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# Improved Beamforming for FSO MISO System over Gamma-Gamma Fading with Pointing Errors

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**Abstract**—In this paper, we study the problem of feedback based beamforming for multiple-input single-output free space optical (FSO) system with pointing errors. For a  $2 \times 1$  FSO system, it is shown by analysis that any arbitrary beamforming scheme performs poorer to the repetition coding scheme but achieves full diversity over the Gamma-Gamma fading with pointing errors. Then we study a beamforming scheme for  $2 \times 1$  FSO system which employs one bit feedback from receiver to the transmitter. Erroneous feedback leads to loss in diversity for this beamforming scheme as established by the bit error rate (BER) analysis. For avoiding the loss in diversity, an improved one bit feedback based beamforming scheme is proposed which outperforms the repetition coding. The average BER of this scheme is obtained by using the order statistic and it is minimized to find the optimized transmit weights for the transmit apertures under the erroneous feedback over Gamma-Gamma fading with pointing errors.

**Index Terms**—Bit error rate, free space optical links, Gamma-Gamma fading, MISO system, subcarrier intensity modulation.

## I. INTRODUCTION

Free space optical (FSO) communication system has been heavily studied along with multiple-input multiple-output (MIMO) technology [1]–[3], because MIMO technology enables the FSO communication to overcome the atmospheric turbulence. The Gamma-Gamma distribution is used for characterizing the FSO links over a wide range of atmospheric turbulence conditions (weak to strong) [4]–[7]. However, for study of a practical FSO system, the *pointing error* along with the atmospheric fluctuations must be considered [8]–[10]. The pointing error can occur with boresight due to deterministic displacement of the laser beam at the receiving aperture, which is more pronounced in the longer distance FSO communication systems like in earth-to-satellite links [10]. The Alamouti space-time block code (STBC) [11] is a useful coding scheme for radio frequency (RF) MIMO system for achieving the transmit diversity. However, the Alamouti code is found to work inferior to the repetition coding in the FSO system employing intensity modulation and direct detection with on/off keying, and subcarrier intensity modulation (SIM) [5], [6], in some recent literature [12]–[14].

The transmit aperture selection performs better than the repetition coding scheme in FSO MIMO system [15]–[19]. Limited feedback based transmit power allocation/beamforming scheme has been studied in [15], for *log-normal* fading FSO links; in this scheme, the transmit apertures are divided into different sizes of partitions and one partition is selected based on (error-free) feedback bits for transmitting signals by using the repetition coding. In [16], a transmit laser selection

scheme is proposed which requires perfect knowledge of the channel gains in the transmitter for the selection of the transmit aperture with maximum channel magnitude. However, in practice, the feedback information *cannot* be received error-free. In [19], a simple *one bit* erroneous feedback based beamforming scheme is discussed for multiple-input single-output (MISO) FSO system. However, in [19] the effect of pointing error is ignored.

In this paper, we consider the problem of erroneous quantized feedback based beamforming in FSO MISO systems, employing SIM binary phase shift keying (BPSK) and operating under the Gamma-Gamma fading with *zero* boresight pointing errors. We first derive an analytical framework for the BER of a  $2 \times 1$  FSO system with arbitrary beamforming over the Gamma-Gamma fading channels with pointing errors. It is figured out based on the BER analysis that any arbitrary beamforming with non-zero weights performs poorer to the repetition coding scheme but achieves the maximum possible diversity over Gamma-Gamma fading with pointing errors. Then we consider the case when only one bit feedback about the instantaneous channel state information (CSI) is available in the transmitter. By using this one bit feedback, the transmitter can employ the best transmit aperture selection scheme; but it is analytically shown that the transmit aperture selection scheme loses diversity with the errors in the feedback bit. For avoiding the loss in the diversity, we propose an improved one bit feedback based scheme for  $2 \times 1$  FSO system; this scheme outperforms the repetition coding scheme with one bit erroneous feedback over the Gamma-Gamma fading with pointing errors.

## II. SYSTEM AND CHANNEL MODEL

### A. System Model

Let us consider a FSO MISO system with  $M$  transmit and a single receive apertures. It is assumed that this system employs the SIM scheme [6]. In a transmission time interval, a BPSK symbol  $s \in \{A, -A\}$  with  $E[|s|^2] = E_s$ , where  $E[\cdot]$  stands for the expectation, is transmitted by all transmit apertures. Before transmission the symbol  $s$  is multiplied by a beamforming vector  $\mathbf{v} = [v_1, v_2, \dots, v_M]^T$  such that  $\sum_{i=1}^M v_i \leq M$ . The received electrical signal at the receive aperture, after optical-to-electrical conversion, can be written as

$$y_i = \frac{\eta}{M} [I_1, I_2, \dots, I_M] [v_1, v_2, \dots, v_M]^T s + e$$

$$= \frac{\eta}{M} \sum_{i=1}^M I_i v_i s + e, \quad (1)$$

where  $e$  is the zero-mean complex-valued additive white Gaussian noise (AWGN) of  $\sigma^2$  variance;  $I_i$  is the real-valued irradiance of the link between the  $i$ -th transmit aperture and the receive aperture, following the Gamma-Gamma distribution with pointing errors [8]; and  $\eta$  denotes the optical-to-electrical conversion coefficient. A maximum-likelihood receiver for the scheme is given by

$$\hat{s} = \min_{\tilde{s} \in \{A, -A\}} \left| y - \frac{\eta}{M} \sum_{i=1}^M I_i v_i \tilde{s} \right|^2. \quad (2)$$

