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Research Article

Mathematical Methods in the Applied Sciences

Received XXXX

(www.interscience.wiley.com) DOI: 10.1002/sim.0000
MOS subject classification:

Determining the conductivity for a non-autonomous hyperbolic operator in a cylindrical domain

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This paper is devoted to the reconstruction of the conductivity coefficient for a non-autonomous hyperbolic operator an infinite cylindrical domain. Applying a local Carleman estimate we prove the uniqueness and a Hölder stability in the determination of the conductivity using a single measurement data on the lateral boundary. Our numerical examples show good reconstruction of the location and contrast of the conductivity function in three dimensions.

Keywords: *Inverse Problem*, *Carleman estimate*, *time and space-dependent coefficient*, *infinite domain*, *hyperbolic equation*

1. Introduction

The present result is based on a recent work [13] dealing with the inverse problem of determining the time-independent isotropic conductivity coefficient $c : \Omega \to \mathbb{R}$ appearing in the hyperbolic partial differential equation $(\partial_t^2 - \nabla \cdot c\nabla)u = 0$, where $\Omega := \omega \times \mathbb{R}$ is an infinite cylindrical domain whose cross section ω is a bounded open subset of $\mathbb{R}^{n-1}$, $n \geq 2$. Our goal is to extend this reconstruction result, based on a finite number of observations, to a more general class of conductivities: time and space-dependent conductivities $\tilde{c}(x, t)$. The reconstruction of time and space-dependent coefficients in an infinite domain with a finite number of observations is very challenging. The approach developed here is to retrieve any arbitrary bounded subpart of the non-compactly supported conductivity $\tilde{c}$

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from one single measurement data taken on a compact subset of the lateral boundary $\Gamma = \partial\Omega = \partial\omega \times (-\infty, \infty)$. Furthermore, a stability inequality is established which links the distance between two sets of coefficients $\tilde{c}_1(x,t)$ and $\tilde{c}_2(x,t)$ with the distance of lateral boundary observation of the Neumann derivative of the solutions u_1 and u_2. First, this stability inequality implies the uniqueness of the determination of the coefficient $\tilde{c}$ and second, we can use it to perform numerical reconstruction using noisy observations because real observations are generally noisy. Since a lot of background appearing in real physical applications involve time and space-dependent conductivities, we extend the results in [13] by considering the following initial boundary value problem

$$\begin{cases} \partial_t^2 u - \nabla \cdot \tilde{c} \nabla u = 0 & \text{in } Q := \Omega \times (0,T), \\ u(\cdot,0) = \theta_0, \ \partial_t u(\cdot,0) = \theta_1 & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma := \Gamma \times (0,T), \end{cases} \tag{1.1}$$

with initial conditions (θ_0, θ_1), where $\tilde{c}$ is the unknown conductivity coefficient we aim to retrieve, and we assume that $\tilde{c}$ is time and space depending in the following form :

$$\tilde{c}(x,t) = c_0(x,t) + c(x), \tag{1.2}$$

where $c_0(x,t)$ is assumed to be known. That means that we consider the case of the perturbation of a general time and space-dependent conductivity by a space-dependent one. Such model is not a direct application of known results and involves several technical difficulties connected to the time dependence which will be detailed later. For a similar general non-stationary media, we can refer to [25] where the authors study an inverse problem for Maxwell's equations.

Several stability results in the inverse problem of determining one or several unknown coefficients of a hyperbolic equation from a finite number of measurements of the solution are available in the mathematics literature, see [7, 8, 10, 11, 12, 18, 19, 21, 22, 23, 26] for example, and their derivation relies on a Carleman inequality specifically designed for hyperbolic systems. All these works concern space-dependent coefficients.

On the other hand, none of these works are associated to numerical simulations whereas the existence of a stability inequality allows to improve the resolution of the minimization problem by choosing more precisely the functional to minimize. Furthermore, the case of the reconstruction of the conductivity coefficient in the divergence form for the hyperbolic operator induces some numerical difficulties, see [3, 6, 14] for details. A lot of papers are dealing with an optimization approach without any theoretical study and in most of the cases, the uniqueness of the associated inverse problem is not proved. On the other hand, we develop in this paper numerical simulations for a problem of the reconstruction of the conductivity coefficient in the form (1.2) based on partial boundary observations similar to whose used in our theoretical result.

In previous works [3] were presented numerical studies of the reconstruction of the space-dependent conductivity function in a hyperbolic equation using backscattered data in three dimensions. Also in [14] were presented numerical simulations of reconstruction of the only space-dependent conductivity function in two-dimensions. In [14] a layer-stripping procedure was used instead of the Lagrangian approach of [3]. However, the time-dependent function as a part of the conductivity function was not considered in the above cited works.

In our numerical examples of this work we tested the reconstruction of a conductivity function that represents a sum of two space-dependent gaussians and one time-dependent function. Since by our assumption the time-dependent function is known inside the domain, then the goal of our numerical experiments is to reconstruct only the space-dependent part of the conductivity function. To do that we used Lagrangian approach together with the domain decomposition finite element/finite difference method of [3]. One of the important points of this work is that in our numerical simulations we applied one non-zero initial condition in the model problem which corresponds well to the uniqueness and stability results of this paper.

