Coloring non-crossing strings

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Abstract

For a family $\mathcal{F}$ of geometric objects in the plane, define $\chi(\mathcal{F})$ as the least integer ℓ such that the elements of $\mathcal{F}$ can be colored with ℓ colors, in such a way that any two intersecting objects have distinct colors. When $\mathcal{F}$ is a set of pseudo-disks that may only intersect on their boundaries, and such that any point of the plane is contained in at most k pseudo-disks, it can be proved that $\chi(\mathcal{F}) \leqslant 3k/2 + o(k)$ since the problem is equivalent to cyclic coloring of plane graphs. In this paper, we study the same problem when pseudo-disks are replaced by a family $\mathcal{F}$ of pseudo-segments (a.k.a. strings) that do not cross. In other words, any two strings of $\mathcal{F}$ are only allowed to "touch" each other. Such a family is said to be k-touching if no point of the plane is contained in more than k elements of $\mathcal{F}$. We give bounds on $\chi(\mathcal{F})$ as a function of k, and in particular we show that k-touching segments can be colored with $k + 5$ colors. This partially answers a question of Hliněný (1998) on the chromatic number of contact systems of strings.

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1 Introduction

For a family $\mathcal{F} = \{S_1, \ldots, S_n\}$ of subsets of a set Ω, the *intersection graph* $G(\mathcal{F})$ of $\mathcal{F}$ is defined as the graph with vertex-set $\mathcal{F}$, in which two vertices are adjacent if and only if the corresponding sets have non-empty intersection.

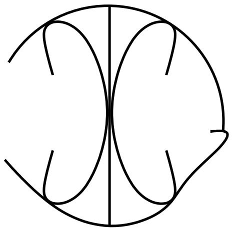
For a graph G, the *chromatic number* of G, denoted $\chi(G)$, is the least number of colors needed in a proper coloring of G (a coloring such that any two adjacent vertices have distinct colors). When talking about a proper coloring of a family $\mathcal{F}$ of subsets of a given set, we implicitly refer to a proper coloring of the intersection graph of $\mathcal{F}$, thus the chromatic number $\chi(\mathcal{F})$ is defined in a natural way.

The chromatic number of families of geometric objects in the plane have been extensively studied since the sixties [2, 13, 18, 19, 20]. Since it is possible to construct sets of pairwise intersecting (straight-line) segments of any size, the chromatic number of sets of segments in the plane is unbounded in general. However, Erdős conjectured that triangle-free intersection graphs of segments in the plane have bounded chromatic number (see [12]). This was recently disproved [23]. The conjecture of Erdős initiated the study of the chromatic number of families of geometric objects in the plane as a function of their *clique number*, the maximum size of subsets of the family that pairwise intersect [8]. In this paper, we consider families of geometric objects in the plane for which the chromatic number only depends on local properties of the families, such as the maximum number of objects containing a given point of the plane.

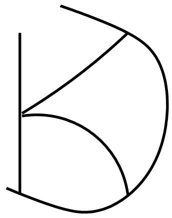
Consider a set $\mathcal{F} = \{\mathcal{R}_1, \ldots, \mathcal{R}_n\}$ of pseudo-disks (subsets of the plane which are homeomorphic to a closed disk) such that the intersection of the interiors of any two pseudo-disks is empty. Let $\mathcal{H}_\mathcal{F}$ be the planar hypergraph with vertex set $\mathcal{F}$, in which the hyperedges are the maximal sets of pseudo-disks whose intersection is non-empty. A proper coloring of $\mathcal{F}$ is equivalent to a coloring of $\mathcal{H}_\mathcal{F}$ in which all the vertices of each hyperedge have distinct colors. If every point is contained in at most k pseudo-disks, Borodin conjectured that there exists such a coloring of $\mathcal{H}_\mathcal{F}$ with at most $\frac{3}{2}k$ colors [3]. It was recently proved that this conjecture holds asymptotically [1] (not only in the plane, but also on any fixed surface). As a consequence, $\mathcal{F}$ can be properly colored with $\frac{3}{2}k+o(k)$ colors.

It seems natural to investigate the same problem when pseudo-disks are replaced by *pseudo-segments*. These are continuous injective functions from $[0, 1]$ to $\mathbb{R}^2$ and are usually referred to as *strings*. Consider a set $\mathcal{S} = \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ of such strings. We will always assume that any two strings intersect in a finite number of points. We say that $\mathcal{S}$ is *touching* if no pair of strings of $\mathcal{S}$ cross, and that it is k*-touching* if furthermore at most k strings can "touch" in any point of the plane, i.e., any point of the plane is contained in at most k strings (see Figure 1(a) for an example).

Note that the family of all touching sets of strings contains all *contact systems of strings*, defined as sets of strings such that the interior of any two strings have empty intersection. In other words, if c is a contact point in the interior of a string s, all the strings containing c distinct from s end at c. In [15], Hliněný studied contact system of strings such that all the strings ending at c leave from the same side of s. Such a



(a)



(b)

Figure 1: (a) A 3-touching set of strings $\mathcal{S}_1$ with $G(\mathcal{S}_1) \cong K_5$. (b) A one-sided 3-contact representation of curves $\mathcal{S}_2$ with $G(\mathcal{S}_2) \cong K_4$.

representation is said to be *one-sided* (see Figure 1(b) for an example). It was proved in [15] that if a contact system of strings is k-touching and every contact point is one-sided, then the strings can be colored with $2k$ colors.

In this paper, our aim is to study k-touching sets of strings in their full generality. Observe that if $\mathcal{S}$ is k-touching, k might be much smaller than the maximum degree of $G(\mathcal{S})$. However, based on the cases of pseudo-disks and contact system of strings, we conjectured the following in the conference version of this paper [5]:

Conjecture 1.1. For some constant $c > 0$, any k-touching set of strings can be colored with ck colors.

This conjecture was subsequently proved by Fox and Pach [9], who showed that any k-touching set of strings can be colored with $6ek+1$ colors (where e is the base of the natural logarithm). In Section 2, we show how to slightly improve their bound for small values of k. We also show that for any odd k, the clique on $\frac{9}{2}(k-1)$ vertices can be represented as a set of k-touching strings, so the best possible constant c in Conjecture 1.1 is between 4.5 and $6e \approx 16.3$.

