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Mathematics of Engineers: Elements for a New History of Numerical Analysis

Dominique Tournès*

Abstract. The historiography of numerical analysis is still relatively poor. It does not take sufficient account of numerical and graphical methods created, used and taught by military and civil engineers in response to their specific needs, which are not always the same as those of mathematicians, astronomers and physicists. This paper presents some recent historical research that shows the interest it would be to examine more closely the mathematical practices of engineers and their interactions with other professional communities to better define the context of the emergence of numerical analysis as an autonomous discipline in the late 19th century.

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1. Introduction

Few recent books have been devoted to the history of numerical analysis. Goldstine [18] was a pioneer. His work focuses primarily on identifying numerical methods encountered in the works of some great mathematicians: Newton, Maclaurin, Euler, Lagrange, Laplace, Legendre, Gauss, Cauchy and Hermite. The main problems are the construction of logarithmic and trigonometric tables necessary for astronomical calculations, Kepler's equation, the Lunar theory and its connection with the calculation of longitudes, the three-body problem and, more generally, the study of perturbations of orbits of planets and comets. Through these problems we assist to the birth of finite difference methods for interpolating functions and calculating quadratures, developments in series or continued fractions for solving algebraic equations and differential equations, and the method of least squares for finding optimal solutions of linear systems with more equations or less equations than unknowns. At the end of the book, a few pages involve Runge, Heun,

*I am grateful to the French National Research Agency, which funded the four-year project "History of Numerical Tables" (2009-2013). A large part of the contents of this paper is issued from this project. I also thank the laboratory SPHERE (UMR 7219, CNRS and University Paris-Diderot), which offered me a good research environment for many years.

Kutta, Moulton, that is to say, some characters who can be considered as being the first applied mathematicians identified as such in the late 19th century and the beginning of the 20th. In Goldstine's survey, numerical analysis is thus the fruit of a few great mathematicians who developed the foundations of today's numerical methods by solving some major problems of astronomy, celestial mechanics and rational mechanics. These numerical methods were then deepened by professional applied mathematicians appearing in the late 19th century, which was the time when numerical analysis, as we know it today, structured itself into an autonomous discipline. In this story, a few areas of inspiration and intervention other than astronomy are sometimes mentioned incidentally, but no engineer is explicitly quoted.

While Goldstine actually begins his history in the 16th century, Chabert [7] gives more depth to the subject by examining numerical algorithms in a variety of texts from various civilizations since Antiquity. Besides the famous previously mentioned problems of astronomy such as Kepler's equation, the determination of orbits of comets, the brightness of stars, etc., there are some references to other domains, for example the theory of vibrating strings or the signal theory. Some engineers are mentioned, in general in connection with secondary points. Only one of them, Cholesky, is quoted for a significant contribution consisting in an original method for solving linear systems (see Section 3). Despite these few openings compared to previous work, most numerical analysis questions addressed in Chabert's book are presented as abstract mathematical problems, out of context.

In a more recent collective book edited by Bultheel and Cools [6], the birth of modern numerical analysis is located precisely in 1947, in a paper of John von Neumann (1903–1957) and Herman Goldstine (1913–2004) [23] which analyzes for the first time in detail the propagation of errors when solving a linear system, in conjunction with the first uses of digital computers. The authors recognize naturally that a lot of numerical calculations were made long before this date in various questions of physics and engineering, but for them the problem of the practical management of calculations made by computer actually founds the field of numerical analysis and this apparently technical problem is at the origin of the considerable theoretical developments that this domain generated since the mid-20th century. In this book written not by historians but by specialists of numerical analysis, it is interesting to note that the accepted actors of the domain do not trace the history of their discipline beyond what characterizes their current personal practices.

In fact, the birth of numerical analysis, in the modern sense of the term, should not be connected to the advent of digital computers, but to the distinction between pure mathematics and applied mathematics (formerly “mixed mathematics”), which is clarified gradually throughout the 19th century with a more and more marked separation between the two domains in scientific journals, institutions and university positions¹. The development of new calculating instruments – be-

¹A very interesting workshop on this subject took place in March 2013 in Oberwolfach, organized by Moritz Epple, Tinne Hoff Kjeldsen and Reinhard Siegmund-Schultze, and entitled “From ‘Mixed’ to ‘Applied’ Mathematics: Tracing an important dimension of mathematics and

fore computers, there were numerical and graphical tables, slide rules, mechanical instruments of integration, desktop calculators, etc. – has also contributed to setting up a new equilibrium between analytical, numerical and graphical methods. This is actually around 1900 that mathematicians began to formulate, in concrete terms, what is meant by “applied mathematics”. Germany, and particularly Göttingen, played a leading role in this international process of institutionalization of applied mathematics as an autonomous domain [26, p. 60–63]. Encouraged by Felix Klein, Carl Runge (1856–1927) and Rudolf Mehmke (1857–1944) assumed in 1901 the editorship of the *Zeitschrift für Mathematik und Physik* and devoted this journal to applied mathematics. In 1904, Runge accepted the first full professorship of applied mathematics at the University of Göttingen. In 1907, German applied mathematicians adopted the following definition:

The essence of applied mathematics lies in the development of methods that will lead to the numerical and graphical solution of mathematical problems.²

Recent research has shown that engineers have constituted a bridge between mathematics and their applications since the 18th century, and that problems encountered in ballistics, strength of materials, hydrodynamics, steam engines, electricity and telephone networks also played an important role in the creation of original numerical and graphical methods of computation. In fact, the mathematical needs of engineers seem very different from those of mathematicians. To illustrate this with a significant example, consider the problem of the numerical solution of equations, a pervasive problem in all areas of mathematics intervention. Léon-Louis Lalanne (1811–1892), a French civil engineer who, throughout his career, sought to develop practical methods for solving equations, wrote what follows as a summary when he became director of the *École des ponts et chaussées*:

The applications have been, until now, the stumbling block of all the methods devised for solving numerical equations, not that, nor the rigor of these processes, nor the beauty of the considerations on which they are based, could have been challenged, but finally it must be recognized that, while continuing to earn the admiration of geometers, the discoveries of Lagrange, Cauchy, Fourier, Sturm, Hermite, etc., did not always provide easily practicable means for the determination of the roots.³

its history” [13].

