity solution of PPDE (4.1) if it is both a sub and a supersolution.

We shall end this section with the following result which will be useful in the PPDE derivation of the value function.

Lemma 4.3. For all $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$ and $\tau \in \mathcal{T}_+^{t, \mathbf{x}}$ we have

$$\underline{\mathcal{E}}_t^n[\tau - t] > 0.$$

Proof. Fix $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$ and $\tau \in \mathcal{T}_+^{t, \mathbf{x}}$ and denote by $(U, V) \in \mathfrak{S}_{t, \mathbf{x}}^2 \times \mathfrak{H}_{t, \mathbf{x}}^2$ the unique solution of the following backward stochastic differential equation on $[t, \tau]$

$$U_s = \tau - t - \int_s^\tau \bar{n} |V_r| dr - \int_s^\tau V_r \cdot dW_r^{t, \mathbf{x}}, s \in [t, \tau], \mathbb{P}_0^{t, \mathbf{x}} - a.s.,$$

where we remind the reader that in our context τ is $\mathcal{F}_\tau^{t,\mathbf{x},o}$ -measurable, and that we can always consider a $\mathbb{P}_0^{t,\mathbf{x}}$ -version of U (resp. V), which we still denote by U (resp. V) for simplicity, and which is $\mathbb{F}^{t,\mathbf{x},o}$ -progressively measurable (resp. predictable).

Let μ be an arbitrary $\mathbb{F}^{t,\mathbf{x},o}$ -predictable process satisfying $|\mu| \leq \bar{n}$, so that for all $z \in \mathbb{R}^d$ we have $-\bar{n}|z| \leq \mu \cdot z$. This implies in particular that $\mathbb{Q}^\mu \in \mathcal{M}^{t,\mathbf{x},\bar{n}}$ with

$$d\mathbb{Q}^\mu := \mathcal{E} \left(\int_t^T \mu_s \cdot dW_s^{t,\mathbf{x}} \right) d\mathbb{P}_0^{t,\mathbf{x}}.$$

Hence, by standard a comparison result for BSDEs (see for instance Theorem 2.2 in [19]) we have

$$U_t \leq \mathbb{E}^{\mathbb{Q}^\mu}[\tau - t].$$

Hence, the arbitrariness of μ implies that

$$U_t \leq \underline{\mathcal{E}}_t^n[\tau - t]. \quad (4.2)$$

On the other hand, let ν^n be defined such that $\nu^n \cdot V = -\bar{n}|V|$ and observe that $|\nu^n| \leq \bar{n}$. Then we have $\mathbb{Q}^n \in \mathcal{M}^{t,\mathbf{x},\bar{n}}$ with

$$d\mathbb{Q}^n := \mathcal{E} \left(\int_t^T \nu_s^n \cdot dW_s^{t,\mathbf{x}} \right) d\mathbb{P}_0^{t,\mathbf{x}}.$$

We hence have, using the fact that U_t is a constant by the Blumenthal 0 – 1 law

$$U_t = \tau - t - \int_t^\tau V_s \cdot (dW_s^{t,\mathbf{x}} - \nu_s^n ds) = \mathbb{E}^{\mathbb{Q}^n}[\tau - t] \geq \underline{\mathcal{E}}_t^n[\tau - t],$$

by definition of $\underline{\mathcal{E}}_t^n$. The last inequality together with (4.2) gives that

$$U_t = \underline{\mathcal{E}}_t^n[\tau - t].$$

Since $\tau > t$, $\mathbb{P}_0^{t,\mathbf{x}}$ -a.s., the result follows by strict comparison (see again Theorem 2.2 in [19]). \square

4.2 The viscosity solution properties

We start with the value function v^n .

Proposition 4.4. *Under Assumptions 2.1 and 2.3, v^n is a viscosity solution of PPDE (4.1).*

Proof. We proceed along the lines of [17, Proof of Proposition 4.4] and split the proof into two steps.

Fix $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$ for the remainder of the proof.

Step 1: We show that $v^n \in C^0([0, T] \times \mathbf{\Lambda}^{0,\mathbf{x}_0})$ and satisfies the dynamic programming principle, for any $\tau \in \mathcal{T}^t$ and any $\theta \in \mathcal{T}^{t,\mathbf{x}}$

$$v^n(t, \mathbf{x}) = \sup_{\nu \in \mathcal{U}^{t,n}} \mathbb{E}^{\mathbb{P}_0^{t,\mathbf{x}}} [v^n(\tau, \mathbf{x} \otimes_t X^{t,\mathbf{x},\nu})] = \sup_{\nu \in \mathcal{U}^{t,n}} \mathbb{E}^{\mathbb{P}_\nu^{t,\mathbf{x}}} [v^n(\theta, \mathbf{x} \otimes_t B^{t,\mathbf{x}_t})] \quad (4.3)$$

The dynamic programming result is actually classical since the controls ν that we consider here take values in a compact subset of \mathbb{R}^d . We refer the reader to Proposition 2.5 and Theorem 3.3 in [20], where we emphasize that their proof of Theorem 3.3 can immediately be extended to the case of μ and σ Lipschitz continuous with linear growth (instead of Lipschitz continuous and bounded), as they only require to have existence of a strong solution to the SDEs considered.

We next show the continuity of v^n . For any $(t, t'; \mathbf{x}, \mathbf{x}') \in [0, T] \times [t, T] \times \mathcal{C}^0 \times \mathcal{C}^0$, we have

$$|v^n(t, \mathbf{x}) - v^n(t', \mathbf{x}')| \leq |v^n(t, \mathbf{x}) - v^n(t, \mathbf{x}')| + |v^n(t, \mathbf{x}') - v^n(t', \mathbf{x}')|.$$

We now estimate separately the two terms in the right-hand side above. We have first using Assumption 2.3, (2.3), (2.4) and the fact that the controls $\nu \in \mathcal{U}^{t,n}$ are bounded by $n\sqrt{d}$

$$\begin{aligned} |v^n(t, \mathbf{x}) - v^n(t, \mathbf{x}')| &\leq \sup_{\nu \in \mathcal{U}^{t,n}} \mathbb{E}^{\mathbb{P}_0^t} \left[\left| U^{t,\mathbf{x}}(X^{t,\mathbf{x},\nu}) - U^{t,\mathbf{x}'}(X^{t,\mathbf{x}',\nu}) \right| \right] \\ &\leq C \sup_{\nu \in \mathcal{U}^{t,n}} \mathbb{E}^{\mathbb{P}_0^t} \left[\left\| \mathbf{x} \otimes_t (X^{t,\mathbf{x},\nu} - X^{t,\mathbf{x}',\nu}) \right\|_{\infty, T}^2 \right]^{\frac{1}{2}} \left(1 + \mathbb{E}^{\mathbb{P}_0^t} \left[\left\| \mathbf{x} \otimes_t X^{t,\mathbf{x},\nu} \right\|_{\infty, T}^{2r} \right]^{\frac{1}{2}} + \mathbb{E}^{\mathbb{P}_0^t} \left[\left\| \mathbf{x}' \otimes_t X^{t,\mathbf{x}',\nu} \right\|_{\infty, T}^{2r} \right]^{\frac{1}{2}} \right) \\ &\leq C_d n^{1\vee r} \left\| \mathbf{x} - \mathbf{x}' \right\|_{\infty, t} \left(1 + \left\| \mathbf{x} \right\|_{\infty, t}^r + \left\| \mathbf{x}' \right\|_{\infty, t}^r \right), \end{aligned}$$

where the constant C_d does not depend on n .

Next, using (4.3) for $\tau = t'$, we compute, using the previous calculation and (2.2)

$$\begin{aligned} |v^n(t, \mathbf{x}') - v^n(t', \mathbf{x}')| &\leq \sup_{\nu \in \mathcal{U}^{t,n}} \mathbb{E}^{\mathbb{P}_0^t} \left[\left| v^n(t', \mathbf{x}' \otimes_t X^{t,\mathbf{x}',\nu}) - v^n(t', \mathbf{x}') \right| \right] \\ &\leq C_d n^{1\vee r} \sup_{\nu \in \mathcal{U}^{t,n}} \mathbb{E}^{\mathbb{P}_0^t} \left[\left\| \mathbf{x}' \otimes_t X^{t,\mathbf{x}',\nu} - \mathbf{x}' \right\|_{\infty, t'}^2 \right]^{\frac{1}{2}} \left(1 + \left\| \mathbf{x}' \right\|_{\infty, t'}^r + \mathbb{E}^{\mathbb{P}_0^t} \left[\left\| \mathbf{x}' \otimes_t X^{t,\mathbf{x}',\nu} \right\|_{\infty, t'}^{2r} \right]^{\frac{1}{2}} \right) \\ &\leq C_d n^{1\vee r+1+r} (t' - t)^{\frac{1}{2}} \left(1 + \left\| \mathbf{x}' \right\|_{\infty, t}^{r+1} + \left\| \mathbf{x}' \right\|_{\infty, t'}^{r+1} \right). \end{aligned}$$

By definition of d_∞ , we have thus obtained that

$$|v^n(t, \mathbf{x}) - v^n(t', \mathbf{x}')| \leq C_d n^{1+r+1\vee r} d_\infty((t, \mathbf{x}), (t', \mathbf{x}')) \left(1 + \left\| \mathbf{x} \right\|_{\infty, t}^{r+1} + \left\| \mathbf{x}' \right\|_{\infty, t'}^{r+1} \right),$$

which proves the continuity of v^n with respect to d_∞ .

Step 2: We show that v^n is a viscosity subsolution to PPDE (4.1).

Assume to the contrary that there $(t, \mathbf{x}; \varphi) \in [0, T] \times \mathcal{C}^0 \times \underline{\mathcal{A}}v^n(t, \mathbf{x})$ s.t. for some $c > 0$

$$-\mathcal{L}^{t,\mathbf{x}}\varphi(t, \mathbf{x}^t) - n\rho(f_t^\top D\varphi(t, \mathbf{x}^t)) \geq 2c > 0.$$

Without loss of generality, we may reduce H in the definition of $\varphi \in \underline{\mathcal{A}}v^n(t, \mathbf{x})$ so that by continuity of all the above maps, we obtain

$$-\mathcal{L}^{t,\mathbf{x}}\varphi(s, B^{t,\mathbf{x}_t}) - n\rho(f_s^\top D\varphi(s, B^{t,\mathbf{x}_t})) \geq c, \text{ on } [t, H], \mathbb{P}_0^{t,\mathbf{x}} - a.s.$$

Furthermore, observe that for each $s \in [t, H]$

$$n\rho(f_s^\top D\varphi(s, B^{t,\mathbf{x}_t})) = \sup_{u \in [0, n]^d} u \cdot (f_s^\top D\varphi(s, B^{t,\mathbf{x}_t})),$$

so that by definition of $\mathcal{U}^{t,n}$ we have for all $\nu \in \mathcal{U}^{t,n}$

$$-\mathcal{L}^{t,\mathbf{x}}\varphi(s, B^{t,\mathbf{x}_t}) - \nu_s \cdot f_s^\top D\varphi(s, B^{t,\mathbf{x}_t}) \geq c, \text{ on } [t, H], \mathbb{P}_0^{t,\mathbf{x}} - a.s. \quad (4.4)$$

Since φ is smooth on $[t, H]$, we can write under $\mathbb{P}_0^{t,\mathbf{x}}$ that

$$\begin{aligned} \varphi(H, B^{t,\mathbf{x}_t}) &= \varphi(t, \mathbf{x}^t) + \int_t^H \mathcal{L}^{t,\mathbf{x}}\varphi(s, B^{t,\mathbf{x}_t}) ds + \int_t^H D\varphi(s, B^{t,\mathbf{x}_t}) \cdot \sigma_s^{t,\mathbf{x}}(B^{t,\mathbf{x}_t}) dW_s^{t,\mathbf{x}} \\ &= \varphi(t, \mathbf{x}^t) + \int_t^H \mathcal{L}^{t,\mathbf{x}}\varphi(s, B^{t,\mathbf{x}_t}) + D\varphi(s, B^{t,\mathbf{x}_t}) \cdot f_s \nu_s ds \\ &\quad + \int_t^H D\varphi(s, B^{t,\mathbf{x}_t}) \cdot \sigma_s^{t,\mathbf{x}}(B^{t,\mathbf{x}_t}) [dW_s^{t,\mathbf{x}} - (\sigma_s^{t,\mathbf{x}})^{-1}(B^{t,\mathbf{x}_t}) f_s \nu_s ds]. \end{aligned}$$

