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# On the Cost of Simulating a Parallel Boolean Automata Network by a Block-Sequential One

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**Abstract.** In this article we study the minimum number  $\kappa$  of additional automata that a Boolean automata network (BAN) associated with a given block-sequential update schedule needs in order to simulate a given BAN with a parallel update schedule. We introduce a graph that we call NECC graph built from the BAN and the update schedule. We show the relation between  $\kappa$  and the chromatic number of the NECC graph. Thanks to this NECC graph, we bound  $\kappa$  in the worst case between  $n/2$  and  $2n/3+2$  ( $n$  being the size of the BAN simulated) and we conjecture that this number equals  $n/2$ . We support this conjecture with two results: the clique number of a NECC graph is always less than or equal to  $n/2$  and, for the subclass of bijective BANs,  $\kappa$  is always less than or equal to  $n/2+1$ .

**Keywords:** Boolean automata networks, intrinsic simulation, block-sequential update schedules.

## 1 Introduction

In this article, we study Boolean automata networks (BANs). A BAN can be seen as a set of two-states automata interacting with each other and evolving in a discrete time. BANs have been first introduced by McCulloch and Pitts in the 1940<sup>s</sup> [12]. They are common representational models for natural dynamical systems like neural or genetic networks [6,9], but they are also computing models with which we can study computability or complexity. In this article we are interested in intrinsic simulations between BANs, *i.e.* simulations that focus on the dynamics rather than the computing power. More concretely, given a BAN  $A$  we want to find a BAN  $B$  satisfying some constraints and that reproduce the dynamics of  $A$ . Intrinsic simulation of BANs has already been used in the 1980<sup>s</sup> [2,7,16,17]. But since then, the notion has not produced much more literature [14,13]. Meanwhile, intrinsic simulation of many other similar objects (cellular automata, tilings, subshifts, self-assembly, etc.) has been really developing since 2000 [3,4,5,8,10,11,15].

A given BAN can be associated with several dynamics, depending on the schedule (*i.e.* the order) we choose to update the automata. In this article, we will consider all block-sequential update schedules: we group automata into blocks, and we update all automata of a block at once, and iterate the blocks sequentially. Among these update schedules are the following classical ones: the parallel one (a unique block composed of  $n$  automata) and the  $n!$  sequential ones ( $n$  blocks of 1 automaton). The pair of a BAN and its update schedule is called a scheduled Boolean automata network (SBAN).

For the last 15 years, people have studied the influence of the update schedules on the dynamics of a BAN [1]. Here, we do the opposite. We take a SBAN, and try to find the smallest SBAN with a constrained update schedule which simulates this dynamics. For example, let  $N$  be a parallel SBAN of size 2 with 2 automata that exchange their values.

There are no SBANs  $N'$  of size 2 with a sequential update schedule which simulates  $N$ . Indeed, when we update the first automaton, we necessarily erase its previous value. If we did not previously save it, we cannot use the value of the first automaton to update the second automaton. Thus,  $N'$  needs an additional automaton to simulate  $N$  under the sequential update schedule constraint. Note that a SBAN  $N$  of size  $n$  with a parallel update schedule can always be simulated by a SBAN  $N'$  of size  $2n$  with a given sequential update schedule. Indeed, we just need to add  $n$  automata which copy all the information from the original automata and then, we compute sequentially the updates of the original automata using the saved information. However, in this article, we will bound more precisely the number of required additional automata, function of  $n$ , in the worst case.

In Section 2, we define BANs and detail the notion of simulation that we use. In Section 3, we consider the dynamics of a BAN  $F$  with automata set  $V$  and the parallel update schedule and we consider a block-sequential update schedule  $W$ . We focus on the minimum number  $\kappa(F, W)$  of additional automata that a SBAN needs to simulate this dynamics with an update schedule identical to  $W$  on  $V$ . In Section 4, we define a graph which connects configurations depending on a BAN  $F$  and a block-sequential update schedule  $W$ . We prove that the chromatic number of this graph gives us the number  $\kappa(F, W)$  defined in the previous section. We also enunciate the following conjecture:  $\kappa(F, W)$  is always less than or equal to  $n/2$ , with  $n$  the size of the BAN  $F$ . In Section 5, we define another graph constructed from the previous graph where we quotient the configuration which have the same image. We prove that the chromatic number of this new graph is always greater than that of the previous graph. Then, we find an upper bound for  $\kappa(F, W)$  depending on  $n$ , the size of  $F$ . In Section 6, we try to support our conjecture by making an upper bound for the clique number of the graph defined in Section 4. Finally, in Section 7, we study  $\kappa(F, W)$  in the case where  $F$  is bijective.

## 2 Definitions and notations

### 2.1 BANs and SBANs

In this article, unless otherwise stated, BANs have a size  $n \in \mathbb{N}$ , which means that they are composed of  $n$  automata numbered from 0 to  $n - 1$ . Usually, we denote this set of automata by  $V = \{0, 1, \dots, n - 1\}$  (which will be abbreviated by  $\llbracket 0, n \rrbracket$ ). Each automaton can take two states in the Boolean set  $\mathbb{B} = \{0, 1\}$ . Notice that for all  $b \in \mathbb{B}$ , we denote by  $\bar{b}$  the negation of the state of  $b$ . In other words,  $\bar{0} = 1$  and  $\bar{1} = 0$ . A configuration a Boolean vector of size  $n$  such that each element of the vector is the state of one automaton of the BAN. In other words, if  $x$  is a configuration, then  $x \in \mathbb{B}^n$  and  $x = (x_0, \dots, x_{n-1})$  with  $x_i$  the state of automaton  $i$  (for all  $i$  in  $V$ ). We also denote by  $\bar{x}$  the negation of  $x$  such that  $\bar{x} = (\bar{x}_0, \dots, \bar{x}_{n-1})$ . And we denote by  $\bar{x}^i$  or  $\bar{x}^I$  the negation of  $x$  respectively restricted to an automaton or a set of automata. So, if  $x' = \bar{x}^I$  then  $\forall i \in V$ , if  $i \in I$  then  $x'_i = \bar{x}_i$  and  $x'_i = x_i$  otherwise. Furthermore,  $\forall I \subseteq V$ , we denote by  $x_I$  the restriction of  $x$  in  $I$ . In other words, if  $I = \{i_1, i_2, \dots, i_p\}$  with  $i_1 < i_2 < \dots < i_p$  then  $x_I = (x_{i_1}, x_{i_2}, \dots, x_{i_p})$ . We also denote by  $x_{\bar{I}}$  the restriction of  $x$  in  $V \setminus I$ . In this article, we only study BANs with block-sequential update schedules. A SBAN  $N = (F, W)$  is characterised by two things:

- a global update function  $F : \mathbb{B}^n \rightarrow \mathbb{B}^n$  which represents the BAN;
- a block-sequential update schedule  $W$ .

The *global update function* of a BAN is the collection of the local update functions of the BAN. As a consequence, we have  $F(x) = (f_0(x), \dots, f_{n-1}(x))$  with  $\forall i \in V$ ,  $f_i : \mathbb{B}^n \rightarrow \mathbb{B}$  the

local update functions of automata  $i$ . We also use the  $I$ -update function  $F_I$  with  $I \subseteq V$  which gives a configuration where the states of automata in  $I$  are updated and the other are not. In other words,  $\forall i \in V, F_I(x)_i = f_i(x)$  if  $i \in I$  and  $x_i$  otherwise. And, for singleton, we simply write  $F_i(x) = F_{\{i\}}(x)$ .

*Remark 1.* It is important not to confuse  $F_I(x)$  and  $F(x)_I$ . The first one is the  $I$ -update function that we have just defined. The second is the configuration  $F(x)$  restricted to  $I$ .

A *block-sequential update schedule* is an ordered partition of  $V$ . The set ordered partition of  $V$  is denoted by  $\vec{\mathcal{P}}(V)$ . Let  $W \in \vec{\mathcal{P}}(V)$ . We make particular use of  $F^W$  defined as  $F^W = F_{W_p} \circ \dots \circ F_{W_0}$ . Let  $x \in \mathbb{B}^n$  be the configuration of the BAN at time  $t$ . Then,  $F^W(x)$  is the configuration of the BAN at the time  $t + 1$ . There are two particular kinds of block-sequential update schedules:

1. the parallel mode where all automata are updated at the same time step. So, we have  $W = [V]$  (i.e.  $|W| = 1$  and  $W_0 = V$ ) and  $F^W = F$ .
2. the sequential modes where automata are updated one at the time. So, we have  $\forall i \in \llbracket 0, p \rrbracket, |W_i| = 1$  and  $|W| = n$ .

For all  $j \in \llbracket 0, p \rrbracket$ , we denote  $W_{<j} = \bigcup_{i=0}^{j-1} W_i$ . In particular, we have  $W_{<0} = \emptyset$  and  $W_{<p} = V$ . For all  $i \in \llbracket 0, p \rrbracket$ , we also denote  $W^{<i} = (W_0, W_1, \dots, W_{i-1})$ . In particular, we have  $W^{<0} = [ ]$  (the empty vector) and  $W^{<p} = W$ .

*Remark 2.* We will often use the two following notations:

- $F^{W^{<j}}$  which is equal to  $F_{W_{j-1}} \circ \dots \circ F_{W_0}$ . It is the function which makes the  $j$  first steps of the transition of the SBAN  $(F, W)$ .
- $F_{W_{<j}}$  which is equal to  $F_{W_0 \cup \dots \cup W_{j-1}}$ . It is the function which updates only the automata in the  $j$  first blocks of  $W$ .