Our three-dimensional numerical simulations show that we can accurately reconstruct location and large contrast of the space-dependent function which is a part of the known time-dependent function. However, the location of this function in the third, x_3 direction, should be still improved. Similarly with [2, 5, 6] we are going to apply an adaptive finite element method to improve the reconstruction of the shape of the space-dependent function obtained in this work.

The outline of the work is the following: in section 2 we prove the uniqueness and stability result for the system (1.1), in section 3 we present our numerical simulations, and in section 4 we summarize the results of our work.

2. Mathematical background and main theoretical result

2.1. Notations and hypothesis

Throughout this article, we keep the following notations: $x = (x', x_n) \in \Omega$ for every $x' := (x_1, \ldots, x_{n-1}) \in \omega$ and $x_n \in \mathbb{R}$. Further, we denote by $|y| := \left(\sum_{i=1}^m y_j^2\right)^{1/2}$ the Euclidean norm of $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, $m \in \mathbb{N}^*$, and we write $\mathbb{S}^{n-1} := \left\{x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1},\ |x'| = 1\right\}$. We write ∂_j for $\partial/\partial x_j$, $j \in \mathbb{N}^*_{n+1} := \{m \in \mathbb{N}^*,\ m \leq n+1\}$. For convenience the time variable t is sometimes denoted by x_{n+1} so that $\partial_t = \partial/\partial t = \partial_{n+1}$. We set $\nabla := (\partial_1, \ldots, \partial_n)^T$, $\nabla_{x'} := (\partial_1, \ldots, \partial_{n-1})^T$ and $\nabla_{x,t} = (\partial_1, \ldots, \partial_n, \partial_t)^T$.

For any open subset D of $\mathbb{R}^m$, $m \in \mathbb{N}^*$, we note $H^p(D)$ the p-th order Sobolev space on D for every $p \in \mathbb{N}$, where $H^0(D)$ stands for $L^2(D)$. We write $\| \cdot \|_{p,D}$ for the usual norm in $H^p(D)$ and we note $H^1_0(D)$ the closure of $C^\infty_0(D)$ in the topology of $H^1(D)$.

Finally, for $d > 0$ we put $\Omega_d := \omega \times (-d, d)$, $Q_d := \Omega_d \times (0, T)$, $\Gamma_d := \partial\omega \times (-d, d)$ and $\Sigma_d := \partial\omega \times (-d, d) \times (0, T)$.

We are interested by the initial boundary value problem (1.1). We shall suppose that $\tilde{c}$ fulfills the ellipticity condition

$$\tilde{c} \geq c_m \text{ in } Q, \tag{2.1}$$

for some positive constant c_m. In order to solve the inverse problem associated with (1.1) we seek solutions belonging to $\cap_{k=3}^4 C^k([0, T]; H^{5-k}(\Omega))$. Following the strategy used in [13] based on the reference [24, Sect. 3, Theorem 8.2], it is sufficient to assume that the coefficient $\tilde{c}$ (resp. c) is in $C^\infty(Q; \mathbb{R})$ (resp. is in $C^\infty(\Omega; \mathbb{R})$) and $\partial\omega \in C^\infty$, to get the required regularity for the solution u of the system (1.1). We note c_M a positive constant fulfilling

$$\|\tilde{c}\|_{W^{4,\infty}(Q)} \leq c_M. \tag{2.2}$$

Since our strategy is based on a Carleman estimate for the hyperbolic system (1.1), it is also required that the condition

$$a' \cdot \nabla_{x'} \tilde{c} \geq \mathfrak{a}_0 \text{ in } Q, \tag{2.3}$$

holds for some $a' = (a_1, \ldots, a_{n-1}) \in \mathbb{S}^{n-1}$ and $\mathfrak{a}_0 > 0$. Hence, given $\omega^\#$ an open subset of $\mathbb{R}^{n-1}$ such that $\partial\omega \subset \omega^\#$, we put $\mathcal{O}_* = \omega^\# \times \mathbb{R}$, and for $c_* \in C^4((\overline{\mathcal{O}_* \cap \Omega}) \times [0, T]; \mathbb{R})$ satisfying

$$c_* \geq c_m \text{ and } a' \cdot \nabla_{x'} c_* \geq \mathfrak{a}_0 \text{ in } (\mathcal{O}_* \cap \Omega) \times (0, T), \tag{2.4}$$

we introduce the set $\Lambda_{\mathcal{O}_*} = \Lambda_{\mathcal{O}_*}(a', \mathfrak{a}_0, c_*, c_m, c_M)$ of admissible conductivity coefficients as

$$\Lambda_{\mathcal{O}_*} := \{\tilde{c} \in W^{4,\infty}(Q; \mathbb{R}) \cap C^1(\overline{Q}; \mathbb{R}) \text{ obeying } (1.2) - (2.3);\ \tilde{c} = c_* \text{ in } (\mathcal{O}_* \cap \Omega) \times (0, T)\}. \tag{2.5}$$