### B. Channel Model

The Gamma-Gamma fading caused by atmospheric turbulence with zero boresight pointing errors is considered for the study. When both the atmospheric turbulence and the pointing errors are considered, the distribution of  $I_i$  is given by [8, Eq. (8)]

$$f_{I_i}(x) = \frac{\alpha\beta\xi^2}{A_0\Gamma(\alpha)\Gamma(\beta)} G_{1,3}^{3,0} \left( \frac{\alpha\beta x}{A_0} \middle| \xi^2 - 1, \alpha - 1, \beta - 1 \right), \quad (3)$$

where  $G_{p,q}^{m,n}(\cdot|\dots)$  is the Meijer-G function [20, Eq. (9.301)] and  $\alpha$  and  $\beta$  depict the atmospheric fluctuations [6]. The effect of the pointing errors is characterized by the following parameters:  $A_0 = [\text{erf}(\nu)]^2$ ,  $\nu = \sqrt{\pi/2} R/w_b$ ,  $\xi^2 = w_e^2/4\sigma_s^2$ ,  $R$  is the radius of the receiver aperture,  $w_b$  is the normalized beamwaist,  $w_e = [\sqrt{\pi}\text{erf}(\nu)w_b^2/(2\nu e^{-\nu^2})]^{1/2}$  is the equivalent beamwaist,  $\sigma_s^2$  is the variance of the Gaussian distributions for both horizontal and vertical buildings' sway, and  $\text{erf}(\cdot)$  denotes the error function.

It is difficult to directly deal with the PDF of (3) due to the presence of the Meijer-G function. Therefore, we aim to have an alternative representation of the PDF of (3), which can render simplified analysis. Let us use the Slater's theorem [21] to express the Meijer-G function in (3) as

$$G_{1,3}^{3,0} \left( z \middle| \begin{matrix} a_1 \\ b_1, b_2, b_3 \end{matrix} \right) = \sum_{h=1}^3 \frac{\prod_{j=1}^3 \Gamma(b_j - b_h)^\dagger}{\Gamma(a_1 - b_h)} z^{b_h} \times {}_1F_2 \left( 1 + b_h - a_1; [1 + b_h - \mathbf{b}]^\dagger; z \right), \quad (4)$$

where  $[\cdot]$  indicates a row vector,  $\mathbf{b} = [b_1, b_2, b_3]$ ,  $(\cdot)^\dagger$  indicates to ignore the terms with  $b_j = b_h$ ,  $\Gamma(\cdot)$  is the Gamma function [20], and  ${}_1F_2(\cdot; \cdot, \cdot; \cdot)$  is the generalized Hypergeometric function [20]. Note that the expansion in (4) is valid only if  $b_j - b_h \notin \mathbb{Z}$  and  $a_1 - b_h \notin \mathbb{Z}$ .

From (3) and (4), and employing the series representation:

$${}_1F_2(a_1; b_1, b_2; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n}{(b_1)_n (b_2)_n} \frac{z^n}{n!}, \quad (5)$$

where  $(x)_n$  is the Pochhammer symbol, we obtain the PDF of

$I_{i,j}$  in series form as

$$f_{I_{i,j}}(x) = X_0 x^{\xi^2-1} + \sum_{n=0}^{\infty} Y_n x^{n+\alpha-1} + \sum_{n=0}^{\infty} Z_n x^{n+\beta-1}, \quad (6)$$

where

$$\begin{aligned} X_0 &= \frac{\xi^2}{\Gamma(\alpha)\Gamma(\beta)} \Gamma(\alpha - \xi^2) \Gamma(\beta - \xi^2) \left( \frac{\alpha\beta}{A_0} \right)^{\xi^2} \\ Y_n &= \frac{\xi^2}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\xi^2 - \alpha) \Gamma(\beta - \alpha)}{\Gamma(1 + \xi^2 - \alpha)} \\ &\times \frac{(\alpha - \xi^2)_n}{(1 + \alpha - \xi^2)_n (1 + \alpha - \beta)_n n!} \left( \frac{\alpha\beta}{A_0} \right)^{n+\alpha} \\ Z_n &= \frac{\xi^2}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\xi^2 - \beta) \Gamma(\alpha - \beta)}{\Gamma(1 + \xi^2 - \beta)} \\ &\times \frac{(\beta - \xi^2)_n}{(1 + \beta - \alpha)_n (1 + \beta - \xi^2)_n n!} \left( \frac{\alpha\beta}{A_0} \right)^{n+\beta}. \end{aligned} \quad (7)$$

*Remark 1:* The proposed series based representation of the PDF of Gamma-Gamma channels with pointing errors, cf. (6), contains two power series. Each series contains summation terms with only exponents of  $x$ . Therefore, it is easy to calculate an integral containing the proposed series representation as compared to a complicated function based representation in (3). It can be easily shown by applying the ratio test for each power series in (6) that the new series representation contains converging power series with infinite radius of convergence.

### III. STUDY OF AN ARBITRARY BEAMFORMING SCHEME FOR $2 \times 1$ GAMMA-GAMMA FADING FSO LINKS WITH POINTING ERRORS

We will derive the analytical average BER performance of an arbitrary beamforming based FSO MISO system with Gamma-Gamma fading and pointing errors, in this section. For simplicity, we concentrate over the  $2 \times 1$  FSO system; however, more generalized results can be obtained by following the method given in this section.

Let us define a random variable (RV) as

$$w_i \triangleq I_i v_i. \quad (8)$$

From (6) and (8), the PDF of  $w_i$  will be

$$f_{w_i}(x) = X_{0,i} x^{\xi^2-1} + \sum_{n=0}^{\infty} Y_{n,i} x^{n+\alpha-1} + \sum_{n=0}^{\infty} Z_{n,i} x^{n+\beta-1}, \quad (9)$$

where  $X_{0,i} = X_0/v_i^{\xi^2}$ ,  $Y_{n,i} = Y_n/v_i^{n+\alpha}$ , and  $Z_{n,i} = Z_n/v_i^{n+\beta}$ . By using (9) in the relation:  $M_{w_i}(s) = \int_0^\infty e^{-sx} f_{w_i}(x) dx$ , the MGF of  $w_i$  can be expressed as