## 2.2 Simulation

Here, we define the notion of simulation used in this article. We consider that a SBAN  $N$  of size  $m$  simulates another SBAN  $N'$  of size  $n$  if there is a projection from  $\mathbb{B}^m$  to  $\mathbb{B}^n$  such that the projection of the update in  $N'$  equals the update in  $N$  of the projection.

**Definition 1.** Let  $F : \mathbb{B}^n \rightarrow \mathbb{B}^n$  and  $F' : \mathbb{B}^m \rightarrow \mathbb{B}^m$  with  $m \geq n$ ,  $V = \llbracket 0, n \rrbracket$  and  $V' = \llbracket 0, m \rrbracket$ ,  $W \in \vec{\mathcal{P}}(V)$  and  $W' \in \vec{\mathcal{P}}(V')$ . Let  $h : V \rightarrow V'$  be an injective function. And let us consider  $\varphi_h : \mathbb{B}^m \rightarrow \mathbb{B}^n$   $x \mapsto (x_{h(i)})_{i \in V}$ . We say that  $(F', W')$   $h$ -simulates  $(F, W)$  and we write  $(F', W') \triangleright^h (F, W)$  if  $\varphi_h \circ F'^{W'} = F^W \circ \varphi_h$ . And we say that  $(F', W')$  simulates  $(F, W)$  and we write  $(F', W') \triangleright (F, W)$  if there is a  $h$  such that  $(F', W') \triangleright^h (F, W)$ .

In this article we often use an *id*-simulation which is a  $h$ -simulation with  $h$  the identity function ( $h(i) = i$ ).

### 3 Number of required additional automata

Here, we focus on finding a block-sequential SBAN  $(F', W')$  which simulates a parallel SBAN  $(F, [V])$ . We could as well study the problem of finding a block-sequential SBAN  $(F', W')$  which simulates another block-sequential SBAN  $(G, W)$ . However, this problem is in fact the same. Indeed, for all block-sequential SBAN  $(G, W)$ , the parallel SBAN  $(G^W, [V])$  *id*-simulate the SBAN  $(G, W)$ .

In this section, we define the main object of this article. Given a global transition function  $F$  and a block-sequential update schedule  $W$ , we consider the smallest SBAN  $(F', W')$  (where  $W'$  extends  $W$  by preserving its order) which simulates  $(F, [V])$  ( $[V]$  being the parallel update schedule). This new SBAN  $(F', W')$  often needs additional automata to simulate the SBAN  $(F, [V])$ . We denote by  $\kappa(F, W)$  this number of additional automata needed. And  $\forall n \in \mathbb{N}$ , we denote by  $\kappa_n$  the maximum of  $\kappa(F, W)$  for any SBAN  $(F, W)$  of size  $n$ . Let  $W \in \overrightarrow{\mathcal{P}}(V)$  be an update schedule. We know that each automaton of a block-sequential SBAN is updated only once, in one step of the update schedule. We denote by  $W(i)$  the step at which  $i$  is updated. More formally,  $\forall i \in V, W(i)$  is the number  $j \in \llbracket 0, p \rrbracket$  such that  $i \in W_j$ .

From an update schedule  $W$  and a BAN of size  $n$ , we define the notion of update schedule extending  $W$  for a bigger BAN of size  $m$ . Let  $V' = \llbracket 0, m \rrbracket$ . Let  $h : V \rightarrow V'$  be an injective function. We denote by  $\mathcal{E}_h(W, V')$  the set of update schedules  $W'$  extending  $W$  such that each  $W'$  preserves the order of  $W$  for the projection by  $h$  of the automata of  $V$ . That is to say, if two automata of  $V$  are updated at the same step in  $W$ , then the projection of these automata are updated in the same step in  $W'$ . Moreover if one is updated before another one, then the projection of these automata in  $V'$  will preserve the same update order in  $W'$ . More formally,  $\mathcal{E}_h(W, V') = \{W' \in \overrightarrow{\mathcal{P}}(V') \mid \forall i \in V, W(i) = W(i') \iff W'(h(i)) = W'(h(i')) \text{ and } W(i) < W(i') \implies W'(h(i)) < W'(h(i'))\}$ .

**Definition 2.**  $\kappa(F, W)$  is the smallest  $k$  such that,  $\exists h : V \rightarrow V'$  an injective function, an update schedule  $W' \in \mathcal{E}_h(W, V')$  extending  $W$ , a BAN  $F' : \mathbb{B}^{n+k} \rightarrow \mathbb{B}^{n+k}$  such that  $(F', W') \triangleright^h (F, [V])$  with  $V' = \llbracket 0, n + k \rrbracket$ . Furthermore,  $\kappa_n$  is the value of  $\kappa(F, W)$  in the worst case among all SBANs. In other words,  $\kappa_n = \max(\{\kappa(F, W) \mid F : \mathbb{B}^n \rightarrow \mathbb{B}^n \text{ and } W \in \overrightarrow{\mathcal{P}}(V)\})$ .

The main objective of this article is to bound the values of  $\kappa_n$ .

### 4 NECCs set and NECC graph

To give an answer to this problem, we introduce a new concept: the not equivalent and confusable configurations or NECCs and the NECC graph. Theorem 1 will show that the logarithm of the chromatic number of the NECC graph of a SBAN and the  $\kappa$  of this SBAN are equal.  $\text{NEC}_F$  or only NEC (the acronym standing for *not equivalent configurations*) is the set of couple of configurations with different images by  $F$ . In other words,

$$\text{NEC}_F = \{(x, x') \in \mathbb{B}^n \times \mathbb{B}^n \mid F(x) \neq F(x')\}$$

. We call *confusable configurations* and denote by  $\text{CC}_{F,W}$  or only CC (the acronym standing for *confusable configurations*) is the set of couples of configurations which become identical when we update  $i$  first blocks of  $W$  (with  $i \in \llbracket 0, p \rrbracket$ ). So we have

$$\text{CC} = \{(x, x') \in \mathbb{B}^n \times \mathbb{B}^n \mid \exists i \in \llbracket 0, p \rrbracket, F_{W_{<i}}(x) = F_{W_{<i}}(x')\}$$

**Definition 3.**  $\text{NECC}_{F,W}$  or only NECC (the acronym standing for not equivalent and confusable configurations) is the set of couples of configurations which are confusable and not equivalent at the same time,  $\text{NECC}_{F,W} = \text{CC}_{F,W} \cap \text{NEC}_F$ .

Also, for all  $x, x' \in \mathbb{B}^n$ , we denote by  $\text{CC}_{F,W}(x, x')$  (or just  $\text{CC}_{F,W}(x, x')$ ) the set of time steps  $i$  which make them confusable. More formally,  $\forall x, x' \in \mathbb{B}^n$ ,  $\text{CC}_{F,W}(x, x') = \{i \in \llbracket 0, p \rrbracket \mid F_{W_{<i}}(x) = F_{W_{<i}}(x')\}$ .

*Remark 3.* We have  $\text{CC}(x, x') = \emptyset$  if and only if  $(x, x') \notin \text{CC}$ .

**Definition 4.** We call NECC graph and denote by  $(\mathbb{B}^n, \text{NECC})$  the graph which has the set of configurations  $\mathbb{B}^n$  as nodes and the set of NECC couples as edges.

We make a particular use of two concepts of graph theory. A *valid coloration* of  $G$  is a coloration of all the nodes of  $G$  such that if there is an edge between two nodes then they do not have the same color. We denote by  $\chi(G)$  the *chromatic number* of the graph  $G$ , namely the minimum number of colors of a valid coloration of  $G$ . We denote the chromatic number of the NECC graph by  $\chi(\text{NECC}) = \chi(\mathbb{B}^n, \text{NECC})$ . We see in Lemma 1 that we can get a valid coloration of the  $\text{NECC}_{F,W}$  graph from the SBAN  $(F', W')$  which simulates  $(F, [V])$ . This coloration does not use more than  $2^k$  colors with  $k$  the number of additional automata of  $F'$ . We color the configuration of the NECC graph using the values of the added automata after the update.

**Lemma 1.** For any BAN  $F : \mathbb{B}^n \rightarrow \mathbb{B}^n$  and any block-sequential update schedule  $W_H$ ,  $\kappa(F, W) \geq \lceil \log_2(\chi(\text{NECC}_{F,W})) \rceil$ .