Furthermore, it is required that θ_0 be in $W^{3,\infty}(\Omega) \cap H^5(\Omega)$ and satisfies

$$-a' \cdot \nabla_{x'} \theta_0 \geq \eta_0 e^{-(1+x_n^2)}, x = (x', x_n) \in \text{ in } \omega_* \times \mathbb{R}, \tag{2.6}$$

for some $\eta_0 > 0$ and some open subset ω_* in $\mathbb{R}^{n-1}$, with C^2 boundary, satisfying

$$\overline{\omega \setminus (\omega^{\#} \cap \omega)} \subset \omega_* \text{ and } \overline{\omega_*} \subset \omega, \tag{2.7}$$

and that exists $M_0 > 0$ such that

$$\|\theta_0\|_{W^{3,\infty}(\Omega)} + \|\theta_0\|_{H^5(\Omega)} \leq M_0. \tag{2.8}$$

2.2. Main result

The following result claims Hölder stability in the inverse problem of determining c in Ω_ℓ, where $\ell > 0$ is arbitrary, from the knowledge of one boundary measurement of the solution to (1.1), performed on Σ_L for $L > \ell$ sufficiently large. The corresponding observation is viewed as a vector of the Hilbert space

$$H(\Sigma_L) := H^3(0, T; L^2(\Gamma_L)),$$

endowed with the norm,

$$\|v\|^2_{H(\Sigma_L)} := \|v\|^2_{H^3(0,T;L^2(\Gamma_L))}, \ v \in H(\Sigma_L).$$

Theorem 2.1. *Assume that $\partial\omega$ is C^5 and let $\mathcal{O}_*$ be a neighborhood of Γ in $\mathbb{R}^{n-1}$. Assume that $c_0 \in C^4\left(\overline{Q};\mathbb{R}\right)$ and $\partial_t c_0(\cdot,0) = \partial_t^3 c_0(\cdot,0) = 0$ in Ω. For $a' = (a_1,\dots,a_{n-1}) \in \mathbb{S}^{n-1}$, $\mathfrak{a}_0 > 0$, $c_m \in (0,1)$, $c_M > c_m$ and $c_* \in C^4\left(\overline{(\mathcal{O}_* \cap \Omega)} \times [0,T];\mathbb{R}\right)$ fulfilling (2.4), pick $c_j(x)$, $j = 1,2$ such that $\tilde{c}_j = c_j + c_0 \in \Lambda_{\mathcal{O}_*}(a', \mathfrak{a}_0, c_*, c_m, c_M)$, defined by (2.5). Further, given $M_0 > 0$, $\eta_0 > 0$ and an open subset $\omega_* \subset \mathbb{R}^{n-1}$ obeying (2.7), let θ_0 fulfill (2.6)-(2.8), and $\theta_1 = 0$.*

Then for any $\ell > 0$ we may find $L > \ell$ and $T > 0$, such that the $\bigcap_{k=0}^{5} C^k([0,T], H^{5-k}(\Omega))$-solution u_j, $j = 1,2$, to (1.1) associated with (θ_0,θ_1), where $\tilde{c}_j$ is substituted for $\tilde{c}$, satisfies

$$\|\tilde{c}_1 - \tilde{c}_2\|_{H^1(\Omega_\ell)} \leq C \left\| \frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} \right\|^{\kappa}_{H(\Sigma_L)}.$$

Here $C > 0$ and $\kappa \in (0,1)$ are two constants depending only on ω, ℓ, M_0, η_0, a', $\mathfrak{a}_0$, c_*, c_m and c_M.

The main difficulties associated to the time dependence of the conductivity coefficient $\tilde{c}$ appear on one hand in the proof of the Carleman estimate for second order hyperbolic operators. On the other hand, the Bukhgeim-Klibanov method uses intensively time differentiation. In the case of time and space-dependent coefficients we have to manage a lot of additive terms with respect to the case of only space-dependent conductivity. Nevertheless, the result obtained in Theorem 2.1 is similar to Theorem 1.1 in p.411 established in [13].

Remark 2.2. *In Theorem 2.1, we have to assume (2.3) for some $a' \in \mathbb{S}^{n-1}$ and $\mathfrak{a}_0 > 0$. (2.3) can be interpreted as a monotonicity condition about the coefficient $\tilde{c}$. Maybe one is able to replace (2.3) by a weaker condition, but it is difficult to search for sharpest conditions for deriving a Carleman estimate, which is our main technical tool. In fact, (2.3) is a sufficient condition for the pseudoconvexity, which is a sharp sufficient condition for a Carleman estimate (e.g., [20]). From the physical viewpoint, by Snell's Law on refraction, (2.3) ensures that enough waves will arrive at a finite portion of the boundary of the cylinder after some time for the determination of unknown principal coefficients $\tilde{c}$. We point out that (2.3) is not satisfied by constant coefficients nor coefficients depending only on x_n. The case that coefficients are constants is trivial since we need to assume that $\tilde{c} = c_*$ in $(\mathcal{O}_* \cap \Omega) \times (0,T)$ for some given c_* by the method we use in this paper. In the case that coefficients depend only on x_n, by Snell's Law on refraction, it is possible that some waves are localized in the middle of the cylinder and never meet the boundary.*