In Section 3, we give improved bounds when any two strings can intersect a bounded number of times. In Section 4, we restrict ourselves to contact systems of strings where any two strings intersect at most once (called 1-intersecting), which were previouly studied by Hliněný [15]. He asked whether there is a constant c such that every one-sided 1-intersecting k-touching contact system of strings is $(k+c)$-colorable. We prove that they are $(\frac{4k}{3}+6)$-colorable, and that every k-touching contact system of straight-line segments is $(k+5)$-colorable. Note that we do not need our contact systems to be one-sided.

Before giving general bounds, let us first mention two classical families of touching strings for which coloring problems are well understood.

If a k-touching set of strings has the property that the interior of each string is disjoint from all the other strings, then each string can be thought of as an edge of some (planar) graph with maximum degree k. By a classical theorem of Shannon, the strings can then be colored with $3k/2$ colors. If moreover, any two strings intersect at most once, then they can be colored with $k+1$ colors by a theorem of Vizing (even with k colors whenever $k \geqslant 7$, using a more recent result of Sanders and Zhao [24]). In this sense, all the problems considered in this article can be seen as a extension of edge-coloring of planar graphs.

An *x-monotone string* is a string such that every vertical line intersects it in at most one point. Alternatively, it can be defined as the curve of a continuous function from an interval of $\mathbb{R}$ to $\mathbb{R}$. Sets of k-touching x-monotone strings are closely related to bar k-visibility graphs. A *bar k-visibility graph* is a graph whose vertex-set consists of horizontal segments in the plane (bars), and two vertices are adjacent if *and only if* there is a vertical segment connecting the two corresponding bars, and intersecting no more than k other bars. It is not difficult to see that the graph of any set of k-touching x-monotone strings is a spanning subgraph of some bar $(k-2)$-visibility graph, while any bar $(k-2)$-visibility graph can be represented as a set of k-touching x-monotone strings. Using this correspondence, it directly follows from [4] that k-touching x-monotone strings are $(6k-6)$-colorable, and that the complete graph on $4k-4$ vertices can be represented as a set of k-touching x-monotone strings. If the left-most point of each x-monotone string intersects the vertical line $x = 0$, then it directly follows from [7] that the strings can be colored with $2k-1$ colors (and the complete graph on $2k-1$ vertices can be represented by k-touching x-monotone strings in this specific way).

2 General bounds

Before proving our first results on the structure of sets of k-touching strings in general, we make two important observations:

Observation 2.1. *The family of intersection graphs of 2-touching strings is exactly the class of planar graphs.*

The class of planar graphs being exactly the class of intersection graphs of 2-touching pseudo-disks (see [17]) it is clear that planar graphs are intersection graphs of 2-touching strings (by taking a connected subset of the boundaries of each pseudo-disk). Furthermore, every intersection graph of 2-touching strings is contained in an intersection graph of 2-touching pseudo-disks, and is thus planar. Indeed, it is easy given a set of 2-touching strings $\mathcal{S} = \{\mathcal{C}_1, \dots, \mathcal{C}_n\}$ to draw a set of 2-touching pseudo-disks $\mathcal{F} = \{\mathcal{R}_1, \dots, \mathcal{R}_n\}$ such that $\mathcal{C}_i \subset \mathcal{R}_i$ for every $i \in [1, n]$.

Observation 2.2. *We can assume without loss of generality that the strings in any set of k-touching strings are polygonal lines (i.e. each string is the union of finitely many straight-line segments) and that no endpoint of a string of $\mathcal{S}$ coincides with an intersection between strings of $\mathcal{S}$.*

To see this, take a set $\mathcal{S}$ of k-touching strings and consider the following graph G: the vertices are the contact points and the endpoints of the strings of $\mathcal{S}$, and the edges connect two points if they are consecutive in some string of $\mathcal{S}$. The resulting graph is planar, but might contain multiple edges. Subdivide each edge once, and observe that the resulting graph H is a simple planar (finite) graph, and each string of $\mathcal{S}$ is the union of some edges of H. Since H is a simple planar graph, by Fáry's theorem [6] it has an equivalent drawing in which all the edges are (straight-line) segments. If some endpoint of a string of $\mathcal{S}$ coincides with an intersection x between strings of $\mathcal{S}$, one can prolong each string ending at x with a small enough segment, and Observation 2.2 follows.

The following result was proved by Fox and Pach [9] (this shows Conjecture 1.1). In the following, e is the base of the natural logarithm.

Theorem 2.3 ([9])**.** *Any k-touching set of strings is $(6ek+1)$-colorable.*

Their theorem is a direct consequence of the following bound on the number of edges in a graph represented by a set of k-touching strings. Their proof is inspired by the probabilistic proof of the Crossing Lemma.

Lemma 2.4 ([9])**.** *Any graph represented by a set of n k-touching strings has less than $3ekn$ edges.*

When $k = 3$, their proof can easily be optimized to show that the number of edges is less than $12n$. Hence, every such graph has a vertex with degree less than 24. These graphs are thus 23-degenerate and have chromatic number at most 24. We now show how to modify their proof to slightly improve this bound.

Theorem 2.5. *Any 3-touching set of strings is 19-colorable.*

This is a direct consequence of the following lemma.

Lemma 2.6. *Any graph represented by a set of n 3-touching strings has at most $\frac{6}{7}(6+\sqrt{22})n \approx 9.16\,n$ edges.*

Proof. Let $\mathcal{S}$ be a set of n 3-touching strings, and let m be the number of edges in the corresponding intersection graph G. By Observation 2.2, we can assume that each string of $\mathcal{S}$ is a polygonal line (a union of finitely many segments) and that no endpoint of a string of $\mathcal{S}$ coincides with an intersection between strings of $\mathcal{S}$. Consider a point p contained in three different strings s_0, s_1, s_2. Then there is a small disk D centered in p that only intersects the strings s_0, s_1, s_2, and is such that the boundary C of D intersects each s_i (for $i = 0, 1, 2$) in exactly two points, say p_i, p'_i. Assume that walking around C in clockwise order, we see $p_0, p'_0, p_1, p'_1, p_2, p'_2$. Then for $i = 0, 1, 2$, we replace $s_i \cap D$ by a new string s'_i between p_i and p'_i as follows. For each $i = 0, 1, 2$, let q_i be a point of D that is after p'_i and before p_{i+1} in clockwise order (with indices modulo 3). For two points x, y, let $S(x, y)$ denote the (straight-line) segment between x and y. Then we replace $s_i \cap D$ (the portion of s_i between p'_i and p_i) by the concatenation of $S(p'_i, q_i)$, $S(q_i, q_{i-1})$, and $S(q_{i-1}, p_i)$ (see Figure 2). Note that the resulting set of strings is still 3-touching, and the intersection graph remains unchanged. Repeating this operation if necessary, we can assume without loss of generality that for any three strings s_0, s_1, s_2 as above, we see $p_0, p'_0, p'_1, p'_2, p_2, p_1$ when walking around C in clockwise order. In this case we say that s_1 is "sandwiched" between s_0 and s_2 at p (see Figure 3, left).

