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# Optimal measurement times for observing a Brownian Motion over a finite period using a Kalman filter

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**Abstract.** This article deals with the optimization of the schedule of measures for observing a random process in time using a Kalman filter, when the length of the process is finite and fixed, and a fixed number of measures are available. The measure timetable plays a critical role for the accuracy of this estimator. Two different criteria of optimality of a timetable (not necessarily regular) are considered: the maximal and the mean variance of the estimator. Both experimental and theoretical methods are used for the problem of minimizing the mean variance. The theoretical methods are based on studying the cost function as a rational function. An analytical formula of the optimal instant of measure is obtained in the case of one measure. Its properties are studied. An experimental solution is given for a particular case with  $n > 1$  measures.

**Keywords:** Random walk, Wiener process, Kalman filter

## 1 Introduction

When a latent phenomenon is observed through different acquisition methods, more information can be acquired than from a single method, but making the most of these measurements is a challenge [5]. This is due to discrepancies in the nature of data, in particular in the sampling. The observer often cannot control the instants of measure and makes regular measures with each of the available sensors. In this case, controlling the delays between measurements with different sensors can lead to a consequent gain in the quality of the estimator [1]. One may also ask: what is the optimal (not necessarily regular) timetable of measurements?

A model, where sensors are active during an interval of time, has been considered [3]. On the other hand, the model with instantaneous measures is considered in this article. In this model one observes a *scalar* continuous latent variable on a *finite* interval of time with noisy sensors, each having access to *only one* measurement at one time instant and having its own measurement noise variance. A Kalman filter based approach for estimating the hidden state is taken in this

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paper, and two ways of evaluating the quality of estimation are considered: the maximal and the mean variance of the estimator over time.

The main theoretical result of this text is the optimal instant of measure (with respect to the mean variance of the estimation error) given by (16) in the case of one measure. The problem whether the variance of the estimator can be maintained below a fixed limit over the whole interval is essentially solved. The main experimental result is the numerical computation of the optimal schedules in a particular case where 2 measures are available.

The paper is organized as follows. The general (multimodal, irregularly scheduled) Kalman estimation model is defined in Section 2. The maximal variance of the estimator is used as the cost function in Section 3. In the formal treatment of this problem, the maximal variance of the estimator is considered as a bound. It is proved that the intuitive algorithm “measure when the error variance reaches the bound” is globally optimal in this sense. An algorithm of this kind has been proposed for a model of linear filtering in discrete time [2]. The more challenging problem, where the mean variance of the estimator is used as the cost function, is stated in Section 4. The only case where the optimal solution can be written in closed form is the case of one measure, and this case is studied in detail in Subsections 4.1 and 4.2.

## 2 Model Description

We assume that the estimation of the system state is done by computing the time evolution of a parameter, and that the variance of the estimation grows linearly between measurements. This simple assumption models the fact that decreasing the measure frequency decreases the accuracy on the system state estimation. In this purpose, we consider a real Brownian motion  $\theta(t)$  ( $t \in [0, T]$ ), satisfying for  $t > s$ ,  $\theta(t) - \theta(s) \stackrel{d}{\sim} \mathcal{N}(0, \sigma^2(t - s))$  i.e., the increments are Gaussian with mean 0 and variance  $\sigma^2(t - s)$ .

Suppose  $n$  sensors can make measurements at moments  $t_1, \dots, t_n$ . It is assumed that each sensor  $k$  returns a measured value equal to  $X_k$  at time  $t_k$  and that  $0 \leq t_1 \leq \dots \leq t_n \leq T$ . No subsequence of the sequence  $(t_1, \dots, t_n)$  is constrained to be regular in any sense. Suppose, the initial state  $\theta(0)$  is a Gaussian random variable of mean  $\bar{\theta}_0$  and variance  $v_0$ . Suppose that  $\theta(0)$ , the measurement noise and the evolution of the Brownian motion  $\theta(t)$  are independent. The Kalman filter framework can apply with the state and measurement equations:

$$\theta(t_k) = \theta(t_{k-1}) + w_k, \quad w_k \stackrel{d}{\sim} \mathcal{N}(0, \sigma^2(t_k - t_{k-1})) \quad (1)$$

$$X_k = \theta(t_k) + n_k, \quad n_k \stackrel{d}{\sim} \mathcal{N}(0, v_k). \quad (2)$$