2.3. A Carleman estimate for second order hyperbolic operators with time-dependent coefficient in cylindrical domains

In this section we establish a global Carleman estimate for the system (1.1) and in view of the inverse problem, we start by time-symmetrizing the solution u of (1.1). Namely, we put

$$u(x,t) := u(x,-t),\ x \in \Omega,\ t \in (-T,0). \tag{2.9}$$

and

$$c_0(x,t) := c_0(x,-t),\ x \in \Omega,\ t \in (-T,0). \tag{2.10}$$

By the condition $\theta_1 = 0$ and $c_0 \in C^4\left(\overline{Q};\mathbb{R}\right)$, $\partial_t c_0(\cdot,0) = \partial_t^3 c_0(\cdot,0) = 0$ in Ω in Theorem 2.1, we can verify that $\tilde{c} \in W^{4,\infty}(\Omega\times(-T,T)) \cap C^1\left(\overline{\Omega}\times[-T,T]\right)$ and $u \in \bigcap_{k=3}^4 C^k([-T,T];H^{5-k}(\Omega))$ for any $c(x)$ such that $\tilde{c}(x,t) = c_0(x,t) + c(x) \in \Lambda_{\mathcal{O}_*}$.

We consider the operator

$$A := A(x,t,\partial) = \partial_t^2 - \nabla\cdot\tilde{c}\nabla + R, \tag{2.11}$$

where R is a first-order partial differential operator with $L^\infty(Q)$ coefficients. For simplicity, we put $Q := \Omega\times(-T,T)$, $\Sigma := \Gamma\times(-T,T)$, and $\Sigma_L := \partial\omega\times(-L,L)\times(-T,T)$ for any $L > 0$, in the remaining part of this section.

We define for every $\delta > 0$, $\gamma > 0$, and $a' \in \mathbb{S}^{n-1}$ fulfilling (2.3), the following weight functions:

$$\psi(x,t) = \psi_\delta(x,t) := |x'-\delta a'|^2 - x_n^2 - t^2 \text{ and } \varphi(x,t) = \varphi_{\delta,\gamma}(x,t) := e^{\gamma\psi(x,t)},\ (x,t)\in Q. \tag{2.12}$$

Proposition 2.3. *Let A be defined by (2.11), where $\tilde{c} \in W^{4,\infty}(Q;\mathbb{R}) \cap C^1\left(\overline{Q};\mathbb{R}\right)$ verifies (2.1)–(2.3), and let ℓ be positive. Then there exist $\delta_0 > 0$ and $\gamma_0 > 0$ such that for all $\delta \geq \delta_0$ and $\gamma \geq \gamma_0$, we may find $L > \ell$, $T > 0$ and $s_0 > 0$ for which the estimate*

$$s\sum_{j=0,1} s^{2(1-j)}\|e^{s\varphi}\nabla_{x,t}^j v\|_{0,Q_L}^2 \leq C\left(\|e^{s\varphi}Av\|_{0,Q_L}^2 + s\sum_{j=0,1} s^{2(1-j)}\|e^{s\varphi}\nabla_{x,t}^j v\|_{0,\partial Q_L}^2\right), \tag{2.13}$$

holds for any $s \geq s_0$ and $v \in H^2(Q_L)$. Here C is a positive constant depending only on ω, a', $\mathfrak{a}_0$, δ_0, γ_0, s_0, c_m and c_M.

Moreover there exists a constant $d_\ell > 0$, depending only on ω, ℓ, δ_0 and γ_0, such that the weight function φ defined by (2.12) satisfies

$$\varphi(x',x_n,0) \geq d_\ell,\ (x',x_n) \in \overline{\omega}\times[-\ell,\ell], \tag{2.14}$$

and we may find $\epsilon \in (0,(L-\ell)/2)$ and $\zeta > 0$ so small that we have:

$$\max_{x\in\overline{\omega}\times[-L,L]} \varphi(x',x_n,t) \leq \tilde{d}_\ell := d_\ell e^{-\gamma\zeta^2},\ |t| \in [T-2\epsilon,T], \tag{2.15}$$

$$\max_{(x',t)\in\overline{\omega}\times[-T,T]} \varphi(x',x_n,t) \leq \tilde{d}_\ell,\ |x_n| \in [L-2\epsilon,L]. \tag{2.16}$$

Proof. We mimic a part of the proof from our previous paper (see Proposition 3.1, p.414 in [13]), first adapting the definition of δ_0 in (2.18) because the time dependence of the conductivity introduces several difficulties. We stress out the main change with respect to the time independent version.

We define first δ_0, L and T and following the notations (3.8) and (3.9) introduced p.414 in [13] we introduce

$$g_\ell(\delta) = \left(\sup_{x'\in\omega}|x'-\delta a'|^2 - \inf_{x'\in\omega}|x'-\delta a'|^2 + \ell^2\right)^{1/2} \tag{2.17}$$

then there exists $\delta_0 > 0$ so large that

$$\delta\mathfrak{a}_0 > \left(\left(1+\frac{2\sqrt{n}}{c_m{}^{1/2}}\right)g_\ell(\delta)+\sqrt{n-1}|\omega|+2\right)c_M+2+\frac{g_\ell(\delta)}{c_m}c_M,\ \delta\geq\delta_0, \tag{2.18}$$

where $\mathfrak{a}_0$, a' are introduced in (2.3).