Since $\nu \in \mathcal{U}^{t,n}$, we have $\mathbb{P}_\nu^{t,\mathbf{x}} \in \mathcal{M}^{t,\mathbf{x},\bar{n}}$ so that by (4.4):

$$\mathbb{E}^{\mathbb{P}_\nu^{t,\mathbf{x}}} [(\varphi - (v^n)^{t,\mathbf{x}})(H, B^{t,\mathbf{x}_t})] \leq -c\underline{\mathcal{E}}_t[H - t] + \varphi(t, \mathbf{x}^t) - \mathbb{E}^{\mathbb{P}_\nu^{t,\mathbf{x}}} [(v^n)^{t,\mathbf{x}}(H, B^{t,\mathbf{x}_t})],$$

and taking the infimum on $\mathbb{P}_\nu^{t,\mathbf{x}} \in \mathcal{M}^{t,\mathbf{x},\bar{n}}$ on the left-hand side and recalling that $\varphi \in \underline{\mathcal{A}}v^n(t, \mathbf{x})$, this gives

$$0 < \underline{\mathcal{E}}_t [(\varphi - (v^n)^{t,\mathbf{x}})(H, B^{t,\mathbf{x}})] \leq -c\underline{\mathcal{E}}_t [H - t] - \mathbb{E}^{\mathbb{P}_\nu^{t,\mathbf{x}}} [(v^n)^{t,\mathbf{x}}(H, B^{t,\mathbf{x}_t}) - (v^n)^{t,\mathbf{x}}(t, \mathbf{x}^t)],$$

and finally, by the dynamic programming principle (4.3), taking the infimum over $\nu \in \mathcal{U}^{t,n}$ on the right-hand side gives

$$0 < -c\underline{\mathcal{E}}_t [H - t],$$

which is a contradiction since by Lemma 4.3 the right-hand side is negative.

Step 3: We show that v^n is a viscosity supersolution to PPDE (4.1).

Assume to the contrary that there $(t, \mathbf{x}; \varphi) \in [0, T] \times \mathcal{C}^0 \times \overline{\mathcal{A}}v^n(t, \mathbf{x})$ such that for some $c > 0$

$$-\mathcal{L}^{t,\mathbf{x}}\varphi(t, \mathbf{x}^t) - n\rho(f_t^\top D\varphi(t, \mathbf{x}^t)) \leq -3c < 0.$$

Observe again that for each $s \in [t, T]$

$$n\rho(f_s^\top D\varphi(s, B^{t,\mathbf{x}})) = \sup_{u \in [0, n]^d} u \cdot (f_s^\top D\varphi(s, B^{t,\mathbf{x}})),$$

so that there is $u_n^* \in [0, n]^d$ such that

$$-\mathcal{L}^{t,\mathbf{x}}\varphi(t, \mathbf{x}^t) - u_n^* \cdot (f_t^\top D\varphi(t, \mathbf{x}^t)) \leq -2c.$$

Without loss of generality, we may reduce H in the definition of $\varphi \in \overline{\mathcal{A}}v^n(t, \mathbf{x})$ so that by continuity, we obtain

$$-\mathcal{L}^{t,\mathbf{x}}\varphi(t, \mathbf{x}^t) - u_n^* \cdot (f_s^\top D\varphi(s, B^{t,\mathbf{x}_t})) \leq -c, \text{ on } [t, H], \mathbb{P}_0^{t,\mathbf{x}} - a.s.$$

Set the constant control of $\mathcal{U}^{t,n}$: $\nu^n \equiv u_n^*$. Hence we have

$$-\mathcal{L}^{t,\mathbf{x}}\varphi(s, B^{t,\mathbf{x}_t}) - \nu_s^n \cdot f_s^\top D\varphi(s, B^{t,\mathbf{x}_t}) \leq -c, \text{ on } [t, H], \mathbb{P}_0^{t,\mathbf{x}} - a.s. \quad (4.5)$$

Since φ is smooth on $[t, H]$, we have, $\mathbb{P}_0^{t,\mathbf{x}} - a.s.$,

$$\begin{aligned} \varphi(H, B^{t,\mathbf{x}}) &= \varphi(t, \mathbf{x}^t) + \int_t^H \mathcal{L}^{t,\mathbf{x}}\varphi(s, B^{t,\mathbf{x}_t}) ds + \int_t^H D\varphi(s, B^{t,\mathbf{x}_t}) \cdot \sigma_s^{t,\mathbf{x}}(B^{t,\mathbf{x}_t}) dW_s^{t,\mathbf{x}} \\ &= \varphi(t, \mathbf{x}^t) + \int_t^H \mathcal{L}^{t,\mathbf{x}}\varphi(s, B^{t,\mathbf{x}_t}) + D\varphi(s, B^{t,\mathbf{x}_t}) \cdot f_s \nu_s^n ds \\ &\quad + \int_t^H D\varphi(s, B^{t,\mathbf{x}_t}) \cdot \sigma_s^{t,\mathbf{x}}(B^{t,\mathbf{x}_t}) [dW_s^{t,\mathbf{x}} - (\sigma_s^{t,\mathbf{x}})^{-1}(B^{t,\mathbf{x}_t}) f_s \nu_s^n ds]. \end{aligned}$$

By (4.5), this gives

$$\mathbb{E}^{\mathbb{P}_\nu^{t,\mathbf{x}}} [(\varphi - (v^n)^{t,\mathbf{x}})(H, B^{t,\mathbf{x}_t})] \geq c\mathbb{E}^{\mathbb{P}_\nu^{t,\mathbf{x}}} [H - t] + \varphi(t, \mathbf{x}^t) - \mathbb{E}^{\mathbb{P}_\nu^{t,\mathbf{x}}} [(v^n)^{t,\mathbf{x}}(H, B^{t,\mathbf{x}_t})].$$

Since $\varphi \in \overline{\mathcal{A}}v^n(t, \mathbf{x})$, we have the equality $\varphi(t, \mathbf{x}^t) = v^n(t, \mathbf{x})$. Moreover, the fact that $\nu^n \in \mathcal{U}^{t,n}$ enables us to use the DPP (4.3) to have

$$\mathbb{E}^{\mathbb{P}_\nu^{t,\mathbf{x}}} [(\varphi - (v^n)^{t,\mathbf{x}})(H, B^{t,\mathbf{x}_t})] \geq c\mathbb{E}^{\mathbb{P}_\nu^{t,\mathbf{x}}} [H - t] > 0.$$

Using (again) the fact that $\nu^n \in \mathcal{U}^{t,n}$ implies that $\mathbb{P}_\nu^{t,\mathbf{x}} \in \mathcal{M}^{t,\mathbf{x},n}$, this contradicts $\varphi \in \overline{\mathcal{A}}v^n(t, \mathbf{x})$. \square

Let us now treat the penalized BSDEs.

Proposition 4.5. *Under Assumptions 2.1, 2.3, 3.2 and 3.6, $u^n(t, \mathbf{x}) := \mathfrak{V}_t^{t,\mathbf{x},n} = \mathcal{Y}_t^{t,\mathbf{x},n}$ is a viscosity solution of PPDE (4.1).*

Proof. We proceed along the lines of [17, Proof of Proposition 4.4] and split the proof into two steps. Fix $(t, \mathbf{x}) \in [0, T] \times \mathbf{\Lambda}$ for the remainder of the proof.

Step 1: We show that $u^n \in C^0([0, T] \times \mathbf{\Lambda}^{0, \mathbf{x}_0})$ and satisfies the following dynamic programming principle, for any $\tau \in \mathcal{T}^{t, \mathbf{x}}$:

$$\mathfrak{Y}^{t, \mathbf{x}, n} = (u^n)^{t, \mathbf{x}}(\tau, B^{t, \mathbf{x}_t}) + \int_{\cdot}^{\tau} n\rho \left[f_s^\top ((\sigma_s^{t, \mathbf{x}})^\top)^{-1} (B^{t, \mathbf{x}}) \mathfrak{Z}_s^{t, \mathbf{x}, n} \right] ds - \int_{\cdot}^{\tau} \mathfrak{Z}_s^{t, \mathbf{x}, n} \cdot dW_s^{t, \mathbf{x}}, \mathbb{P}_0^{t, \mathbf{x}}\text{-a.s.} \quad (4.6)$$

First of all, since $n\rho^\varepsilon$ is Lipschitz-continuous and nul at 0 since Assumption 2.3 holds, by standard stability results on BSDEs (see e.g. [19]), for any $n \geq 1$, there is a constant C_n (which may vary from line to line) such that for all $(t, \mathbf{x}, \mathbf{x}') \in [0, T] \times (\mathcal{C}^0)^2$

$$\mathbb{E}^{\mathbb{P}_0^t} \left[\sup_{t \leq s \leq T} |\mathfrak{Y}_s^{t, \mathbf{x}, n}|^2 + \int_t^T \|\mathfrak{Z}_s^{t, \mathbf{x}, n}\|^2 ds \right] \leq C_n \left(1 + \|\mathbf{x}\|_{\infty, t}^{2(r+1)} \right), \quad (4.7)$$

$$\mathbb{E}^{\mathbb{P}_0^t} \left[\sup_{t \leq s \leq T} |\mathfrak{Y}_s^{t, \mathbf{x}, n} - \mathfrak{Y}_s^{t, \mathbf{x}', n}|^2 + \int_t^T \|\mathfrak{Z}_s^{t, \mathbf{x}, n} - \mathfrak{Z}_s^{t, \mathbf{x}', n}\|^2 ds \right] \leq C_n \|\mathbf{x} - \mathbf{x}'\|_{\infty, t}^2 \left(1 + \|\mathbf{x}\|_{\infty, t}^{2r} + \|\mathbf{x}'\|_{\infty, t}^{2r} \right). \quad (4.8)$$

In particular, this gives the following regularity

$$|u^n(t, \mathbf{x})| \leq C_n \left(1 + \|\mathbf{x}\|_{\infty, t}^{r+1} \right) \quad (4.9)$$

$$\text{and } |u^n(t, \mathbf{x}) - u^n(t, \mathbf{x}')| \leq C_n \|\mathbf{x} - \mathbf{x}'\|_{\infty, t} \left(1 + \|\mathbf{x}\|_{\infty, t}^r + \|\mathbf{x}'\|_{\infty, t}^r \right). \quad (4.10)$$

By standard arguments in the BSDE theory (this would be simply the tower property for conditional expectations if ρ were equal to 0, and the result can easily be generalized using the fact that solutions to BSDEs with Lipschitz drivers can be obtained via Picard iterations), we have the following dynamic programming principle, for any $t < t' \leq T$

$$\mathfrak{Y}^{t, \mathbf{x}, n} = (u^n)^{t, \mathbf{x}}(t', B^{t, \mathbf{x}_t}) + \int_{\cdot}^{t'} n\rho \left[f_s^\top ((\sigma_s^{t, \mathbf{x}})^\top)^{-1} (B^{t, \mathbf{x}_t}) \mathfrak{Z}_s^{t, \mathbf{x}, n} \right] ds - \int_{\cdot}^{t'} \mathfrak{Z}_s^{t, \mathbf{x}, n} \cdot dW_s^{t, \mathbf{x}}, \mathbb{P}_0^{t, \mathbf{x}}\text{-a.s.} \quad (4.11)$$