By the theory of Kalman filtering (see [4]), the maximum likelihood estimate  $\hat{\theta}_{t_k}^{t_k}$  of  $\theta(t_k)$  and its variance  $\Gamma_{t_k}^{t_k}$  are defined by the following recursive equations:

$$\begin{cases} \hat{\theta}_{t_k}^{t_k} = \hat{\theta}_{t_k}^{t_{k-1}} + K(t_k) (X_k - \hat{\theta}_{t_k}^{t_{k-1}}) & (3) \\ \hat{\theta}_{t_k}^{t_{k-1}} = \hat{\theta}_{t_{k-1}}^{t_{k-1}} & (4) \\ \Gamma_{t_k}^{t_k} = \Gamma_{t_k}^{t_{k-1}} - K(t_k) \Gamma_{t_k}^{t_{k-1}} & (5) \\ K(t_k) = \Gamma_{t_k}^{t_{k-1}} (\Gamma_{t_k}^{t_{k-1}} + v_k)^{-1} & (6) \\ \Gamma_{t_k}^{t_{k-1}} = \Gamma_{t_{k-1}}^{t_{k-1}} + \sigma^2(t_k - t_{k-1}), & (7) \end{cases}$$

where  $\hat{\theta}_{t_k}^{t_l}$  ( $l \in \{k-1, k\}$ ) is the maximum likelihood estimate of  $\theta(t_k)$  conditionally to the data available at time  $t_l$ , and  $\Gamma_{t_k}^{t_l}$  is the variance of the estimate  $\hat{\theta}_{t_k}^{t_l}$ .  $K(t_k)$  is the Kalman gain used for the update at time  $t_k$ . In order for (7) to make sense for  $k=1$ , define  $t_0=0$  and  $\Gamma_{t_0}^{t_0}=v_0$ .

Remark that, by (5),(6), using the fact that all quantities are scalar,

$$\Gamma_{t_k}^{t_k} = \Gamma_{t_k}^{t_{k-1}} - \frac{\left(\Gamma_{t_k}^{t_{k-1}}\right)^2}{\Gamma_{t_k}^{t_{k-1}} + v_k} = \frac{\left(\Gamma_{t_k}^{t_{k-1}} + v_k\right) \Gamma_{t_k}^{t_{k-1}} - \left(\Gamma_{t_k}^{t_{k-1}}\right)^2}{\Gamma_{t_k}^{t_{k-1}} + v_k} = \frac{v_k \Gamma_{t_k}^{t_{k-1}}}{v_k + \Gamma_{t_k}^{t_{k-1}}}, \quad (8)$$

which is equivalent (by (7)) to

$$\left(\Gamma_{t_k}^{t_k}\right)^{-1} = v_k^{-1} + \left(\Gamma_{t_k}^{t_{k-1}}\right)^{-1} = v_k^{-1} + \left(\Gamma_{t_{k-1}}^{t_{k-1}} + \sigma^2(t_k - t_{k-1})\right)^{-1}. \quad (9)$$

Therefore, each  $\Gamma_{t_k}^{t_k}$  is a rational function of  $\sigma^2, t_1, \dots, t_k, v_0, \dots, v_k$ .

For each  $t \in [0, T]$ , denote  $v(t)$  the variance of  $\hat{\theta}(t)$ , i.e. the variance when the last measurement was taken plus the uncertainty due to the time without new feedbacks. It equals:

$$v(t) = \Gamma_{t_k}^{t_k} + \sigma^2(t - t_k) \text{ where } k = \max\{i | t_i \leq t\}. \quad (10)$$

$v(t)$  is a piecewise linear function composed of line intervals of slope  $\sigma^2$ . Two examples of functions  $v(t)$  are shown Figure 1. In the first example,  $v_1, v_2, v_3$  are equal, in the second example they are different.

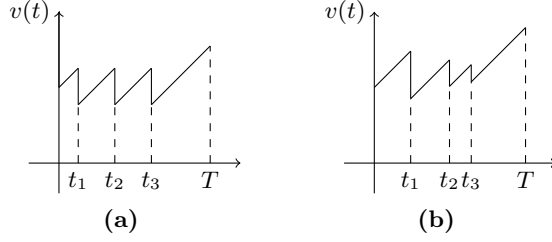
In these and in all the other examples of this article, the value  $\sigma^2 = 1$  has been taken. This can be done without loss of generality.