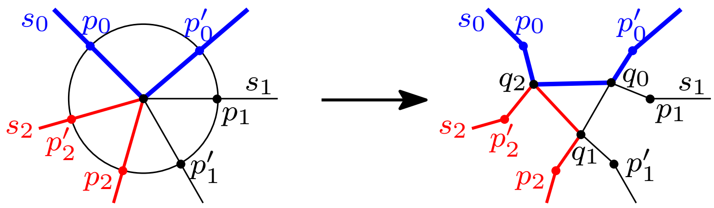


Figure 2: A local modification turning one 3-touching point into three 2-touching points, without changing the intersection graph.

We now define two spanning subgraphs G_1 and G_0 of G as follows. Two strings a and c are adjacent in G_1 if they intersect and for every intersection point p of a and c, there exists a third string b that is sandwiched between a and c at p. For every pair of strings a and c adjacent in G_1, let $P_1(a, c)$ be an arbitrarily chosen intersection point between a and c. By definition, there is a unique string that is sandwiched between a and c at $P_1(a, c)$.

Two strings a and c are adjacent in G_0 if they are adjacent in G but not in G_1, i.e., if there exists an intersection point $P_0(a, c)$ of a and c such that either $P_0(a, c)$ is not contained in another string or $P_0(a, c)$ is contained in a third string b that is not sandwiched between a and c. For $i = 0, 1$, the edge-set of G_i is denoted by E_i, and the cardinality of E_i is denoted by m_i.

Claim 2.7. $7m_0 \geqslant m + 6n$

For each edge $ab \in E_1$, assume that ab gives a charge of 1 to the unique string c sandwiched between a and b at $P_1(a, b)$. Let $N_0(c)$ be the set of neighbors of c in G_0. The total charge $\rho(c)$ received by c is at most the number of pairs of vertices $x, y \in N_0(c)$ such that in $\mathcal{S} \setminus \{c\}$, $P_0(x, y)$ is a 2-contact point. If we modify the set of strings intersecting c so as to only preserve those 2-touching points (all the other pairs of strings are made disjoint), we obtain a planar graph with vertex-set $N_0(c)$ and with at least $\rho(c)$ edges. It follows that $\rho(c) \leqslant 3|N_0(c)| - 6$. Summing for all strings c, we obtain that the total charge $\sum_{c\in\mathcal{S}} \rho(c) = m_1 \leqslant 6m_0 - 6n$. Since $m_0 + m_1 = m$, we have $7m_0 \geqslant m + 6n$, as claimed.

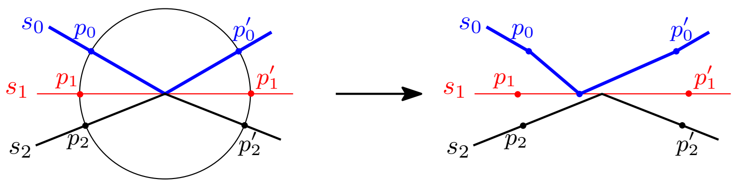


Figure 3: A local modification preserving adjacency in the graph G_0.

We are now ready to prove the lemma. We select each string of $\mathcal{S}$ uniformly at random with probability p (to be chosen later). In the subset of chosen strings, we slightly modify each 3-touching point x as follows. There is a small disk D with boundary C centered in x that only intersects the three strings s_0, s_1, s_2 containing x, and (with the same notation as before), we can assume without loss of generality that we see $p_0, p_0', p_1', p_2', p_2, p_1$ when walking around C in clockwise order. Then for some point q on s_1, close to x, we replace $s_0 \cap D$ by the concatenation of the segments $S(p_0, q)$ and $S(q, p_0')$ (see Figure 3). Note that this preserves all edges in G_0. Let $\mathcal{S}'$ be the new set of strings. For each edge $ab \in E_0$, the probability that ab appears as an edge in $\mathcal{S}'$ is p^2, and for each edge $ab \in E_1$, the probability that ab appears as an edge in $\mathcal{S}'$ is at least the probability that a and b were both selected and the string c sandwiched between a and b at $P_1(a, b)$ was not selected, which is $p^2(1-p)$. The expected number of vertices of the graph represented by $\mathcal{S}'$ is then pn and its expected number of edges is at least

$$\begin{aligned} m_0p^2 + m_1p^2(1-p) &= p^2\left(m_0 + m_1(1-p)\right) \\ &= p^2\left(m_0p + m(1-p)\right) \\ &\geqslant \tfrac{p^2}{7}\left((m+6n)p + 7m(1-p)\right) \\ &\geqslant \tfrac{p^2}{7}(m(7-6p) + 6pn) \end{aligned}$$

By Observation 2.1, the graph is planar and therefore, $\frac{p^2}{7}(m(7-6p)+6pn) \leqslant 3pn$. This can be rewritten as $m \leqslant n\,\frac{21-6p^2}{p(7-6p)}$. Taking $p = 3-\sqrt{11/2}$, we obtain $m \leqslant \frac{6}{7}(6+\sqrt{22})n$. $\square$

The ideas of Lemma 2.6 can be used to slightly improve the multiplicative constant in Theorem 2.3 for all k. Since our improvement is minor (we obtain a bound of $6k \times 2.686$ instead of $6k \times 2.718$), we omit the details.

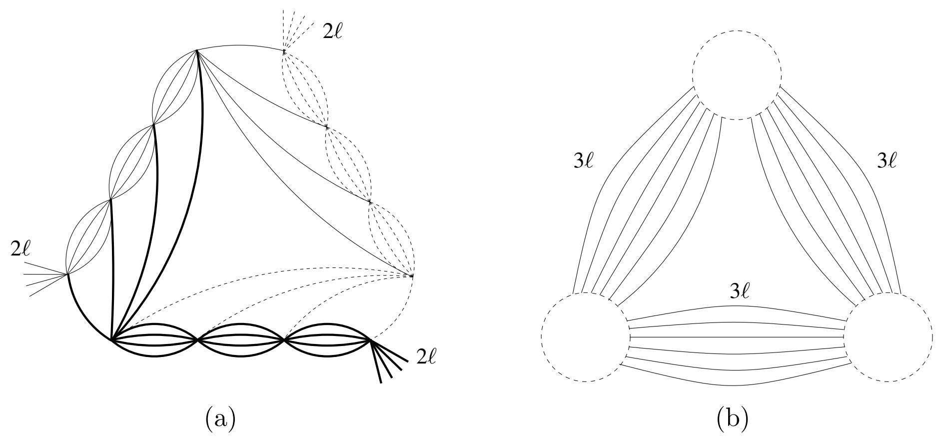


Figure 4: (a) The construction of a 2ℓ-sun. (b) A set $\mathcal{S}$ of k-touching strings requiring $\lceil \frac{9k}{2} \rceil - 5$ distinct colors (each dashed circle represents a 2ℓ-sun).