$$M_{w_i}(s) = \tilde{X}_{0,i} s^{-\xi^2} + \sum_{n=0}^{\infty} \tilde{Y}_{n,i} s^{-n-\alpha} + \sum_{n=0}^{\infty} \tilde{Z}_{n,i} s^{-n-\beta}, \quad (10)$$

where  $\tilde{X}_{0,i} = \Gamma(\xi^2)X_{0,i}$ ,  $\tilde{Y}_{n,i} = \Gamma(n + \alpha)Y_{n,i}$ , and  $\tilde{Z}_{n,i} = \Gamma(n + \beta)Z_{n,i}$ . Let us consider  $2 \times 1$  FSO system and we can define another RV as

$$w \triangleq I_1 v_1 + I_2 v_2 = w_1 + w_2. \quad (11)$$

After observing that  $I_1$  and  $I_2$  are independent, the MGF of  $w$  can be written from (10) as

$$\begin{aligned} M_w(s) &= \sum_{n=0}^{\infty} A_{n,1} s^{-n-\alpha-\xi^2} + \sum_{n=0}^{\infty} A_{n,2} s^{-n-\beta-\xi^2} \\ &+ \sum_{n=0}^{\infty} A_{n,3} s^{-n-2\alpha} + \sum_{n=0}^{\infty} A_{n,4} s^{-n-\alpha-\beta} \\ &+ \sum_{n=0}^{\infty} A_{n,5} s^{-n-2\beta} + A_{0,6} s^{-2\xi^2}. \end{aligned} \quad (12)$$

In (12),  $A_{n,1} = \tilde{X}_{0,1}\tilde{Y}_{n,2} + \tilde{X}_{0,2}\tilde{Y}_{n,1}$ ,  $A_{n,2} = \tilde{X}_{0,1}\tilde{Z}_{n,2} + \tilde{X}_{0,2}\tilde{Z}_{n,1}$ ,  $A_{n,3} = \tilde{Y}_{n,1} * \tilde{Y}_{n,2}$ ,  $A_{n,4} = \tilde{Y}_{n,1} * \tilde{Z}_{n,2} + \tilde{Z}_{n,1} * \tilde{Y}_{n,2}$ ,  $A_{n,5} = \tilde{Z}_{n,1} * \tilde{Z}_{n,2}$ ,  $A_{0,6} = \tilde{X}_{0,1}\tilde{X}_{0,2}$ , and  $*$  denotes the convolution. The PDF of  $w$  can be obtained by taking the inverse Laplace transform of (12)

$$\begin{aligned} f_w(x) &= \sum_{n=0}^{\infty} \tilde{A}_{n,1} x^{n+\alpha+\xi^2-1} + \sum_{n=0}^{\infty} \tilde{A}_{n,2} x^{n+\beta+\xi^2-1} \\ &+ \sum_{n=0}^{\infty} \tilde{A}_{n,3} x^{n+2\alpha-1} + \sum_{n=0}^{\infty} \tilde{A}_{n,4} x^{n+\alpha+\beta-1} \\ &+ \sum_{n=0}^{\infty} \tilde{A}_{n,5} x^{n+2\beta-1} + \tilde{A}_{0,6} x^{2\xi^2-1}, \end{aligned} \quad (13)$$

where  $\tilde{A}_{n,1} = A_{n,1}/\Gamma(n+\alpha+\xi^2)$ ,  $\tilde{A}_{n,2} = A_{n,2}/\Gamma(n+\beta+\xi^2)$ ,  $\tilde{A}_{n,3} = A_{n,3}/\Gamma(n+2\alpha)$ ,  $\tilde{A}_{n,4} = A_{n,4}/\Gamma(n+\alpha+\beta)$ ,  $\tilde{A}_{n,5} = A_{n,5}/\Gamma(n+2\beta)$ , and  $\tilde{A}_{0,6} = A_{0,6}/\Gamma(2\xi^2)$ .

Using (1) and (11), we have the input-output (I/O) relation for  $2 \times 1$  FSO system:

$$y_i = \frac{\eta}{2} w s + e. \quad (14)$$

The instantaneous BER for the I/O relation of (14) is given as [22]

$$Pe(w, \bar{\gamma}) = Q\left(\sqrt{2\bar{\gamma}w}\right). \quad (15)$$

Here  $Q(\cdot)$  is the q-function and  $\bar{\gamma} = \eta^2 E_s / (4\sigma^2)$  denotes the average signal-to-noise ratio (SNR) per diversity branch. By using the relation that  $Q(x) = (1/2)\text{erfc}(x/\sqrt{2})$ , where  $\text{erfc}(\cdot)$  is the complementary error function, we have the average BER of the scheme as

$$Pe(\bar{\gamma}) = \frac{1}{2} \int_0^{\infty} \text{erfc}(\sqrt{\bar{\gamma}x}) f_w(x) dx. \quad (16)$$

Let us first state a useful relation:

$$\int_0^{\infty} x^{a-1} \text{erfc}(bx) dx = \frac{\Gamma(\frac{a+1}{2})}{ab^a \sqrt{\pi}}. \quad (17)$$