Further, since ω is bounded and $a' \neq 0_{\mathbb{R}^{n-1}}$ by (2.3), we may as well assume upon possibly enlarging δ_0, that we have in addition $c_m^{1/2}\inf_{x'\in\overline{\omega}}|x'-\delta a'| > g_\ell(\delta)$ for all $\delta \geq \delta_0$. This and (2.18) yield that there exists $\vartheta > 0$ so small that the two following inequalities

$$\delta\mathfrak{a}_0 - \left(L+\sqrt{n-1}|\omega|+2\left(1+\frac{\sqrt{n}T}{c_m{}^{1/2}}\right)\right)c_M - 2 - \frac{T}{c_m}c_M > 0, \tag{2.19}$$

and

$$c_m^{1/2}\inf_{x'\in\overline{\omega}}|x'-\delta a'| > T, \tag{2.20}$$

hold simultaneously for every L and T in $(g_\ell(\delta), g_\ell(\delta)+\vartheta)$, uniformly in $\delta \geq \delta_0$.
We note that the time dependence of $\tilde{c}$ implies that we need to reformulate the inequalities (2.18) and (2.19) with respect to the similar one, (3.9) and (3.10) in [13].

Now, we come back to the proof of (2.13). Our approach is based on Isakov [20, Theorem 3.2.1']. We put $\mathbf{x} := (x,t)$ for $(x,t)\in Q_L$ and $\nabla_{\mathrm{x}} = (\partial_1,\dots,\partial_n,\partial_{n+1})^T$. We also write $\xi' = (\xi_1,\dots,\xi_{n-1})^T \in \mathbb{R}^{n-1}$, $\xi = (\xi_1,\dots,\xi_n)^T \in \mathbb{R}^n$ and $\tilde{\xi} = (\xi_1,\dots,\xi_n,\xi_{n+1})^T \in \mathbb{R}^{n+1}$. We call A_2 the principal part of the operator A, that is $A_2 = A_2(\mathbf{x},\partial) = \partial_t^2 - \tilde{c}(x,t)\Delta$, and denote its symbol by $A_2(\mathbf{x},\tilde{\xi}) = \tilde{c}(x,t)|\xi|^2 - \xi_{n+1}^2$, where $|\xi| = \left(\sum_{j=1}^n \xi_j^2\right)^{1/2}$. Since $A_2(\mathbf{x},\nabla_{\mathrm{x}}\psi(\mathbf{x})) = 4\left(\tilde{c}(x,t)(|x'-\delta a'|^2+x_n^2) - x_{n+1}^2\right)$ for every $\mathbf{x}\in\overline{Q}_L$, we have

$$A_2(\mathbf{x},\nabla_{\mathrm{x}}\psi(\mathbf{x})) > 0,\ \mathbf{x}\in\overline{Q}_L, \tag{2.21}$$

by (2.1) and (2.20). For all $\mathbf{x}\in\overline{Q}_L$ and $\tilde{\xi}\in\mathbb{R}^{n+1}$, put

$$J(\mathbf{x},\tilde{\xi}) = J = \sum_{j,k=1}^{n+1}\frac{\partial A_2}{\partial\xi_j}\frac{\partial A_2}{\partial\xi_k}\partial_j\partial_k\psi + \sum_{j,k=1}^{n+1}\left(\left(\partial_k\frac{\partial A_2}{\partial\xi_j}\right)\frac{\partial A_2}{\partial\xi_k} - (\partial_k A_2)\frac{\partial^2 A_2}{\partial\xi_j\partial\xi_k}\right)\partial_j\psi, \tag{2.22}$$

where we write ∂_j, $j\in\mathbb{N}_{n+1}^*$, instead of $\partial/\partial x_j$, and x_{n+1} stands for t. We assume that

$$A_2(\mathbf{x},\tilde{\xi}) = \tilde{c}(x,t)|\xi|^2 - \xi_{n+1}^2 = 0,\ x\in\overline{\Omega},\ t\in(0,T),\ \tilde{\xi}\in\mathbb{R}^{n+1}\backslash\{0\}, \tag{2.23}$$

and that

$$\nabla_{\tilde{\xi}}A_2(x,\tilde{\xi})\cdot\nabla_{\mathrm{x}}\psi(\mathbf{x}) = 4\left[\tilde{c}(x,t)(\xi'\cdot(x'-\delta a') - \xi_n x_n) + \xi_{n+1}x_{n+1}\right] = 0, \tag{2.24}$$

for $\mathbf{x}\in\overline{Q}_L$, $\tilde{\xi}\in\mathbb{R}^{n+1}\backslash\{0\}$ and we shall prove that $J(\mathbf{x},\tilde{\xi}) > 0$ for any $(\mathbf{x},\tilde{\xi})\in\overline{Q}_L\times\{\mathbb{R}^{n+1}\setminus\{0\}\}$.