²“Das Wesen der angewandten Mathematik liegt in der Ausbildung und Ausübung von Methoden zur numerischen und graphischen Durchführung mathematischer Probleme” (quoted in [27, p. 724]).

³“Les applications ont été, jusqu’à ce jour, la pierre d’achoppement de tous les procédés imaginés pour la résolution des équations numériques, non pas que, ni la rigueur de ces procédés, ni la beauté des considérations sur lesquelles ils se fondent, en aient reçu la moindre atteinte; mais enfin il bien reconnaître que, sans cesser de mériter l’admiration des géomètres, les découvertes de Lagrange, de Cauchy, de Fourier, de Sturm, d’Hermite, etc., n’ont pas fourni toujours des moyens facilement praticables pour la détermination des racines” [20, p. 1487].

Lalanne says that as politely as possible, but his conclusion is clear: the methods advocated by mathematicians are not satisfactory. These methods are complicated to understand, long to implement and sometimes totally impracticable for ground engineers, foremen and technicians, who, moreover, did not always receive a high-level mathematical training.

Given such a situation, 19th-century engineers were often forced to imagine by themselves the operational methods and the calculation tools that mathematicians could not provide them. The objectives of the engineer are not the same as those of the mathematician, the physicist or the astronomer: the engineer rarely needs high accuracy in his calculations, he is rather sensitive to the speed and simplicity of their implementation, especially since he often has to perform numerous and repetitive operations. He also needs methods adapted for use on the ground, and not just for use at the office. Finally, priority is given to methods that avoid performing calculations by oneself, methods that provide the desired result directly through a simple reading of a number on a numerical or graphical table, on a diagram, on a curve or on the dial of a mechanical instrument.

In this paper, I would want to show, through some examples from recent historical research, that the engineers, so little mentioned so far in the historiography of numerical analysis, have contributed significantly throughout the 19th century to the creation of those numerical and graphical methods that became an autonomous discipline around 1900. More than that, I shall underline that their practical methods have sometimes been at the origin of new theoretical problems that also inspired pure mathematicians.

2. From Civil Engineering to Nomography

The 19th century is the moment of the first industrial revolution, which spreads throughout the Western world at different rates in different countries. Industrialization causes profound transformations of society. In this process, the engineering world acquires a new identity, marked by its implications in the economic development of industrial states and the structuration of new professional relationships that transcend national boundaries. Linked to the Industrial Revolution, enormous computational requirements appeared during the 19th century in all areas of engineering sciences and caused an increasing mathematization of these sciences. This led naturally to the question of engineering education: how were engineers prepared to use high-level mathematics in their daily work and, if necessary, to create by themselves new mathematical tools?

The French model of engineering education in the early 19th century is that of the *École polytechnique*, founded in 1794⁴. Although it had initially the ambition to be comprehensive and practice-oriented, this school quickly promoted a high-level teaching dominated by mathematical analysis. This theoretical teaching was then completed, from the professional point of view, by two years in application

⁴On the professional milieu of French engineers during the 19th century and the *École polytechnique*, see the papers by Bruno Belhoste and Konstantinos Chatzis ([2], [9]).

schools with civil and military purposes. Such a training model, which subordinates practice to theory, has produced a corporation of “scholarly engineers” capable of using the theoretical resources acquired during their studies to achieve an unprecedented mathematization of the engineering art. This model is considered to have influenced the creation of many polytechnic institutes throughout Europe and to the United States.

A paradigmatic example of a corpus of mathematical tools, constituting an autonomous knowledge which was created from scratch by engineers themselves to meet their needs, is that of nomography⁵. The main purpose of nomography is to construct graphical tables to represent any relationship between three variables, and, more generally, relationships between any number of variables. Among the “Founding Fathers” of nomography, four were students at the *École polytechnique*: Lalanne, Charles Lallemand (1857–1938), Maurice d’Ocagne (1862–1938) and Rodolphe Soreau (1865–1936). The only exception in this list is the Belgian engineer Junius Massau (1852–1909), an ancient student and then professor at the school of civil engineering of the University of Ghent, but, in this school of civil engineering, the training was comparable to that of the *École polytechnique*, with high-level courses of mathematics and mechanics.

During the years 1830–1860, the sector of public works experiences a boom in France and more generally in Europe. The territories of the different countries are covered progressively by vast networks of roads, canals, and, after 1842, of railways. These achievements require many tedious calculations of surfaces of “cut and fill” on cross-sections of the ground. Cut and fill is the process of earthmoving needed to construct a road, a canal or a railway. You have to cut land where the ground level is too high and then transport this land to fill the places where the ground level is too low. And to calculate roughly the volume of land to be transported, you have to decompose this volume in thin vertical slices, evaluate the area of each slice and sum all these elementary areas.

Civil engineers tried different methods of calculation more or less expeditious. Some, like Gaspard-Gustave Coriolis (1792–1843), have calculated numerical tables giving the surfaces directly based on a number of features of the road and its environment. Other engineers, especially in Germany and Switzerland, designed and built several kinds of planimeters, that is mechanical instruments used to quickly calculate the area of any plane surface. These planimeters, which concretize the continuous summation of infinitesimal surfaces, had significant applications in many other scientific fields beyond cuts and fills. Still others, like Lalanne, have imagined replacing numerical tables by graphical tables, cheaper and easier to use. It is within this framework that nomography developed itself and was deepened throughout the second half of the 19th century.