In particular $\mathfrak{Y}_s^{t, \mathbf{x}, n} = (u^n)^{t, \mathbf{x}}(s, B^{t, \mathbf{x}_t})$ for any $s \in [t, T]$. It then follows

$$\begin{aligned} |u^n(t, \mathbf{x}) - u^n(t', \mathbf{x})| &= \left| \mathbb{E}^{\mathbb{P}_0^{t, \mathbf{x}}} \left[\mathfrak{Y}_t^{t, \mathbf{x}, n} - \mathfrak{Y}_{t'}^{t, \mathbf{x}, n} + (u^n)^{t, \mathbf{x}}(t', B^{t, \mathbf{x}_t}) - u^n(t', \mathbf{x}) \right] \right| \\ &\leq \mathbb{E}^{\mathbb{P}_0^{t, \mathbf{x}}} \left[\int_t^{t'} |n\rho(f_s^\top ((\sigma_s^{t, \mathbf{x}})^\top)^{-1} (B^{t, \mathbf{x}_t}) \mathfrak{Z}_s^{t, \mathbf{x}, n})| ds \right] + \mathbb{E}^{\mathbb{P}_0^{t, \mathbf{x}}} [|(u^n)^{t, \mathbf{x}}(t', B^{t, \mathbf{x}_t}) - u^n(t', \mathbf{x})|] \\ &\leq \mathbb{E}^{\mathbb{P}_0^{t, \mathbf{x}}} \left[\int_t^{t'} |n\rho(f_s^\top ((\sigma_s^{t, \mathbf{x}})^\top)^{-1} (B^{t, \mathbf{x}_t}) \mathfrak{Z}_s^{t, \mathbf{x}, n})| ds \right] \\ &\quad + C_n \left(\sup_{t \leq s \leq t'} \|\mathbf{x}_s - \mathbf{x}_t\| + \mathbb{E}^{\mathbb{P}_0^{t, \mathbf{x}}} \left[\sup_{t \leq s \leq t'} \|B_s^{t, \mathbf{x}_t} - \mathbf{x}_t\|^2 \right]^{\frac{1}{2}} \right) \left(1 + \|\mathbf{x}\|_{\infty, t'}^r + \mathbb{E}^{\mathbb{P}_0^{t, \mathbf{x}}} \left[\sup_{t \leq s \leq t'} \|B_s^{t, \mathbf{x}_t}\|^2 \right]^{\frac{1}{2}} \right) \\ &\leq \mathbb{E}^{\mathbb{P}_0^{t, \mathbf{x}}} \left[\int_t^{t'} |n\rho(f_s^\top ((\sigma_s^{t, \mathbf{x}})^\top)^{-1} (B^{t, \mathbf{x}_t}) \mathfrak{Z}_s^{t, \mathbf{x}, n})| ds \right] + C_n d_{\infty}((t, \mathbf{x}); (t', \mathbf{x})) \left(1 + \|\mathbf{x}\|_{\infty, t'}^r \right), \end{aligned} \quad (4.12)$$

where the last line follows from (2.2). Observe from (4.7) combined with Assumption 2.1 and the Lipschitz-continuity of ρ^ε that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_0^{t, \mathbf{x}}} \left[\int_t^{t'} |n\rho(f_s^\top ((\sigma_s^{t, \mathbf{x}})^\top)^{-1} (B^{t, \mathbf{x}_t}) \mathfrak{Z}_s^{t, \mathbf{x}, n})| ds \right] &= \mathbb{E}^{\mathbb{P}_0^t} \left[\int_t^{t'} |n\rho(f_s^\top ((\sigma_s^{t, \mathbf{x}})^\top)^{-1} (B^t) \mathfrak{Z}_s^{t, \mathbf{x}, n})| ds \right] \\ &\leq C_n \left(1 + \|\mathbf{x}\|_{\infty, t'}^{r+1} \right) (t' - t)^{\frac{1}{2}}. \end{aligned}$$

Plugging the latter into (4.12) and since $\sqrt{t' - t} \leq d_\infty((t, \mathbf{x}); (t', \mathbf{x}))$, this gives finally

$$|u^n(t, \mathbf{x}) - u^n(t', \mathbf{x})| \leq C_n \left(1 + \|\mathbf{x}\|_{\infty, t'}^{r+1}\right) d_\infty((t, \mathbf{x}); (t', \mathbf{x})).$$

Finally, the regularity in time that we just proved, allows us to classically extend the dynamic programming principle in (4.11) to stopping times, giving (4.6) (the result is clear for stopping times taking finitely many values, and the general result follows by the usual approximation of stopping times by decreasing sequences of stopping times with finitely many values).

Step 2: We conclude the proof.

Without loss of generality, we prove only the viscosity subsolution, the supersolution being obtained similarly. Assume to the contrary that there is $(t, \mathbf{x}; \varphi) \in [0, T] \times \mathcal{C}^0 \times \underline{\mathcal{A}}u^n(t, \mathbf{x})$ such that

$$2c := -\mathcal{L}^{t, \mathbf{x}}\varphi(t, \mathbf{x}^t) - n\rho\left(f_t^\top D\varphi(t, \mathbf{x}^t)\right) > 0.$$

Let H be the hitting time corresponding to the definition of $\varphi \in \underline{\mathcal{A}}u^n(t, \mathbf{x})$. By continuity of φ and ρ , reducing H if necessary, we deduce that

$$-\mathcal{L}^{t, \mathbf{x}}\varphi(s, B^{t, \mathbf{x}^t}) - n\rho\left(f_s D\varphi(s, B^{t, \mathbf{x}^t})\right) \geq c > 0, \quad s \in [t, H].$$

By the DPP (4.6) and the smoothness of φ , we have under $\mathbb{P}_0^{t, \mathbf{x}}$, we have

$$\begin{aligned} & (\varphi - (u^n)^{t, \mathbf{x}})_H(\cdot, B^{t, \mathbf{x}^t}) - (\varphi - (u^n)^{t, \mathbf{x}})_t(\cdot, \mathbf{x}^t) \\ &= \int_t^H \mathcal{L}^{t, \mathbf{x}}\varphi(s, B^{t, \mathbf{x}^t}) ds + \int_t^H \sigma_s^{t, \mathbf{x}}(B^{t, \mathbf{x}^t}) D\varphi(s, B^{t, \mathbf{x}^t}) \cdot dW_s^{t, \mathbf{x}} \\ & \quad + \int_t^H n\rho\left(f_s^\top \left((\sigma_s^{t, \mathbf{x}})^\top\right)^{-1}(B^{t, \mathbf{x}^t}) \mathfrak{Z}_s^{t, \mathbf{x}, n}\right) ds - \int_t^H \mathfrak{Z}_s^{t, \mathbf{x}, n} \cdot dW_s^{t, \mathbf{x}} \\ & \leq - \int_t^H \left[c + n \left(\rho\left(f_s^\top D\varphi(s, B^{t, \mathbf{x}^t})\right) - \rho\left(f_s^\top \left((\sigma_s^{t, \mathbf{x}})^\top\right)^{-1}(B^{t, \mathbf{x}^t}) \mathfrak{Z}_s^{t, \mathbf{x}, n}\right) \right) \right] ds \\ & \quad + \int_t^H \sigma_s^{t, \mathbf{x}}(B^{t, \mathbf{x}^t}) \left(D\varphi(s, B^{t, \mathbf{x}^t}) - (\sigma_s^{t, \mathbf{x}})^{-1}(B^{t, \mathbf{x}^t}) \mathfrak{Z}_s^{t, \mathbf{x}, n} \right) \cdot dW_s^{t, \mathbf{x}} \\ & = -c(H - t) + \int_t^H \alpha_s^n \cdot \left(D\varphi(s, B^{t, \mathbf{x}^t}) - (\sigma_s^{t, \mathbf{x}})^{-1}(B^{t, \mathbf{x}^t}) \mathfrak{Z}_s^{t, \mathbf{x}, n} \right) ds \\ & \quad + \int_t^H \sigma_s^{t, \mathbf{x}}(B^{t, \mathbf{x}^t}) \left(D\varphi(s, B^{t, \mathbf{x}^t}) - (\sigma_s^{t, \mathbf{x}})^{-1}(B^{t, \mathbf{x}^t}) \mathfrak{Z}_s^{t, \mathbf{x}, n} \right) \cdot dW_s^{t, \mathbf{x}} \\ & = c(t - H) + \int_t^H \left(D\varphi(s, B^{t, \mathbf{x}^t}) - (\sigma_s^{t, \mathbf{x}})^{-1}(B^{t, \mathbf{x}^t}) \mathfrak{Z}_s^{t, \mathbf{x}, n} \right) \cdot \left(\sigma_s^{t, \mathbf{x}}(B^{t, \mathbf{x}^t}) dW_s^{t, \mathbf{x}} + \alpha_s^n ds \right), \end{aligned}$$

where $|\alpha^n| \leq n\|f\| ds \otimes d\mathbb{P}_0^{t, \mathbf{x}}$ -a.e. By Girsanov's Theorem, we then have that there is $\mathbb{Q} \in \mathcal{M}^{t, \mathbf{x}, \bar{n}}$ such that $\int \sigma_s^{t, \mathbf{x}}(B^{t, \mathbf{x}^t}) dW_s^{t, \mathbf{x}} + \alpha_s^n ds$ is a \mathbb{Q} -Brownian motion. The above inequality holds then \mathbb{Q}^n -a.s. so that

$$(\varphi - (u^n)^{t, \mathbf{x}})_t(\cdot, \mathbf{x}^t) \geq \mathbb{E}^{\mathbb{Q}^n} [(\varphi - (u^n)^{t, \mathbf{x}})_H(\cdot, B^{t, \mathbf{x}^t}) + c(H - t)] > \mathbb{E}^{\mathbb{Q}^n} [(\varphi - (u^n)^{t, \mathbf{x}})_H(\cdot, B^{t, \mathbf{x}^t})],$$

which is in contradiction with the definition of $\varphi \in \underline{\mathcal{A}}u^n(t, \mathbf{x})$. \square

4.3 The main result

Define, for any $\mathbf{x} \in \mathcal{C}^0$, the following subset of $C^0([0, T] \times \mathbf{\Lambda}^{0, \mathbf{x}_0})$

$$\begin{aligned} C_2^0([0, T] \times \mathbf{\Lambda}^{0, \mathbf{x}_0}) := & \left\{ u \in C^0([0, T] \times \mathbf{\Lambda}^{0, \mathbf{x}_0}), \text{ s.t. for any } (t, \tilde{\mathbf{x}}) \in [0, T] \times \mathcal{C}^0, \right. \\ & \left. u^{t, \mathbf{x}} \text{ is continuous in time } \mathbb{P}_0^{t, \tilde{\mathbf{x}}} - a.s., u^{t, \tilde{\mathbf{x}}} \in \mathfrak{S}_{t, \tilde{\mathbf{x}}}^2 \right\}. \end{aligned}$$

We now recall the following comparison theorem from [34] (see their Theorem 4.1), adapted to our context.

Theorem 4.6 ([34]). *Let u, v in $C_2^0([0, T] \times \Lambda^{0, \mathbf{x}_0})$ be respectively viscosity subsolution and supersolution of PPDE (4.1). If $u(T, \cdot) \leq v(T, \cdot)$, then $u \leq v$ on $[0, T] \times \mathcal{C}^0$.*

Our first main result is then

Theorem 4.7. *Let Assumptions 2.1, 2.3, 3.2 and 3.6 hold. Then, for any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$, we have*

$$v(t, \mathbf{x}) = \mathcal{Y}_t^{t, \mathbf{x}} = \mathfrak{Y}_t^{t, \mathbf{x}}.$$

Proof. From Proposition 4.4 and Proposition 4.5, we know that for every $n \geq 1$, v^n and u^n are viscosity solutions of PPDE (4.1). Since it is clear by all our estimates that $v^n, u^n \in C_2^0([0, T] \times \Lambda^{0, \mathbf{x}_0})$, and since $v^n(T, \cdot) = u^n(T, \cdot)$, by Theorem 4.6 we deduce that

$$v^n(t, \mathbf{x}) = \mathcal{Y}_t^{t, \mathbf{x}, n} = \mathfrak{Y}_t^{t, \mathbf{x}, n}.$$

By Lemma 2.7, (3.8) and (3.9), it then suffices to let n go to $+\infty$. \square

5 Extension to degenerate diffusions

5.1 The setting

The result of the previous section is fundamentally based on the non-degeneracy of the diffusion matrix σ . Our main purpose here is to extend our general representation to cases where σ is allowed to degenerate. As will be clear later on, the type of degeneracy we will consider will be rather specific, but it will nonetheless be particularly well-suited for the applications we have in mind. Before stating our results, we need to introduce some notations.