To this end we notice that the first sum in the right hand side of (2.22) reads

$$\langle \mathrm{Hess}(\psi)\nabla_{\tilde{\xi}}A_2, \nabla_{\tilde{\xi}}A_2\rangle = 8\left(\tilde{c}^2(|\xi'|^2-\xi_n^2) - \xi_{n+1}^2\right),$$

and that

$$\begin{aligned}\textstyle\sum_{j,k=1}^{n+1}\left(\left(\partial_k\frac{\partial A_2}{\partial\xi_j}\right)\frac{\partial A_2}{\partial\xi_k} - (\partial_k A_2)\frac{\partial^2 A_2}{\partial\xi_j\partial\xi_k}\right)\partial_j\psi &= 2\tilde{c}\left(2(\nabla\tilde{c}\cdot\xi)(\nabla\psi\cdot\xi) - (\nabla\tilde{c}\cdot\nabla\psi)|\xi|^2\right)\\ &\quad -4(\partial_{n+1}\tilde{c})\xi_{n+1}(\nabla\psi\cdot\xi) + 2(\partial_{n+1}\tilde{c})(\partial_{n+1}\psi)|\xi|^2,\end{aligned}$$

since from the time dependence of $\tilde{c}$,

$$\left(\partial_k \frac{\partial A_2}{\partial \xi_j}\right) \frac{\partial A_2}{\partial \xi_k} - (\partial_k A_2) \frac{\partial^2 A_2}{\partial \xi_j \partial \xi_k} \neq 0$$

if either j or k is equivalent to $n+1$.

Therefore we have

$$\begin{aligned} J &= 4\left[2\tilde{c}^2(|\xi'|^2 - \xi_n^2) - \left(2 + (x' - \delta a') \cdot \nabla_{x'} \tilde{c} - x_n \partial_n \tilde{c}\right) \xi_{n+1}^2 - 2x_{n+1}\xi_{n+1}(\nabla \tilde{c} \cdot \xi)\right] \\ &\quad + 8\frac{\partial_{n+1}\tilde{c}}{\tilde{c}} x_{n+1}\xi_{n+1}^2 - 4t(\partial_{n+1}\tilde{c})|\xi|^2 \end{aligned}$$

from (2.23)-(2.24). Further, in view of (2.23) we have

$$\tilde{c}^2(|\xi'|^2 - \xi_n^2) \geq -\tilde{c}^2|\xi|^2 \geq -\tilde{c}\, \xi_{n+1}^2$$

and

$$|\nabla \tilde{c} \cdot \xi| \leq |\nabla \tilde{c}||\xi| \leq (|\nabla \tilde{c}|/\tilde{c}^{1/2})|\xi_{n+1}|,$$

and the additive term with respect to the paper [13] in p.416 is underestimated by

$$-4\frac{T}{\tilde{c}}|\partial_{n+1}\tilde{c}|\xi_{n+1}^2,$$

then,

$$J \geq 4\left[\delta a' \cdot \nabla_{x'}\tilde{c} - \left(x' \cdot \nabla_{x'}\tilde{c} - x_n\partial_n\tilde{c} + 2\tilde{c} + 2T\frac{|\nabla \tilde{c}|}{\tilde{c}^{1/2}} + 2 + \frac{T}{\tilde{c}}|\partial_{n+1}\tilde{c}|\right)\right]\xi_{n+1}^2. \tag{2.25}$$

Here we used the fact that $x_{n+1} = t \in [0, T]$. Due to (2.1)-(2.3), the right hand side of (2.25) is lower bounded, up to the multiplicative constant $4\xi_{n+1}^2$, by the left hand side of (2.19). Since ξ_{n+1} is non zero by (2.2) and (2.23), then we obtain $J(\mathbf{x}, \tilde{\xi}) > 0$ for all $(\mathbf{x}, \tilde{\xi}) \in \overline{Q}_L \times \{\mathbb{R}^{n+1} \setminus \{0\}\}$. With reference to (2.21), we may apply [20, Theorem 3.2.1'], getting two constants $s_0 = s_0(\gamma) > 0$ and $C > 0$ such that (2.13) holds for any $s \geq s_0$ and $v \in H^2(Q_L)$. The end of the proof is similar to the third part in the proof of Proposition 3.1 in p.416 of [13].

□

Now we can derive from Proposition 2.3 a global Carleman estimate for the solution to the boundary value problem

$$\left\{ \begin{array}{ll} \partial_t^2 u - \nabla \cdot \tilde{c} \nabla u = f & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \end{array} \right. \tag{2.26}$$

where $f \in L^2(Q)$. To this purpose we introduce a cut-off function $\chi \in C^2(\mathbb{R}; [0, 1])$, such that

$$\chi(x_n) := \left\{ \begin{array}{ll} 1 & \text{if } |x_n| < L - 2\epsilon, \\ 0 & \text{if } |x_n| \geq L - \epsilon, \end{array} \right. \tag{2.27}$$

where ϵ is the same as in Proposition 2.3, and we set

$$u_\chi(x, t) := \chi(x_n)u(x, t) \text{ and } f_\chi(x, t) := \chi(x_n)f(x, t), \; (x, t) \in Q.$$