### 3 Controlling the Maximal Variance

In this section, the observer chooses the measurement instants  $t_1, \dots, t_n$  so that the variance  $v(t)$  of the maximum likelihood estimator of  $\theta(t)$  does not exceed a fixed bound  $V$ :

$$\forall t \in [0, T] \quad v(t) \leq V. \quad (11)$$

The question is: how large can the length  $T$  of the process be so that the constraint (11) can be satisfied? The following lemma answers this question.



**Fig. 1.** The function  $v(t)$  in particular cases. In **(a)**,  $v_0 = \frac{1}{2}$ ,  $v_1 = v_2 = v_3 = 1$ ,  $T = 1$ ,  $\sigma^2 = 1$  and  $t_1 = 0.128$ ,  $t_2 = 0.369$ ,  $t_3 = 0.611$ . In **(b)**,  $v_0 = \frac{1}{2}$ ,  $v_1 = 1$ ,  $v_2 = 2$ ,  $v_3 = 3$ ,  $T = 1$ ,  $\sigma^2 = 1$  and  $t_1 = 0.241$ ,  $t_2 = 0.494$ ,  $t_3 = 0.641$ . The values of  $v_1, v_2, v_3$  control the differences of the variance before and after the measurement.

**Lemma 1.** *Suppose that  $v_0 \leq V$ . The constraint (11) can be satisfied if and only if*

$$T \leq \frac{V - v_0}{\sigma^2} + \left( \sum_{k=1}^n \delta(V, v_k) \right) \text{ where } \sigma^2 \delta(V, v) = V - \frac{Vv}{V + v}. \quad (12)$$

One can remark that the time defined by (12) increases when  $V$  increases and decreases when any of  $v_i$  increases (i.e., if the sensors or the estimate of the initial state are less accurate).

The proof of Lemma 1 (see Appendix) also implies that the intuitive algorithm “measure when  $v(t) \geq V$  is reached” (which consists in iteratively applying (32) to define the instants of measure) keeps the estimation error variance bounded by  $V$  during as long time as possible. Remark that this algorithm is also optimal in this sense when  $v_0 > V$ . Indeed, in this case the first measure has to be done at the instant 0 in order to achieve  $v(0) \leq V$ .

It is interesting to remark that if  $v_1 = \dots = v_n$ , this algorithm defines a regular sampling. Conversely, if measurement accuracies differ, the optimal solution leads to irregular sampling.

## 4 Controlling the Mean Variance

In this section, the observer chooses the measurement instants  $t_1, \dots, t_n$  so that the mean of the variance  $v(t)$  of the maximum likelihood estimator of  $\theta(t)$  is minimal. This implies that the following cost function is to be minimized under the constraint  $0 \leq t_1 \leq t_2 \leq \dots \leq t_n \leq T$ :

$$\begin{aligned} J_{\sigma^2, T, v_0, v_1, \dots, v_n}(t_1, \dots, t_n) &= \int_0^T v(t) dt \\ &= \frac{\sigma^2 t_1^2}{2} + v_0 t_1 + \frac{\sigma^2 (t_2 - t_1)^2}{2} + \Gamma_{t_1}^{t_2} (t_2 - t_1) + \dots + \frac{\sigma^2 (T - t_n)^2}{2} + \Gamma_{t_n}^{T} (T - t_n). \end{aligned} \quad (13)$$

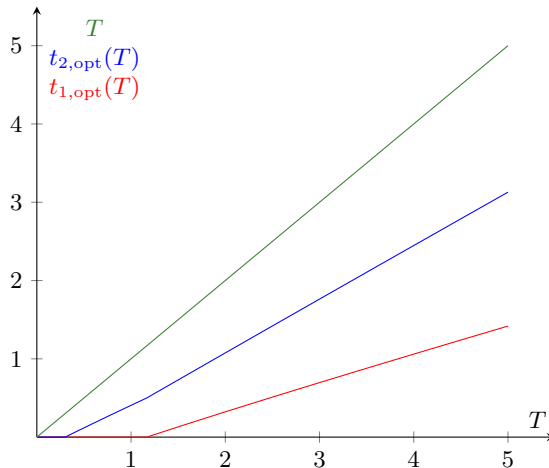
One can remark that the cost function (13) is rational in its  $2n+3$  parameters  $\sigma^2, T, v_0, \dots, v_n, t_1, \dots, t_n$ .