*Proof.* Let  $h : V \rightarrow V'$  injective,  $W' \in \mathcal{E}_h(W, V')$ , let  $p = |W|$ ,  $p' = |W'|$  and  $F' : \mathbb{B}^{n+k} \rightarrow \mathbb{B}^{n+k}$  such that  $(F', W') \triangleright^h (F, [V])$ . We prove that  $k \geq \lceil \log_2(\chi(\text{NECC})) \rceil$ . Let  $V_x = \{h(i) \mid i \in V\}$ ,  $V_y = V' \setminus V_x$  and  $y = [0]^k$ . First, let us prove that if  $(x, x') \in \text{NECC}$  then  $F'(z)_{V_y} \neq F'(z')_{V_y}$  with  $z_{V_y} = z'_{V_y} = y$ ,  $\varphi_h(z) = x$  and  $\varphi_h(z') = x'$ . Let  $(x, x') \in \text{NECC}$ . For the sake of contradiction, suppose we have  $F'(z)_{V_y} = F'(z')_{V_y}$ . Since  $(x, x') \in \text{NECC}$ , we have  $F(x) \neq F(x')$  and  $\exists j \in \llbracket 0, p \rrbracket$ ,  $F_{W_{<j}}(x) = F_{W_{<j}}(x')$ . Let  $j' \in \llbracket 0, p' \rrbracket$  be the smallest number such that  $\forall i \in W_{<j}$ ,  $h(i) \in W'_{<j'}$ . Let  $Z = F'^{W'_{<j'}}(z) = F'_{W'_{j'}} \circ \dots \circ F'_{W'_0}(z)$  and  $Z' = F'^{W'_{<j'}}(z') = F'_{W'_{j'}} \circ \dots \circ F'_{W'_0}(z')$ . We have  $z_{V_y} = y = z'_{V_y}$  and  $F(z)_{V_y} = F(z')_{V_y}$  by hypothesis. Thus,  $Z_{V_y} = Z'_{V_y}$ . Furthermore, we have  $\varphi_h(Z_{V_x}) = F_{W_{<j}}(x) = F_{W_{<j}}(x') = \varphi_h(Z'_{V_x})$ . As a result,  $Z_{V_x} = Z'_{V_x}$  and  $Z = Z'$ . Consequently,  $F'(z) = F'_{W'_{p-1}} \circ \dots \circ F'_{W'_0}(z) = F'_{W'_{p-1}} \circ \dots \circ F'_{W'_{j'}}(Z)$  and  $F'(z') = F'_{W'_{p-1}} \circ \dots \circ F'_{W'_0}(z') = F'_{W'_{p-1}} \circ \dots \circ F'_{W'_{j'}}(Z')$ . We have then  $F'(z) = F'(z')$  (because  $Z = Z'$ ). However,  $(x, x') \in \text{NEC}$ . Thus,  $F'(z)_{V_x} = F(x) \neq F(x') = F'(z')_{V_x}$ . As a consequence, we have also  $F'(z) \neq F'(z')$ . There is a contradiction. Consequently, if  $(x, x') \in \text{NECC}$  then  $F(z)_{V_y} \neq F(z')_{V_y}$ . In other words,  $\{F(z)_{V_y} \mid z_{V_y} = y\}$  has at least  $\chi(\text{NECC})$  different values. To encode these values, we need to have  $k \geq \lceil \log_2(\chi(\text{NECC})) \rceil$ . So  $\kappa(F, W) \geq \log_2(\chi(\text{NECC}))$ .  $\square$

We see in Lemma 2 that we can get a SBAN  $(F', W')$  which simulates  $(F, [V])$  from a valid coloration of the  $\text{NECC}_{F,W}$  graph.

**Lemma 2.** For any BAN  $F : \mathbb{B}^n \rightarrow \mathbb{B}^n$  and any block-sequential update schedule  $W$ ,  $\kappa(F, W) \leq \lceil \log_2(\chi(\text{NECC}_{F,W})) \rceil$

*Proof.* Let  $k = \lceil \log_2(\chi(\text{NECC})) \rceil$ . We take  $W'$  such that first, we update the  $k$  last nodes, and after, we update as  $W$ :  $W' = (\{n\}, \{n+1\}, \dots, \{n+k-1\}, W_0, W_1, \dots, W_{p-1})$ . Let  $V_x = \llbracket 0, n \llbracket$  and  $V_y = \llbracket n, n+k \llbracket$ . Let  $\text{color} : \mathbb{B}^n \rightarrow \mathbb{N}$  be a minimum coloration of the NECC graph. For all  $x \in \mathbb{B}^n$ , let  $\text{COLOR}(x)$  be the number  $\text{color}(x)$  encoded with a Boolean vector of size  $k$ . It is possible to encode it with  $k$  Booleans because with  $k$  bits we can encode  $2^k$  values and  $k = \lceil \log_2(\chi(\text{NECC})) \rceil$  so we can encode at least  $\chi(\text{NECC})$  values and we have  $|\{\text{color}(x) | x \in \mathbb{B}^n\}| = \chi(\text{NECC})$ . For all  $z'$  in  $\mathbb{B}^{n+k}$ , let  $x' \in \mathbb{B}^n$  and  $y' \in \mathbb{B}^k$  be respectively the first and the second parts of  $z'$  (we denote by  $z' = x' || y'$ ). For all  $j \in \llbracket 0, p \llbracket$ . Let  $A_j(z') = \{F(x) | x \in \mathbb{B}^n \text{ and } \text{COLOR}(x) = y' \text{ and } F_{W_{<j}}(x) = x'\}$ . We can prove that  $|A_j| \leq 1$ . For the sake of contradiction suppose  $\exists F(x'), F(x'') \in A$ ,  $F(x') \neq F(x'')$ . Then,  $(x', x'') \in \text{NEC}$ . However,  $\exists j \in \llbracket 0, p \llbracket | F_{W_{<j}}(x') = x = F_{W_{<j}}(x'')$ . Then,  $(x', x'') \in \text{CC}$ . So  $(x', x'') \in \text{NECC}$ . However,  $\text{COLOR}(x') = y = \text{COLOR}(x'')$ . There is a contradiction because if  $(x', x'') \in \text{NECC}$  then  $\text{COLOR}(x') \neq \text{COLOR}(x'')$ . Let  $\forall i \in \llbracket n, n+k \llbracket$ ,  $F'(z)_i = \text{COLOR}(x)_{i-n}$ . Let  $F' : \mathbb{B}^{n+k} \rightarrow \mathbb{B}^{n+k}$  such that for all  $z' \in \mathbb{B}^{n+k}$ ,  $F_{V_y}(z') = \text{COLOR}(x)$  and  $\forall j \in \llbracket 0, p \llbracket$ ,  $F'(z')_{W'_{k+j}} = F(x)_{W_j}$  if  $A_j(z') = \{F(x)\}$  or  $[0]^{W_j}$  otherwise. Now, we show that  $\forall z = x || y \in \mathbb{B}^{n+k}$ ,  $F'^{W'}(z)_{V_x} = F(x)$ . Let  $z = x || y \in \mathbb{B}^{n+k}$ . Let us show that  $\forall i \in \llbracket 0, p \llbracket$ ,  $F'^{W'^{k+j}}(z)_{V_x} = F_{W_{<j}}(x)$ . Let  $j = 0$ . We have  $F'^{W'^{<k+j}}(z)_{V_x} = F'^{W'^{<k}}(z)_{V_x} = x$  (because in the  $k$  first steps of  $W'$  we only update the automata of  $V_y$ ) and  $F_{W_{<j}}(x) = F_{W_{<0}}(x) = x$ . So  $F'^{W'^{k+j}}(z)_{V_x} = F_{W_{<j}}(z_{V_x})$ . Let  $j \in \llbracket 0, p \llbracket$ . Let us say that  $F'^{W'^{<k+j}}(z)_{V_x} = F_{W_{<j}}(x)$ . Let  $z' = F'^{W'^{<k+j}}(z)$ . We have  $F'^{W'^{<k+j+1}}(z)_{V_x} = F'_{W'^{k+j+1}}(z')$ . Thus,  $F'^{W'^{<k+j}}(z)_{V_x \setminus W'_{k+j+1}} = z'_{V_x \setminus W'_{k+j+1}} = F_{W_{<j}}(x)_{V_x \setminus W_{j+1}} = F_{W_{<j+1}}(x)_{V_x \setminus W_{j+1}}$ . Furthermore,  $\text{COLOR}(x) = F(z)_{V_y} = z'_{V_y}$ , and by induction hypothesis,  $F_{W_{<j}}(x) = F'^{W'^{k+j}}(z) = z'_{V_x}$ . Thus,  $F(x) \in A_j(z')$ . As a consequence,  $F'^{W'^{<k+j+1}}(z)_{W'_{k+j+1}} = F'_{W'^{k+j+1}}(z')_{W'_{k+j+1}} = F(x)_{W'_{k+j+1}} = F(x)_{W_{j+1}}$ . As a result,  $F'^{W'^{<k+j}}(z)_{V_x} = F_{W_{<j+1}}(x)$ . Consequently,  $\forall z = x || y \in \mathbb{B}^{n+k}$ ,  $F'^{W'}(z)_{V_x} = F(x)$ . Thus,  $(F', W') \triangleright^{id} (F, [V])$ . Finally,  $\kappa(F, W) \leq \lceil \log_2(\chi(\text{NECC})) \rceil$ .  $\square$

Lemma 1 and Lemma 2 show that there is an equivalence between a coloration of the  $\text{NECC}_{F,W}$  graph and a SBAN  $(F', W')$  which simulates  $(F, [V])$ . More precisely, from the number of colors of the NECC graph, we can upper bound the number of required additional automata of the SBAN. And reciprocally.

**Theorem 1.** *For any BAN  $F : \mathbb{B}^n \rightarrow \mathbb{B}^n$  and any block-sequential update schedule  $W$ ,  $\kappa(F, W) = \lceil \log_2(\chi(\text{NECC}_{F,W})) \rceil$ .*

In Lemma 3, using the example of  $n/2$  automata which exchange their values, we find a lower bound for  $\kappa_n$ . We use the fact that if we take the good update schedule  $W$ , this  $\text{NECC}_{F,W}$  graph has a big clique number.