We now show that the constant c in Conjecture 1.1 is at least $\frac{9}{2}$.

Theorem 2.8. *For every odd $k \geqslant 1$, $k = 2\ell + 1$, there exists a set of k-touching strings $\mathcal{S}_k$ such that the strings of $\mathcal{S}_k$ pairwise touch and such that $|\mathcal{S}_k| = 9\ell = 9(k-1)/2$. Thus $\chi(\mathcal{S}_k) = 9\ell = \lceil \frac{9k}{2} \rceil - 5$.*

Proof. Consider n touching strings $s_1, \dots, s_n$ that all intersect n points $c_1, \dots, c_n$ in the same order (see the set of bold strings in Figure 4(a) for an example when $n = 4$), and call this set of strings an *n-braid.* For some $\ell > 0$, take three 2ℓ-braids $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, and for $i = 1, 2, 3$, connect each of the strings of $\mathcal{S}_i$ to a different intersection point of $\mathcal{S}_{i+1}$ (with indices taken modulo 3), while keeping the set of strings touching (see Figure 4(a)). We call this set of touching strings a *2ℓ-sun.* Observe that a 2ℓ-sun contains 6ℓ strings that pairwise intersect, and that each intersection point contains at most $2\ell + 1$ strings. Moreover, each of the 6ℓ strings has an end that is incident to the infinite face.

We now consider three 2ℓ-suns $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$. Each of them has 6ℓ strings with an end incident to the outerface. For each $i = 1, 2, 3$, we arbitrarily divide the strings leaving $\mathcal{R}_i$ into two sets of 3ℓ consecutive strings, say $R_{i,i+1}$ and $R_{i,i-1}$. For each $i = 1, 2, 3$, we now take the strings of $R_{i,i+1}$ and $R_{i+1,i}$ by pairs (one string in $R_{i,i+1}$, one string in $R_{i+1,i}$), and identify these two strings into a single string. This can be made in such a way that the resulting set of $(6 \times 3\ell)/2 = 9\ell$ strings is still $(2\ell+1)$-touching (see Figure 4(b), where the three 2ℓ-suns are represented by dashed circles, and only the portion of the strings leaving the suns is displayed for the sake of clarity). Hence we obtain a k-touching set of $\lceil \frac{9k}{2} \rceil - 5$ strings that pairwise intersect, as desired. ☐

3 μ-intersecting strings

Let $\mathcal{S}$ be a k-touching set of strings. The set $\mathcal{S}$ is said to be *μ-intersecting* if any two strings intersect in at most μ points. We denote by $H(\mathcal{S})$ the multigraph associated to $\mathcal{S}$: the vertices of $H(\mathcal{S})$ are the strings of $\mathcal{S}$, and two strings with t common points correspond to two vertices connected by t edges in $H(\mathcal{S})$. Note that the intersection graph $G(\mathcal{S})$ of $\mathcal{S}$ is the simple graph underlying $H(\mathcal{S})$.

We prove the following result (which only supersedes Theorem 2.3 for $\mu \leqslant 5$).

Theorem 3.1. *Any k-touching set $\mathcal{S}$ of μ-intersecting strings can be properly colored with $3\mu k$ colors.*

Again, the proof is based on an upper bound on the number of edges of such graphs.

Lemma 3.2. *If $\mathcal{S}$ is a k-touching set of n μ-intersecting strings, then $H(\mathcal{S})$ (and so, $G(\mathcal{S})$) has less than $\frac{3}{2}\mu k n$ edges.*

Proof. Let n denote the number of strings of $\mathcal{S}$, and let N denote the number of intersection points of $\mathcal{S}$. Let us denote by $d(c)$ the number of strings containing an intersection point c (for any c, $2 \leqslant d(c) \leqslant k$ by definition).

By Observation 2.2, we can assume that each string of $\mathcal{S}$ is a polygonal line (a union of finitely many segments) and that no endpoint of a string of $\mathcal{S}$ coincides with an intersection between strings of $\mathcal{S}$. Let us slightly modify $\mathcal{S}$ in order to obtain a set $\mathcal{S}'$ of 2-touching and μ-intersecting strings. For that, repeat the following operation while there exists an intersection point c with $d(c) > 2$. There is a small disk D centered in c such that D only intersects the strings containing c, and for any such string s, $s \cap D$ is the union of two segments. Pick a string s_1 containing c such that the angle between its two segments at c is minimal, and let s_2 be a string containing c, distinct from s_1, such that a face f of $D - \mathcal{S}$ is bounded by s_1, s_2 and the boundary of C. Let p_1, p_1' be the intersection of s_1 and the boundary of D, and let q be a point of $s_2 \cap f$ distinct from c (and close to c). Then we replace $D \cap s_1$ by the concatenation of the straight-line segments $S(p_1, q)$ and $S(q, p_1')$ (see Figure 5). This is similar to the second modification used in Theorem 2.5 and illustrated in Figure 3, which was restricted to the case where c is contained in exactly three strings.

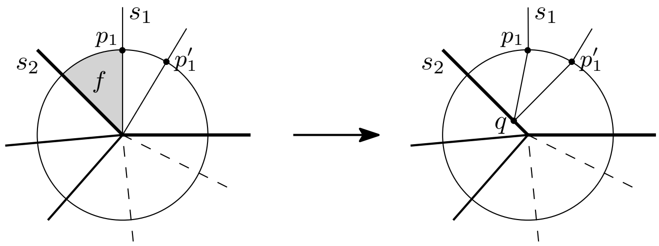


Figure 5: A local modification reducing the number of strings containing a given point.