Suppose that this function is minimized in a unique point

$$(t_{1,\text{opt}}(\sigma^2, T, v_0, v_1, \dots, v_n), \dots, t_{n,\text{opt}}(\sigma^2, T, v_0, v_1, \dots, v_n)). \quad (14)$$

These values are the optimal measurement instants. We can wonder where these instants are located, and especially if some of them are equal to zero. The minimizer is indeed unique in the case  $n = 1$ , which is proved in Subsection 4.2.

We are also interested in the behavior of the optimal measurement times as functions of  $T$ : monotonicity, asymptotic properties, etc. The dependency of the optimal instants on  $T$  for  $n=2$  and fixed values of  $\sigma^2, v_0, v_1, v_2$  is shown Figure 2. It suggests that the optimal instants vary continuously, are monotonically increasing and are close to piecewise-linear functions of  $T$ . It also suggests that some optimal instants are located at zero when  $T$  is small enough. When  $T$  is large, the optimal instants are approximately equally spaced in time. These properties are proved for  $n = 1$  measure in Subsections 4.1 and 4.2.



**Fig. 2.** Experimentally computed values of  $t_{1,\text{opt}}(1, T, 1, 1, 1)$  and  $t_{2,\text{opt}}(1, T, 1, 1, 1)$  ( $n=2$ ) for  $T$  varying from 0.01 to 5. For example,  $t_{1,\text{opt}}(1, 3, 1, 1, 1) = 0.696$  and  $t_{2,\text{opt}}(1, 3, 1, 1, 1) = 1.763$  define irregularly spaced measures.

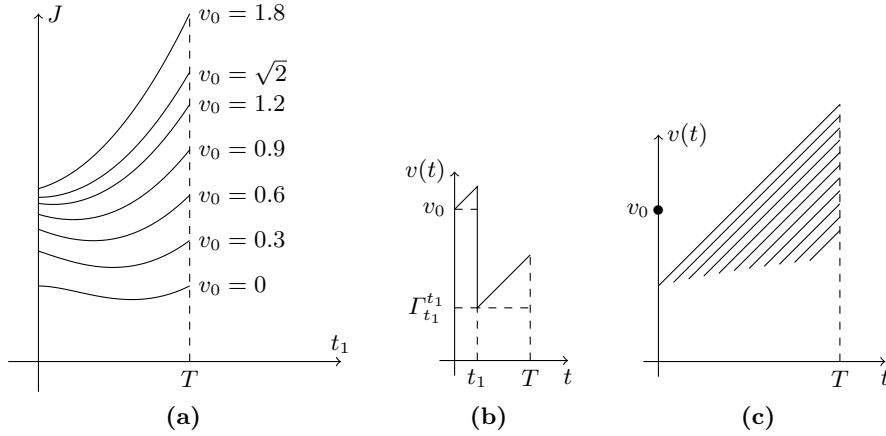
#### 4.1 The Optimal Instant in Case of One Measure: Qualitative Results.

In this and the next subsections, the above problem is studied for the particular case where  $n=1$  measure can be performed. All questions listed above are solved in terms of explicit formulas in Section 4.2.

The cost function (13) takes the form

$$J_{\sigma^2, T, v_0, v_1}(t_1) = \frac{\sigma^2 t_1^2}{2} + v_0 t_1 + \frac{\sigma^2 (T - t_1)^2}{2} + \frac{(\sigma^2 t_1 + v_0) v_1 (T - t_1)}{\sigma^2 t_1 + v_0 + v_1}. \quad (15)$$

Its behavior is shown Figure 3, **(a)**. Remark that the RHS term in equation (15) can be split into two terms: the "rectangular term"  $\left(v_0 t_1 + \frac{(\sigma^2 t_1 + v_0) v_1 (T - t_1)}{\sigma^2 t_1 + v_0 + v_1}\right)$  and the "triangular term"  $\left(\frac{\sigma^2 t_1^2}{2} + \frac{\sigma^2 (T - t_1)^2}{2}\right)$ , respectively accounting for the contributions of the rectangular and triangular shaped area in the integral of  $v(t)$ , and shown on Figure 3, **(b)**. Minimizing the cost function  $J_{T, v_0, v_1}(t_1)$  constitutes a tradeoff between minimizing these two terms.