For every $n \in \mathbb{N} \setminus \{0\}$ and any $t \in [0, T)$, we consider uniform partitions of the interval $[t, T]$ by $\{t_k^{t, n} := t + k(T - t)n^{-1}, k = 0, \dots, n\}$. We also define for every $0 \leq k \leq n$ and every $(s, x_{0:k}) \in [t, T] \times (\mathbb{R}^d)^{k+1}$, the linear interpolator $i_k : (\mathbb{R}^d)^{k+1} \rightarrow \mathcal{C}^0$ by

$$i_k(x_{0:k})(s) = \frac{n}{T-t} \sum_{i=0}^{k-1} \left((t_{i+1}^{t, n} - s)x_i + (s - t_i^{t, n})x_{i+1} \right) \mathbf{1}_{[t_i^{t, n}, t_{i+1}^{t, n}]}(s).$$

Our main assumption now becomes

Assumption 5.1. *Assumption 2.1(i), (ii), (iii) hold and*

(iv') For any $p > 0$, there exist progressively measurable maps $\eta^p : [0, T] \times \mathcal{C}^0 \rightarrow \mathbb{R}_-$ with linear growth, that is there exists some $C > 0$ such that

$$0 \leq -\eta_t^p(\mathbf{x}) \leq C(1 + \|\mathbf{x}\|_{\infty, t}),$$

and a deterministic map $M : [0, T] \rightarrow \mathbb{S}^d$, such that M_t is symmetric positive for every $t \in [0, T]$, the maps $\mathbf{x} \mapsto \eta_t^p(\mathbf{x})$ are concave for every $t \in [0, T]$, the sequence $(\eta^p)_{p \geq 0}$ is non-decreasing, and such that for any $p \geq 0$ and any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$, the matrix $\sigma_t^{\eta^p}(\mathbf{x})$ is an invertible matrix such that $(\sigma_t^{\eta^p})^{-1}(\mathbf{x})f$ is uniformly bounded in (t, \mathbf{x}) , where

$$\sigma_t^{\eta^p}(\mathbf{x}) := \eta_t^p(\mathbf{x})M_t + \sigma_t(\mathbf{x}).$$

(v) The matrix $M_t \sigma_t^\top(\mathbf{x}) + \sigma_t(\mathbf{x})M_t$ is symmetric negative, for every $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$.

(vi) The maps U, μ and σ are such that U is concave and for every $n \geq 1$, and $0 \leq k \leq n - 1$, for every $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$, every $\{(\alpha_{i,j}, \beta_{i,j}, \gamma_{i,j}) \in \mathbb{R}^d \times \mathbb{R}_+^ \times \mathbb{R}^d, 1 \leq i \leq n - k, 0 \leq j \leq n - k\}$, and every*

$(x_{0:k}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \lambda) \in (\mathbb{R}^d)^{k+1} \times \mathcal{C}^t \times \mathcal{C}^t \times [0, 1]$, we have

$$\begin{aligned} & U\left(i_n\left(x_{0:k-1}, \mathbf{w}^\lambda(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}; 0, 0), \sum_{i=0}^1 \mathbf{w}^\lambda(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}; i, 1), \dots, \sum_{i=0}^{n-k} \mathbf{w}^\lambda(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}; i, n-k)\right)\right) \\ & \geq U\left(i_n\left(x_{0:k-1}, \mathbf{z}^\lambda(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}; 0, 0), \sum_{i=0}^1 \mathbf{z}^\lambda(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}; i, 1), \dots, \sum_{i=0}^{n-k} \mathbf{z}^\lambda(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}; i, n-k)\right)\right) \end{aligned}$$

where

$$\begin{aligned} \mathbf{w}^\lambda(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}; i, \ell) &:= \alpha_{i,\ell} + \beta_{i,\ell} \mu_{t_{k-1+i}}^{t,x}(\lambda \tilde{\mathbf{x}} + (1-\lambda)\tilde{\mathbf{y}}) + \left(\eta_{t_{k-1+i}}^{t,\mathbf{x}} M_{t_{k-1+i}}^{t,n} + \sigma_{t_{k-1+i}}^{t,\mathbf{x}}\right) (\lambda \tilde{\mathbf{x}} + (1-\lambda)\tilde{\mathbf{y}}) \gamma_{i,\ell}, \\ \mathbf{z}^\lambda(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}; i, \ell) &:= \alpha_{i,\ell} + \beta_{i,\ell} \left(\lambda \mu_{t_{k-1+i}}^{t,x}(\tilde{\mathbf{x}}) + (1-\lambda) \mu_{t_{k-1+i}}^{t,x}(\tilde{\mathbf{y}})\right) + \gamma_{i,\ell} \lambda \left(\eta_{t_{k-1+i}}^{t,\mathbf{x}} M_{t_{k-1+i}}^{t,n} + \sigma_{t_{k-1+i}}^{t,\mathbf{x}}\right) (\tilde{\mathbf{x}}) \\ &\quad + \gamma_{i,\ell} (1-\lambda) \left(\eta_{t_{k-1+i}}^{t,\mathbf{x}} M_{t_{k-1+i}}^{t,n} + \sigma_{t_{k-1+i}}^{t,\mathbf{x}}\right) (\tilde{\mathbf{y}}) \end{aligned}$$

Remark 5.2. This assumption deserves a certain number of comments.

- (iv') is here in order to ensure that the degenerate matrix σ becomes invertible when it is suitably perturbed. Of course, our ultimate goal here is to assume that η^p converges to 0 and to approximate the solution of our problem with degenerate diffusion as the corresponding limit. We also emphasize that this assumption implies in particular that for any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$ and any $p \geq p'$

$$\sigma_t^{\eta^p}(\mathbf{x}) - \sigma_t^{\eta^{p'}}(\mathbf{x}) = (\eta_t^p(\mathbf{x}) - \eta_t^{p'}(\mathbf{x})) M_t,$$

which is a symmetric positive matrix. Hence, the sequence σ^{η^p} is non-decreasing for the usual order on symmetric positive matrices.

- (v) and (vi) are actually here mainly so that the results of Lemma 5.3 below hold for a certain function f involving U (see the proof of Proposition 5.4 below). They take a particularly complicated form for two reasons. First, our setting is fully non-Markovian, and second, it is also multidimensional. Indeed, as can be checked directly, if $d = 1$ and $\mathbf{x} \mapsto \sigma_t(\mathbf{x})$ is linear, then we only need to assume that μ is concave and f non-decreasing for (5.1) below to hold. Similarly, (vi) is somehow a concavity assumption on U , μ and σ . Indeed, if again $d = 1$ and if U were Markovian, then a sufficient condition for (vi) to hold is that U is non-decreasing, μ is concave and σ and η are linear.

Our strategy of proof here is to start by obtaining a monotonicity result, with respect to the parameter p , for the solution of our control problem with diffusion coefficient σ^p . Such a result will be based on convex order type arguments. More precisely, we follow the strategy outlined by Pagès [31] and start by proving the result in a discrete-time setting, this is Proposition 5.4, which can then be extended to continuous-time through weak convergence arguments. Though the strategy of proof is the same as in [31], our proofs are more involved mainly due to the fact that, unlike in [31], our framework is fully non-Markovian and multidimensional.

Lemma 5.3. *Let Assumption 5.1 hold. Fix $n \geq 1$. For every $(t, s, \mathbf{x}, \tilde{\mathbf{x}}, \lambda, u) \in [0, T] \times [t, T] \times \mathcal{C}^0 \times \mathcal{C}^t \times \mathbb{R}_+^* \times \mathbb{R}$, for every $k = 0, \dots, n-1$, and for every Borel map $f : \mathbb{R}^d \rightarrow \mathbb{R}$ with polynomial growth, define the following operators*

$$\begin{aligned} & Q_{t,\mathbf{x},\lambda}^{k+1}(f)(\tilde{\mathbf{x}}, u) \\ & := \mathbb{E}^{\mathbb{P}_0^t} \left[f\left(\tilde{\mathbf{x}}_{t_k}^{t,n} + \lambda \left(\mu_{t_k}^{t,\mathbf{x}}(\tilde{\mathbf{x}}) + f_{t_k}^{t,n} \nu_{t_k}^{t,n}\right) + \left(u M_{t_k}^{t,n} + \sigma_{t_k}^{t,\mathbf{x}}(\tilde{\mathbf{x}})\right) \left(B_{t_{k+1}}^{t,n} - B_{t_k}^{t,n}\right)\right) \middle| \mathcal{F}_{t_k}^{t,n} \right]. \end{aligned}$$

If f is concave and s.t. for every $(t, s, \mathbf{x}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \alpha, \beta, \gamma, \eta) \in [0, T] \times [t, T] \times \mathcal{C}^0 \times \mathcal{C}^t \times \mathcal{C}^t \times \mathbb{R}^d \times \mathbb{R}_+^* \times \mathbb{R}^d \times [0, 1]$,

$$\begin{aligned} & f(\alpha + \beta \mu_s^{t,x}(\eta \tilde{\mathbf{x}} + (1 - \eta) \tilde{\mathbf{y}}) + \sigma_s^{t,\mathbf{x}}(\eta \tilde{\mathbf{x}} + (1 - \eta) \tilde{\mathbf{y}}) \gamma) \\ & \geq f(\alpha + \beta(\eta \mu_s^{t,x}(\tilde{\mathbf{x}}) + (1 - \eta) \mu_s^{t,x}(\tilde{\mathbf{y}})) + (\eta \sigma_s^{t,\mathbf{x}}(\tilde{\mathbf{x}}) + (1 - \eta) \sigma_s^{t,x}(\tilde{\mathbf{y}})) \gamma), \end{aligned} \quad (5.1)$$

then the map $(\tilde{\mathbf{x}}, u) \mapsto Q_{t,\mathbf{x},\lambda}^{k+1}(f)(\tilde{\mathbf{x}}, u)$ is concave, and the map $u \mapsto Q_{t,\mathbf{x},\lambda}^{k+1}(f)(\tilde{\mathbf{x}}, u)$ is non-decreasing on \mathbb{R}_- .

Proof. The fact that the operators $Q_{t,\mathbf{x},\gamma}^{k+1}$ are well-defined is clear from the polynomial growth of f , the linear growth of μ and σ , and the fact that B^t has moments of any order under \mathbb{P}_0^t . Then, the concavity of $Q_{t,\mathbf{x},\gamma}^{k+1}(f)$ is an immediate corollary of the concavity assumptions on f , as well as (5.1).

Then, since f has polynomial growth, Feynman-Kac's formula implies that

$$Q_{t,\mathbf{x},\lambda}^{k+1}(f)(\tilde{\mathbf{x}}, u) = v(t_k^{t,n}, B_{t_k}^{t,n}),$$

where $v : [t_k^{t,n}, t_{k+1}^{t,n}] \times \mathbb{R}^d \rightarrow \mathbb{R}$ is the unique viscosity solution of the PDE

$$\begin{cases} -v_s - \frac{1}{2} \text{Tr} \left[\left(u M_{t_k}^{t,n} + \sigma_{t_k}^{t,\mathbf{x}}(\tilde{\mathbf{x}}) \right) \left(u M_{t_k}^{t,n} + (\sigma_{t_k}^{t,\mathbf{x}})^\top(\tilde{\mathbf{x}}) \right) v_{xx} \right] = 0, & \text{on } [t_k^{t,n}, t_{k+1}^{t,n}] \times \mathbb{R}^d, \\ v(t_{k+1}^{t,n}, x) = f(\tilde{\mathbf{x}}_{t_k}^{t,n} + \lambda(\mu_{t_k}^{t,\mathbf{x}}(\tilde{\mathbf{x}}) + f_{t_k}^{t,n} \nu_{t_k}^{t,n}) + (u M_{t_k}^{t,n} + \sigma_{t_k}^{t,\mathbf{x}}) x), & x \in \mathbb{R}^d. \end{cases}$$