First principles of nomography. The departure point of nomography lies in the fact that a relationship between three variables α , β and γ can be considered, under certain conditions, as the result of the elimination of two auxiliary variables

⁵This Section is an abridged and synthetic version of developments contained in my papers [30], [32] and [34].

x and y between three equations, each containing only one of the initial variables. One can then represent the equation by three sets of lines in the plane x - y , one of them parametrized by α , the second by β and the third by γ . On this kind of graphical table, called a “concurrent-line abaque”, a solution of the equation corresponds to an intersection point of three lines.

Isolated examples of graphical translation of double-entry tables are found already in the first half of the 19th century, mainly in the scope of artillery, but this is especially Lalanne who gave a decisive impetus to the theory of graphical tables. In 1843, he provided consistent evidence that any law linking three variables can be graphed in the same manner as a topographic surface using its marked level lines. His ideas came to a favorable moment. Indeed, the Act of June 11, 1842 had decided to establish a network of major railway lines arranged in a star from Paris. To run the decision quickly, one felt the need for new ways of evaluating the considerable earthworks to be carried out. In 1843, the French government sent to all engineers involved in this task a set of graphical tables for calculating the areas of cut and fill on the profile of railways and roads.

Curves other than straight lines are difficult to construct on paper. For this reason, Lalanne imagined the use of non-regular scales on the axes for transforming curves into straight lines. By analogy with the well-known optical phenomenon previously used by certain painters, he called “anamorphosis” this general transformation process. After Lalanne, the graphical tables resting on the principle of concurrent lines spread rapidly until becoming, in the third quarter of the 19th century, very common tools in the world of French engineers.

Massau succeeded Lalanne to enrich the method and its scope of applications. For that, he introduced a notion of generalized anamorphosis, seeking what are the functions that can be represented using three pencils of lines. Massau put in evidence that a given relationship between three variables can be represented by a concurrent-straight-line abaque if, and only if, it can be put into the form of a determinant of the type

$$\begin{vmatrix} f_1(\alpha) & f_2(\alpha) & f_3(\alpha) \\ g_1(\beta) & g_2(\beta) & g_3(\beta) \\ h_1(\gamma) & h_2(\gamma) & h_3(\gamma) \end{vmatrix} = 0.$$

These determinants, called “Massau determinants”, played an important role in the subsequent history of nomography; they are encountered in research until today. As an application of this new theory, Massau succeeded in simplifying Lalanne’s abaques for cuts and fills. With Massau’s publications, the theory of abaques was entering into a mature phase, but at the same time a new character intervened to orient this theory towards a new direction.

From concurrent-line abaques to alignment nomograms. In 1884, when he is only 22 years old, d’Ocagne observes that most of the equations encountered in practice can be represented by an abaque with three systems of straight lines and that three of these lines, each taken in one system, correspond when they meet into a point. His basic idea is then to construct by duality, by substituting the use

of tangential coordinates to that of punctual coordinates, a figure in correlation with the previous one: each line of the initial chart is thus transformed into a point, and three concurrent lines are transformed into three aligned points. The three systems of marked straight lines become three marked curves. Through this transformation, a concurrent-straight-line abaque becomes an “alignment abaque”, which is easier to use.

A given relationship between three variables is representable by an alignment abaque if, and only if, it can be put into the form of a Massau determinant, because it is clear that the problem of the concurrency of three straight lines and the problem of the alignment of three points, dual to each other, are mathematically equivalent. As his predecessors, d’Ocagne applied immediately his new ideas to the problem of cuts and fills, actually one of the main problems of civil engineering.

After this first achievement in 1891, d’Ocagne deepened the theory and applications of alignment abaques until the publication of a large treatise in 1899, the famous *Traité de nomographie*, which became for a long time the reference book of the new discipline. A little later, he introduced the generic term “nomogram” to replace “abaque”, and the science of graphical tables became “nomography”. From there, alignment nomograms were quickly adopted by many engineers for the benefit of the most diverse applications. At the turn of the 20th century, nomography was already an autonomous discipline well established in the landscape of applied sciences.

Mathematical implications of nomography. The mathematical practices of engineers are often identified only as “applications”, which is equivalent to consider them as independent from the development of mathematical knowledge in itself. In this perspective, the engineer is not supposed to develop a truly mathematical activity. We want to show, through the example of nomography, that this representation is somewhat erroneous: it is easy to realize that the engineer is sometimes a creator of new mathematics, and, in addition, that some of the problems which he arises can in turn irrigate the theoretical research of mathematicians.

Firstly, the problem of general anamorphosis, that is to say, of characterizing the relationships between three variables that can be put in the form of a Massau determinant, has inspired many theoretical research to mathematicians and engineers: Cauchy, Saint-Robert, Massau, Lecornu, and Duporcq have brought partial responses to this problem before that in 1912 the Swedish mathematician Thomas Hakon Gronwall (1877–1932) gives a complete solution resulting in the existence of a common integral to two very complicated partial differential equations. But, as one can easily imagine, this solution was totally inefficient, except in very simple cases.

After Gronwall, other mathematicians considered the problem of anamorphosis in a different way, with a more algebraic approach that led to study the important theoretical problem of linear independence of functions of several variables. These mathematicians, like Kellogg in the US, wanted to find a more practical solution not involving partial differential equations. A complete and satisfactory solution was finally found by the Polish mathematician Mieczyslaw Warmus (1918–

2007). In his Dissertation of 1958, Warmus defined precisely what is a nomographic function, that is a function of two variables that can be represented by an alignment nomogram, and classified nomographic functions through homography into 17 equivalence classes of Massau determinants. Moreover, he gave an effective algorithm for determining if a function is nomographic and, if true, for representing it explicitly as a Massau determinant.

Beyond the central problem of nomographic representation of relationships between three variables, which define implicit functions of two variables, there is the more general problem of the representation of functions of three or more variables. Engineers have explored various ways in this direction, the first consisting in decomposing a function of any number of variables into a finite sequence of functions of two variables, which results in the combined use of several nomograms with three variables, each connected to the next by means of a common variable.