**Fig. 3.** **(a)**:  $J_{\sigma^2, T, v_0, v_1}(t_1)$  as function of  $v_0$  and  $t_1$ . The parameters are  $v_1 = 1, T = 1, \sigma^2 = 1$ . The cost function is minimized at  $t_1 = 0$  if and only if  $v_0 \geq \sqrt{2}$ . **(b)**: An example of a function  $v(t)$  showing the geometric interpretation of the rectangular and the triangular terms of the expression (15) of the integral cost function. **(c)**: The dependency of the function  $v(t)$  in the interval  $t \in [t_1, T]$  on the choice of  $t_1$ . In this example,  $n = 1, v_0 = v_1 = 1, T = 1.2, \sigma^2 = 1$ , and  $t_1$  takes values in  $[0, 1]$ . Each straight line represents one possible function  $v(t)$ . The slopes of all lines equal  $\sigma^2$  and their left endpoints have coordinates  $(t_1, \Gamma_{t_1}^{t_1})$ .

Different situations are possible as it can be seen on Figure 3, **(a)**. One can define the **regime 1** as the set of situations when  $t_1 = 0$  is the optimum. Similarly, define the **regime 2** as the set of situations where the optimal  $t_1$  is in the interior of the interval  $[0, T]$ . Then, the optimal  $t_1$  is the point where the derivative of the cost function (15) vanishes. Its value is given by (20). Remark that in the regime 2, the optimal  $t_1$  can be larger than  $\frac{T}{2}$ .

The distinction between these two regimes is justified by the following analytic property: the function, which defines the optimal  $t_1$ ,

$$\begin{aligned} t_{1,\text{opt}}(\sigma^2, T, v_0, v_1) &= \arg \min_{t_1} J_{\sigma^2, T, v_0, v_1}(t_1) \\ &= \max \left( 0, \frac{-3v_0 - 3v_1 + \sigma^2 T + \sqrt{(\sigma^2 T + v_0 + 5v_1)^2 - (4v_1)^2}}{4\sigma^2} \right), \end{aligned} \quad (16)$$

is differentiable everywhere except at the border between the regions which correspond to the two regimes.

#### 4.2 Derivation and Properties of (16).

One can decide whether a local minimum is achieved at  $t_1=0$  by computing the corresponding partial derivative:

$$\frac{\partial J_{\sigma^2, T, v_0, v_1}(t_1)}{\partial t_1} \Big|_{t_1=0} = \frac{v_0}{v_0 + v_1} \left( v_0 - \sigma^2 T \left( \frac{v_1}{v_0 + v_1} + 1 \right) \right). \quad (17)$$

The partial derivative at a point  $t_1=t'$  has a similar expression:

$$\begin{aligned} \frac{\partial J_{\sigma^2, T, v_0, v_1}(t_1)}{\partial t_1} \Big|_{t_1=t'} &= \frac{v_0 + \sigma^2 t'}{v_0 + v_1 + \sigma^2 t'} \times \\ &\quad \left( v_0 + \sigma^2 t' - \sigma^2 (T - t') \left( \frac{v_1}{v_0 + v_1 + \sigma^2 t'} + 1 \right) \right). \end{aligned} \quad (18)$$

Remark that the RHS of (18) is a product of two increasing (with respect to  $t'$ ) factors, the first of which  $\left( \frac{v_0 + \sigma^2 t'}{v_0 + v_1 + \sigma^2 t'} \right)$  is positive. Therefore, the locus of nonnegativity of  $\frac{\partial J_T(t_1)}{\partial t_1} \Big|_{t_1=t'}$  is an interval of the form  $[t_{1,\text{opt}}, T]$ , where  $t_{1,\text{opt}}$  may equal zero or be strictly positive.