**Lemma 3.**  $\forall n \in \mathbb{N}$ ,  $\kappa_n \geq \lfloor n/2 \rfloor$ .

*Proof.* We suppose that  $n$  is even. However, if it is not, we just have to add a useless automaton and the result is the same. Let us consider the BAN  $F$  such that:  $\forall i \in \llbracket 0, \frac{n}{2} \llbracket$ ,  $f_i(x) = x_{i+n/2}$  and  $\forall i \in \llbracket \frac{n}{2}, n \llbracket$ ,  $f_i(x) = x_{i-n/2}$ . We also consider the simple sequential update schedule  $W$ . Let  $X = \{x \in \mathbb{B}^n | x_{\llbracket n/2, n \llbracket} = [0]^{n/2}\}$ , and  $x, x' \in X$  such that  $x \neq x'$ . When we update the first half of the automata,  $x$  and  $x'$  both become the configuration full of 0. Then, for  $i = n/2$ , we have  $F_{W_{<i}}(x) = [0]^n = F_{W_{<i}}(x')$ . Thus,  $(x, x') \in \text{CC}$ . We also have  $x \neq x'$ . So  $\exists i \in \llbracket n/2, n \llbracket$  such that  $x_i \neq x'_i$  and  $f_{i+n/2}(x) = x_i$

and  $f_{i+n/2}(x') = x'_i$ . Consequently,  $f_{i+n/2}(x) \neq f_{i+n/2}(x')$ . Then,  $F(x) \neq F(x')$  and  $(x, x') \in \text{NECC}$ . As a result, we have  $(x, x') \in \text{NECC}$ . We know that  $X$  is a clique. Moreover,  $X$  is a clique of size  $2^{n/2}$ . Thus, the chromatic number of the NECC graph is at least  $2^{n/2}$  and  $\kappa(F, W) \geq n/2$ . So  $\forall n \in \mathbb{N}, \kappa_n \geq n/2$ .  $\square$

We conjecture that  $\lfloor n/2 \rfloor$  is the upper bound as well. This conjecture had not been proven yet, but Theorem 3 supports it by giving an upper bound to the clique number of a NECC graph.

*Conjecture 1.*  $\forall n \in \mathbb{N}, \kappa_n \leq \lfloor n/2 \rfloor$ .

## 5 INECC graph

In this section, we define the INECC graph which is the NECC graph after we quotiented its configurations which have the same image. We can prove that INECC has a bigger chromatic number than the NECC graph, find an upper bound of the INECC graph chromatic number and then have an upper bound of the NECC graph chromatic number as well.

**Definition 5.** *The INECC graph nodes are the image of the configurations of the NECC graph that is to say:  $\{F(x) \mid x \in \mathbb{B}^n\}$ . Two images  $y$  and  $y'$  are in relation in the INECC graph if they have each one a fiber that are in relation in the NECC graph. That is to say if  $\exists x, x' \in \mathbb{B}^n$  such that  $F(x) = y, F(x') = y'$  and  $(x, x') \in \text{NECC}$  then  $y$  and  $y'$  are in relation in the INECC graph.*

Now we prove that we can use a valid coloration of the INECC graph to color a NECC graph.

**Lemma 4.** *We have  $\chi(\text{INECC}) \geq \chi(\text{NECC})$ .*

*Proof.* We partition the configurations into sets of equivalent configurations (*i.e.* configurations which have the same image)  $E_1, E_2, \dots, E_k$ . And we denote by  $Y_i \in \mathbb{B}^n$  the image of the configurations of  $E_i$  for each  $i \in \llbracket 0, k \rrbracket$ . In other words,  $\forall i \in \llbracket 0, k \rrbracket, \forall x \in E_i, F(x) = Y_i$ . Let  $\text{color} : \llbracket 0, k \rrbracket \rightarrow \mathbb{N}^*$  be an optimal coloration of the INECC graph. In the NECC graph, we can color all the configurations of a set  $E_i$  by the color of  $Y_i$  in the INECC graph. Let  $x, x' \in \mathbb{B}^n$ . If  $x$  and  $x'$  have the same color:

1. either  $x$  and  $x'$  are in the same set  $E_i$ , and then  $(x, x') \notin \text{NECC}$  because they are equivalent;
2. or they are in two distinct sets  $E_i$  and  $E_{i'}$ . In this case  $(x, x') \notin \text{NECC}$  otherwise  $Y_i$  and  $Y_{i'}$  would be connected in the INECC graph and they would have different colors.

So, the coloration is a valid coloration and does not need more colors than the INECC graph coloration:  $\chi(\text{INECC}) \geq \chi(\text{NECC})$ .  $\square$

*Remark 4.* We can see that if we take two SBANs  $(F, W)$  and  $(F, W')$  with  $W'$  a sequentialised version of  $W$  (*i.e.* an update schedule that breaks the blocks of  $W$  into blocks of size 1), the chromatic number of the NECC graph of  $(F, W)$  are always greater than or equal to the chromatic number of the NECC graph of  $(F, W')$ . Indeed, the set of edges of the NECC graph of  $(F, W)$  is included in the set of edges of the NECC graph of  $(F, W')$ , thus the chromatic number of the latter is greater. Furthermore, the same reasoning applies to the INECC graph. As a result, if we want to find an upper bound to the chromatic number of the NECC or INECC graph, we can restrict our study to SBAN with sequential update schedule.



*Remark 5.* We can see that if we have a SBAN  $(F, W)$ , with  $W$  a sequential update schedule, we can find another SBAN  $(F', W')$  with  $W'$  the simple update schedule  $(\{0\}, \{1\}, \dots, \{n-1\})$  which will have the same NECC and INECC graphs up to permutation. As a consequence, their NECC and INECC graphs chromatic number are equal. Thus, if we want to find an upper bound to the chromatic number of the NECC or INECC graph, we can restrict our study to SBAN with the simple sequential update schedule  $(\{0\}, \{1\}, \dots, \{n-1\})$ .

We now find an upper bound to the INECC graph chromatic number by defining a colouring method of the graph based on a greedy algorithm.

**Lemma 5.**  $\chi(\text{INECC}) \leq 2^{2n/3+2}$ .

*Proof.* We consider the BAN  $F : \mathbb{B}^n \rightarrow \mathbb{B}^n$  and the simple sequential update schedule  $W = (\{0\}, \{1\}, \dots, \{n-1\})$ . We partition the configurations into sets of equivalent configurations  $E_1, E_2, \dots, E_k$ . We denote by  $Y_i \in \mathbb{B}^n$  the images of the configurations of  $E_i$  for each  $i \in \llbracket 1, k \rrbracket$ . In other words,  $\forall i \in \llbracket 1, k \rrbracket, \forall x \in E_i, F(x) = Y_i$ . We denote the neighbour of the  $i^{\text{th}}$  image by  $V(i)$ . So

$$V(i) = \{i' \mid \exists x \in E_i, x' \in E_{i'}, (x, x') \in \text{NECC}\}$$

. The degree of the  $i^{\text{th}}$  image is denoted by  $D(i) = |V(i)|$ . We sort the images by decreasing degree so that  $\forall i < i', D(i) \geq D(i')$ . To choose the color of  $Y_i$ , we apply a greedy algorithm: we use the smallest color not already used by a neighbour of  $Y_i$ .  $\text{color}(Y_i) = \min(\mathbb{N}^* \setminus \{\text{color}(Y_{i'}) \mid i' < i \text{ and } i' \in V(i)\})$ .

We now prove that it is a proper colouring. Let us prove that if  $(Y_i, Y_{i'}) \in \text{INECC}$  then  $\text{color}(Y_i) \neq \text{color}(Y_{i'})$ . Let  $(Y_i, Y_{i'}) \in \text{INECC}$ . With no loss of generality, let us say that  $i' < i$ . By definition of INECC,  $\exists (x, x') \in \text{NECC}$  such that  $F(x) = Y_i$  and  $F(x') = Y_{i'}$ . So  $i' \in V(i)$ , and by definition of color,  $\text{color}(Y_i) \neq \text{color}(Y_{i'})$ . As a consequence, that is a proper colouring. Now, let  $c$  be the biggest color used and  $k'$  the index of (one of) the images which have  $c$  as color. We have  $c \leq D(E_{k'}) + 1$  and  $c \leq k'$ .  $\forall i$ , we denote by  $\ell_i = \lfloor \log_2(D(E_i) + 1) \rfloor$  and  $\ell = \ell_{k'}$ . Since  $c \leq D(E_{k'}) + 1$ , we have  $c \leq 2^{\ell+1}$ . Consider

$$L(i) = V(i) \setminus \{i' \mid Y_{i \llbracket 0, n-\ell_i+1 \rrbracket} = Y_{i' \llbracket 0, n-\ell_i+1 \rrbracket}\}.$$

We have  $|\{i' \mid Y_{i \llbracket 0, n-\ell_i+1 \rrbracket} = Y_{i' \llbracket 0, n-\ell_i+1 \rrbracket}\}| \leq 2^{\ell_i-1}$ . We know that  $i \in \{i' \mid Y_{i' \llbracket 0, n-\ell_i+1 \rrbracket} = Y_{i \llbracket 0, n-\ell_i+1 \rrbracket}\}$ . So  $L(i) = (V(i) \cup \{i\}) \setminus \{i' \mid Y_{i' \llbracket 0, n-\ell_i+1 \rrbracket} = Y_{i \llbracket 0, n-\ell_i+1 \rrbracket}\}$ . We also know that  $i \notin V(i)$ . As a consequence,  $|V(i) \cup \{i\}| = D(E_i) + 1 \geq 2^{\ell_i}$ . Then,  $|L(i)| \geq 2^{\ell_i} - 2^{\ell_i-1}$ . So  $|L(i)| \geq 2^{\ell_i-1}$ .