This linear PDE classically satisfies a comparison theorem, and v is concave in x because of the concavity of f . Moreover, the diffusion part of the PDE rewrites, as a quadratic functional of u

$$u^2 \text{Tr} \left[M_{t_k}^{t,n} v_{xx} \right] + u \text{Tr} \left[\left(M_{t_k}^{t,n} (\sigma_{t_k}^{t,\mathbf{x}})^\top(\tilde{\mathbf{x}}) + \sigma_{t_k}^{t,\mathbf{x}}(\tilde{\mathbf{x}}) M_{t_k}^{t,n} \right) v_{xx} \right] + \text{Tr} \left[\sigma_{t_k}^{t,\mathbf{x}}(\tilde{\mathbf{x}}) (\sigma_{t_k}^{t,\mathbf{x}})^\top(\tilde{\mathbf{x}}) \right].$$

Since M_t^2 is symmetric positive, $M_t \sigma_t^\top(\mathbf{x}) + \sigma_t(\mathbf{x}) M_t$ is symmetric negative and v_{xx} is symmetric negative as well, the above is actually non-decreasing for $u \in \mathbb{R}_-$. The same then holds for $Q_{t,\mathbf{x},\lambda}^{k+1}(f)(\tilde{\mathbf{x}}, u)$ by comparison. \square

Proposition 5.4. *Let Assumptions 2.3 and 5.1 hold and fix some $q \geq p > 0$. For every $n \in \mathbb{N} \setminus \{0\}$, $\mathbf{x} \in \mathcal{C}^0$, let us define recursively $(X_k^{t,\mathbf{x},n})_{0 \leq k \leq n}$ and $(Y_k^{t,\mathbf{x},n})_{0 \leq k \leq n}$ by $X_0^{t,\mathbf{x},n} = Y_0^{t,\mathbf{x},n} = \mathbf{x}_t$, and for $0 \leq k \leq n - 1$*

$$\begin{aligned} X_{k+1}^{t,\mathbf{x},n} &= X_k^{t,\mathbf{x},n} + \left(\mu_{t_k}^{t,\mathbf{x}}(\bar{X}_k^{t,\mathbf{x},n}) + f_{t_k}^{t,n} \nu_{t_k}^{t,n} \right) (t_{k+1}^{t,n} - t_k^{t,n}) \\ &\quad + \left(\eta_{t_k}^{p,t,\mathbf{x}}(\bar{X}_k^{t,\mathbf{x},n}) M_{t_k}^{t,n} + \sigma_{t_k}^{t,\mathbf{x}}(\bar{X}_k^{t,\mathbf{x},n}) \right) (B_{t_{k+1}}^{t,n} - B_{t_k}^{t,n}), \\ Y_{k+1}^{t,\mathbf{x},n} &= Y_k^{t,\mathbf{x},n} + \left(\mu_{t_k}^{t,\mathbf{x}}(\bar{Y}_k^{t,\mathbf{x},n}) + f_{t_k}^{t,n} \nu_{t_k}^{t,n} \right) (t_{k+1}^{t,n} - t_k^{t,n}) \\ &\quad + \left(\eta_{t_k}^{q,t,\mathbf{x}}(\bar{Y}_k^{t,\mathbf{x},n}) M_{t_k}^{t,n} + \sigma_{t_k}^{t,\mathbf{x}}(\bar{Y}_k^{t,\mathbf{x},n}) \right) (B_{t_{k+1}}^{t,n} - B_{t_k}^{t,n}), \end{aligned}$$

where $(\bar{X}_k^{t,\mathbf{x},n})_{0 \leq k \leq n}$ and $(\bar{Y}_k^{t,\mathbf{x},n})_{0 \leq k \leq n}$ are defined as the following piecewise linear interpolations

$$\bar{X}_k^{t,\mathbf{x},n} := i_k(X_{0:k}^{t,\mathbf{x},n}), \quad \bar{Y}_k^{t,\mathbf{x},n} := i_k(Y_{0:k}^{t,\mathbf{x},n}).$$

Then, we have

$$\mathbb{E}^{\mathbb{P}_0^t} \left[U^{t,\mathbf{x}} \left(i_n \left(X_{0:n}^{t,n,\mathbf{x}} \right) \right) \right] \leq \mathbb{E}^{\mathbb{P}_0^t} \left[U^{t,\mathbf{x}} \left(i_n \left(Y_{0:n}^{t,n,\mathbf{x}} \right) \right) \right].$$

Proof. Let $\Delta^{t,n} := t_{k+1}^{t,n} - t_k^{t,n} = (T - t)/n$, and consider the following martingales, for $0 \leq k \leq n$,

$$M_k := \mathbb{E}^{\mathbb{P}_0^t} \left[U^{t,\mathbf{x}} \left(i_n \left(X_{0:n}^{t,n,\mathbf{x}} \right) \right) \middle| \mathcal{F}_{t_k}^{t,n} \right], \quad N_k := \mathbb{E}^{\mathbb{P}_0^t} \left[U^{t,\mathbf{x}} \left(i_n \left(Y_{0:n}^{t,n,\mathbf{x}} \right) \right) \middle| \mathcal{F}_{t_k}^{t,n} \right],$$

which are well-defined, since U has polynomial growth and we know from Lemma 2.2 that $X_{0:n}^{t,n,\mathbf{x}}$ and $Y_{0:n}^{t,n,\mathbf{x}}$ have moments of any order. For any $k = 0, \dots, n$, we also define the following sequences of functions from \mathbb{R}^{k+1} to \mathbb{R} , for $k = 0, \dots, n-1$, by backward induction, for any $x_{0:k} \in (\mathbb{R}^d)^{k+1}$

$$\begin{aligned} \Phi_n &:= U^{t,\mathbf{x}} \circ i_n, \quad \Phi_k(x_{0:k}) := Q_{t,\mathbf{x},\Delta^{t,n}}^{k+1}(\Phi_{k+1}(x_{0:k}, \cdot)) \left(i_k(x_{0:k}), \eta_{t_k}^p(i_k(x_{0:k})) \right), \\ \Psi_n &:= U^{t,\mathbf{x}} \circ i_n, \quad \Psi_k(x_{0:k}) := Q_{t,\mathbf{x},\Delta^{t,n}}^{k+1}(\Psi_{k+1}(x_{0:k}, \cdot)) \left(i_k(x_{0:k}), \eta_{t_k}^q(i_k(x_{0:k})) \right). \end{aligned}$$

It is immediate by definition of $X^{t,\mathbf{x},n}$ and $Y^{t,\mathbf{x},n}$ that we have for every $0 \leq k \leq n$

$$M_k = \Phi_k(X_{0:k}^{t,\mathbf{x},n}) \quad \text{and} \quad N_k = \Psi_k(Y_{0:k}^{t,\mathbf{x},n}).$$

Let us now show that the maps Φ_k and Ψ_k are concave for every $k = 0, \dots, n$, and that they verify that for any $0 \leq i \leq k-1$, $(x_{0:n}, \tilde{x}_{0:n}, \eta) \in (\mathbb{R}^d)^{n+1} \times (\mathbb{R}^d)^{n+1} \times [0, 1]$, for any $\{(\alpha_{m,l}, \beta_{m,l}, \gamma_{m,l}) \in \mathbb{R}^d \times \mathbb{R}_+^* \times \mathbb{R}^d, i \leq m \leq k-1, 0 \leq l \leq k-i-1\}$, we have for $\varphi = \Phi, \Psi$

$$\begin{aligned} &\varphi_k \left(x_{0:i}, \alpha_{i,0} + \beta_{i,0} \mu_{t_i}^{t,\mathbf{x}}(i_i(\eta x_{0:i} + (1-\eta)\tilde{x}_{0:i})) + \sigma_{t_i}^{t,\mathbf{x}}(i_i(\eta x_{0:i} + (1-\eta)\tilde{x}_{0:i}))\gamma_{i,0}, \right. \\ &\quad \sum_{j=i}^{i+1} \left(\alpha_{j,1} + \beta_{j,1} \mu_{t_j}^{t,\mathbf{x}}(i_j(\eta \hat{x}_{0:j} + (1-\eta)\hat{\tilde{x}}_{0:j})) + \sigma_{t_j}^{t,\mathbf{x}}(i_j(\eta \hat{x}_{0:j} + (1-\eta)\hat{\tilde{x}}_{0:j}))\gamma_{j,1} \right), \dots, \\ &\quad \sum_{j=i}^{k-1} \left(\alpha_{j,k-i-1} + \beta_{j,k-i-1} \mu_{t_j}^{t,\mathbf{x}}(i_j(\eta \hat{x}_{0:j} + (1-\eta)\hat{\tilde{x}}_{0:j})) + \sigma_{t_j}^{t,\mathbf{x}}(i_j(\eta \hat{x}_{0:j} + (1-\eta)\hat{\tilde{x}}_{0:j}))\gamma_{j,k-i-1} \right) \Big) \\ &\geq \varphi_k \left(x_{0:i}, \alpha_{i,0} + \beta_{i,0} \left(\eta \mu_{t_i}^{t,\mathbf{x}}(i_i(x_{0:i})) + (1-\eta) \mu_{t_i}^{t,\mathbf{x}}(i_i(\tilde{x}_{0:i})) \right) \right. \\ &\quad \left. + \left(\eta \sigma_{t_i}^{t,\mathbf{x}}(i_i(x_{0:i})) + (1-\eta) \sigma_{t_i}^{t,\mathbf{x}}(i_i(\tilde{x}_{0:i})) \right) \gamma_{i,0} \right), \\ &\quad \sum_{j=i}^{i+1} \left(\alpha_{j,1} + \beta_{j,1} \left(\eta \mu_{t_j}^{t,\mathbf{x}}(i_j(\hat{x}_{0:j})) + (1-\eta) \mu_{t_j}^{t,\mathbf{x}}(i_j(\hat{\tilde{x}}_{0:j})) \right) \right. \\ &\quad \left. + \left(\eta \sigma_{t_j}^{t,\mathbf{x}}(i_j(\hat{x}_{0:j})) + (1-\eta) \sigma_{t_j}^{t,\mathbf{x}}(i_j(\hat{\tilde{x}}_{0:j})) \right) \gamma_{j,1} \right), \dots, \\ &\quad \sum_{j=i}^{k-1} \left(\alpha_{j,k-i-1} + \beta_{j,k-i-1} \left(\eta \mu_{t_j}^{t,\mathbf{x}}(i_j(\hat{x}_{0:j})) + (1-\eta) \mu_{t_j}^{t,\mathbf{x}}(i_j(\hat{\tilde{x}}_{0:j})) \right) \right. \\ &\quad \left. + \left(\eta \sigma_{t_j}^{t,\mathbf{x}}(i_j(\hat{x}_{0:j})) + (1-\eta) \sigma_{t_j}^{t,\mathbf{x}}(i_j(\hat{\tilde{x}}_{0:j})) \right) \gamma_{j,k-i-1} \right) \Big), \end{aligned} \tag{5.2}$$

where \hat{x} and $\hat{\tilde{x}}$ are defined recursively, for $w := x, \tilde{x}$, by

$$\begin{cases} \hat{w}_l := w_l, \quad 0 \leq l \leq i, \\ \hat{w}_{l+1} := \sum_{j=i}^l \left(\alpha_{j,l-i} + \beta_{j,l-i} \mu_{t_j}^{t,\mathbf{x}}(i_j(\hat{w}_{0:j})) + \sigma_{t_j}^{t,\mathbf{x}}(i_j(\hat{w}_{0:j}))\gamma_{j,l-i} \right), \quad i \leq l \leq k-1. \end{cases}$$

We only prove the result for Φ_k , the other one being exactly similar. We argue by backward induction. When $k = n$, the result is obvious since U is concave and Assumption 5.1(vi) holds. Let us assume that the properties holds for Φ_{k+1} for some $k \leq n-1$. Then, let us now show that the map $x_{0:k} \mapsto$

$Q_{t,\mathbf{x},\Delta^{t,n}}^{k+1}(\Phi_{k+1}(x_{0:k}, \cdot))(i_k(x_{0:k}), u)$ is concave for any $u \in \mathbb{R}$. We actually have

$$\begin{aligned} & Q_{t,\mathbf{x},\Delta^{t,n}}^{k+1}(\Phi_{k+1}(x_{0:k}, \cdot))(i_k(x_{0:k}), u) \\ &= \mathbb{E}^{\mathbb{P}_0^t} \left[\Phi_{k+1} \left(x_{0:k}, x_k + \Delta^{t,n} \left(\mu_{t_k}^{t,\mathbf{x}}(i_k(x_{0:k})) + f_{t_k}^{t,n} \nu_{t_k}^{t,n} \right) \right. \right. \\ & \quad \left. \left. + \left(u M_{t_k}^{t,n} + \sigma_{t_k}^{t,\mathbf{x}}(i_k(x_{0:k})) \right) \left(B_{t_{k+1}}^t - B_{t_k}^t \right) \middle| \mathcal{F}_{t_k}^{t,n} \right]. \end{aligned}$$

Therefore, the concavity is immediate from the induction hypothesis on Φ_{k+1} (both the concavity and Inequality (5.2))

Now, we know that $\eta_t^p(\cdot)$ is concave and non-positive, and, from Lemma 5.3, that the map $u \mapsto Q_{t,\mathbf{x},\Delta^{t,n}}^{k+1}(\Phi_{k+1}(x_{0:k}, \cdot))(i_k(x_{0:k}), u)$ is non-decreasing on \mathbb{R}_- . This therefore proves the concavity of Φ_k . Moreover, Φ_k inherits (5.2) directly from Φ_{k+1} by its definition as an expectation of Φ_{k+1} .