Consequently, two different behaviors of the cost function are possible. In the first case (regime 1), it is increasing near  $t_1 = 0$  (its derivative at zero (17) is nonnegative). Then, the cost function  $J_T(t_1)$  is increasing and convex on the whole interval  $[0, T]$ , and its global minimum is  $t_{1,\text{opt}}(T) = 0$ . According to (17), this situation corresponds to

$$T \leq T_{\text{crit}} = \frac{v_0}{\sigma^2 \left( \frac{v_1}{v_0 + v_1} + 1 \right)}. \quad (19)$$

Remark that  $T_{\text{crit}}$  is an increasing function of  $v_0$  and a decreasing function of  $v_1$  and of  $\sigma^2$ .

Intuitively, this behavior is observed when  $T$  is small or  $v_0$  is large, which means that the prior information is poor. In this case, it is penalizing not to take a measure immediately in order to get better information. More formally, the rectangular term has an order of magnitude  $O(T)$  when  $T$  tends to zero,



while the triangular term has an order of magnitude  $O(T^2)$ . Therefore, when  $T$  is small enough, choosing  $t_1=0$  should minimize both the rectangular term and the sum.

In the second case (regime 2), the cost function is decreasing near  $t_1 = 0$ . This is observed when (19) does not hold, i.e.  $T$  is large or  $v_0$  is small. Then, the minimum of the cost function is reached at the only nonzero point  $t_{1,\text{opt}}$ , where its derivative (18) equals zero. By equating the derivative (18) to zero, one gets the following expressions for  $t_{1,\text{opt}}$  and for  $T$ :

$$t_{1,\text{opt}} = \frac{-3v_0 - 3v_1 + \sigma^2 T + \sqrt{(\sigma^2 T + v_0 + 5v_1)^2 - (4v_1)^2}}{4\sigma^2} \quad (20)$$

and

$$\sigma^2 T = 2\sigma^2 t_{1,\text{opt}} + v_0 - v_1 + \frac{2v_1^2}{v_0 + \sigma^2 t_{1,\text{opt}} + 2v_1}. \quad (21)$$

Remark that the function  $t_{1,\text{opt}}(T)$  defined by (20) is concave and increasing. When  $T \rightarrow \infty$ , one gets the asymptotic expansion

$$t_{1,\text{opt}}(T) = \frac{\sigma^2 T + v_1 - v_0}{2\sigma^2} + o\left(\frac{1}{T}\right), \quad (22)$$

the function being always smaller than its asymptote:

$$t_{1,\text{opt}}(T) < \frac{\sigma^2 T + v_1 - v_0}{2\sigma^2}. \quad (23)$$

The following intuitive argument can be given for the order of magnitude of the optimal instant:  $t_{1,\text{opt}}(T) \sim \frac{T}{2}$  (by (22)). When  $T$  is large, the triangular terms become more important than the ‘‘rectangular terms’’. Therefore, the minimum of the sum should be close to the value  $\frac{T}{2}$ , which minimizes the triangular term.

Using (19) and (20), it is easy to check that

$$t_{1,\text{opt}}(T_{\text{crit}}) = \frac{-3v_0 - 3v_1 + \sigma^2 T_{\text{crit}} + \sqrt{(\sigma^2 T_{\text{crit}} + v_0 + 5v_1)^2 - (4v_1)^2}}{4\sigma^2} = 0, \quad (24)$$

i.e., both formulas of regime 1 and regime 2 coincide if the values of the parameters lie on the boundary. This proves (16).

Remark that the dependence of  $t_{1,\text{opt}}$  in  $\sigma^2$  and  $T$  is simplified by the relation

$$t_{1,\text{opt}}\left(\frac{\sigma^2}{\alpha}, \alpha T, v_0, v_1\right) = \alpha t_{1,\text{opt}}(\sigma^2, T, v_0, v_1), \quad (25)$$

therefore, the ratio  $t_{1,\text{opt}}/T$  depends only on  $\sigma^2 T$ ,  $v_0$  and  $v_1$ .

## 5 Conclusion and Perspectives

A simple model is studied, where the variance about the system parameters (here a single parameter) evolving over a finite period of time grows linearly in

the absence of measure. The properties of the optimal measure timetable are studied.

In Section 4, the particular case, where the instant of exactly 1 measure is to be chosen, is studied in detail. The system can behave in one of the two different regimes. If the duration of the process is larger than a critical value, the optimal instant is inside the interval and is asymptotically close (when  $T$  is large) to  $\frac{T}{2}$ . If the duration of the process is smaller than the critical value, the sampling at time 0 turns out to be optimal. These results are in a closed form, therefore they are more precise than the results obtained for the model considered in [3].