Such a practical concern was echoed unexpectedly in the formulation of the Hilbert's 13th problem, one of the famous 23 problems that were presented at the International Congress of Mathematicians in 1900 [19]. The issue, entitled "Impossibility of the solution of the general equation of the 7th degree by means of functions of only two arguments", is based on the initial observation that up to the sixth degree, algebraic equations are nomographiable.

Indeed, up to the fourth degree, the solutions are expressed by a finite combination of additions, subtractions, multiplications, divisions, square root extractions and cube root extractions, that is to say, by functions of one or two variables. For the degrees 5 and 6, the classical Tschirnhaus transformations lead to reduced equations whose solutions depend again on one or two parameters only. The seventh degree is then the first actual problem, as Hilbert remarks:

Now it is probable that the root of the equation of the seventh degree is a function of its coefficients which does not belong to this class of functions capable of nomographic construction, i. e., that it cannot be constructed by a finite number of insertions of functions of two arguments. In order to prove this, the proof would be necessary that the equation of the seventh degree is not solvable with the help of any continuous functions of only two arguments [19, p. 462].

In 1901, d'Ocagne had found a way to represent the equation of the seventh degree by a nomogram involving an alignment of three points, two being carried by simple scales and the third by a double scale. Hilbert rejected this solution because it involved a mobile element. Without going into details, we will retain that there has been an interesting dialogue between an engineer and a mathematician reasoning in two different perspectives. In the terms formulated by Hilbert, it was only in 1957 that the 13th problem is solved negatively by Vladimir Arnold (1937–2010), who proved to everyone's surprise that every continuous function of three variables could be decomposed into continuous functions of two variables only.

3. From Topography to Linear Systems

The French military engineer André-Louis Cholesky (1875–1918) offers us the occasion of a perfect case study. Before 1995, not many details were known on his life. In 1995 (120 years after his birth), the documents about him kept in the archives of the army at the Fort de Vincennes (near Paris) were open to the public. In 2003, we had the chance that a grandson of Cholesky, Michel Gross, donated the personal archives of his grandfather to the *École polytechnique*⁶.

Cholesky was born on 15 October 1875, in Montguyon, a village near Bordeaux, in the south-west of France. In October 1895, he was admitted to the *École polytechnique* and, two years later, he was admitted as a *sous-lieutenant* at the *École d'application de l'artillerie et du génie* in Fontainebleau. He had to spend one year at the school and then to serve for one year in a regiment of the army. There he had courses on artillery, fortification, construction, mechanics, topography, etc.

Cholesky as a topographer. Between 1902 and 1906, he was sent to Tunisia and then to Algeria for several missions. In 1905, he was assigned to the Geographical Service of the Staff of the Army. In this service, there were a section of geodesy and a section of topography. Around 1900, following the revision of the meridian of Paris, the extent of the meridian of Lyon and a new cadastral triangulation of France had been decided. These missions were assigned to the section of geodesy together with the establishment of the map of Algeria, and a precise geometric leveling of this country. The problem of the adjustment (or compensation) of networks (corrections to be brought to the measured angles) concerned many officers of the Geographical Service, eager to find a simple, fast and accurate method. According to Commandant Benoît, one of his colleagues, it was at this occasion that Cholesky imagined his method for solving the equations of conditions by the method of least squares.

Cholesky is representative of these “scholarly engineers” of whom we spoke above. Due to his high-level mathematical training, he was able to work with efficiency and creativity in three domains: as a military engineer, specialized in artillery and in topography, able to improve and optimize the methods used on the ground at this time; as a mathematician able to create new algorithms when it is necessary; and as a teacher (because in parallel to his military activities, he participated during four years to the teaching by correspondence promoted by the *École spéciale des travaux publics* founded in Paris by Léon Eyrolles).

Concerning topography, Cholesky is well known among topographers for a leveling method of his own: the method of double-run leveling (*double cheminement* in French). Leveling consists in measuring the elevation of points with respect to a surface taken as reference. This surface is often the geoid in order to be able to draw level curves, also called “contour lines”. Double-run leveling consists in conducting simultaneously two separate survey traverses, very close to each other,

⁶Claude Brezinski has classified these archives and published a lot of papers about the life and work of Cholesky: see [3], [4] and [5]. For this Section, I found a lot of information in these papers.

and comparing the results so as to limit the effects of some instrumental defects. This method is still taught and used today.

Cholesky's method for linear systems. As said before, Cholesky is a good example of an engineer creating a new mathematical method and a new algorithm of calculation for his own needs. Cholesky's method for linear systems is actually an important step in the history of numerical analysis. A system of linear equations has infinitely many solutions when the number of unknowns is greater than the number of equations. Among all possible solutions, one looks for the solution minimizing the sum of the squares of the unknowns. This is the case in the compensation of triangles in topography which interested Cholesky. The method of least squares is very useful and is much used in many branches of applied mathematics (geodesy, astronomy, statistics, etc.) for the treatment of experimental data and fitting a mathematical model to them. This method was published for the first time by Legendre in 1806. Its interpretation as a statistical procedure was given by Gauss in 1809.

As it is known, the least square method leads to a system with a symmetric positive definite matrix. Let us describe Cholesky's method to solve such a system. Let A be a symmetric positive definite matrix. It can be decomposed as $A = LL^T$, where L is a lower triangular matrix with positive diagonal elements, which are computed by an explicit algorithm. Then the system $Ax = b$ can be written as $LL^T x = b$. Setting $y = L^T x$, we have $Ly = b$. Solving this lower triangular system gives the vector y . Then x is obtained as the solution of the upper triangular system $L^T x = y$.

What was the situation before Cholesky? When the matrix A is symmetric, Gauss method makes no use of this property and needs too many arithmetical operations. In 1907, Otto Toeplitz showed that an Hermitian matrix can be factorized into a product LL^* with L lower triangular, but he gave no rule for obtaining the matrix L . That is precisely what Cholesky did in 1910. Cholesky's method was presented for the first time in 1924 in a note published in the *Bulletin géodésique* by commandant Benoît, a French geodesist who knew Cholesky well, but the method remained unknown outside the circle of French military topographers. Cholesky method was rebirth by John Todd who taught it in his numerical analysis course at King's College in London in 1946 and thus made it known. When Claude Brezinski classified Cholesky's papers in 2003, he discovered the original unpublished manuscript where Cholesky explained his method⁷. The manuscript of 8 pages is dated 2 December 1910. That was an important discovery for the history of numerical analysis.