Each intersection point c in $\mathcal{S}$ corresponds to a set X_c of intersection points in $\mathcal{S}'$. Let N' be the number of intersection points in $\mathcal{S}'$. Since $\mathcal{S}'$ is 2-touching, N' is also the number of edges of the multigraph $H(\mathcal{S}')$. By construction, each X_c has size exactly $d(c)-1$, hence $N' = \sum_c |X_c| = \sum_c (d(c)-1)$. By Observation 2.1, the graph $G(\mathcal{S}')$ and the multigraph $H(\mathcal{S}')$ are planar, and since $\mathcal{S}'$ is μ-intersecting, $H(\mathcal{S}')$ is a planar multigraph in which each edge has multiplicity at most μ, therefore it contains $N' \leqslant (3n-6)\mu$ edges. As $d(c) \leqslant k$ for any intersection point c in $\mathcal{S}$, we have

$$\sum_c d(c)(d(c)-1) \;\leqslant\; kN' \;\leqslant\; (3n-6)\mu k \;<\; 3\mu k n.$$

Finally, since the number of edges of $H(\mathcal{S})$ is precisely $\frac{1}{2}\sum_c d(c)(d(c)-1)$, we have that $H(\mathcal{S})$ has less than $\frac{3}{2}\mu k n$ edges, as desired. □

In particular, if a k-touching set $\mathcal{S}$ of strings is such that any two strings intersect in at most one point, Theorem 3.1 yields a bound of $3k$ for the chromatic number of $\mathcal{S}$. We suspect that it is far from tight:

Conjecture 3.3. There is a constant $c > 0$, such that every k-touching set of 1-intersecting strings can be colored with $k + c$ colors.

In the next section, we show that this conjecture holds for k-touching (straight-line) segments, a special case of 1-intersecting strings. It is interesting to note that even though the bound for k-touching μ-intersecting graphs in Conjecture 1.1 and Theorem 2.3 does not depend on μ, the chromatic number of these graphs has some connection with μ: sets of strings with $\mu = 1$ have chromatic number at most $3k$, whereas there exists sets of strings with large μ and chromatic number at least $\frac{9}{2}(k-1)$.

Figure 6: (a) A 3-touching set of 1-intersecting strings requiring 7 colors (b) A k-touching set of 1-intersecting strings requiring $k+2$ colors (here $k = 4$).

Note that the constant c in Conjecture 3.3 is at least 4. Figure 6(a) depicts a 3-touching set of seven 1-intersecting strings, in which any two strings intersect. Hence, this set requires seven colors. However this construction does not extend to k-touching sets with $k \geqslant 4$, it might be that the constant is smaller for higher k. In Figure 6(b), the k-touching set $\mathcal{S}_k$ contains $k+2$ non-crossing strings, and is such that any two strings intersect. Hence, $k+2$ colors are required in any proper coloring.

4 1-Intersecting contact system of strings

In this section, all the sets of strings we consider are 1-intersecting (any two strings intersect in at most one point). An interesting example of 1-intersecting set of strings is any family of non-crossing (straight-line) segments in the plane. Such a family is also known as a *contact system of segments*, and has been studied in [10], where the authors proved that any bipartite planar graph has a contact representation with horizontal and vertical segments.

More generally, a *contact system of strings* is a family of strings such that the interiors of the strings are pairwise non-intersecting. In other words, if c is a contact point in the interior of a string s, all the strings containing c and distinct from s end at c. A contact point p is a *peak* if every string containing p has an end at p. Otherwise, that is if p is an interior point of a string s and an end for all the other strings containing p, p is *flat*. A flat contact point p is *one-sided* if all the strings ending at p are on the same side of the unique string whose interior contains p. A contact system of strings in which every flat contact point is one-sided is also said to be one-sided.

It was proved by Hliněný [16] that the intersection graph of any one-sided 2- or 3-touching set of segments is planar. Note that as a 2-touching set of segments is always one-sided, it is also 4-colorable. In [22], Ossona de Mendez proved that it is NP-complete to determine whether a 2-touching set of segments is 3-colorable.

In [15], Hliněný studied the clique and chromatic numbers of one-sided k-touching contact systems of strings. He proved that the maximal clique in this class is K_{k+1} and that the graphs in this class are $2k$-colorable. He also asked the following: is there a constant c such that if a contact system of strings is k-touching, 1-intersecting, and one-sided, then it is $(k+c)$-colorable? Note that Conjecture 3.3 would imply a positive answer to this question.

In the first part of this section, we prove that 1-intersecting and k-touching contact systems of strings are $\big(\lceil \frac{4}{3}k \rceil + 6\big)$-colorable. In the second part, we show that any k-touching contact system of segments is $(k+5)$-colorable. Note that we do not assume the contact systems to be one-sided (but we also show that adding this assumption slightly improves the additive constants in our results).

Theorem 4.1. *For any $k \geqslant 3$, any 1-intersecting k-touching contact system of strings can be colored with $\lceil \frac{4}{3}k \rceil + 6$ colors.*

As in Theorem 2.3, the result is a consequence of a bound on the degeneracy of these graphs:

Lemma 4.2. *For any $k \geqslant 3$, if $\mathcal{S}$ is a 1-intersecting k-touching contact system of strings, then $G(\mathcal{S})$ contains a vertex of degree at most $\lceil \frac{4}{3}k \rceil + 5$.*

Proof. Assume that there is a counterexample, i.e. a 1-intersecting k-touching contact system $\mathcal{S}$ of n strings such that $G(\mathcal{S})$ has minimum degree at least $\lceil \frac{4}{3}k \rceil + 6$. In particular, $G(\mathcal{S})$ has $m \geqslant n(\frac{2}{3}k + 3)$ edges. We take a counterexample for which n is minimal, and with respect to this, m is maximal. Observe that $G(\mathcal{S})$ is connected, since otherwise by minimality of n, some connected component would have a vertex of degree at most $\lceil \frac{4}{3}k \rceil + 5$, a contradiction. Observe also that if some string of $\mathcal{S}$ has at most one contact point, then the corresponding vertex of $G(\mathcal{S})$ has degree at most $k - 1 \leqslant \lceil \frac{4}{3}k \rceil + 5$, a contradiction. This implies that every string of $\mathcal{S}$ has at least two contact points. As a consequence, we can also assume that the two ends of each string of $\mathcal{S}$ are contact points (if not, delete the portion of a string between a free end and its first contact point).

Let $H(\mathcal{S})$ be the plane graph whose vertices are the contact points of $\mathcal{S}$, whose edges link two contact points if and only if they are consecutive on a string of $\mathcal{S}$, and whose faces are the connected regions of $\mathbb{R}^2 \setminus \mathcal{S}$.