We have  $\forall i' \in L(i), \forall x' \in E_{i'}, \exists x \in E_i, (x, x') \in \text{NECC}$ . However,  $\forall x \in E_i, \{x' \in E_{i'} \mid i' \in L(i) \text{ and } (x, x') \in \text{NECC}\} \subseteq \{x' \mid x_{\llbracket n-\ell_i+1, n \rrbracket} = x'_{\llbracket n-\ell_i+1, n \rrbracket}\}$ . So  $\forall x \in E_i, |\{x' \in E_{i'} \mid i' \in L(i) \text{ and } (x, x') \in \text{NECC}\}| \leq 2^{n-\ell_i+1}$ . So  $\forall i, |E_i| \geq 2^{\ell_i-1}/2^{n-\ell_i+1}$ . So  $\forall i, |E_i| \geq 2^{2\ell_i}/2^{n+2}$ . Furthermore,  $\sum_{i=1}^{k'} |E_i| \leq 2^n$  and  $\forall i \leq k', |E_i| \geq 2^{2\ell_i}/2^{n+2} \geq 2^{2\ell}/2^{n+2}$ . So  $k' \times 2^{2\ell}/2^{n+2} \leq 2^n$ .

So  $k' \leq 2^{2n+2}/2^{2\ell}$ . Then,  $c \leq 2^{2n+2}/2^{2\ell}$ . However, we have also  $c \leq 2^{\ell+1}$ . An upper bound for  $c$  is reached when  $2^{\ell+1} = 2^{2n+2}/2^{2\ell}$  (see Figure 1). In other words, when  $2^{3\ell} = 2^{2n+1}$ . In other words, when  $2^\ell = 2^{(2n+1)/3}$ . Then,  $c \leq 2^{(2n+1)/3+1}$ . Then,  $c \leq 2^{2n/3+2}$ . Furthermore,  $\chi(\text{INECC}) \leq c$ . As a result,  $\chi(\text{INECC}) \leq 2^{2n/3+2}$ .  $\square$

From lemma 4 and Lemma 5, we can deduce an upper bound for the chromatic number of a NECC graph. Furthermore, using the relation between the chromatic number of a  $\text{NECC}_{F,W}$  graph and  $\kappa(F, W)$ , we can find an upper bound for  $\kappa_n$ .

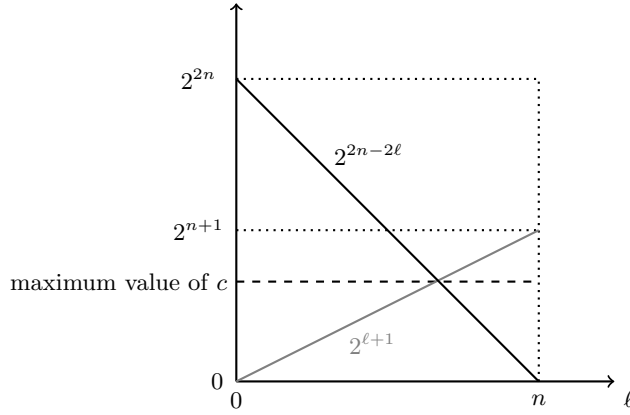


Fig. 1: Upper bound for  $c$ .

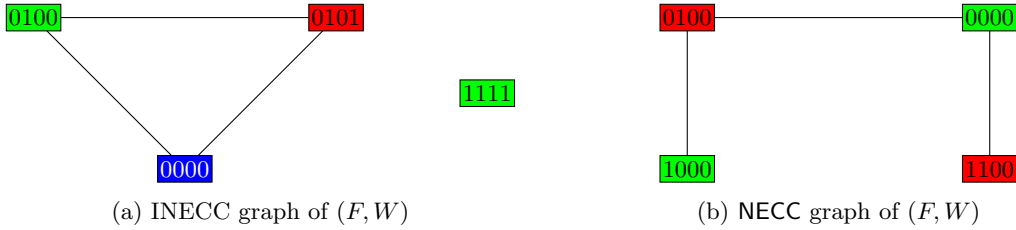


Fig. 2: INECC and NECC graphs of  $(F, W)$ .

**Theorem 2.**  $\forall n \in \mathbb{N}, \kappa_n \leq 2n/3 + 2$ .

*Proof.* Let  $F : \mathbb{B}^n \rightarrow \mathbb{B}^n$  and  $W \in \vec{\mathcal{P}}(V)$ . Thanks to Lemma 4 and Lemma 5, we know that  $\chi(\text{NECC}_{F,W}) \leq \chi(\text{INECC}_{F,W})$  and  $\chi(\text{INECC}_{F,W}) \leq 2^{2n/3+2}$ . As a consequence  $\chi(\text{NECC}_{F,W}) \leq 2^{2n/3+2}$  and  $\log_2(\chi(\text{NECC}_{F,W})) \leq 2n/3 + 2$  and then  $\kappa(F, W) \leq 2n/3 + 2$ . We have  $\forall F : \mathbb{B}^n \rightarrow \mathbb{B}^n$  and  $W \in \vec{\mathcal{P}}(V)$ ,  $\kappa(F, W) \leq 2n/3 + 2$ . By definition of  $\kappa_n$ , we have  $\kappa_n \leq 2n/3 + 2$ .  $\square$

*Remark 6.* The chromatic number of the INECC graph gives an upper bound of the NECC graph. However, the NECC graph can have a smaller chromatic number. For instance, let us consider the following BAN. Let  $F : \mathbb{B}^4 \rightarrow \mathbb{B}^4$  such that  $F((0, 0, 0, 0)) = (0, 0, 0, 0)$ ,  $F((1, 1, 0, 0)) = (0, 0, 0, 0)$ ,  $F((1, 0, 0, 0)) = (0, 1, 0, 0)$ ,  $F((0, 1, 0, 0)) = (0, 1, 0, 1)$ , and for all other  $x \in \mathbb{B}^4$ ,  $F(x) = (1, 1, 1, 1)$ . And let  $W$  be the simple sequential schedule  $(\{0\}, \{1\}, \{2\}, \{3\})$ . Figures 2a and 2b respectively show that the chromatic number of the INECC and NECC graphs are **3** and **2**.

So, even if the worst INECC graph had a chromatic number equal to  $2^{2n/3}$ , it would not disprove the conjecture. We can still hope that the worst NECC graph has a better chromatic number. Indeed, we can color some equivalent configurations differently.

## 6 Clique Number in the NECC graph

The *clique number* of a graph  $G$  (denoted by  $\omega(G)$ ) is the size of the biggest clique of  $G$ . We denote by  $\omega(\text{NECC})$  the clique number of the NECC graph. In this part, we find the maximum value that  $\omega(\text{NECC})$  can get. It is important because we know that the

chromatic number is bigger than the clique number. So, if in a NECC graph the clique number is bigger than  $2^{n/2}$  then the chromatic number is bigger as well and the conjecture is wrong. However, if the clique number is smaller than  $2^{n/2}$  then we cannot deduce anything about the conjecture.

Lemma 6 proves that the set of steps at which two configurations are confusable is an interval.

**Lemma 6.** *Let  $(x, x') \in \text{CC}$ . Let  $I = \text{CC}(x, x')$ . If  $a = \min(I)$  and  $b = \max(I)$  then  $I = \llbracket a, b \rrbracket$ .*

*Proof.* Since  $a = \min(I)$  and  $b = \max(I)$ , we have  $I \subseteq \llbracket a, b \rrbracket$ . For the sake of contradiction, let us suppose that  $\exists j \in \llbracket a, b \rrbracket$  such that  $j \notin I$ . Let  $j$  be the smallest such number. So  $F_{W_{<j}}(x) \neq F_{W_{<j}}(x')$ . We know that  $j \neq a$  because  $a \in I$ . Then,  $j-1 \in \llbracket a, b \rrbracket$ . Furthermore,  $j-1$  does not valid this propriety (because  $j$  is the smallest number which valid it). As a consequence,  $F_{W_{<j-1}}(x) = F_{W_{<j-1}}(x')$  and  $F_{W_{<j}}(x) \neq F_{W_{<j}}(x')$ . So  $F(x)_{W_{j-1}} \neq F(x')_{W_{j-1}}$ . Furthermore,  $F_{W_{<b}}(x)_{W_{j-1}} = F(x)_{W_{j-1}}$  because  $j \leq b$  (and then  $W_{j-1} \subseteq W_{<b}$ ) and  $F_{W_{<b}}(x')_{W_{j-1}} = F(x')_{W_{j-1}}$ . So  $F_{W_{<b}}(x)_{W_{j-1}} \neq F_{W_{<b}}(x')_{W_{j-1}}$  and thus  $F_{W_{<b}}(x) \neq F_{W_{<b}}(x')$ . As a consequence,  $b \notin I$  which is a contradiction. As a result,  $I = \llbracket a, b \rrbracket$ .  $\square$

Lemma 7 shows that if two configurations are confusable with a third configuration at one step, then they are also confusable between themselves at this step.