From (13), (16), and (17), we obtain the average BER of the scheme:

$$\begin{aligned} Pe(\bar{\gamma}) &= \frac{1}{2} \sum_{n=0}^{\infty} \tilde{A}_{n,1} \frac{\Gamma(\frac{n+\alpha+\xi^2+1}{2})}{(n+\alpha+\xi^2)\sqrt{\pi}\bar{\gamma}^{\frac{n+\alpha+\xi^2}{2}}} \\ &+ \frac{1}{2} \sum_{n=0}^{\infty} \tilde{A}_{n,2} \frac{\Gamma(\frac{n+\beta+\xi^2+1}{2})}{(n+\beta+\xi^2)\sqrt{\pi}\bar{\gamma}^{\frac{n+\beta+\xi^2}{2}}} \\ &+ \frac{1}{2} \sum_{n=0}^{\infty} \tilde{A}_{n,3} \frac{\Gamma(\frac{n+1}{2} + \alpha)}{(n+2\alpha)\sqrt{\pi}\bar{\gamma}^{\frac{n}{2} + \alpha}} \\ &+ \frac{1}{2} \sum_{n=0}^{\infty} \tilde{A}_{n,4} \frac{\Gamma(\frac{n+\alpha+\beta+1}{2})}{(n+\alpha+\beta)\sqrt{\pi}\bar{\gamma}^{\frac{n+\alpha+\beta}{2}}} \\ &+ \frac{1}{2} \sum_{n=0}^{\infty} \tilde{A}_{n,5} \frac{\Gamma(\frac{n+1}{2} + \beta)}{(n+2\beta)\sqrt{\pi}\bar{\gamma}^{\frac{n}{2} + \beta}} \\ &+ \frac{1}{2} \tilde{A}_{0,6} \frac{\Gamma(\frac{1}{2} + \xi^2)}{(2\xi^2)\sqrt{\pi}\bar{\gamma}^{\xi^2}}. \end{aligned} \quad (18)$$

The BER performance of the considered SIM scheme can be characterized at high SNR by using two parameters: coding gain ( $C_g$ ) and diversity gain ( $\delta$ ). The coding gain depicts the relative horizontal shift of the BER versus SNR plots on the log-log scale; whereas, the diversity gain indicates the slope of the decay of these plots at high SNR. The standard definition of the asymptotic BER is

$$\lim_{\bar{\gamma} \rightarrow \infty} Pe(\bar{\gamma}) \approx (C_g \bar{\gamma})^{-\delta}. \quad (19)$$

In the considered scheme, the term corresponding to the smallest exponent of the average SNR  $\bar{\gamma}$  in the power series in (18) dominates the BER, at high SNR. Therefore, after substituting  $n = 0$  in (18) and keeping the terms dominating the BER performance at very high SNR, we get the asymptotic BER of the scheme:

$$\begin{aligned} \lim_{\bar{\gamma} \rightarrow \infty} Pe(\bar{\gamma}) &= \frac{(\Gamma(\delta))^2 \Gamma(\delta + \frac{1}{2})}{4\delta \sqrt{\pi} \Gamma(2\delta) \prod_{i=1}^2 v_i^{\delta}} \\ &\times \left( \frac{(\alpha\beta)^{\delta} P(\alpha, \beta, \xi^2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(1+\xi^2-\delta)A_{0,6}^{\delta}} \right)^2 \frac{1}{\bar{\gamma}^{\delta}}, \end{aligned} \quad (20)$$

where  $\delta = \min(\alpha, \beta, \xi^2)$  and  $P(\alpha, \beta, \xi^2) = \Gamma(\alpha - \delta)^{\dagger} \Gamma(\beta - \delta)^{\dagger} \Gamma(\xi^2 - \delta)^{\dagger}$ . It can be easily shown that the diversity of a single FSO link based communication system is  $\delta = \min(\alpha, \beta, \xi^2)$ . Therefore, the diversity of the considered beam-forming scheme is twice of that of a single FSO link based system. Further, it can be seen from (20) that in order to have a tolerable BER we should choose the weights such that  $v_i \neq 0$ . So let us make the following remark.

**Remark 2:** The arbitrary beamforming based  $2 \times 1$  FSO system achieves full diversity of  $\min(\alpha, \beta, \xi^2)$ , if the beam-forming vector  $\mathbf{v} = [v_1, v_2]$  is a full vector<sup>1</sup>.

<sup>1</sup>does not contain any zero entry.

The coding gain of the beamforming scheme can be given from (20) after some algebra as

$$C_g = \frac{(4\delta\sqrt{\pi}\Gamma(2\delta))^{1/\delta} A_0^2 \prod_{i=1}^2 v_i}{\left((\Gamma(\delta))^2 \Gamma(\delta + \frac{1}{2})\right)^{1/\delta} (\alpha\beta)^2} \times \left(\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(1+\xi^2-\delta)}{P(\alpha, \beta, \xi^2)}\right)^{2/\delta}. \quad (21)$$

As shown in (21) that  $C_g$  is proportional to the product of  $v_1$  and  $v_2$ , i.e.,  $\prod_{i=1}^2 v_i$ . Therefore, in order to maximize the coding gain in (21), we need to maximize this product. Since  $v_i > 0$  and there is a constraint that  $v_1 + v_2 = 2$ , the product  $\prod_{i=1}^2 v_i$  will be maximized when  $v_1 = v_2 = 1$ . The uniform weighing stands for the spatial repetition coding.

*Remark 3:* No arbitrary weighting based beamforming scheme can perform better than the spatial repetition coding scheme in  $2 \times 1$  FSO system over the Gamma-Gamma fading links with pointing errors.

This observation is similar to [19], where study of arbitrary beamforming is performed for a  $2 \times 1$  FSO system over the Gamma-Gamma fading and *no* pointing error.

#### IV. STUDY OF ONE BIT FEEDBACK BASED BEAMFORMING SCHEME FOR $2 \times 1$ GAMMA-GAMMA FADING FSO LINKS WITH POINTING ERRORS