Corollary 2.4. *Let $f \in L^2(Q)$. Then, under the conditions of Proposition 2.3, there exist two constants $s_* > 0$ and $C > 0$, depending only on ω, ℓ, M_0, η_0, a', $\mathfrak{a}_0$, c_m and c_M, such that the estimate*

$$s \sum_{j=0,1} s^{2(1-j)} \|e^{s\varphi} \nabla_{x,t}^j u\|_{0,Q_L}^2 \leq C \left(\|e^{s\varphi} f\|_{0,Q_L}^2 + s^3 e^{2s\tilde{d}_\ell} \|u\|_{1,Q_L}^2 + s \sum_{j=0,1} s^{2(1-j)} \|e^{s\varphi} \nabla_{x,t}^j u_\chi\|_{0,\Sigma_L}^2 \right),$$

holds for any solution u to (2.26), uniformly in $s \geq s_*$.

For the proof see Corollary 3.2 in p.417 of [13].

2.4. Inverse problem

In this subsection we introduce the linearized inverse problem associated with (1.1) and relate the first Sobolev norm of the conductivity to some suitable initial condition of this boundary problem.

Namely, given $\tilde{c}_i \in \Lambda_{\mathcal{O}_*}$ for $i = 1, 2$, we note u_i the solution to (1.1) where $\tilde{c}_i$ is substituted for $\tilde{c}$, suitably extended to $(-T, 0)$ in accordance with (2.10). Thus, putting

$$c := \tilde{c}_1 - \tilde{c}_2 \text{ and } f_c := \nabla \cdot (c \nabla u_2), \tag{2.28}$$

it is clear from (1.1) that the function $u := u_1 - u_2$ is solution to the linearized system

$$\begin{cases} \partial_t^2 u - \nabla \cdot (\tilde{c}_1 \nabla u) = f_c & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(\cdot, 0) = \partial_t u(\cdot, 0) = 0 & \text{in } \Omega. \end{cases} \tag{2.29}$$

Note that $\tilde{c}_1$ is time-dependent and this time dependence of a coefficient in the system (2.29) implies to rewrite carefully the method used in [13].

By differentiating k-times (2.29) with respect to t, for $k = 1, 2, 3$, we see that $u^{(k)} := \partial_t^k u$ is solution to

$$\begin{cases} \partial_t^2 u^{(k)} - \nabla \cdot (\tilde{c}_1 \nabla u^{(k)}) + R_k(\partial_t u^{(1)}, \partial_t u^{(2)}, \cdots, \partial_t u^{(k)}, \nabla u^{(0)}, \nabla u^{(1)}, \cdots, \nabla u^{(k-1)}) \\ \qquad = P_k(f_c^{(0)}, f_c^{(1)}, \cdots, f_c^{(k)}) \text{ in } Q, \\ \quad u^{(k)} = 0 \text{ on } \Sigma, \end{cases} \tag{2.30}$$

with $f_c^{(j)} := \partial_t^j f_c = \nabla \cdot (c \nabla u_2^{(j)})$, $u_2^{(j)} = \partial_t^j u_2, j = 0, 1, 2, ..., k$, where R_k and P_k stand for generic zero-order operators with coefficients in $W^{3-k,\infty}(Q)$. The result (2.30) can be proved by the method of induction.

Proof. In fact, differentiating (2.29) with respect to t, we have $u^{(1)} = 0$ on Σ, and

$$\begin{aligned} \partial_t^2 u^{(1)} - \nabla \cdot \left(\tilde{c}_1 \nabla u^{(1)}\right) &= \nabla \cdot \left(\left(\frac{\partial_t \tilde{c}_1}{\tilde{c}_1} \right) \tilde{c}_1 \nabla u \right) + f_c^{(1)} \\ &= \nabla \left(\frac{\partial_t \tilde{c}_1}{\tilde{c}_1} \right) \cdot (\tilde{c}_1 \nabla u) + \frac{\partial_t \tilde{c}_1}{\tilde{c}_1} \{ \nabla \cdot (\tilde{c}_1 \nabla u) \} + f_c^{(1)} \text{ in } Q. \end{aligned}$$

Then using (2.29), we obtain

$$\begin{aligned} &\partial_t^2 u^{(1)} - \nabla \cdot \left(\tilde{c}_1 \nabla u^{(1)}\right) - \nabla \left(\frac{\partial_t \tilde{c}_1}{\tilde{c}_1} \right) \cdot \left(\tilde{c}_1 \nabla u^{(0)} \right) \\ &\quad = \frac{\partial_t \tilde{c}_1}{\tilde{c}_1} \left(\partial_t^2 u - f_c \right) + f_c^{(1)} = \frac{\partial_t \tilde{c}_1}{\tilde{c}_1} \partial_t u^{(1)} - \frac{\partial_t \tilde{c}_1}{\tilde{c}_1} f_c^{(0)} + f_c^{(1)} \text{ in } Q. \end{aligned}$$

Therefore, noting $c, \tilde{c}_1 \in \Lambda_{\mathcal{O}_*}(a', \mathfrak{a}_o, c_*, c_m, c_M)$, $u_2 \in \bigcap_{k=0}^5 C^k([0, T], H^{5-k}(\Omega))$, (2.1) and (2.5), we see that (2.30) holds for $k = 1$ where R_1 and P_1 stand for zero-order operators with coefficients in $W^{2,\infty}(Q)$.