**Lemma 7.** *Let  $x, x', x'' \in \mathbb{B}^n$ . We have  $\text{CC}(x, x') \cap \text{CC}(x, x'') \subseteq \text{CC}(x', x'')$ .*

*Proof.* Let  $i \in \text{CC}(x, x') \cap \text{CC}(x, x'')$ . So we have  $i \in \text{CC}(x, x')$  and  $i \in \text{CC}(x, x'')$ . Then,  $F_{W_{<i}}(x) = F_{W_{<i}}(x')$  and  $F_{W_{<i}}(x) = F_{W_{<i}}(x'')$ . As a consequence,  $F_{W_{<i}}(x') = F_{W_{<i}}(x'')$  and then  $i \in \text{CC}(x', x'')$ . We have  $\forall i \in \text{CC}(x, x') \cap \text{CC}(x, x''), i \in \text{CC}(x', x'')$ . We conclude that  $\text{CC}(x, x') \cap \text{CC}(x, x'') \subseteq \text{CC}(x', x'')$ .  $\square$

Lemma 8 shows that if two configurations are confusable with a third configuration, the two are not confusable if and only if they are never confusable with the third at the same steps.

**Lemma 8.** *Let  $x, x', x'' \in \mathbb{B}^n$  such that:  $(x, x') \in \text{CC}$  and  $(x, x'') \in \text{CC}$ . We have:  $\text{CC}(x, x') \cap \text{CC}(x, x'') \neq \emptyset \iff (x', x'') \in \text{CC}$ .*

*Proof.* Let us suppose that  $\text{CC}(x, x') \cap \text{CC}(x, x'') \neq \emptyset$ . By Lemma 7 we know we have:  $\text{CC}(x, x') \cap \text{CC}(x, x'') \subseteq \text{CC}(x', x'')$ . So  $\text{CC}(x', x'') \neq \emptyset$ . As a result,  $(x', x'') \in \text{CC}$ . Now, let us suppose we have  $(x', x'') \in \text{CC}$ . Let  $\llbracket a, b \rrbracket = \text{CC}(x, x')$  and  $\llbracket a', b' \rrbracket = \text{CC}(x, x'')$ . For the sake of contradiction, let us consider that  $\text{CC}(x, x') \cap \text{CC}(x', x'') = \emptyset$ . So  $\llbracket a, b \rrbracket \cap \llbracket a', b' \rrbracket = \emptyset$ . With no loss of generality, let us consider that  $b < a'$ . So  $0 \leq a \leq b < a' \leq b' \leq p-1$  (with  $p = |W|$ ). Let  $j \in \text{CC}(x', x'')$ . Thus,  $F_{W_{<j}}(x') = F_{W_{<j}}(x'')$ . We can show that  $j \notin \llbracket a, b \rrbracket$  and  $j \notin \llbracket a', b' \rrbracket$ . Indeed, if  $j \in \llbracket a, b \rrbracket$ , then  $j \in \text{CC}(x, x')$  and  $F_{W_{<j}}(x) = F_{W_{<j}}(x')$ . So  $F_{W_{<j}}(x) = F_{W_{<j}}(x'')$  (because, by definition of  $j$ , we have  $F_{W_{<j}}(x') = F_{W_{<j}}(x'')$ ) and, as a consequence,  $j \in \text{CC}(x, x'')$  and thus  $j \in \text{CC}(x, x') \cap \text{CC}(x, x'')$ . As a result,  $\text{CC}(x, x') \cap \text{CC}(x, x'') \neq \emptyset$ . There is a contradiction, so  $j \notin \llbracket a, b \rrbracket$ . Similarly, we can prove that  $j \notin \llbracket a', b' \rrbracket$ . Now, let us prove that  $j \notin \llbracket 0, a \rrbracket$ . For the sake of contradiction let us say that  $j \in \llbracket 0, a \rrbracket$ . Then,  $\exists j' \in \llbracket j, a \rrbracket$ ,  $F(x')_{W_{j'}} \neq F(x'')_{W_{j'}}$ . Otherwise we would have  $F_{W_{<a}}(x'') = F_{W_{<a}}(x') = F_{W_{<a}}(x)$  and then  $\llbracket a, b \rrbracket \cap \llbracket a', b' \rrbracket \neq \emptyset$ . Furthermore, we know that  $F_{W_{<a}}(x') = F_{W_{<a}}(x)$  (because  $a \in \text{CC}(x, x')$ ) and  $W_{j'} \subseteq W_{<a}$  (because  $j' < a$ ) so  $F(x')_{W_{j'}} = F(x)_{W_{j'}}$  and thus  $F(x'')_{W_{j'}} \neq F(x)_{W_{j'}}$ . As a consequence,  $F_{W_{<a'}}(x')_{W_{j'}} \neq$

$F_{W_{<a'}}(x)_{W_{j'}}$  (because  $W_{j'} \subseteq W_{<a'}$  because  $j' < a < a'$ ). So  $a' \notin \text{CC}(x, x'')$ . That is a contradiction, so  $j \notin \llbracket 0, a \rrbracket$ . Now, let us prove that  $j \notin \llbracket b, a' \rrbracket \cup \llbracket b', p \rrbracket$ . If  $j \in \llbracket b, a' \rrbracket \cup \llbracket b', p \rrbracket$  then  $j > b$ . We know that  $F(x)_{W_b} \neq F(x')_{W_b}$  (else we would have  $F_{W_{<b+1}}(x) \neq F_{W_{<b+1}}(x')$  and then  $b+1 \in \text{CC}(x, x')$ ). However, we have  $F(x)_{W_b} = F(x'')_{W_b}$  because  $W_b \subseteq W_{<a'}$  because  $b < a'$ . So  $F(x')_{W_b} \neq F(x'')_{W_b}$ . Thus,  $F_{W_{<j}}(x') \neq F_{W_{<j}}(x'')$  because  $W_{<j}$  because  $b < j$ . That is a contradiction. As a consequence,  $j \notin \llbracket b, a' \rrbracket \cup \llbracket b', p \rrbracket$ . As a result,  $j$  does not exist. Thus,  $\text{CC}(x', x'') = \emptyset$ . Finally,  $(x', x'') \notin \text{CC}$ .  $\square$

Lemma 9 shows that all cliques of the NECC graph have at least one step during which all the configurations of the clique are simultaneously confusable.

**Lemma 9.** *Let  $X$  be a clique of the NECC graph. Then,  $\exists i, \forall x, x' \in X, i \in \text{CC}(x, x')$ .*

*Proof.* Let  $x \in X$  and  $k = |X|$ . Let  $X_1, X_2, \dots, X_k$  be the configuration of  $X$ . So we have  $X = \{X_1, X_2, \dots, X_k\}$ . Let  $I_1 = \text{CC}(x, X_1), I_2 = \text{CC}(x, X_2), \dots, I_k = \text{CC}(x, X_k)$ . Let  $I = I_1 \cap I_2 \cap \dots \cap I_k$ . We can prove that all intervals intersect each other two by two. In other words,  $\forall i, i' \in \llbracket 0, k \rrbracket, I_i \cap I_{i'} \neq \emptyset$ . Let us assume, for the sake of contradiction that there are disjoint intervals. In other words, we would have  $x', x'' \in X$  such that  $\text{CC}(x, x') \cap \text{CC}(x, x'') = \emptyset$ . So by Lemma 8, we would have  $(x', x'') \in \text{CC}$ . However,  $x', x'' \in X$  so  $(x', x'') \in \text{CC}$ . There is a contradiction so all intervals intersect each other two by two. And we know that if a set of intervals intersect each other two by two then they have an interval in common. So  $I \neq \emptyset$ .

Let  $i \in I$ . Now we prove that  $\forall x', x'' \in X, i \in \text{CC}(x', x'')$ . Let  $x', x'' \in X$ . We have  $i \in \text{CC}(x, x')$  and  $i \in \text{CC}(x, x'')$ . Consequently,  $F_{W_{<i}}(x) = F_{W_{<i}}(x')$  and  $F_{W_{<i}}(x) = F_{W_{<i}}(x'')$ . Thus,  $F_{W_{<i}}(x') = F_{W_{<i}}(x'')$ . As a result,  $i \in \text{CC}(x', x'')$ . As a consequence,  $\forall x', x'' \in X, i \in \text{CC}(x', x'')$ .  $\square$

Using Lemma 9, Theorem 3 shows that the clique number of any NECC graph is less than or equal to  $2^{n/2}$ .

**Theorem 3.**  $\omega(\text{NECC}) \leq 2^{\lfloor n/2 \rfloor}$ .

*Proof.* Let  $X$  be the biggest clique of the NECC graph,  $x \in X$  and  $i$  such that  $\forall x, x' \in X, i \in \text{CC}(x, x')$  (Thanks to Lemma 9, we know there is one). In other words,  $\forall x' \in X, F_{W_{<i}}(x') = F_{W_{<i}}(x)$ . So  $\forall x, x' \in X, x_{\overline{W_{<i}}} = x'_{\overline{W_{<i}}}$  and  $F(x)_{W_{<i}} = F(x')_{W_{<i}}$ . Let  $x \in X$ . There are 2 cases:

- $|W_{<i}| < n/2$ . Then, we have  $|\overline{W_{<i}}| \geq n/2$ . Thus,  $|\{x' | x'_{\overline{W_{<i}}} = x_{\overline{W_{<i}}}\}| < 2^{n/2}$ . And since  $X \subseteq \{x' | x'_{\overline{W_{<i}}} = x_{\overline{W_{<i}}}\}$ , we have  $|X| < 2^{n/2}$ .
- $|W_{<i}| \geq n/2$  then, we have  $\{F(x') | x' \in X\} \subseteq \{x' | F(x')_{W_{<i}} = F(x)_{W_{<i}}\}$  and  $|\{F(x') | F(x')_{W_{<i}} = F(x)_{W_{<i}}\}| \leq 2^{n/2}$ . And since all configurations of  $X$  are not equivalent, we have  $\forall x, x' \in X, x \neq x' \rightarrow F(x) \neq F(x')$ . Thus,  $|X| \leq |\{F(x') | x' \in X\}|$ . As a consequence,  $|X| \leq 2^{n/2}$ .