In this section, we study a beamforming scheme for  $2 \times 1$  system which utilizes one bit feedback from the receiver for determining the transmit weights. The effect of error in the feedback is analyzed on the BER performance of this simple scheme. Let the transmitter can use one of the following beamforming vectors [19]:

$$\mathbf{v}_1 = [a, 2-a]^T \text{ or } \mathbf{v}_2 = [2-a, a]^T, \quad (22)$$

for some constant  $a$  that satisfies  $0 \leq a \leq 2$ . The choice of the beamforming vector depends upon the one bit feedback received from the receiver. Provided that there is no error in the feedback, then the best strategy is to use  $\mathbf{v}_1$  if  $I_1 > I_2$  and  $\mathbf{v}_2$  otherwise, where  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are given in (22). The best choice in this case for  $a$  is  $a = 2$ . However, in practice perfect error-free feedback is not possible. If there is an error in decoding the feedback bit, then the transmitter chooses the wrong aperture with  $\min(I_1, I_2)$ . This would lead to increase in the error probability of the receiver. Let  $P_c$  denotes the probability of correct detection of the feedback bit received by the transmitter. Therefore, the average BER under the erroneous feedback will be

$$Pe(\bar{\gamma}) = \frac{P_c}{2} \int_0^\infty \text{erfc}(2\sqrt{\bar{\gamma}}x) f_t(x) dx + \frac{(1-P_c)}{2} \int_0^\infty \text{erfc}(2\sqrt{\bar{\gamma}}x) f_u(x) dx, \quad (23)$$

where  $t = \max(I_1, I_2)$  and  $u = \min(I_1, I_2)$ . By observing that  $I_i$ s are independent and identically distributed (i.i.d), from order statistics [23],  $t$  and  $u$  are distributed as

$$\begin{aligned} f_t(x) &= 2F_{I_1}(x)f_{I_1}(x) \\ f_u(x) &= 2f_{I_1}(x) - 2F_{I_1}(x)f_{I_1}(x), \end{aligned} \quad (24)$$

where  $F_{I_1}(x)$  denotes the cumulative distribution function of  $I_1$ . Substituting these distributions in (23) and after some algebra, it can be shown that

$$Pe(\bar{\gamma}) = (2P_c - 1) \int_0^\infty \text{erfc}(2\sqrt{\bar{\gamma}}x) F_{I_1}(x)f_{I_1}(x) dx + (1 - P_c) \int_0^\infty \text{erfc}(2\sqrt{\bar{\gamma}}x) f_{I_1}(x) dx. \quad (25)$$

After using the relation:  $F_{I_1}(x) = \int_0^x f_{I_1}(x) dx$  and some algebra, the average BER of the beamforming scheme with feedback error is given by

$$\begin{aligned} Pe(\bar{\gamma}) &= (2P_c - 1) \sum_{n=0}^\infty B_{n,1} \frac{\Gamma(\frac{n+\alpha+\xi^2+1}{2})}{(n+\alpha+\xi^2)\sqrt{\pi}(4\bar{\gamma})^{\frac{n+\alpha+\xi^2}{2}}} \\ &+ (2P_c - 1) \sum_{n=0}^\infty B_{n,2} \frac{\Gamma(\frac{n+\beta+\xi^2+1}{2})}{(n+\beta+\xi^2)\sqrt{\pi}(4\bar{\gamma})^{\frac{n+\beta+\xi^2}{2}}} \\ &+ (2P_c - 1) \sum_{n=0}^\infty B_{n,3} \frac{\Gamma(\frac{n+1}{2} + \alpha)}{(n+2\alpha)\sqrt{\pi}(4\bar{\gamma})^{\frac{n}{2} + \alpha}} \\ &+ (2P_c - 1) \sum_{n=0}^\infty B_{n,4} \frac{\Gamma(\frac{n+\alpha+\beta+1}{2})}{(n+\alpha+\beta)\sqrt{\pi}(4\bar{\gamma})^{\frac{n+\alpha+\beta}{2}}} \\ &+ (2P_c - 1) \sum_{n=0}^\infty B_{n,5} \frac{\Gamma(\frac{n+1}{2} + \beta)}{(n+2\beta)\sqrt{\pi}(4\bar{\gamma})^{\frac{n}{2} + \beta}} \\ &+ (2P_c - 1) B_{0,6} \frac{\Gamma(\frac{1}{2} + \xi^2)}{(2\xi^2)\sqrt{\pi}(4\bar{\gamma})^{\xi^2}} \\ &+ (1 - P_c) X_0 \frac{\Gamma(\frac{1+\xi^2}{2})}{\xi^2\sqrt{\pi}(4\bar{\gamma})^{\xi^2/2}} \\ &+ (1 - P_c) \sum_{n=0}^\infty Y_n \frac{\Gamma(\frac{1+n+\alpha}{2})}{(n+\alpha)\sqrt{\pi}(4\bar{\gamma})^{(n+\alpha)/2}} \\ &+ (1 - P_c) \sum_{n=0}^\infty Z_n \frac{\Gamma(\frac{1+n+\beta}{2})}{(n+\beta)\sqrt{\pi}(4\bar{\gamma})^{(n+\beta)/2}}, \end{aligned} \quad (26)$$

where  $B_{n,1} = X_0\tilde{Y}_n + \tilde{X}_0Y_n$ ,  $B_{n,2} = X_0\tilde{Z}_n + \tilde{X}_0Z_n$ ,  $B_{n,3} = Y_n * \tilde{Y}_n$ ,  $B_{n,4} = Y_n * \tilde{Z}_n + Z_n * \tilde{Y}_n$ ,  $B_{n,5} = Z_n * \tilde{Z}_n$ ,  $B_{0,6} = X_0\tilde{X}_0$ ,  $\tilde{X}_0 = X_0/\xi^2$ ,  $\tilde{Y}_n = Y_n/(n+\alpha)$ , and  $\tilde{Z}_n = Z_n/(n+\beta)$ . By substituting  $n = 0$  in (26) and keeping the terms with the lowest power of  $\bar{\gamma}$ , we get the asymptotic BER of the beamforming scheme with erroneous feedback:

$$\begin{aligned} &\lim_{\bar{\gamma} \rightarrow \infty} Pe(\bar{\gamma}) \\ &= (1 - P_c) \frac{\xi^2\Gamma(\delta + \frac{1}{2})(\alpha\beta)^{2\delta}P(\alpha, \beta, \xi^2)}{2\delta\sqrt{\pi}A_0^{2\delta}\Gamma(\alpha)\Gamma(\beta)} \frac{1}{(4\bar{\gamma})^\delta}, \end{aligned} \quad (27)$$

where  $\delta = \min(\alpha/2, \beta/2, \xi^2/2)$  represents the diversity order of the beamforming scheme with erroneous feedback, which is same as that of a single link based FSO system.