4. From Ballistics to Differential Equations

The main problem of exterior ballistics is to determine the trajectory of a projectile launched from a cannon with a given angle and a given velocity. The differential

⁷This manuscript has been published in 2005 in the *Revue d'histoire des mathématiques* [3].

equation of motion involves the gravity g , the velocity v and the tangent inclination θ of the projectile, and the air resistance $F(v)$, which is an unknown function of v :⁸

$$g \, d(v \cos \theta) = vF(v) \, d\theta.$$

To calculate their firing tables and to adjust their cannons, the artillerymen have used for a long time the assumption that the trajectory is parabolic, but this was not in agreement with the experiments. Newton was the first to research this topic taking into account the air resistance. In his *Principia* of 1687, he solved the problem with the hypothesis of resistance proportional to the velocity, and he got quite rough approximations when the resistance is proportional to the square of the velocity. After Newton, Jean Bernoulli discovered the general solution in the case of resistance proportional to any power of the velocity, but his solution, published in the *Acta Eruditorum* of 1719, was not convenient for numerical computation.

This problem of determining the ballistic trajectory for a given law of air resistance is particularly interesting because it stands at the crossroads of two partly contradictory concerns: on the one hand, the integration of the differential equation of motion is a difficult problem which interests the mathematicians from the point of view of pure analysis; on the other hand, the artillerymen on the battlefield must quickly determine the firing angle and the initial velocity of their projectile in order to attain a given target, and for that practical purpose they need firing tables precise and easy to use. This tension between theoreticians, generally called ballisticians, and practitioners, described rather to be artillerymen, is seen in all synthesis treatises of the late 19th and early 20th century. I shall content myself with one quotation to illustrate this tension. In 1892, in the French augmented edition of his main treatise, Francesco Siacci (1839–1907), a major figure in Italian ballistics, writes:

Our intention is not to present a treatise of pure science, but a book of immediate usefulness. Few years ago ballistics was still considered by the artillerymen and not without reason as a luxury science, reserved for the theoreticians. We tried to make it practical, adapted to solve the firing questions fast, as exactly as possible, with economy of time and money.⁹

By these words, Siacci condemns a certain type of theoretical research as a luxury, but he also condemns a certain type of experimental research that accumulates numerous and expensive firings and measurements without obtaining convincing results.

⁸In fact, the problem is more complex because we must take into account other factors like the variations of the atmospheric pressure and temperature, the rotation of the Earth, the wind, the geometric form of the projectile and its rotation around its axis, etc. However these effects could be often neglected in the period considered here, because the velocities of projectiles remained small.

⁹“Notre intention d’ailleurs n’est pas de présenter un traité de science pure, mais un ouvrage d’utilité immédiate. Il y a peu d’années que la balistique était encore considérée par les artilleurs et non sans raison comme une science de luxe, réservée aux théoriciens. Nous nous sommes efforcé de la rendre pratique, propre à résoudre les questions de tir rapidement, facilement, avec la plus grande exactitude possible, avec économie de temps et d’argent” [25, p. x].

Of course, the problem of integrating the ballistic equation is difficult. Many, many attempts have been done to treat this equation mathematically with the final objective of constructing firing tables. We can organize these attempts throughout two main strategies, one analytical and one numerical.

Analytical approach of the ballistic differential equation. The analytical strategy consists in integrating the differential equation in finite terms or, alternatively, by quadratures. Reduction to an integrable equation can be achieved in two ways: 1) choose an air resistance law so that the equation can be solved in finite form (if the air resistance is not known with certainty, why not consider abstractly, formally, some potential laws of air resistance, leaving it to the artillerymen to choose after among these laws according to their needs?); 2) if a law of air resistance is needed through experience or by tradition, it is then possible to change the other coefficients of the equation to make it integrable, with, of course, the risk that modifying the equation could also modify the solution in a significant way. Fortunately, in the same time of theoretical mathematical research, there have been many experimental studies to determine empirically the law of air resistance and the equation of the ballistic curve. Regular confrontations took place between the results of the theoreticians and those of the practitioners.

In 1744, D'Alembert restarts the problem of integrability of the equation, which had not advanced since the Bernoulli's memoir of 1719. He finds four new cases of integrability: $F(v) = a + bv^n$, $F(v) = a + b \ln v$, $F(v) = av^n + R + bv^{-n}$, $F(v) = a(\ln v)^2 + R \ln v + b$. D'Alembert's work went relatively unnoticed at first. In 1782, Legendre found again the case $F(v) = a + bv^2$, without quoting D'Alembert. In 1842, Jacobi found the case $F(v) = a + bv^n$ to generalize Legendre's results, quoting Legendre, but still ignoring D'Alembert. After studying this case in detail, Jacobi notes also that the problem is integrable for $F(v) = a + b \ln v$, but he does not study further this form, because, he says, it would be abhorrent to nature (it's hard indeed to conceive an infinite resistance when velocity equals zero). Jacobi puts the equations in a form suitable for the use of elliptic integrals. Several ballisticians like Greenhill, Zabudski, MacMahon, found here inspiration to calculate ballistic tables in the case of air resistance proportional to the cube or to the fourth power of the velocity. These attempts contributed to popularizing elliptic functions among engineers and were quoted in a lot of treatises about elliptic functions.