Finally, let us prove, again by backward induction, that for every $k = 0, \dots, n$, $\Phi_k \leq \Psi_k$. The result is obvious by definition for $k = n$. Assume now that for some $k \leq n - 1$, we have $\Phi_{k+1} \leq \Psi_{k+1}$. Then, for any $x_{0:k} \in \mathbb{R}^{k+1}$

$$\begin{aligned} \Phi_k(x_{0:k}) &= Q_{t,\mathbf{x},\Delta^{t,n}}^{k+1}(\Phi_{k+1}(x_{0:k}, \cdot)) \left(i_k(x_{0:k}), \eta_{t_k}^p(i_k(x_{0:k})) \right) \\ &\leq Q_{t,\mathbf{x},\Delta^{t,n}}^{k+1}(\Phi_{k+1}(x_{0:k}, \cdot)) \left(i_k(x_{0:k}), \eta_{t_k}^q(i_k(x_{0:k})) \right) \\ &\leq Q_{t,\mathbf{x},\Delta^{t,n}}^{k+1}(\Psi_{k+1}(x_{0:k}, \cdot)) \left(i_k(x_{0:k}), \eta_{t_k}^q(i_k(x_{0:k})) \right) = \Psi_k(x_{0:k}), \end{aligned}$$

where we have used successively the fact that $u \mapsto Q_{t,\mathbf{x},\Delta^{t,n}}^{k+1}(\Phi_{k+1}(x_{0:k}, \cdot))(i_k(x_{0:k}), u)$ is non-decreasing on \mathbb{R}_- (remember that $\eta^p \leq \eta^q$) and the induction hypothesis. To conclude, it suffices to take $k = 0$ to obtain $\Phi_0(\mathbf{x}_t) \leq \Psi_0(\mathbf{x}_t)$, which is equivalent by the martingale property of M and N to

$$\mathbb{E}^{\mathbb{P}_0^t} \left[U^{t,\mathbf{x}} \left(i_n \left(X_{0:n}^{t,n,\mathbf{x}} \right) \right) \right] \leq \mathbb{E}^{\mathbb{P}_0^t} \left[U^{t,\mathbf{x}} \left(i_n \left(Y_{0:n}^{t,n,\mathbf{x}} \right) \right) \right].$$

□

We can now state the main technical result of this section.

Proposition 5.5. *Let Assumptions 2.3 and 5.1 hold. For any $p > 0$, denote by $X^{t,\mathbf{x},\nu,p}$ the solution to the SDE (2.6) with diffusion matrix σ^p instead of σ , and let*

$$v^p(t, \mathbf{x}) := \sup_{\nu \in \mathcal{U}^t} \mathbb{E}^{\mathbb{P}_0^t} \left[U \left(\mathbf{x} \otimes_t X^{t,\mathbf{x},\nu,p} \right) \right].$$

Then, for any $q \geq p > 0$, we have for any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$

$$v^p(t, \mathbf{x}) \leq v^q(t, \mathbf{x}).$$

Proof. By Proposition 5.4, we know that if we replace $X^{t,\mathbf{x},\nu,p}$ and $X^{t,\mathbf{x},\nu,q}$ by their Euler scheme, then the expectation of U of these Euler schemes are ordered. We can then follow exactly the arguments of the proofs of Lemma 2.2 and Theorem 2.1 in [31], using in particular the continuity we have assumed for U , as well as the fact that the genuine Euler scheme for a non-Markovian SDE converges to the solution of the SDE for the uniform topology on \mathcal{C}^0 , to extend this result and obtain

$$\mathbb{E}^{\mathbb{P}_0^t} \left[U \left(\mathbf{x} \otimes_t X^{t,\mathbf{x},\nu,p} \right) \right] \leq \mathbb{E}^{\mathbb{P}_0^t} \left[U \left(\mathbf{x} \otimes_t X^{t,\mathbf{x},\nu,q} \right) \right],$$

from which the result is clear. □

Our main result is then that with degenerate volatility, the singular stochastic control problem can be represented as an infimum of solution of constrained BSDEs.

Theorem 5.6. *Let Assumptions 2.3, 3.2, 3.6 and 5.1 hold, with σ^p instead of σ , and assume in addition that*

$$\sup_{(t,\mathbf{x}) \in [0,T] \times \mathcal{C}^0} |\eta_t^p(\mathbf{x})| \xrightarrow{p \rightarrow +\infty} 0.$$

Then, we have

$$v(t, \mathbf{x}) = \lim_{p \rightarrow +\infty} \uparrow v^p(t, \mathbf{x}) = \sup_{p > 0} v^p(t, \mathbf{x}) = \sup_{p > 0} \mathcal{Y}_t^{t,\mathbf{x},p} = \sup_{p > 0} \mathfrak{Y}_t^{t,\mathbf{x},p},$$

where $\mathcal{Y}^{t,\mathbf{x},p}$ and $\mathfrak{Y}^{t,\mathbf{x},p}$ are defined as $\mathcal{Y}^{t,\mathbf{x}}$ and $\mathfrak{Y}^{t,\mathbf{x}}$ with σ^p instead of σ .

Proof. Since σ^p satisfies all the required assumptions, by Theorem 4.7 and Proposition 5.5, the only equality that we have to prove is the first one. But it is a simple consequence of classical estimates for SDEs and the uniform convergence we have assumed for η^p . \square

Remark 5.7. The representation we have just obtained involves a supremum of solutions of constrained BSDEs. Formally speaking, such an object is close in spirit to so-called constrained second order BSDEs, as introduced by Fabre in her Phd thesis [21]. Indeed, the supremum over p could be seen as a supremum over a family of probability measures, such that under these measures the canonical process has the same law as a continuous martingale whose quadratic variation has density $\sigma^p(\sigma^p)^\top$. To prove such a relationship rigorously is a very interesting problem, which however falls outside the scope of this paper.

5.2 Utility maximization with transaction costs for non-Markovian dynamics

5.2.1 The setup

Let us consider the following framework. Let us fix $d = 3$, and for a given $\lambda > 0$

$$f := \begin{pmatrix} 1 & -(1+\lambda) & 0 \\ -(1+\lambda) & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let us also be given the following bounded and progressively measurable maps r , μ^S and σ^S from the space of continuous functions (from $[0, T]$ to \mathbb{R}) to \mathbb{R} . For any $\mathbf{x} \in \mathcal{C}^0$, we let

$$\mu_t(\mathbf{x}) := \begin{pmatrix} r_t(\mathbf{x}^3) \mathbf{x}_t^1 \\ \mu_t^S(\mathbf{x}^3) \mathbf{x}_t^2 \\ 0 \end{pmatrix}, \quad \sigma_t(\mathbf{x}) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma_t^S(\mathbf{x}^3) \mathbf{x}_t^2 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_t := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \eta_t^p(\mathbf{x}) := -\frac{1}{p}.$$

Then, we have

$$(\sigma_t^p)^{-1}(\mathbf{x}) = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & p\sigma_t^S(\mathbf{x}^3) \mathbf{x}_t^2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and } (\sigma_t^p)^{-1}(\mathbf{x})f = -pf.$$

Moreover, we have

$$M_t \sigma_t(\mathbf{x}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so that Assumption 5.1(iv'),(v) are satisfied. The dynamics of the 3 coordinates of $X^{t,\mathbf{x},\nu,p}$ are then given by

$$\begin{cases} X_s^{t,\mathbf{x},\nu,p,1} = \mathbf{x}_t^1 + \int_t^s r_u^{t,\mathbf{x}^3} (X_s^{t,\mathbf{x},\nu,p,3}) X_u^{t,\mathbf{x},\nu,p,1} du - \frac{1}{p} B_s^{t,1} + \int_t^s (\nu_u^1 - (1+\lambda)\nu_u^2) du, \\ X_s^{t,\mathbf{x},\nu,p,2} = \mathbf{x}_t^2 + \int_t^s X_u^{t,\mathbf{x},\nu,p,2} \left((\mu^S)_u^{t,\mathbf{x}^3} (X_u^{t,\mathbf{x},\nu,p,3}) du + (\sigma^S)_u^{t,\mathbf{x}^3} (X_u^{t,\mathbf{x},\nu,p,3}) dB_u^{t,3} \right) \\ \quad - \frac{1}{p} B_s^{t,2} + \int_t^s (\nu_u^2 - (1+\lambda)\nu_u^1) du, \\ X_s^{t,\mathbf{x},\nu,p,3} = \mathbf{x}_t^3 + B_s^{t,3}. \end{cases}$$

Under this form, the limit when p goes to $+\infty$ of the above system is exactly the dynamics of a portfolio position in a market with transaction costs, as in [15, 37] for instance. More precisely, the financial market considered consists of a riskless asset with (random) short rate r and a risky asset S whose (non-Markovian) dynamics is, under \mathbb{P}_0^t ,

$$\frac{dS_u}{S_u} = (\mu^S)_u^{t,\mathbf{x}^3} (B^{t,3}) du + (\sigma^S)_u^{t,\mathbf{x}^3} (B^{t,3}) dB_u^{t,3}.$$

Moreover, transactions between the risky and the riskless asset incur a proportional transaction cost of size λ . Then, $X^{t,\mathbf{x},\nu,p,1}$ above can be interpreted as the total amount of money invested in the riskless asset by an investor since time t , while $X^{\mathbf{x},\nu,p,2}$ is the total amount of money invested in the risky asset, and the controls ν^1 and ν^2 record respectively the transactions from the risky to the riskless asset and from the riskless to the risky asset. Finally, the only role played by $X^{t,\mathbf{x},\nu,p,3}$ is to represent the Brownian motion driving the randomness in r , μ and σ .

5.2.2 The result

We will actually use the result proved in Theorem 5.6 conditionally on $X^{t,\mathbf{x},\nu,p,3}$ (that is to say that we consider conditional expectations with respect to $\sigma(X_s^{t,\mathbf{x},\nu,p,3}, t \leq s \leq T)$ instead of simple expectations). It can be checked readily that all our arguments still go through in this case. Moreover, the drifts and volatility in the dynamics of $X^{t,\mathbf{x},\nu,p,1}$ and $X^{t,\mathbf{x},\nu,p,2}$ then become (conditionally) linear. We then choose a particular specification for the map U

$$U(\mathbf{x}) =: \mathcal{U}(\mathbf{x}^3, \ell(\mathbf{x}_T^1, \mathbf{x}_T^2)),$$

where the so-called liquidation function ℓ is defined by

$$\ell(x, y) := x + \frac{y^+}{1 + \lambda} - (1 + \lambda)y^-, \quad (x, y) \in \mathbb{R}^2,$$

and the map \mathcal{U} is assumed to be a (random) utility function, which is increasing and strictly concave with respect to its second-variable, as well as locally-Lipschitz continuous with polynomial growth, so that Assumption 2.3 is satisfied. Then, remembering Remark 5.2 above, we know that (conditionally), Assumption 5.1(vi) is also satisfied. Therefore, we can apply our result to obtain that the value function, which corresponds in this case to that of the utility maximization problem in finite horizon with transaction costs can be represented as a supremum of solutions of constrained BSDEs. But there is more. Indeed, in this case the constraint can be read

$$-pf\mathcal{Z}_s^{t,\mathbf{x}} \in K_s, \quad dt \times d\mathbb{P} \text{ a.e.},$$

which, by definition of K and since $p > 0$, is actually equivalent to $-f\mathcal{Z}_s^{t,\mathbf{x}} \in K_s$. Therefore, the solution to the constrained BSDE is actually independent of p . Therefore, the value function can actually be represented as the solution of another BSDE, with a modified constraint as above.