*Remark 4:* The  $2 \times 1$  FSO beamforming system with  $a = 2$ , can achieve only the diversity of a single link based FSO system, for any value of  $P_c < 1$ .

**Remark 4** is very useful as it indicates that the error performance of a simple beamforming leading to the transmit



aperture selection is very sensitive to the feedback errors. Even very small values of the feedback error has a potential to deteriorate the diversity performance of the FSO system. These observations are similar to drawn in [19] for  $2 \times 1$  FSO MISO system *without pointing error*. However, in this study, we have found that these useful observations are also valid for a  $2 \times 1$  FSO MISO system with pointing errors.

## V. IMPROVED ONE BIT FEEDBACK BASED BEAMFORMING SCHEME FOR $2 \times 1$ GAMMA-GAMMA FADING FSO LINKS WITH POINTING ERRORS

As indicated by **Remark 2**, for avoiding the loss in the diversity of the one bit feedback based beamforming scheme,  $a$  should be chosen such that  $0 \neq a \neq 2$ . This condition will ensure that both weights are non-zero and full diversity is guaranteed irrespective of the error in the feedback. However, we need an optimized value of  $a$  to minimize the average BER of the scheme. Therefore, the modified beamforming scheme is more generalized than the simple transmit aperture selection scheme. The I/O relation for the scheme can be written as

$$y = \frac{\eta}{2}zs + e. \quad (28)$$

In (28),  $z$  is the effective channel and has two possibilities depending upon the feedback:

$$\begin{aligned} z|_{\text{correct}} &= [t, u]\mathbf{v}_1 \\ &= at + (2-a)u \\ z|_{\text{wrong}} &= [t, u]\mathbf{v}_2 \\ &= (2-a)t + au. \end{aligned} \quad (29)$$

From (28) and (29), the conditional (conditioned on  $I_1$  and  $I_2$ ) BER of the scheme can be written after some algebra as

$$\begin{aligned} P_e(I_1, I_2, \bar{\gamma}) &= P_c Q(\sqrt{2\bar{\gamma}}(at + (2-a)u)) \\ &+ (1 - P_c) Q(\sqrt{2\bar{\gamma}}((2-a)t + au)). \end{aligned} \quad (30)$$

By using the bound  $Q(x) \leq (1/2)e^{-x^2/2}$  [24, Fig. 3.1] in (30), we get

$$\begin{aligned} P_e(I_1, I_2, \bar{\gamma}) &\leq \frac{P_c}{2} e^{-(at + (2-a)u)^2 \bar{\gamma}} \\ &+ \frac{(1 - P_c)}{2} e^{-((2-a)t + au)^2 \bar{\gamma}}. \end{aligned} \quad (31)$$

Since  $t > 0$ ,  $u > 0$ , and  $0 < a < 2$ , from (31) we get the following upperbound on the conditional BER:

$$\begin{aligned} P_e(I_1, I_2, \bar{\gamma}) &\leq \frac{P_c}{2} e^{-(a^2 t^2 + (2-a)^2 u^2) \bar{\gamma}} \\ &+ \frac{(1 - P_c)}{2} e^{-((2-a)^2 t^2 + a^2 u^2) \bar{\gamma}}. \end{aligned} \quad (32)$$

Under the observation that  $I_1$  and  $I_2$  are identically distributed, an upperbound of the average BER of the scheme can be

written from (32) as

$$\begin{aligned} P_e(\bar{\gamma}) &\leq P_c \underbrace{\int_0^\infty \int_0^x e^{-(a^2 x^2 + (2-a)^2 y^2) \bar{\gamma}} f_{I_1}(x) f_{I_2}(y) dy dx}_{J_1} \\ &+ (1 - P_c) \underbrace{\int_0^\infty \int_0^x e^{-((2-a)^2 x^2 + a^2 y^2) \bar{\gamma}} f_{I_1}(x) f_{I_2}(y) dy dx}_{J_2}. \end{aligned} \quad (33)$$

Note that the exact solution of the integrals in (33) can be obtained by using the proposed series representation given in (6), but it would be a cumbersome expression containing many power series. It is very difficult to optimize this power series based expression for finding an optimized value of  $a$ . Therefore, we proposed to use the following asymptotic PDF for finding a simplified asymptotic upperbound of the average BER:

$$f_{I_1}(x) = Ax^{\delta-1}, \quad (34)$$

where  $\delta = \min(\alpha, \beta, \xi^2)$  and

$$A = \frac{\xi^2 P(\alpha, \beta, \xi^2)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(1 + \xi^2 - \delta)} \left( \frac{\alpha\beta}{A_0} \right)^\delta. \quad (35)$$

The asymptotic PDF is obtained from (6) by substituting  $n = 0$  and writing the remaining terms compactly.

Let us consider to solve the integral given in  $J_1$  term in (33) first. After some algebra and using (34), we get

$$\begin{aligned} J_1 &= \frac{A^2}{2((2-a)^2 \bar{\gamma})^{\delta/2}} \int_0^\infty x^{\delta-1} e^{-a^2 x^2 \bar{\gamma}} \\ &\times \gamma\left(\frac{\delta}{2}, (2-a)^2 \bar{\gamma} x^2\right) dx. \end{aligned} \quad (36)$$

After substituting  $h = x^2$  and using the relation:

$$\begin{aligned} \int_0^\infty h^{d-1} e^{-ph} \gamma(g, ch) dh &= \frac{c^g \Gamma(d+g)}{gp^{d+g}} \\ &\times {}_2F_1\left(g, d+g; g+1; -\frac{c}{p}\right), \end{aligned} \quad (37)$$

where  ${}_2F_1(\cdot, \cdot; \cdot; w)$  is the Gauss hypergeometric function, in (36), we get

$$J_1 = \frac{A^2 \Gamma(\delta)}{2\delta a^{2\delta} \bar{\gamma}^\delta} {}_2F_1\left(\frac{\delta}{2}, \delta; \frac{\delta}{2} + 1; -\frac{(2-a)^2}{a^2}\right). \quad (38)$$