One goal of the future research is to find the optimal measurement instants when the number of measures is  $n > 1$ . Partial analytic results, which explain some properties of Figure 2, are available. These extend the case  $n = 1$ . The main method is the following: in a subproblem where the instants  $t_1, \dots, t_{n-1}$  are fixed, one can find the optimal instant  $t_n$  using the results of this article, then check whether the given instants  $t_1, \dots, t_{n-1}$  are locally optimal.

In this problem, the order of the measures is fixed. It is also possible to allow it to vary. The main property of this problem is the fact that the cost function is no longer rational, but piecewise-rational.

Another objective of the future research is to consider more complex models than the real Brownian motion considered presently.

## Appendix: Proof of Lemma 1.

The following intuitively clear result is important for the proof.

**Lemma 2.** *The variance  $\Gamma_{t_1}^{t_1} = \frac{1}{v_1 + \sigma^2 t_1 + v_0}$  (see (9) for  $k = 1$ ) satisfies*

$$\frac{\partial \Gamma_{t_1}^{t_1}}{\partial t_1} < \sigma^2. \quad (26)$$

The proof is done by a direct computation. An interpretation of this inequality is the following: in a setting with  $n = 1$  measure, the variance of the estimator of  $\theta(T)$  equals  $v(T) = \sigma^2(T - t_1) + \Gamma_{t_1}^{t_1}$  and decreases when the instant  $t_1$  of the measure approaches  $T$ . This is represented Figure 3, (c).

We prove Lemma 1 by induction on  $n$  (the number of measures). If  $n = 0$ , the function  $v(t)$  has a simple form:  $v(t) = v_0 + \sigma^2 t$ . Therefore, the constraint (11) is expressed as  $v(T) = v_0 + \sigma^2 T \leq V$ , which is equivalent to (12).

Now we are going to prove the Lemma for  $n + 1$  measures supposing that it is valid for  $n$  measures. The first instant  $t_1$  must be chosen in the interval  $[0, \frac{V - v_0}{\sigma^2}]$  in order for (11) to hold for  $t \in [0, t_1[$ . The function  $v(t)$  in the interval  $[t_1, T]$  is also defined by (3)-(10), but with the following parameters:

$$n_{\text{new}} = n, \quad T_{\text{new}} = T_{\text{old}} - t_1 \quad (27)$$

$$v_{0,\text{new}} = \Gamma_{t_1,\text{old}}^{t_1} = \frac{(v_{0,\text{old}} + \sigma^2 t_{1,\text{old}})v_{1,\text{old}}}{v_{0,\text{old}} + \sigma^2 t_{1,\text{old}} + v_{1,\text{old}}}, \quad (28)$$

$$v_{k,\text{new}} = v_{k+1,\text{old}} \text{ and } t_{k,\text{new}} = t_{k+1,\text{old}} - t_1 \text{ for } k \in \{1, \dots, n\} \quad (29)$$

By applying the induction hypothesis to the new function, we obtain that the constraint (11) can be satisfied if and only if

$$\sigma^2(T - t_1) \leq V - \Gamma_{t_1}^{t_1} + \sigma^2 \left( \sum_{k=2}^{n+1} \delta(V, v_k) \right) \quad (30)$$

(all quantities here are the “old” ones, i.e. with respect to the initial problem), which is equivalent to

$$\Gamma_{t_1}^{t_1} - \sigma^2 t_1 \leq V - \sigma^2 T + \sigma^2 \left( \sum_{k=2}^{n+1} \delta(V, v_k) \right). \quad (31)$$

The RHS of (31) is independent of  $t_1$ . By Lemma 2, the LHS is a strictly decreasing function of  $t_1$ . Therefore, (31) can be satisfied if and only if it holds for

$$t_1 = \frac{V - v_0}{\sigma^2}. \quad (32)$$

After replacing  $t_1$  by its value (32) in (31) and applying the definition (12) of  $\delta(V, v_1)$ , one gets

$$\sigma^2 T \leq V - v_0 + \sigma^2 \left( \sum_{k=1}^{n+1} \delta(v_k) \right), \quad (33)$$

which is Lemma 1.

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