As far as we know, such a result is completely new in the literature, even in the Markovian case. Moreover, as pointed out in the recent paper [26], the non-Markovian case has actually never been studied using stochastic control and PDE tools, the only approach in the literature being convex duality. We thus believe that our approach achieves a first step allowing to tackle this difficult problem. Let us nonetheless point out a gap in our approach. If we wanted to cover completely the problem of transaction costs, we should have added state constraints in our original stochastic control problem. Indeed, those are inherent to the problem of transaction costs, in order to avoid bankruptcy issues (though this is actually a lesser issue when the time horizon is finite, as in our case). We have chosen not to do so so as not to complicate even more our arguments, but we believe that they could be also used in this setting,

albeit with possibly important modifications. In particular, the full dynamic programming principle that we used does not seem to be proved in such a general framework in the literature, when state constraints are present (see however [8, 9]), and it is not completely clear in which sense the equality between v^{sing} and v will then hold.

6 Applications: DPP and regularity for singular stochastic control

6.1 Dynamic programming principle

Notice that in all our study, we never actually proved that the dynamic programming principle actually held for the singular stochastic control problem defining v (or v_{sing}). However, it is an easy consequence of our main result.

Theorem 6.1. *For any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$, for any $\tau \in \mathcal{T}^t$ and any $\theta \in \mathcal{T}^{t, \mathbf{x}}$, we have*

$$v(t, \mathbf{x}) = \sup_{\nu \in \mathcal{U}^t} \mathbb{E}^{\mathbb{P}_0^t} [v(\tau, \mathbf{x} \otimes_t X^{t, \mathbf{x}, \nu})] = \sup_{\nu \in \mathcal{U}^t} \mathbb{E}^{\mathbb{P}_\nu^{t, \mathbf{x}}} [v(\theta, \mathbf{x} \otimes_t B^{t, \mathbf{x}_t})]$$

Proof. It is an immediate consequence of the dynamic programming principle satisfied by the penalized BSDEs (4.6) and the convergence of penalized BSDEs to the minimal solution of the constrained BSDE. \square

6.2 Regularity results

In this section, we show how our representation can help to obtain *a priori* regularity results for the value function of singular stochastic control problems. Such results in that level of generality are, as far as we know, the first available in the literature.

The main idea of the proof is that as soon as one knows that the value function of the singular control problem is associated to a constrained BSDE, one can use the fact that such BSDEs are actually linked to another different singular stochastic control problem, which is actually simpler to study. Such a representation is not new and was already the crux of the arguments of Cvitanić, Karatzas and Soner [13]. It has also been used very recently in [7] to obtain the first regularity results in the literature for constrained BSDEs. For the sake of simplicity, and since this is not the heart of our article, we will concentrate on the Markovian set-up for this application, and leave the more general case to future research².

Let us define the map $\delta : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ such that for any $t \in [0, T]$, $\delta_t(\cdot)$ is the so-called support function of the set K_t , that is to say

$$\delta_t(u) := \sup \{k \cdot u, k \in K_t\}.$$

Notice that since the zero vector in \mathbb{R}^d belongs to K_t for any $t \in [0, T]$, it is clear that δ is non-negative. This section requires the following additional assumptions.

Assumption 6.2. (i) *The maps U , μ and σ are Markovian, that is to say, abusing notations slightly*

$$U(\mathbf{x}) = U(\mathbf{x}_T), \mu_t(\mathbf{x}) = \mu_t(\mathbf{x}_t), \sigma_t(\mathbf{x}) = \sigma_t(\mathbf{x}_t), \quad \text{for any } \mathbf{x} \in \mathcal{C}^0.$$

(ii) *the map $t \mapsto f_t$ does not depend on t , and thus δ as well.*

(iii) *If one defines the so-called face-lift of U by*

$$\widehat{U}(x) := \sup_{u \in \mathbb{R}^d} \{U(x + f_T u) - \delta(u)\}, \quad x \in \mathbb{R}^d,$$

²We would like to point out the reader to the recent work in preparation [6] which will actually extend the results of [7] to the non-Markovian case

then we have for some constant $C > 0$ and any $(x, x') \in \mathbb{R}^d \times \mathbb{R}^d$

$$\left| \widehat{U}(x) - \widehat{U}(x') \right| \leq C \|x - x'\|.$$

The main result of this section is

Theorem 6.3. *Let Assumptions 2.1, 2.3, 3.2, 3.6 and 6.2 hold. Then, there is a constant $C > 0$ such that for any $(t, t', \mathbf{x}, \mathbf{x}') \in [0, T] \times [t, T] \times \mathcal{C}^0 \times \mathcal{C}^0$*

$$|v(t, \mathbf{x}) - v(t, \mathbf{x}')| \leq C \|\mathbf{x}_t - \mathbf{x}'_t\|, \quad \left| v(t, \mathbf{x}) - \mathbb{E}^{\mathbb{P}_0^t}[v(t', \mathbf{x})] \right| \leq C(1 + \|\mathbf{x}_t\|)(t' - t)^{1/2}.$$

The remaining of this section is dedicated to the proof of this result. We shall make a strong use of the connection with constrained BSDEs established before.

6.2.1 Another singular control problem

For any $t \in [0, T]$, let us consider the following set of controls

$$\mathcal{V}_b^t := \left\{ (u_s)_{t \leq s \leq T}, \text{ which are } \mathbb{R}^d\text{-valued, } \mathbb{F}^t\text{-predictable and bounded.} \right\}.$$

For any $(t, x) \in [0, T] \times \mathbb{R}^d$, we define

$$Y_t^x := \sup_{u \in \mathcal{V}_b^t} \mathbb{E}^{\mathbb{P}_0^t} \left[U(X_T^{t,x,u}) - \int_t^T \delta(u_s) ds \right],$$

where $X^{t,x,u}$ is the unique strong solution on $(\Omega^t, \mathcal{F}_T^{t,o}, \mathbb{P}_0^t)$ of the following SDE

$$X^{t,x,u} = x + \int_t^\cdot \mu_s(X_s^{t,x,u}) ds + \int_t^\cdot f_s u_s ds + \int_t^\cdot \sigma_s(X_s^{t,x,u}) dB_s^t, \quad \mathbb{P}_0^t\text{-a.s.}$$

This value function is always well-defined since u is bounded and δ is non-negative. Our first step is to show that one can actually replace the map U above by its facelift. It is a version of Proposition 3.1 of [7] for our setting.

Lemma 6.4. *Let Assumptions 2.1, 2.3 and 6.2 hold. Then, for any $t < T$, we have*

$$Y_t^x = \sup_{u \in \mathcal{V}_b^t} \mathbb{E}^{\mathbb{P}_0^t} \left[\widehat{U}(X_T^{t,x,u}) - \int_t^T \delta(u_s) ds \right].$$

Proof. Clearly, we have $\widehat{U} \geq U$, so that one inequality is trivial. Next, fix some $u \in \mathcal{V}_b^t$. For some $\varepsilon_o > 0$ small enough and any $\varepsilon \in (0, \varepsilon_o)$, we define the following element of \mathcal{V}_b^t

$$u_s^\varepsilon := u_s \mathbf{1}_{[t, T-\varepsilon]}(s) + \frac{\iota}{\varepsilon} \mathbf{1}_{[T-\varepsilon, T]}(s),$$

where ι is any bounded and $\mathcal{F}_{T-\varepsilon_o}^t$ -measurable random variable. Then, we have by the tower property for expectations and the Markov property for SDEs (see for instance [12]) that

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}_0^t} \left[\mathbb{E}^{\mathbb{P}_0^{T-\varepsilon}} \left[U \left(X_T^{T-\varepsilon, X_{T-\varepsilon}^{t,x,u}, (u^\varepsilon)^{T-\varepsilon, B^t}} \right) - \int_{T-\varepsilon}^T \delta((u^\varepsilon)_s^{T-\varepsilon, B^t}) ds \right] - \int_t^{T-\varepsilon} \delta(u_s) ds \right] \\ &= \mathbb{E}^{\mathbb{P}_0^t} \left[U(X_T^{t,x,u^\varepsilon}) - \int_t^T \delta(u_s^\varepsilon) ds \right], \end{aligned}$$

which implies that

$$Y_t^x \geq \mathbb{E}^{\mathbb{P}_0^t} \left[\mathbb{E}^{\mathbb{P}_0^{T-\varepsilon}} \left[U \left(X_T^{T-\varepsilon, X_{T-\varepsilon}^{t,x,u}, (u^\varepsilon)^{T-\varepsilon, B^t}} \right) - \int_{T-\varepsilon}^T \delta((u^\varepsilon)_s^{T-\varepsilon, B^t}) ds \right] - \int_t^{T-\varepsilon} \delta(u_s) ds \right]. \quad (6.1)$$

Next, we claim that, at least along a subsequence, we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \mathbb{E}^{\mathbb{P}_0^t} \left[\mathbb{E}^{\mathbb{P}_0^{T-\varepsilon}} \left[U \left(X_T^{T-\varepsilon, X_{T-\varepsilon}^{t,x,u}, (u^\varepsilon)^{T-\varepsilon, B^t}} \right) - \int_{T-\varepsilon}^T \delta((u^\varepsilon)_s^{T-\varepsilon, B^t}) ds \right] - \int_t^{T-\varepsilon} \delta(u_s) ds \right] \\ &= \mathbb{E}^{\mathbb{P}_0^t} \left[U(X_T^{t,x,u} + f_T \iota) - \delta(\iota) - \int_t^T \delta(u_s) ds \right]. \end{aligned} \quad (6.2)$$

Indeed, by (an easy extension of) (2.4), we first have for any $p \geq 2$

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_0^t} \left[\left\| X_T^{T-\varepsilon, X_{T-\varepsilon}^{t,x,u}, u^{T-\varepsilon, B^t}} + f\iota - X_T^{T-\varepsilon, X_{T-\varepsilon}^{t,x,u}, (u^\varepsilon)^{T-\varepsilon, B^t}} \right\|^p \right] &\leq C_p \mathbb{E}^{\mathbb{P}_0^t} \left[\left\| f\iota + \int_{T-\varepsilon}^T f \left(u_s^{T-\varepsilon, B^t} - \frac{\iota}{\varepsilon} \right) ds \right\|^p \right] \\ &\leq C_p \int_{T-\varepsilon}^T \mathbb{E}^{\mathbb{P}_0^t} \left[\left\| f u_s^{T-\varepsilon, B^t} \right\|^p \right] ds. \end{aligned}$$

Furthermore, by (2.2), we have \mathbb{P}_0^t -a.s.

$$\mathbb{E}^{\mathbb{P}_0^{T-\varepsilon}} \left[\left\| X_T^{T-\varepsilon, X_{T-\varepsilon}^{t,x,u}, u^{T-\varepsilon, B^t}} - X_{T-\varepsilon}^{t,x,u} \right\|^p \right] \leq C_p \varepsilon^{\frac{1}{2}} \left(1 + \left\| X_{T-\varepsilon}^{t,x,u} \right\|^p \right) + C_p \mathbb{E}^{\mathbb{P}_0^{T-\varepsilon}} \left[\left\| \int_{T-\varepsilon}^T u_s^{T-\varepsilon, B^t} ds \right\|^p \right],$$

which implies by the tower property

$$\mathbb{E}^{\mathbb{P}_0^t} \left[\left\| X_T^{T-\varepsilon, X_{T-\varepsilon}^{t,x,u}, u^{T-\varepsilon, B^t}} - X_{T-\varepsilon}^{t,x,u} \right\|^p \right] \leq C_p \varepsilon^{\frac{1}{2}} \left(1 + \mathbb{E}^{\mathbb{P}_0^t} \left[\left\| X_{T-\varepsilon}^{t,x,u} \right\|^p \right] \right) + C_p \mathbb{E}^{\mathbb{P}_0^t} \left[\left\| \int_{T-\varepsilon}^T u_s ds \right\|^p \right], \quad \mathbb{P}_0^t\text{-a.s.}$$

Hence, passing to a subsequence if necessary, and using the continuity of the paths of $X^{t,x,u}$, we deduce from the above inequalities that

$$X_T^{T-\varepsilon, X_{T-\varepsilon}^{t,x,u}, (u^\varepsilon)^{T-\varepsilon, B^t}} \text{ converges to } X_T^{t,x,u} + f\iota, \quad \mathbb{P}_0^t\text{-a.s. and in } L^p(\mathbb{P}_0^t).$$

By continuity of U , we deduce that the following convergence holds \mathbb{P}_0^t -a.s. and in $L^p(\mathbb{P}_0^t)$

$$U \left(X_T^{T-\varepsilon, X_{T-\varepsilon}^{t,x,u}, (u^\varepsilon)^{T-\varepsilon, B^t}} \right) - \int_{T-\varepsilon}^T \delta((u^\varepsilon)_s^{T-\varepsilon, B^t}) ds \longrightarrow U(X_T^{t,x,u} + f\iota) - \delta(\iota).$$

Then, this implies by the tower property that the following convergence holds in $L^1(\mathbb{P}_0^t)$

$$\mathbb{E}^{\mathbb{P}_0^{T-\varepsilon}} \left[U \left(X_T^{T-\varepsilon, X_{T-\varepsilon}^{t,x,u}, (u^\varepsilon)^{T-\varepsilon, B^t}} \right) - \int_{T-\varepsilon}^T \delta((u^\varepsilon)_s^{T-\varepsilon, B^t}) ds \right] \longrightarrow U(X_T^{t,x,u} + f\iota) - \delta(\iota),$$

which implies the desired claim (6.2).