Let p_i and f_i be the number of contact points of exactly i strings of $\mathcal{S}$ that are respectively peaks and flat. Let us denote by c the total number of contact points, and note that $c = \sum_{i=2}^{k}(p_i + f_i)$. By counting the number of ends of a string of $\mathcal{S}$ in two different ways, we obtain that:

$$2n = \sum_{i=2}^{k} ip_i + \sum_{i=2}^{k}(i-1)f_i \tag{1}$$

Consider a one-sided flat contact point p and let s be the unique string such that p is an interior point of s. If we draw a small open disk D containing p, a unique face f of $H(\mathcal{S})$ has the property that $f \cap D$ is incident to s, and to no other string containing p. We denote this face f of $H(\mathcal{S})$ by $F(p)$. Remark that since $\mathcal{S}$ is 1-intersecting, any face f of $H(\mathcal{S})$ contains at least $|F^{-1}(f)|+3$ vertices, thus at least $|F^{-1}(f)|$ edges can be added to $H(\mathcal{S})$ (inside f) with the property that $H(\mathcal{S})$ remains planar. Hence in total, one can add as many edges to $H(\mathcal{S})$ as the number of one-sided contact points, while keeping $H(\mathcal{S})$ planar. Since every flat 2-contact point is one-sided, and every planar graph on c vertices has at most $3c-6$ edges, it follows that $H(\mathcal{S})$ has at most $3c-6-f_2$ edges. As a consequence, the sum of the degrees of the vertices of $H(\mathcal{S})$ is:

$$\sum_{i=2}^{k} ip_i + \sum_{i=2}^{k}(i+1)f_i \leqslant 2\cdot(3c-6-f_2).$$

By the definition of c, this is equivalent to:

$$\sum_{i=2}^{k}(i-6)p_i + \sum_{i=2}^{k}(i-5)f_i \leqslant -12-2f_2 \tag{2}$$

Since any pair of strings in $\mathcal{S}$ intersects at most once, the number of edges in $G(\mathcal{S})$ satisfies the following equation.

$$m = \sum_{i=2}^{k}\binom{i}{2}(p_i+f_i) \tag{3}$$

Our goal is to use (1), (2), and (3) to prove that m is less than $n(\frac{2}{3}k+3)$, which is a contradiction. To prove this, we will see that in order to maximize m, we have to set all the values of f_i and p_i to zero, except for p_2, f_3, p_k, f_k (in other words, the weight has to be concentrated on the extremal variables). Once this is proved, bounding m will be significantly simpler.

Let us consider the linear program (LP1) defined on variables p_i and f_i with values in $\mathbb{R}^+$ such that the equation (1) and the inequality (2) are satisfied, and such that the value m defined by (3) is maximized. Here n is considered to be a constant (it is not a variable of the linear program). Note that the solution m^* of this problem is clearly an upper bound of the number of edges of $G(\mathcal{S})$.

Claim 4.3. *The optimal solutions of (LP1) are such that* $f_2 = 0$.

If $f_2 \neq 0$, take a small $\epsilon > 0$ and replace f_2 by $f_2-\epsilon$ and f_3 by $f_3+\epsilon/2$. Then (1) remains valid, inequality (2) still holds (both sides are increased by 2ϵ), while (3) is increased by $\epsilon/2$. This concludes the proof of the claim.

Claim 4.4. *The optimal solutions of (LP1) are such that* $f_i = 0$*, for every* $4 \leqslant i \leqslant k-1$.

If for some $4 \leqslant i \leqslant k-1$, $f_i \neq 0$, choose a small $\epsilon > 0$ and replace (i) f_3 by $f_3 + \epsilon\,\frac{(k-i)}{k-3}$; (ii) f_i by $f_i - \epsilon$; and (iii) f_k by $f_k + \epsilon\,\frac{i-3}{k-3}$. Then (1) remains valid, inequality (2) still holds (the left-hand side and the right-hand side remain unchanged), while the value of m in (3) is increased by $\frac{\epsilon}{k-3}(3(k-i)-\binom{i}{2}(k-3)+\binom{k}{2}(i-3))$. The function $g : i \mapsto 3(k-i)-\binom{i}{2}(k-3)+\binom{k}{2}(i-3)$ is a (concave) parabola with $g(4) = \binom{k}{2}-3k+6 > 0$ (recall that $4 \leqslant i \leqslant k-1$, so $k \geqslant 5$) and $g(k) = 0$, so it is positive in the interval $[4, k-1]$. This concludes the proof of the claim.

Claim 4.5. *The optimal solutions of (LP1) are such that* $p_i = 0$*, for every* $3 \leqslant i \leqslant k-1$.

If for some $3 \leqslant i \leqslant k-1$, $p_i \neq 0$, choose a small $\epsilon > 0$ and replace (i) p_2 by $p_2 + \epsilon\,\frac{(k-i)}{k-2}$; (ii) p_i by $p_i - \epsilon$; and (iii) p_k by $p_k + \epsilon\,\frac{i-2}{k-2}$. Then (1) remains valid, inequality (2) still holds (the left-hand side and the right-hand side remain unchanged), while the value of m in (3) is increased by $\frac{\epsilon}{k-2}((k-i)-\binom{i}{2}(k-2)+\binom{k}{2}(i-2))$. The function $g : i \mapsto (k-i)-\binom{i}{2}(k-2)+\binom{k}{2}(i-2)$ is a (concave) parabola with $g(3) = \binom{k}{2}-2k+3 > 0$ (recall that $3 \leqslant i \leqslant k-1$, so $k \geqslant 4$) and $g(k) = 0$, so it is positive in the interval $[3, k-1]$. This concludes the proof of the claim.