The solution of the  $J_2$  term in (33) can be obtained by replacing  $a$  with  $2-a$  in (38), as

$$J_2 = \frac{A^2 \Gamma(\delta)}{2\delta (2-a)^{2\delta} \bar{\gamma}^\delta} {}_2F_1\left(\frac{\delta}{2}, \delta; \frac{\delta}{2} + 1; -\frac{a^2}{(2-a)^2}\right). \quad (39)$$

The BER upperbound of the scheme can be obtained by substituting (38) and (39) in (33):

$$\begin{aligned} P_e(\bar{\gamma}) &\leq \frac{P_c A^2 \Gamma(\delta)}{2\delta a^{2\delta} \bar{\gamma}^\delta} {}_2F_1\left(\frac{\delta}{2}, \delta; \frac{\delta}{2} + 1; -\frac{(2-a)^2}{a^2}\right) \\ &+ \frac{(1 - P_c) A^2 \Gamma(\delta)}{2\delta (2-a)^{2\delta} \bar{\gamma}^\delta} {}_2F_1\left(\frac{\delta}{2}, \delta; \frac{\delta}{2} + 1; -\frac{a^2}{(2-a)^2}\right). \end{aligned} \quad (40)$$

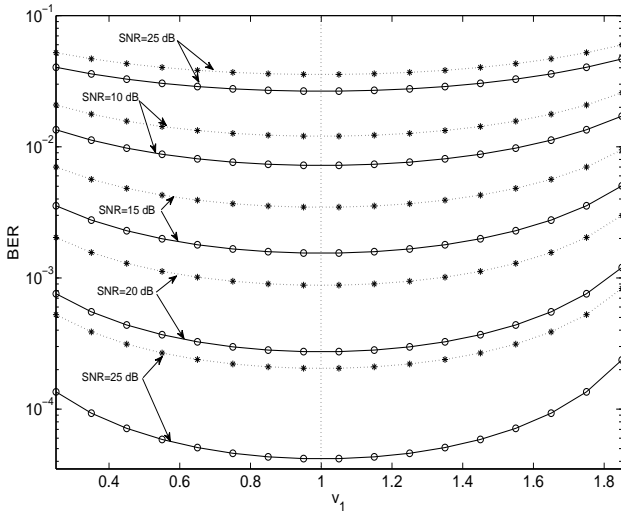


Fig. 1. BER versus  $v_1$  plots of the arbitrary beamforming scheme of Section III in  $2 \times 1$  FSO system over Gamma-Gamma fading channel with strong  $\alpha = 4.0$  and moderate  $\beta = 1.9$  atmospheric turbulence and pointing errors ( $\xi^2 = 4.5$ ,  $A_0 = 1$ ).

*Remark 5:* It can be seen from (40) that the proposed one bit feedback based beamforming scheme achieves the maximum possible spatial diversity of  $\min(\alpha, \beta, \xi^2)$ .

The optimized value of  $a$  can be found by numerically minimizing the asymptotic upperbound of the BER given in (40). It will be shown in the Section VII that the proposed one bit feedback based beamforming scheme provides improvement in the BER performance as compared to the repetition coding in  $2 \times 1$  FSO system with pointing error, despite of the imperfect feedback.

## VI. NUMERICAL RESULTS

A detailed discussion upon the proposed quantized feedback based beamforming technique is provided in this section. The SIM BPSK scheme is employed. The analytical plots are obtained by using the derived theoretical results in the previous sections; the simulation results are obtained by simulating the Gamma-Gamma fading channels with pointing errors through MATLAB<sup>TM</sup>. The derived power series based expressions are truncated to a finite number of terms for obtaining the analytical BER results. In all figures, if not stated otherwise, the SNR denotes the SNR per diversity branch, i.e.,  $\bar{\gamma} = \eta^2 E_s / (M^2 \sigma^2)$ , where  $M$  is the number of transmit apertures.

### A. Performance Evaluation of Arbitrary Beamforming Scheme

In Fig. 1, we plot the BER values of the arbitrary beamforming scheme given in (1) for a  $2 \times 1$  FSO system under the moderate ( $\alpha = 4.0$ ,  $\beta = 1.9$ ) and strong ( $\alpha = 4.2$ ,  $\beta = 1.4$ ) atmospheric turbulence, by varying the weight of the first transmit aperture, i.e.,  $0 < v_1 < 2$  with the constraint that  $v_1 + v_2 = 2$ . The pointing error is characterized by  $\xi^2 = 4.5$  and  $A_0 = 1$ . The BER values are obtained by using (18). The SNR is varied from 5-25 dB in the figure. As seen from the figure that  $v_1 = v_2 = 1$  minimizes the value of the BER under both atmospheric turbulences at all SNR values considered in

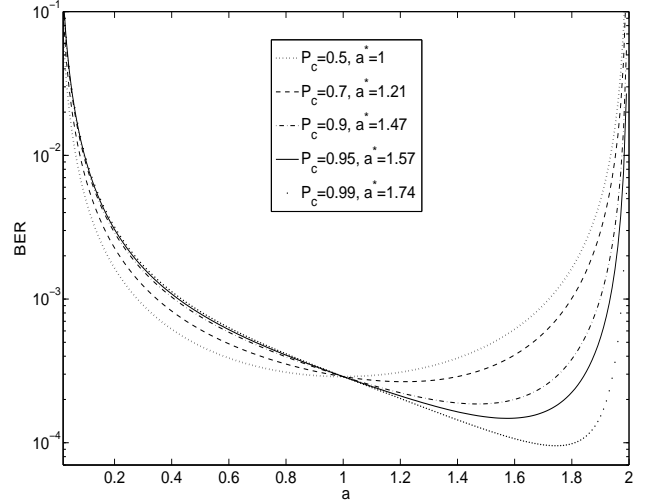


Fig. 2. Optimization of the weight value  $a$  by minimization of BER upperbound of (40) for  $P_c = 0.99, 0.95, 0.9, 0.7, 0.5$  over moderate atmospheric turbulence ( $\alpha = 4.0$ ,  $\beta = 1.9$ ) with pointing error parameters  $\xi^2 = 1.7$  and  $A_0 = 1$ ;  $a^*$  denotes the optimized value of  $a$  which minimizes the BER upperbound.

the figure. This fact corroborates the **Remark 3** that repetition coding outperforms any arbitrary beamforming scheme.