During the 19th century, there is a parallelism between the increasing speeds of bullets and cannonballs, and the appearance of new instruments to measure these speeds. Ballisticians are then conducted to propose new air resistance laws for certain intervals of speeds. In 1921, Carl Julius Cranz (1858–1945) gives an impressive list of 37 empirical laws of air resistance actually used to calculate tables at the end of the 19th century. Thus, theoretical developments, initially free in D'Alembert's hands, led to tables that were actually used by the artillerymen. The fact that some functions determined by artillerymen from experimental measurements fell within the scope of integrable forms has reinforced the idea that it might be useful to continue the search for such forms. It is within this context that Siacci

resumed the theoretical search for integrable forms of the law of resistance. In two papers published in 1901, he places himself explicitly in D'Alembert's tradition. He multiplies the differential equation by various multipliers and seeks conditions for these multipliers are integrant factors. He discovers several integrable equations, including one new integrable Riccati equation. This study leads to eight families of air resistance laws, some of which depend on four parameters. In his second article, he adds two more families to his list.

The question of integrability by quadratures of the ballistic equation is finally resolved in 1920 by Jules Drach (1871–1949), a brilliant mathematician who has contributed much in Galois theory of differential equations in the tradition of Picard, Lie, and Vessiot. Drach puts the ballistic equation in a new form that allows him to apply a theory developed in 1914 for a certain class of differential equations, which he found all cases of reduction. Drach exhausts therefore the problem from the theoretical point of view, by finding again all integrability cases previously identified. As you might expect, the results of this long memoir of 94 pages are very complicated. They were greeted without enthusiasm by the ballisticians, who did not see at all how to transform them into practical applications.

Another way was explored by theoreticians who accepted Newton's law of the square of the velocity, and tried to act on other terms of the ballistic equation to make it integrable. In 1769, the military engineer Jean-Charles de Borda (1733–1799) proposes to assume that the medium density is variable and to choose, for this density, a function that does not stray too far from a constant and makes the equation integrable. Borda makes three assumptions about the density, the first adapted to small angles of fire, the second adapted to large angles of fire, and the third for the general case, by averaging between the previous ones and by distinguishing the ascending branch and the descending branch of the curve.

Legendre deepens Borda's ideas in his *Dissertation sur la question de balistique*, with which he won in 1782 the prize of the Berlin Academy. The question chosen for the competition was: "Determine the curve described by cannonballs and bombs, by taking the air resistance into account; give rules to calculate range that suit different initial speeds and different angles of projection." Legendre puts the ballistic equation in a form similar to that used by Euler, with the slope of the tangent as independent variable. After commenting Euler's method by successive arcs (see below), considered too tiresome for numerical computation, Legendre suggests two ideas of the same type as those of Borda, with a result which is then satisfactory for the entire curve, and not only at the beginning of the trajectory. With these methods, Legendre manages to calculate ten firing tables that will be considered of high quality and will permit him to win the prize of the Berlin Academy. After Legendre, many other people, for example Siacci at the end of the 19th century, have developed similar ideas to obtain very simple, general, and practical methods of integration.

Direct numerical integration of the differential equation. The second strategy for integrating the ballistic differential equation belongs to numerical analysis. It contains three main procedures: 1) calculate the integral by successive

small arcs; 2) develop the integral into an infinite series and keep the first terms; 3) construct graphically the integral curve.

Euler is truly at the starting point of the calculation of firing tables in the case of the square of the velocity. In 1753, Euler resumes Bernoulli's solution and put it in a form that will be convenient for numerical computation. He takes the slope p of the tangent as principal variable. All the other quantities are expressed in function of p by means of quadratures. The integration is done by successive arcs: each small arc of the curve is replaced by a small straight line, whose inclination is the mean of the inclinations at the extremities of the arc. To give an example, Euler calculates a single table, the one corresponding to a firing angle of 55. With this numerical table, he constructs by points the corresponding trajectory. A little later, Henning Friedrich von Grävenitz (1744–1764), a Prussian officer, performs the calculations of the program conceived by Euler. He published firing tables in Rostock in 1764. In 1834, Jacob Christian Friedrich Otto, another military officer, publishes new tables in Berlin, because he finds that those of Grävenitz are insufficient. To answer better the problem encountered in practice by artillerymen, he reverses the table taking the range as the given quantity and the initial velocity as the unknown quantity. Moreover, he calculates a lot more elements than Grävenitz to facilitate interpolation. Otto's tables will experience a great success and will be in use until the early 20th century.

Another approach is that of series expansions. In the second half of the 18th century and early 19th, we are in the era of calculation of derivations and algebraical analysis. The expression of solutions by infinite series whose law of the formation of terms is known, is considered to be an acceptable way to solve a problem exactly, despite the philosophical question of the infinite and the fact that the series obtained, sometimes divergent or slowly convergent, do not always allow an effective numerical computation. In 1765, Johann Heinrich Lambert (1728–1777) is one of the first to express as series the various quantities involved in the ballistic problem. On his side, the engineer Jacques-Frédéric Français (1775–1833) applies the calculation of derivations. He identifies a number of new formulas in the form of infinite series whose law of the formation of the successive terms is explicitly given. However, he himself admits that these formulas are too complicated for applications.

Let us mention finally graphical approaches providing to the artillerymen an easy and economic tool. In 1767, recognizing that the series calculated in his previous memoir are unusable, Lambert constructs a set of curves from Grävenitz's ballistic tables. In France, an original approach is due to Alexander-Magnus d'Obenheim (1752-1840), another military engineer. His idea was to replace the numerical tables by a set of curves carefully constructed by points calculated with great precision. These curves are drawn on a portable instrument called the "gunner board" ("planchette du canonier" in French). The quadrature method used to construct these curves is highly developed. Obenheim employs a method of Newton-Cotes type with a division of each interval into 24 parts. In 1848, Isidore Didion (1798–1878), following Poncelet's ideas, constructs ballistic curves that are not a simple graphic representation of numerical tables, but are obtained directly

from the differential equation by a true graphical calculation: he obtains the curve by successive arcs of circles, using at each step a geometric construction of the center of curvature. Artillery was thus the first domain of engineering science in which graphical tables, called “abaques” in French, were commonly used (see Section 2). One of the major advantages of graphical tables is their simplicity and rapidity of utilization, that is important on the battlefield when the enemy is firing against you!