It follows from the previous claims that $c = p_2 + f_3 + p_k + f_k$, so (2) gives $-4p_2 + (k-6)p_k - 2f_3 + (k-5)f_k < 0$. Therefore,

$$(k-6)(p_k + f_k) < 4p_2 + 2f_3.$$

By equation (1) and the previous claims, we have $2n = 2p_2 + kp_k + 2f_3 + (k-1)f_k$. Hence,

$$2p_2 + f_3 \leqslant 2p_2 + 2f_3 = 2n - kp_k - (k-1)f_k \leqslant 2n - (k-1)(p_k + f_k),$$

which implies $(k-6)(p_k + f_k) < 4n - 2(k-1)(p_k + f_k)$. This can be rewritten as

$$(3k-8)(p_k + f_k) < 4n.$$

Now, by equation (3),

$$\begin{aligned} m = p_2 + 3f_3 + \binom{k}{2}(p_k + f_k) &\leqslant \frac{3}{2}(2n - (k-1)(p_k + f_k)) + \binom{k}{2}(p_k + f_k) \\ &\leqslant 3n + (p_k + f_k)(-\frac{3}{2}(k-1) + \binom{k}{2}) \\ &< 3n + \frac{4n}{3k-8}(3k-8)\frac{k}{6} \quad \text{(since } k \geqslant 3\text{)} \\ &< n\left(\tfrac{2}{3}k + 3\right). \end{aligned}$$

We obtain that the graph $G(\mathcal{S})$ has less than $n(\frac{2}{3}k+3)$ edges, which is a contradiction. ☐

If the contact system we consider is one-sided, the argument we used for flat 2-intersection points while establishing (2) in the previous proof works for all flat points, and it follows that $H(\mathcal{S})$ has at most $3c - 6 - \sum_{i=2}^{k} f_i$ edges. Consequently, inequality (2) can be replaced by the following stronger inequality:

$$\sum_{i=2}^{k}(i-6)p_i + \sum_{i=2}^{k}(i-3)f_i \leqslant -12 \tag{4}$$

Let (LP2) be the new linear program. It is not difficult to check that Claims 4.3 and 4.4 remain satisfied. Moreover, it can be proved that in some optimal solution of (LP2), we also have $f_3 = 0$. Similar computations give that in this case the graph $G(\mathcal{S})$ contains a vertex of degree at most $\lceil \frac{4}{3}k \rceil + 1$. As a consequence:

Theorem 4.6. *For any $k \geqslant 3$, any 1-intersecting one-sided k-touching contact system of strings can be colored with $\lceil \frac{4}{3}k \rceil + 2$ colors.*

We now show how to modify the proof of Theorem 4.1 to prove that k-touching contact systems of segments are $(k+5)$-colorable. For technical reasons, we will consider instead the case of *extendible* contact system of strings, which we define next.

A *pseudo-line* is the homeomorphic image of a straight line in the plane. An *arrangement of pseudo-lines* is a set of pseudo-lines such that any two of them intersect at most once, and when they do they cross each other. We say that a contact system of strings is *extendible* if there is an arrangement of pseudo-lines, such that each string s of the contact system is contained in a distinct pseudo-line of the arrangement (this pseudo-line is called *the support* of s). Observe that a contact system of segments is clearly extendible. On the other hand, it was proved by de Fraysseix and Ossona de Mendez [11] that any extendible contact system of strings can be "stretched", i.e. continuously changed to a contact system of segments, while keeping the same underlying intersection graph.

We start with two observations about extendible contact systems of strings (the first one is similar to the first part of Observation 2.2).

Observation 4.7. *Let $\mathcal{S}$ be an extendible contact system of strings, and let $\mathcal{L}$ be a corresponding arrangement of pseudo-lines. We can assume without loss of generality that each string of $\mathcal{S}$ is the union of finitely many segments, and each pseudo-line of $\mathcal{L}$ is the union of two rays (affine images of $\{0\} \times [0, +\infty)$ in $\mathbb{R}^2$) and finitely many segments.*

The proof is similar to that of Observation 2.2. We consider the plane graph G whose vertices are the endpoints of the strings of $\mathcal{S}$ and the intersection points of $\mathcal{S} \cup \mathcal{L}$, and whose edges connect two points if they are consecutive in some string of $\mathcal{S}$ or some pseudo-line of $\mathcal{L}$. Since any two pseudo-lines intersect at most once, the resulting graph H is a simple planar (finite) graph, and by Fáry's theorem [6] it has an equivalent drawing in which all the edges are (straight-line) segments. We can then replace each end of a pseudo-line by finitely many segments and a ray, without changing $G(\mathcal{S})$, and Observation 4.7 follows.

Observation 4.8. *Let $\mathcal{S}$ be an extendible contact system of strings, and let $\mathcal{L}$ be a corresponding arrangement of pseudo-lines. We can assume without loss of generality that for any strings $s_1, s_2, s_3 \in \mathcal{S}$, if their supporting pseudo-lines intersect in a point p, then s_1, s_2, s_3 also intersect in p.*

Consider a point p contained in $\ell_1, \ell_2, \ell_3 \in \mathcal{L}$, and such that the string $s_1 \in \mathcal{S}$ supported by ℓ_1 does not contain p. We now show how to modify ℓ_1 so that it avoids p, without changing the other elements of $\mathcal{L}$ and the elements of $\mathcal{S}$. By Observation 4.7, we can assume that a small disk D centered in p only intersects the elements of $\mathcal{S} \cup \mathcal{L}$ containing

p, and the intersection of each of these elements with D consists of the union of at most two segments. By the definition of a pseudo-line arrangement, the pseudo-lines intersecting p pairwise cross. Note that ℓ_1 cuts the boundary of D in two arcs, say a, a'. We then modify ℓ_1 by replacing $D \cap \ell_1$ by a. The set $\mathcal{S}$ is not modified, and the pseudo-lines intersecting p still pairwise cross, so we obtain a new pseudo-line arrangement. Repeating this operation if necessary, we finally obtain a pseudo-line arrangement satisfying Observation 4.8.

We can now prove the main result of this section.

Theorem 4.9. *For any $k \geqslant 3$, any k-touching extendible contact system of strings can be properly colored with $k+5$ colors.*

As before, the result is a consequence of a bound on the degeneracy of the corresponding intersection graphs:

Lemma 4.10. *For any $k \geqslant 3$, if $\mathcal{S}$ is a k-touching extendible contact system of strings, then $G(\mathcal{S})$ contains a vertex of degree at most $k+4$.*

Proof. The proof is similar to that of Theorem 4.1. We consider a counterexample $\mathcal{S}$ consisting of n k-touching extendible strings. Since $G(\mathcal{S})$ has minimum degree at least $k+5$, $G(\mathcal{S})$ has $m \geqslant \frac{1}{2}(k+5)\,n$ edges. Again, we take n minimal, and with respect to this, m maximal. As before, we can assume that $G(\mathcal{S})$ is connected and that the two ends of each string are contact points.

We again consider the plane graph $H(\mathcal{S})$ whose vertices are the contact points of $\mathcal{S}$, whose edges link two contact points if and only if they are consecutive on a string of $\mathcal{S}$, and whose faces are the connected regions of $\mathbb{R}^2 \setminus \mathcal{S}$. Let p_i and f_i be the number of contact points of exactly i strings of $\mathcal{S}$ that are respectively peaks and flat. Let $p = \sum_{i=2}^{k} p_i$ and $c = \sum_{i=2}^{k}(p_i + f_i)$.