### B. Performance of the Proposed One Bit Feedback Based Beamforming Scheme in $2 \times 1$ FSO MISO System

In this subsection, we compare the repetition coding, trivial transmit aperture selection scheme, and the proposed one bit feedback based beamforming scheme (discussed in Section V) for  $2 \times 1$  FSO MISO system. It is assumed that there is a possibility of error in the feedback bit which is characterized by the probability of correct detection  $P_c \leq 1$ . The BER performance of the aforementioned schemes is evaluated over moderate atmospheric turbulence with pointing error parameters  $\xi^2 = 1.7$  and  $A_0 = 1$ .

The BER upperbound given in (40) is used to obtain Fig. 2, where the values of BER upperbound are plotted for  $0 < a < 2$  to find the optimized values of  $a$  for different feedback errors in a  $2 \times 1$  FSO system. The optimized values of  $a$  for different values of  $P_c$  over the moderate atmospheric turbulence are calculated by minimizing the BER upperbound; the SNR value is taken as 36 dB. It can be observed from the figure that the optimized value of  $a$ , i.e.,  $a^*$  moves towards two with increasing value of  $P_c$ . This is as expected intuitively.

Fig. 3 shows the BER versus SNR plots for various schemes for  $2 \times 1$  FSO system. The analytical BER versus SNR plots for the best transmit aperture selection based scheme and repetition coding are shown in the figure. In addition, the simulated performance of the proposed one bit feedback based beamforming scheme (discussed in Section V) is shown with optimized beamforming vectors in the figure. It is assumed that the feedback bit can be erroneously decoded by the transmitter. The values of probability of correct detection of the feedback bit are taken as  $P_c = 0.99, 0.95$ . The optimized beamforming

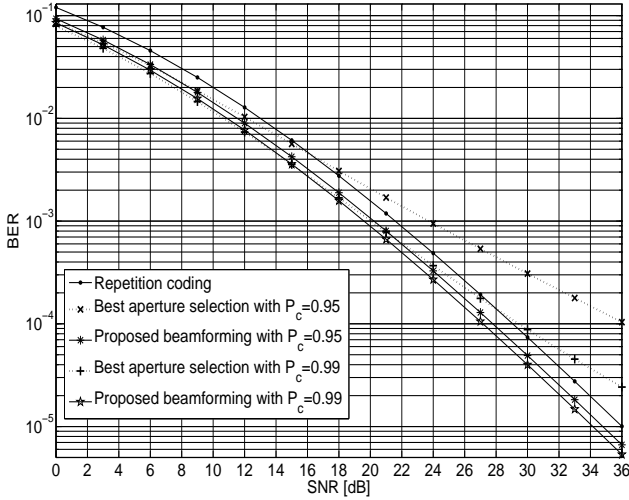


Fig. 3. The simulated BER versus SNR performance of the proposed one bit feedback scheme with optimized beamforming vectors (given in Section V) in  $2 \times 1$  FSO system employing SIM BPSK over moderate atmospheric turbulence perturbed Gamma-Gamma fading channels with pointing error parameters  $\xi^2 = 1.7$  and  $A_0 = 1$ . The analytical BER versus SNR plots of the repetition coding and best transmit aperture selection based scheme with errors in the feedback is also shown. The values of probability of correct detection of the feedback bit is taken as  $P_c = 0.95, 0.99$ .

vectors for the proposed one bit feedback based scheme for different values of  $P_c$  are obtained from (22) by using the values of  $\alpha^*$ , given in Fig. 2. The analytical values of the BER of the transmit aperture selection scheme with erroneous feedback bit are obtained by using (26) and those for the repetition coding are found by setting  $v_1 = v_2 = 1$  in (18). It can be seen from the figure that the trivial best transmit aperture selection based scheme is actually very sensitive to the feedback errors and its BER performance drastically degrades even for small errors in feedback. For example, for  $P_c = 0.95$ , the beamforming scheme performs poorer than the repetition coding for  $\text{SNR} > 16$  dB. Further, it can be seen from the figure that for very small error in feedback, i.e.,  $P_c = 0.99$ , the trivial scheme loses diversity performance. On the other hand, the proposed one bit feedback based beamforming scheme achieves the same diversity order as that of the repetition coding scheme irrespective of the error in the feedback bit. Therefore, the proposed improved one bit beamforming scheme given in Section V enables the  $2 \times 1$  FSO system to achieve the full diversity (that of the repetition coding) despite of the feedback errors, as shown in the figure. In addition, the proposed scheme outperforms the repetition coding at all SNR values considered in Fig. 3. Moreover, for meaningful values of the BER, the proposed scheme significantly outperforms the transmit aperture selection scheme as can be seen from the figure. For example at  $\text{BER} = 10^{-4}$ , the proposed scheme provides approximately 8 dB and 2 dB SNR gains as compared to the best aperture selection scheme and the repetition coding for  $P_c = 0.95$  and  $P_c = 0.99$ , respectively. Further, the plots in the figure corroborate the diversity analysis results obtained in Sections III-V.

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