In all cases, we have  $|X| \leq 2^{n/2}$ . So  $\omega(\text{NECC}) \leq 2^{n/2}$ .  $\square$

This result supports Conjecture 1 because the NECC graphs with the biggest chromatic number we succeeded to build are graph with big clique number. It seems we reached the limit of this technique.

## 7 Class of bijective BANs

In this part, we study BAN whose global transition function is bijective. That is to say, we study BAN whose dynamic with a parallel update schedule is only composed of recurrent configurations. For this class of BAN we can prove a result which is really close to the conjecture. However, to prove this result, we first need to prove two general results.

The first one is that if two configurations are confusable then either the first parts of the two images are equal or the second parts of the two configurations are.

**Lemma 10.** *If  $W = (0, 1, \dots, n)$  then  $\forall (x, x') \in CC, F(x)_{\llbracket 0, n/2 \rrbracket} = F(x')_{\llbracket 0, n/2 \rrbracket}$  or  $x_{\llbracket n/2, n \rrbracket} = x'_{\llbracket n/2, n \rrbracket}$ .*

*Proof.* Let  $(x, x') \in CC$ . So  $\exists i \in \llbracket 0, n \rrbracket, F_{\llbracket 0, i \rrbracket}(x) = F_{\llbracket 0, i \rrbracket}(x')$ . Let  $i$  be the smallest such number. So we have  $F(x)_{\llbracket 0, i \rrbracket} = F(x')_{\llbracket 0, i \rrbracket}$  and  $x_{\llbracket i, n \rrbracket} = x'_{\llbracket i, n \rrbracket}$ . Two cases:

- $i \leq n/2$ . Then,  $\llbracket n/2, n \rrbracket \subseteq \llbracket i, n \rrbracket$ . Thus,  $x_{\llbracket n/2, n \rrbracket} = x'_{\llbracket n/2, n \rrbracket}$ .
- $i \geq n/2$ . Then,  $\llbracket 0, n/2 \rrbracket \subseteq \llbracket 0, i \rrbracket$ . Thus,  $F(x)_{\llbracket 0, n/2 \rrbracket} = F(x')_{\llbracket 0, n/2 \rrbracket}$ .

□

The second general result, and a simple consequence of Lemma 10 is that if we take the neighbours of a configuration in a NECC graph, and we take the set of images of these configurations when we apply  $F$ , then this set has less than  $2^{n/2+1} - 2$  elements.

**Lemma 11.** *If  $W' = (0, 1, \dots, n)$  then  $\forall x \in \mathbb{B}^n, |\{F(x') \mid (x, x') \in \text{NECC}\}| \leq 2^{n/2+1} - 2$ .*

*Proof.* Let  $x \in \mathbb{B}^n$ . According to Lemma 10:  $\forall x' \in \mathbb{B}^n F(x)_{\llbracket 0, n/2 \rrbracket} = F(x')_{\llbracket 0, n/2 \rrbracket}$  or  $x_{\llbracket n/2, n \rrbracket} = x'_{\llbracket n/2, n \rrbracket}$ . Then,  $\{x' \mid (x, x') \in \text{NECC}\} \subseteq \{x' \mid x_{\llbracket n/2, n \rrbracket} = x'_{\llbracket n/2, n \rrbracket}\} \cup \{x' \mid F(x)_{\llbracket 0, n/2 \rrbracket} = F(x')_{\llbracket 0, n/2 \rrbracket}\}$ . So  $\{F(x') \mid (x, x') \in \text{NECC}\} \subseteq \{F(x') \mid x_{\llbracket n/2, n \rrbracket} = x'_{\llbracket n/2, n \rrbracket}\} \cup \{F(x') \mid F(x)_{\llbracket 0, n/2 \rrbracket} = F(x')_{\llbracket 0, n/2 \rrbracket}\}$ . Thus,  $|\{F(x') \mid (x, x') \in \text{NECC}\}| \leq |\{F(x') \mid x_{\llbracket n/2, n \rrbracket} = x'_{\llbracket n/2, n \rrbracket}\}| + |\{F(x') \mid F(x)_{\llbracket 0, n/2 \rrbracket} = F(x')_{\llbracket 0, n/2 \rrbracket}\}|$ . We have  $|\{F(x') \mid F(x)_{\llbracket 0, n/2 \rrbracket} = F(x')_{\llbracket 0, n/2 \rrbracket}\}| \leq 2^{n/2}$ . Furthermore,  $|\{x' \mid x_{\llbracket n/2, n \rrbracket} = x'_{\llbracket n/2, n \rrbracket}\}| \leq 2^{n/2}$ . As a consequence,  $|\{F(x') \mid x_{\llbracket n/2, n \rrbracket} = x'_{\llbracket n/2, n \rrbracket}\}| \leq 2^{n/2}$ . So  $|\{F(x') \mid (x, x') \in \text{NECC}\}| \leq 2^{n/2+1}$ . Furthermore,  $F(x) \in \{F(x') \mid x_{\llbracket n/2, n \rrbracket} = x'_{\llbracket n/2, n \rrbracket}\}$  and  $F(x) \in \{F(x') \mid F(x)_{\llbracket 0, n/2 \rrbracket} = F(x')_{\llbracket 0, n/2 \rrbracket}\}$  but  $F(x) \notin \{F(x') \mid (x, x') \in \text{NECC}\}$ . So  $|\{F(x') \mid (x, x') \in \text{NECC}\}| \leq 2^{n/2+1} - 2$ . □

Using the fact that we are talking about a bijective function, and thanks to Lemma 11, we bound the degree of every configuration in the NECC graph. Then, we deduce a bound for the chromatic number of the NECC and then a bound for  $\kappa$ .

**Theorem 4.** *If  $F : \mathbb{B}^n \rightarrow \mathbb{B}^n$  is a bijective function then  $\kappa(F, W) \leq n/2 + 1$ .*

*Proof.* Let  $F : \mathbb{B}^n \rightarrow \mathbb{B}^n$  be a bijective function.  $\forall x \in \mathbb{B}^n$ , let  $d(x)$  be the degree of  $x$  in the NECC graph. In other words,  $\forall x, d(x) = |\{x' \mid (x, x') \in \text{NECC}\}|$ . Let  $x \in \mathbb{B}^n$  be the configuration with the biggest degree. We know by Lemma 11 that  $|\{F(x') \mid (x, x') \in \text{NECC}\}| \leq 2^{n/2+1} - 2$ . However, since  $F$  is a bijective function, we have  $|\{F(x') \mid (x, x') \in \text{NECC}\}| = |\{x' \mid (x, x') \in \text{NECC}\}|$ . And then  $d(x) \leq 2^{n/2+1} - 2$ . So  $\chi(\text{NECC}) \leq 2^{n/2+1} - 1$ . Thus,  $\log_2(\chi(\text{NECC})) \leq \frac{n}{2} + 1$ . As a result,  $\kappa(F, W) \leq \frac{n}{2} + 1$ . □

## 8 Conclusion and future research

In this article, we were interested in the minimal number ( $\kappa$ ) of additional automata that a SBAN with a sequential update schedule needs to simulate another given one with a parallel update schedule. The maximum value that  $\kappa$  can take for all SBAN of size  $n$  is denoted by  $\kappa_n$ . To answer this matter we introduced the concept of NECC graph, a graph built from the SBAN. We proved that the log of the chromatic number of this graph and the  $\kappa$  of a SBAN were equals. We achieve to bound  $\kappa_n$  in the intervals  $[n/2, 2n/3 + 2]$  and we conjectured that  $\kappa_n$  is equal to  $n/2$ . And, to support this conjecture, we showed that the maximum clique number that a NECC graph can have is equal to  $2^{n/2}$ . That means that the NECC graph of a SBAN which would have a  $\kappa$  greater than  $n/2$  would have a NECC graph with a chromatic number greater than the clique number. Finally, we showed that the conjecture is true (up to one extra automaton) if we restrain to SBAN whose global transition function is bijective.

More work is left to do to bound  $\kappa_n$  more precisely. There is a related problem where, given a SBAN with a parallel update schedule, we search the number of additional automata needed for a SBAN with any sequential update schedule (that is to say, we do not impose any order on the update schedule) to simulate the first SBAN. We can see that for some BAN this number is really smaller that when we impose an order. We can take the example used in Lemma 3. The BAN has  $n/2$  couple of automata that exchange their values. If the mandatory order is to update one automaton only of every couple of automata and then the other we need  $n/2$  additional automata. But if the order is free then we can update all couple of automata one at the time and do with only one additional automaton. The problem to find an upper bound better than  $\kappa_n$  is still open.