Then, by (6.1) and (6.2) we deduce that for random variable ι which is bounded and $\mathcal{F}_{T-\varepsilon_0}^t$ -measurable, we have

$$Y_t^x \geq \mathbb{E}^{\mathbb{P}_0^t} \left[U(X_T^{t,x,u} + f\iota) - \delta(\iota) - \int_t^T \delta(u_s) ds \right], \quad (6.3)$$

and the same statement holds for any ι which is bounded and \mathcal{F}_{T-}^t -measurable by arbitrariness of ε_0 . Now, since the map $(x, \iota) \mapsto U(x + f\iota) - \delta(\iota)$ is Borel measurable, we can argue as in the proof of Proposition 3.1 in [7] to obtain the existence for any $\varepsilon > 0$ of a Borel measurable map $x \mapsto \iota_\varepsilon(x)$ such that

$$\widehat{U}(X_T^{t,x,u}) \leq U(X_T^{t,x,u} + f\iota_\varepsilon(X_T^{t,x,u})) - \delta(\iota_\varepsilon(X_T^{t,x,u})) + \varepsilon.$$

Then, if we define

$$\iota_{n,\varepsilon} := \iota_\varepsilon(X_T^{t,x,u}) \mathbf{1}_{|\iota_\varepsilon(X_T^{t,x,u})| \leq n},$$

$\iota_{\varepsilon, n}$ is bounded and \mathcal{F}_{T-} -measurable by continuity of the paths $X^{t,x,u}$. Hence by (6.3) we have, using the fact that δ_T is null at 0

$$Y_t^x \geq \mathbb{E}^{\mathbb{P}_0^t} \left[\widehat{U}(X_T^{t,x,u}) \mathbf{1}_{|\iota_{\varepsilon}(X_T^{t,x,u})| \leq n} + U(X_T^{t,x,u}) \mathbf{1}_{|\iota_{\varepsilon}(X_T^{t,x,u})| > n} - \int_t^T \delta_s(u_s) ds \right] - \varepsilon.$$

Then the required result follows by letting first n go to infinity and dominated convergence (remember that \widehat{U} is Lipschitz and $X^{t,x,u}$ has moments of any order), and then ε go to 0. \square

The next result is Proposition 3.3 of [7] in our framework

Lemma 6.5. *Let Assumptions 2.1, 2.3 and 6.2 hold. We have for any $(t, x) \in [0, T] \times \mathbb{R}^d$*

$$Y_t^x = \operatorname{essup}_{\iota \in \mathbb{L}_{\infty}(\mathcal{F}_t)} \left\{ Y_t^{x+f\iota} - \delta(\iota) \right\}, \text{ a.s.,}$$

where $\mathbb{L}_{\infty}(\mathcal{F}_t)$ is the set of \mathbb{R}^d -valued, bounded and \mathcal{F}_t -measurable random variables.

Proof. First of all, one inequality is trivial by taking a constant control $\iota = 0$. Then, the following dynamic programming principle holds classically for any $0 \leq t \leq s \leq T$

$$Y_t^x = \sup_{u \in \mathcal{V}_b^t} \mathbb{E}^{\mathbb{P}_0^t} \left[Y_s^{X_s^{t,x,u}} - \int_t^s \delta(u_r) dr \right]. \quad (6.4)$$

Fix now some $(t, x) \in [0, T] \times \mathbb{R}^d$, some $\iota \in \mathbb{L}_{\infty}(\mathcal{F}_t)$, some $\varepsilon > 0$ small enough, and define

$$u^\varepsilon := \frac{1}{\varepsilon} \iota \mathbf{1}_{[t, t+\varepsilon]} \in \mathcal{V}_b^t.$$

By the dynamic programming principle, we have,

$$Y_t^x \geq \mathbb{E}^{\mathbb{P}_0^t} \left[Y_{t+\varepsilon}^{X_{t+\varepsilon}^{t,x,u^\varepsilon}} - \int_t^{t+\varepsilon} \delta(u_r^\varepsilon) dr \right] = \mathbb{E}^{\mathbb{P}_0^t} \left[Y_{t+\varepsilon}^{X_{t+\varepsilon}^{t,x,u^\varepsilon}} - \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \delta(\iota) dr \right]. \quad (6.5)$$

Next, by Lemma 6.4 and the tower property we have for any $u \in \mathcal{V}_b^t$

$$\mathbb{E}^{\mathbb{P}_0^t} \left[Y_{t+\varepsilon}^{X_{t+\varepsilon}^{t,x,u^\varepsilon}} \right] \geq \mathbb{E}^{\mathbb{P}_0^t} \left[\widehat{U} \left(X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,u^\varepsilon}, u^{t+\varepsilon, B^t}} \right) - \int_{t+\varepsilon}^T \delta(u_r) dr \right].$$

Be definition of u^ε , we have, using similar arguments as in the proof of Lemma 6.4, that along a subsequence if necessary,

$$\mathbb{E}^{\mathbb{P}_0^t} \left[\widehat{U} \left(X_T^{t+\varepsilon, X_{t+\varepsilon}^{t,x,u^\varepsilon}, u^{t+\varepsilon, B^t}} \right) - \int_{t+\varepsilon}^T \delta(u_r) dr \right] \xrightarrow{\varepsilon \rightarrow 0} \mathbb{E}^{\mathbb{P}_0^t} \left[\widehat{U} \left(X_T^{t,x+f\iota, u} \right) - \int_t^T \delta(u_r) dr \right].$$

Thus, we deduce from passing to the limit in (6.5) that

$$Y_t^x \geq \mathbb{E}^{\mathbb{P}_0^t} \left[\widehat{U} \left(X_T^{t,x+f\iota, u} \right) - \delta(\iota) - \int_t^T \delta(u_r) dr \right],$$

which implies by Lemma 6.4 and arbitrariness of $u \in \mathcal{V}_b^t$

$$Y_t^x \geq Y_t^{x+f\iota} - \delta(\iota),$$

hence the desired result. \square

We can now give our main result of this section.

Proposition 6.6. *Let Assumptions 2.1, 2.3 and 6.2 hold. Then, there is some constant $C > 0$ such that for any $0 \leq t \leq s < T$, any $(x, x') \in \mathbb{R}^d \times \mathbb{R}^d$*

$$\left| Y_t^x - Y_t^{x'} \right| \leq C \|x - x'\|, \quad \left| Y_t^x - \mathbb{E}^{\mathbb{P}_0^t}[Y_s^x] \right| \leq C(1 + \|x\|)(s - t)^{\frac{1}{2}}.$$

Proof. The first result is an immediate consequence of Lemma 6.4, the fact that \widehat{U} is Lipschitz continuous, and classical estimates on the solutions to SDE satisfied by $X^{t,x,u}$.

Next, we have by (6.4), the regularity in x we just proved and (2.2)

$$Y_t^x \geq \mathbb{E}^{\mathbb{P}_0^t} \left[Y_s^{X_s^{t,x,0}} \right] \geq \mathbb{E}^{\mathbb{P}_0^t} [Y_s^x] - C \mathbb{E}^{\mathbb{P}_0^t} [\|X_s^{t,x,0} - x\|] \geq \mathbb{E}^{\mathbb{P}_0^t} [Y_s^x] - C(1 + \|x\|)(s - t)^{1/2}.$$

Then, we have for any $u \in \mathcal{V}_b^t$, using the fact that $t \mapsto \delta_t(\cdot)$ is non-increasing and sublinear, as well as Lemma 6.5

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_0^s} \left[U \left(X_T^{s, X_s^{t,x,u}, u^{s,B^t}} \right) - \int_s^T \delta(u_r) dr \right] - \int_t^s \delta(u_r) dr &\leq Y_s^{X_s^{t,x,u}} - \int_t^s \delta(u_r) dr \\ &\leq Y_s^{X_s^{t,x,u}} - \delta \left(\int_t^s u_r dr \right) \\ &\leq Y_s^{X_s^{t,x,u} - f \int_t^s u_r dr}. \end{aligned}$$

Then by the tower property, we have, taking expectations under \mathbb{P}_0^t on both sides

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_0^t} \left[U(X_T^{t,x,u}) - \int_t^T \delta(u_r) dr \right] &\leq \mathbb{E}^{\mathbb{P}_0^t} \left[Y_s^{X_s^{t,x,u} - f \int_t^s u_r dr} \right] \\ &\leq \mathbb{E}^{\mathbb{P}_0^t} [Y_s^x] + \mathbb{E}^{\mathbb{P}_0^t} \left[\left\| X_s^{t,x,u} - f \int_t^s u_r dr - x \right\| \right] \\ &\leq \mathbb{E}^{\mathbb{P}_0^t} [Y_s^x] + C(1 + \|x\|)(s - t)^{1/2}, \end{aligned}$$

where we used the fact that in the expression $X_s^{t,x,u} - f \int_t^s u_r dr - x$ the control u actually disappears. By definition of Y_t^x , this ends the proof. \square

6.2.2 Weak formulation and the main result

For any $(t, x) \in [0, T] \times \mathbb{R}^d$ and $u \in \mathcal{V}_b^t$, we now define the following \mathbb{P}_0^t -equivalent measure

$$\frac{d\mathbb{P}^{t,x,u}}{d\mathbb{P}_0^t} = \mathcal{E} \left(\int_t^\cdot (\sigma_s)^{-1} (X^{t,x,0}) f u_s \cdot dB_s^t \right).$$

The weak formulation of the control problem is defined as

$$Y_t^{w,x} := \sup_{u \in \mathcal{V}_b^t} \mathbb{E}^{\mathbb{P}^{t,x,u}} \left[U(X^{t,x,0}) - \int_t^T \delta(u_s) \right], \text{ for any } (t, x) \in [0, T] \times \mathbb{R}^d.$$

The following proposition is a simple consequence of Remark 3.8 and Theorem 4.5 of [20].

Proposition 6.7. *For any $(t, x) \in [0, T] \times \mathbb{R}^d$, we have $Y_t^x = Y_t^{w,x}$.*

We can now proceed to the

Proof of Theorem 6.3. By Theorem 4.1 of [7], we have for any $(t, \mathbf{x}) \in [0, T] \times \mathcal{C}^0$, $Y_t^{w,\mathbf{x}(t)} = \mathcal{Y}_t^{t,\mathbf{x}}$ defined as the first component of the constrained BSDE (3.1)-(3.2). It then suffices to apply Theorem 4.7 together with Proposition 6.6. \square

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