In conclusion, throughout the 18th and 19th centuries, there has been an interesting interaction between analytic theory of differential equations, numerical and graphical integration, and empirical experimental research. Mathematicians, ballisticians and artillerymen, although part of different worlds, collaborated and inspired each other regularly. All this led, however, to a relative failure, both experimentally to find a good law of air resistance, and mathematically to find a simple solution of the ballistic differential equation.

Mathematical research on the ballistic equation has nevertheless played the role of a laboratory where the modern numerical analysis was able to develop. Mathematicians have indeed been able to test on this recalcitrant equation all possible approaches to calculate the solution of a differential equation. There is no doubt that these tests, joined with the similar ones conceived by astronomers for the differential equations of celestial mechanics, have helped to organize the domain into a separate discipline around 1900. In parallel with celestial mechanics, ballistics certainly played an important role in the construction of modern Runge-Kutta and Adams-Bashforth methods for numerically integrating ordinary differential equations.

5. From Hydraulics to Dynamical Systems

Concerning another aspect of the theory of differential equations, it should be noticed that the classification of singular points obtained by Poincaré had occurred earlier in the works of at least two engineers who dealt with hydraulic problems¹⁰. As early as 1924, Russian historians reported a similar classification in a memoir of Nikolai Egorovich Zhukovsky (1847–1921) dated 1876 on the kinematics of liquids. Dobrovolsky published a reproduction of Zhukovsky’s diagrams in 1972 in the *Revue d’histoire des sciences* [10]. In what Zhukovsky called “critical points”, we recognize the so-called saddles, nodes, focuses and centers.

The second engineer is the Belgian Junius Massau, already encountered above about nomography. Considered as the creator of graphical integration, he developed elaborate techniques to construct precisely the integral curves of differential equations [29]. From 1878 to 1887, he published a large memoir on graphical integration [22], with the following objectives:

The purpose of this memoir is to present a general method designed to replace all the calculations of the engineer by graphic operations. [...]

¹⁰A more developed version of this Section can be found in my paper [31]. On Junius Massau, see also [29]. For a general survey on graphical integration of differential equations, see [28].

In what follows, we will always represent functions by curves; when we say ‘to give or to find a function’, it will mean giving or finding graphically the curve that represents it.¹¹

Book VI, the last book of the memoir, is devoted to applications in hydraulics. Massau examines the motion of liquids in pipes and canals. Among these specialized developments, a general and theoretical statement on graphic integration of first order differential equations appears. The entire study of a differential equation rests on the preliminary construction of the loci of points where integral curves have the same slope. Massau calls such a locus an “isocline”. The isoclines (under the Latin name of “directrices”) had already been introduced by Jean Bernoulli in 1694 as a universal method of construction of differential equations, particularly useful in the numerous cases in which the equations cannot be integrated by quadratures. Once enough isoclines are carefully drawn, one takes an arbitrary point A on the first curve and one constructs a polygon of integration $ABCD$, the successive sides of which have the slopes associated with isoclines and the successive summits of which are taken in the middle of the intervals between isoclines. Massau explains that you can easily obtain, by properly combining the directions associated to successive isoclines, graphical constructions equivalent to Newton-Cotes quadrature formulas, whereas the same problem would be difficult to solve numerically because of the implicit equations that appear at each step of the calculation. In fact, numerical algorithms of order greater than 2 will be discovered only at the turn of the 20th century by the German applied mathematicians Runge, Heun and Kutta.

The construction of the integral curves from isoclines is another way of studying globally a differential equation. In contrast to Poincaré’s abstract approach, Massau’s diagram both gives a global description and a local description of the curves. This diagram is both an instrument of numerical calculation – the ordinates of a particular integral curve can be measured with accuracy sufficient for the engineer’s needs – and a heuristic tool for discovering properties of the differential equation. For example, Massau applies this technique to hydraulics in studying the permanent motion of water flowing in a canal. He is interested in the variations of depth depending on the length of the canal, in the case of a rectangular section the width of which is growing uniformly. The differential equation to be solved is very complicated. With his elaborate graphical technique, Massau constructs isoclines and studies the behavior of the integral curves. He discovers that there is what he calls an “asymptotic point” : the integral curves approaching this point are turning indefinitely around it.

Massau then develops a theoretical study of singular points from isoclines. For a differential equation $F(x, y, y') = 0$, he considers the isoclines $F(x, y, \alpha) = 0$ as the projections on the plane (x, y) of the contour lines of the surface of equation $F(x, y, z) = 0$, and the integral curves as the projections of certain curves drawn on this surface. By geometric reasoning in this three-dimensional framework, Massau

¹¹L’objet de ce mémoire est d’exposer une méthode générale ayant pour but de remplacer les calculs de l’ingénieur par des opérations graphiques. [...] Dans ce qui va suivre, nous représenterons toujours les fonctions par des courbes; quand nous dirons donner ou trouver une fonction, cela voudra dire donner ou trouver graphiquement la courbe qui la représente [22, p. 13–16].

finds the same results as Poincaré concerning the singular points, but in a very different manner. He starts with the case where isoclines are convergent straight lines. In the general case, when isoclines pass by the same point, Massau studies the integral curves around this point by replacing the isoclines by their tangents. A singular point is always called a “focus”. The special case that we call “focus” today is the only one to receive a particular name, that of “asymptotic point”. Massau determines very carefully the various possible positions around a focus by considering the number of straight-line solutions passing through this point. In Massau’s reasoning, the isoclines play the same role as Poincaré’s arcs without contact to guide the path of integral curves. By using a graphical technique developed at first as a simple technique of numerical calculation, Massau also succeeds in a qualitative study, the purpose of which is the global layout of the integral curves and the description of their properties.