Furthermore, we could study the issue presented in this article with other kinds of update schedule (which update many times each automata for instance) or other kinds of simulations (where many automata can represent one simulated automaton for example).

These results could also help to create new SBANs, the smallest possible, which would work the same way as other SBANs with different update schedule. Associated with the concept of functional modularity, we could also use them to replace a little functional module which have an unexpected behaviour in some situations by another module more robust to schedule variations.

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## A Appendix

In Lemma 12 we show that, for a couple  $(F, W)$ , if we take the block sequential mode  $W$  and we transform it into a sequential one, the  $\kappa$  can only increase. Differently stated, if we want to study  $\kappa_n$  we can study only SBAN with sequential update schedule.

**Lemma 12.** *Let  $F : \mathbb{B}^n \rightarrow \mathbb{B}^n$ ,  $W \in \overrightarrow{\mathcal{P}}(V)$  and  $W' \in \overrightarrow{\mathcal{P}}(V)$  be a sequential schedule such that if  $i$  is updated before  $i'$  in  $W$  it be updated before  $i'$  in  $W'$ . Then, we have  $\kappa(F, W') \geq \kappa(F, W)$ .*

*Proof.* We show that  $\text{NECC}_{F,W} \subseteq \text{NECC}_{F,W'}$ . Let  $(x, x') \in \text{NECC}_{F,W}$ . We have  $F(x) \neq F(x')$  and  $\exists i \in \llbracket 0, p \llbracket F_{W_{<i}}(x) = F_{W_{<i}}(x')$ . Let  $i' = |W_{<i}|$ . We have  $W'[i'] = W_{<i}$ . So  $F(x) \neq F(x')$  and  $\exists i \in \llbracket 0, p' \llbracket F_{W'[i']}(x) = F_{W'[i']}(x')$ . Then,  $(x, x') \in \text{NECC}_{F,W'}$ . So  $\text{NECC}_{F,W} \subseteq \text{NECC}_{F,W'}$ . Then,  $\chi(\text{NECC}_{F,W'}) \geq \chi(\text{NECC}_{F,W})$ . Finally,  $\kappa(F, W') \geq \kappa(F, W)$ .  $\square$

**Definition 6.** *Let  $T \in \overrightarrow{\mathcal{P}}(V)$ . We denote by  $H_h(T) \in \overrightarrow{\mathcal{P}}(V)$  the update schedule such that if  $W = H_h(T)$  then:*

- $|W| = |T|$ ;
- $\forall i \in \llbracket 0, |T| \llbracket, |W_i| = |T_i|$ ;
- $\forall i \in \llbracket 0, |T| \llbracket, \forall j \in W_i, h(j) \in T_i$ .

Lemma 13 says that if two SBANs are equivalent up to permutation and if they have the same configurations up to the permutation and if we update the first  $i$  blocks of automata of the two SBANs then the configurations stay equivalents up to the permutation.

**Lemma 13.** *Let  $h : V \rightarrow V$  be a bijective function,  $T \in \overrightarrow{\mathcal{P}}(V)$ ,  $G : \mathbb{B}^n \rightarrow \mathbb{B}^n$ ,  $W = H_h(T)$  and  $F : \mathbb{B}^n \rightarrow \mathbb{B}^n$  such that  $\varphi_h \circ G = F \circ \varphi_h$ . We have  $\forall j \in \llbracket 0, p \llbracket, \varphi_h \circ G_{T_{<j}} = F_{W_{<j}} \circ \varphi_h$ .*

*Proof.* Let  $j \in \llbracket 0, p \llbracket$ , let  $x \in \mathbb{B}^n$ . Let us prove that:  $\varphi_h \circ G_{T_{<j}}(x) = F_{W_{<j}} \circ \varphi_h(x)$ . Let  $i \in V$ . There are two cases:

- $i \notin W_{<j}$ . So  $F_{W_{<j}}(\varphi_h(x))_i = \varphi_h(x)_i = x_{h(i)}$ .  
Furthermore,  $h(i) \notin T_{<j}$ . Thus,  $(\varphi_h \circ G_{T_{<j}}(x))_i = G_{T_{<j}}(x)_{h(i)} = x_{h(i)}$ . Consequently,  $(F_{W_{<j}} \circ \varphi_h(x))_i = (\varphi_h \circ G_{T_{<j}}(x))_i$ .
- $i \in W_{<j}$ . So  $F_{W_{<j}}(\varphi_h(x))_i = (F \circ \varphi_h(x))_i = (\varphi_h \circ G(x))_i = G(x)_{h(i)}$ . Furthermore,  $h(i) \in T_{<j}$ . Consequently,  $(\varphi_h \circ G_{T_{<j}}(x))_i = (G_{T_{<j}}(x))_{h(i)} = G(x)_{h(i)}$ . As a result,  $(F_{W_{<j}} \circ \varphi_h(x))_i = (\varphi_h \circ G_{T_{<j}}(x))_i$ .

So  $\forall i \in V, (F_{W_{<j}} \circ \varphi_h(x))_i = (\varphi_h \circ G_{T_{<j}}(x))_i$ . Thus,  $F_{W_{<j}} \circ \varphi_h = \varphi_h \circ G_{T_{<j}}$ .  $\square$

Lemma 14 says that if two SBANs are equivalent up to permutation, then their NECC graph are equivalent up to permutation.

**Lemma 14.** *Let  $h : V \rightarrow V$  be a bijective function,  $T \in \overrightarrow{\mathcal{P}}(V)$ ,  $G : \mathbb{B}^n \rightarrow \mathbb{B}^n$ ,  $W = H_h(T)$  and  $F : \mathbb{B}^n \rightarrow \mathbb{B}^n$  such that  $\varphi_h \circ G = F \circ \varphi_h$ .  $(x, x') \in \text{NECC}_{G,T} \iff (\varphi_h(x), \varphi_h(x')) \in \text{NECC}_{F,W}$ .*



*Proof.*  $(x, x') \in \text{NECC}_{G,T} \iff (x, x') \in \text{NEC}_G$  and  $(x, x') \in \text{CC}_{G,T}$   
 $\iff G(x) \neq G(x')$  and  $\exists j \in \llbracket 0, |W| \rrbracket \llbracket G_{T_{<j}}(x) = G_{T_{<j}}(x') \rrbracket$   
 $\iff G(x) \neq G(x')$  and  $\exists j \in \llbracket 0, |W| \rrbracket \llbracket \varphi_h \circ G_{T_{<j}}(x) = \varphi_h \circ G_{T_{<j}}(x') \rrbracket$   
 $\iff F(x) \neq F(x')$  and  $\exists j \in \llbracket 0, |W| \rrbracket \llbracket F_{W_{<j}} \circ \varphi_h(x) = F_{W_{<j}} \circ \varphi_h(x') \rrbracket$   
 $\iff (\varphi_h(x), \varphi_h(x')) \in \text{NEC}_F$  and  $(\varphi_h(x), \varphi_h(x')) \in \text{CC}_{F,W}$   
 $\iff (\varphi_h(x), \varphi_h(x')) \in \text{NECC}_{F,W}$ .

□

Lemma 15 says that if two SBANs are equivalent up to permutation, then their  $\kappa$  are equivalent up to permutation.

**Lemma 15.** *Let  $h : V \rightarrow V$  be a bijective function,  $T \in \vec{\mathcal{P}}(V)$ ,  $G : \mathbb{B}^n \rightarrow \mathbb{B}^n$ ,  $W = H_h(T)$  and  $F : \mathbb{B}^n \rightarrow \mathbb{B}^n$  such that  $\varphi_h \circ G = F \circ \varphi_h$ . So we have  $\kappa(F, W) = \kappa(G, T)$ .*

*Proof.* Lemma 14 says that the NECC graph of  $(F, W)$  and  $(G, T)$  are equivalent up to permutation (It is only a projection by  $\varphi_h$ ). Thus, they have the same chromatic number. As a result,  $\kappa(F, W) = \kappa(G, T)$  □

Using the fact that the sequential update schedule is the one with the biggest  $\kappa$ , and Lemma 15 which says that we can do a permutation of a SBAN without changing its  $\kappa$ , we see that we can compute  $\kappa_n$  with a SBAN with the simple sequential update schedule as update schedule.

**Lemma 16.** *Let  $W$  be the simple sequential update schedule ( $W = (\{0\}, \{1\}, \dots, \{n-1\})$ ). There is a  $F : \mathbb{B}^n \rightarrow \mathbb{B}^n$  such that:  $\kappa_n = \kappa(F, W)$ .*

*Proof.* Let  $G : \mathbb{B}^n \rightarrow \mathbb{B}^n$  and  $T \in \vec{\mathcal{P}}(V)$  a sequential update schedule such that  $\kappa(G, T) = \kappa_n$ . We know we can take  $T$  sequential because of Lemma 12.

Let  $h : V \rightarrow V$  be the bijective function such that:  $\forall i \in V, h(i) = j$  such that  $T_j = \{i\}$ . Let  $F : \mathbb{B}^n \rightarrow \mathbb{B}^n$  such that  $\varphi_h \circ G = F \circ \varphi_h$  and  $W = H_h(T)$ .  $W$  is the simple sequential update schedule.

And by Lemma 15, we know that  $\kappa(F, W) = \kappa(G, T)$ . As a result,  $\kappa(F, W) = \kappa_n$ . □