Recall that any face f of $H(\mathcal{S})$ contains at least $|F^{-1}(f)| + 3$ vertices (where $F^{-1}(f)$, defined in the proof of Lemma 4.2, is the number of flat contact points "incident" to f), thus at least $|F^{-1}(f)|$ edges can be added to $H(\mathcal{S})$ (inside f) with the property that $H(\mathcal{S})$ remains planar. We now show that, moreover, the vertices corresponding to the peaks of $\mathcal{S}$ all lie on the outerface of $H(\mathcal{S})$. This directly implies that we can add $f_2 + p - 3$ edges to $H(\mathcal{S})$, while keeping $H(\mathcal{S})$ planar (as a consequence, $H(\mathcal{S})$ has at most $3c - 6 - (f_2 + p - 3) = 3c - 3 - f_2 - p$ edges).

Indeed, if some peak x of $\mathcal{S}$ is not incident to the outerface, we choose a string s containing x and prolong s after x (following the pseudo-line supporting s) until it hits some other segment s' (note that since $\mathcal{S}$ is extendible, the pseudo-lines supporting s and s' intersect at most once, and thus s and s' did not intersect previously). Let $\mathcal{S}'$ be the new contact system of strings. It is still extendible, and by Observation 4.8 we can assume that $s \cap s'$ is only contained in s and s', so the new contact point is 2-touching. Consequently, this new contact system is extendible and k-touching, which contradicts the maximality of m.

It follows that inequality (2) in the proof of Theorem 4.1 can be replaced by:

$$\sum_{i=2}^{k}(i-4)p_i + \sum_{i=2}^{k}(i-5)f_i \leqslant -6 - 2f_2 \tag{5}$$

We consider the linear program (LP3) defined on variables p_i and f_i with values in $\mathbb{R}^+$ such that the equation (1) and the inequality (5) are satisfied, and such that the value m defined by (3) is maximized.

The coefficients of the variables f_i being the same in (2) and (5), Claim 4.3 (the optimal solutions of (LP3) are such that $f_2 = 0$) and Claim 4.4 (the optimal solutions of (LP3) are such that $f_i = 0$, for every $4 \leqslant i \leqslant k-1$) are still satisfied. Claim 4.5 is now replaced by the following stronger claim:

Claim 4.11. *The optimal solutions of (LP3) are such that $p_i = 0$, for every $2 \leqslant i \leqslant k$.*

If for some $2 \leqslant i \leqslant k$, $p_i \neq 0$, choose a small $\epsilon > 0$ and replace p_i by $p_i - \epsilon$, and f_i by $f_i + \epsilon \frac{i}{i-1}$. Then (1) remains valid, inequality (5) still holds (the left-hand side is decreased by $\frac{4\epsilon}{i-1}$ and the right-hand side remains unchanged), while the value of m in (3) is increased by $\frac{\epsilon i}{2}$. This concludes the proof of the claim.

It follows from Claims 4.3, 4.4, and 4.11 that $c = f_3 + f_k$, so (5) gives $(k-5)f_k < 2f_3$. By equation (1), $2f_3 = 2n - (k-1)f_k$, which implies $(k-5)f_k < 2n - (k-1)f_k$. This can be rewritten as

$$(k-3)f_k < n.$$

Now, by equation (3),

$$\begin{aligned} m = 3f_3 + \binom{k}{2} f_k &\leqslant \frac{3}{2}(2n - (k-1)f_k) + \binom{k}{2} f_k \\ &\leqslant 3n + f_k \tfrac{(k-1)(k-3)}{2} \\ &< n(\tfrac{k-1}{2} + 3). \end{aligned}$$

We obtain that the graph $G(\mathcal{S})$ has less than $\frac{n}{2}(k+5)$ edges, which is a contradiction. □

It follows that intersection graphs of k-touching segments are $(k+4)$-degenerate and then $(k+5)$-colorable. Note that the $(k+4)$-degeneracy may not be tight for every k. Indeed, we only know graphs that are not $(k+3)$-degenerate for $k \leqslant 6$. Those graphs are obtained in the following way for $k = 6$: First consider the segments in a straight-line drawing of a planar triangulation with minimum degree 5 and maximum degree 6, such that any degree five vertex is at distance at least two from the outerface, and such that any two vertices of degree five are at distance at least three apart. Such a graph can be obtained from the icosahedron by applying several times the following operation: subdivide every edge once, and inside each face, add 3 edges connecting the 3 (newly created) vertices of degree 2. It follows from Euler's formula that there are precisely 12 vertices of degree 5 in the triangulation, and therefore precisely 60 segments whose ends respectively touch a 5- and a 6-contact points. Let $s_1, \ldots, s_{60}$ be these segments. Those are the only segments that touch less than 10 other segments (they only touch 9 of them). To make each of them touch one more segment, prolong successively the segments $s_1, s_2, \ldots, s_{60}$ by their end that is at the 6-contact point, until reaching another segment (we can assume that the drawing of the triangulation is such that no line contains two edges incident to the same vertex, therefore each of the segments $s_1, \ldots, s_{60}$ hits an interior point of another segment).

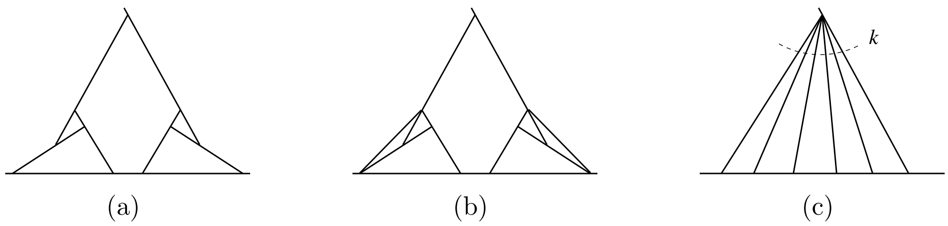


Figure 7: (a) A 2-touching set of segments requiring 4 colors (b) A 3-touching set of segments requiring 5 colors (c) A k-touching set of segments requiring $k+1$ colors.

Figure 7 depicts k-touching sets of segments requiring $k+2$ colors, for $k = 2, 3$. However it does not appear to be trivial to extend this construction for any $k \geqslant 4$. Note that Figure 7(b) also shows that there are intersection graphs of 3-contact representations of segments (with two-sided contact points) that are not planar (this remark also appears in [14]).

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