Knowing that Massau published his Book VI in 1887, is it possible that he had previously read Poincaré’s memoir and that he was inspired in it? It is not very probable because, in fact, Massau had already presented a first version of his Book VI on December 3, 1877, at the Charleroi section of the Association of the engineers of Ghent University, as is shown by the monthly report of this association. Further, the vocabulary, the notations and the demonstrations used by Massau are clearly different from those of Poincaré. In particular, Massau constantly works with the isoclines, a notion about which Poincaré never speaks. Finally, Massau, who quotes many people whose work is related to his, never quotes Poincaré.

Clearly, Massau and Zhukovsky are part of a geometric tradition that survived since the beginning of Calculus within engineering and applied mathematics circles. In this tradition one kept on constructing equations with graphical computation and mechanical devices, as theoretical mathematicians came to prefer the analytical approach. In this story, it is interesting to notice the existence of these two currents without an apparent link between them, the one among academic mathematicians, the other among engineers, with similar results that have been rediscovered several times independently.

6. Conclusion

In previous Sections, I presented some examples, mainly during the second half of the 19th century and the early 20th, that illustrate how civil and military engineers have been strongly engaged in the mathematical activity of their time. The examples that I have chosen are directly related to my own research, but we could mention some other recent works going in the same direction.

David Aubin [1] and Alan Gluchoff [17] have studied the scientific and social context of ballistics during and around the First World War, the one in France with the case of the Polygone de Gâvre, a famous ballistic research center situated in Brittany, and the other in the United States with the Aberdeen Proving Grounds, which was the prominent firing range in America. These papers prolong what I have presented in Section 4 and put in evidence similar collaborations and tensions

between two major milieus, the one of artillerymen, that is military engineers and officers in the military schools and on the battlefield, and the other one of mathematicians that were called to solve difficult theoretical problems. The new firing situations encountered during the First World War (fire against planes, fire over long distances through air layers of widely varying densities, etc.) generated new theoretical problems impossible to solve analytically and thus favored the creation of new numerical algorithms such as Adams-Moulton methods for ordinary differential equations.

Kostas Chatzis ([2], [8]) has studied the professional milieu of 19th century French engineers from the sociological and economic point of view. In particular, he has reviewed the conditions of diffusion of graphical statics, first in France, then in Germany and Italy, and again in France in the late 19th century. Graphical statics was an extensively used calculation tool, for example for the construction of metallic bridges and buildings such as the famous Eiffel Tower in Paris. Its development is closely linked to that of descriptive geometry and projective geometry. For her part, Marie-José Durand-Richard ([11], [12]) has examined the mathematical machines designed by engineers between Babbage's machine and the first digital computer. These machines, which include planimeters, integragraphs and differential analyzers, have played a major role in solving differential equations encountered in many areas. Among the most important of them is the polar planimeter of Jakob Amsler (1823–1912), the integragraph of Abdank-Abakanowicz (1852–1900), the harmonic analyzer of Lord Kelvin (1824–1907) and the large differential analyzers of Vannevar Bush (1890–1974) in the United States and Douglas Rainer Hartree (1897–1958) in Great-Britain. The technical and industrial design of these machines has contributed to the development of new numerical and graphical methods, but also to some advances in logic and information theory, as seen in the work of Claude Elwood Shannon (1916–2001). During and after the Second World War, all this knowledge has been transferred to the first computers like ENIAC. More generally, Renate Tobies ([26], [27]) has explored the relationships between mathematics, science, industry, politics and society, taking as support of her work the paradigmatic case of Iris Runge (1888–1966), a Carl Runge's daughter, who was a mathematician working for Osram and Telefunken corporations.

In the early 20th century, the emerging applications of electricity became a new field of research for engineers, who were then faced with nonlinear differential equations with complex behavior. Jean-Marc Ginoux, Christophe Letellier and Loïc Petitgirard ([21], [14], [15], [16]) have studied the history of oscillatory phenomena produced by various electrical devices. Balthazar Van der Pol (1889–1959) is one of the major figures in this field. Using Massau's techniques of graphical integration (see Section 5), in particular the method of isoclines, Van der Pol studied the oscillations in an electric circuit with a triode, and succeeded in describing the continuous passage from sinusoidal oscillations to quasi-aperiodic oscillations, which he called "relaxation oscillations". A little later, Aleksander Andronov (1901–1952) established a correspondence between the solution of the differential system given by Van der Pol to characterize the oscillations of the triode and the concept of limit cycle created by Poincaré, thus connecting the investigations of engineers

to those of mathematicians. In his thesis, Jean-Marc Ginoux [14] lists carefully all the engineering works on this subject between 1880 and 1940.

Loïc Petitgirard [24] is also interested in another engineer mathematician struggling with nonlinear differential equations: Nicolas Minorsky (1885-1970), an engineer of the Russian Navy trained at the Naval Academy in St. Petersburg. Minorsky was a specialist in the design, stabilization and control of ships. In his naval research during the years 1920-1930, he was confronted with theoretical problems related to nonlinear differential equations, and established mathematical results adapted to maritime issues. He also conceived a system of analog computing in connection with the theory of nonlinear oscillations and the stability theory, emphasizing that the theories produced by mathematicians like Poincaré remain incomplete without computational tools to implement them.

All these recent works demonstrate a large entanglement between the milieus of civil engineers, military engineers, physicists, astronomers, applied mathematicians and pure mathematicians (of course, these categories were far from watertight). It seems necessary to take all of them into account if we want to rethink the construction of knowledge in the domain of numerical analysis and if we want to avoid the historical bias of the projection into the past of contemporary conceptions of the discipline. A new history remains to be written, which would not focus only on a few major authors and some high-level mathematical algorithms, but also on the actors of the domain in the broad sense of the term, and on the numerical and graphical methods actually performed by users on the ground or at the office. A good start to this problem could be, among others, to identify, classify and analyze the mathematical texts contained in the many engineering journals published in Europe and elsewhere since the early 19th century. This could allow to characterize more precisely the mathematical knowledge created and used by engineers, and to study the circulation of this knowledge between the professional circles of engineers and other groups of actors involved in the development of mathematical ideas and practices.

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