We assume that (2.30) holds for $k = 1, 2$, where R_k and P_k stand for generic zero-order operators with coefficients in $W^{3-k,\infty}(Q)$. Differentiating (2.30) for k with respect to t, we have $u^{(k+1)} = 0$ on Σ, and

$$\begin{aligned}\partial_t^2 u^{(k+1)} - \nabla\cdot(\tilde{c}_1\nabla u^{(k+1)}) &= \nabla\cdot\left(\left(\frac{\partial_t\tilde{c}_1}{\tilde{c}_1}\right)\tilde{c}_1\nabla u^{(k)}\right)\\ &\quad - \tilde{R}_{k+1}(\partial_t u^{(1)},\partial_t u^{(2)},\cdots,\partial_t u^{(k+1)},\nabla u^{(0)},\nabla u^{(1)},\cdots,\nabla u^{(k)})\\ &\quad + \tilde{P}_{k+1}(f_c^{(0)},f_c^{(1)},\cdots,f_c^{(k+1)})\\ &= \nabla\left(\frac{\partial_t\tilde{c}_1}{\tilde{c}_1}\right)\cdot\left(\tilde{c}_1\nabla u^{(k)}\right) + \frac{\partial_t\tilde{c}_1}{\tilde{c}_1}\left\{\nabla\cdot\left(\tilde{c}_1\nabla u^{(k)}\right)\right\}\\ &\quad - \tilde{R}_{k+1}(\partial_t u^{(1)},\partial_t u^{(2)},\cdots,\partial_t u^{(k+1)},\nabla u^{(0)},\nabla u^{(1)},\cdots,\nabla u^{(k)})\\ &\quad + \tilde{P}_{k+1}(f_c^{(0)},f_c^{(1)},\cdots,f_c^{(k+1)})\ \text{ in } Q.\end{aligned}$$

where $\tilde{R}_{k+1}$ and $\tilde{P}_{k+1}$ stand for generic zero-order operators with coefficients in $W^{2-k,\infty}(Q)$. Then using (2.30) for k and noting $\partial_t^2 u^{(k)} = \partial_t u^{(k+1)}$, we can obtain

$$\begin{aligned}\partial_t^2 u^{(k+1)} - \nabla\cdot(\tilde{c}_1\nabla u^{(k+1)}) &= \nabla\left(\frac{\partial_t\tilde{c}_1}{\tilde{c}_1}\right)\cdot\left(\tilde{c}_1\nabla u^{(k)}\right) + \frac{\partial_t\tilde{c}_1}{\tilde{c}_1}\partial_t u^{(k+1)}\\ &+ \frac{\partial_t\tilde{c}_1}{\tilde{c}_1}R_k(\partial_t u^{(1)},\partial_t u^{(2)},\cdots,\partial_t u^{(k)},\nabla u^{(0)},\nabla u^{(1)},\cdots,\nabla u^{(k-1)})\\ &- \frac{\partial_t\tilde{c}_1}{\tilde{c}_1}P_k(f_c^{(0)},f_c^{(1)},\cdots,f_c^{(k)})\\ &- \tilde{R}_{k+1}(\partial_t u^{(1)},\partial_t u^{(2)},\cdots,\partial_t u^{(k+1)},\nabla u^{(0)},\nabla u^{(1)},\cdots,\nabla u^{(k)})\\ &+ \tilde{P}_{k+1}(f_c^{(0)},f_c^{(1)},f_c^{(2)},\cdots,f_c^{(k+1)})\\ &\triangleq -R_{k+1}(\partial_t u^{(1)},\partial_t u^{(2)},\cdots,\partial_t u^{(k+1)},\nabla u^{(0)},\nabla u^{(1)},\cdots,\nabla u^{(k)})\\ &+ P_{k+1}(f_c^{(0)},f_c^{(1)},\cdots,f_c^{(k+1)})\ \text{ in } Q.\end{aligned}$$

Then we see that (2.30) holds for $k+1$ where R_{k+1} and P_{k+1} stand for generic zero-order operators with coefficients in $W^{2-k,\infty}(Q)$. Therefore, noting $c, \tilde{c}_1 \in \Lambda_{\mathcal{O}_*}(a', \mathfrak{a}_{\mathrm{o}}, c_*, c_m, c_M)$, $u_2 \in \bigcap_{k=0}^5 C^k([0,T], H^{5-k}(\Omega))$, (2.1) and (2.5), we see that (2.30) holds for $k = 1,2,3$. Hence the proof of (2.30) is complete. □

In this part, a lot of additive terms appear in the source term due to the time dependence of the conductivity $\tilde{c}_i$ and we have to manage them now very precisely. We keep the notations of Corollary 2.4. In particular, for any function v, we denote $\chi(x_n)\cdot v$ by v_χ, where $\chi(x_n)$ is defined in (2.27). Upon multiplying both sides of the identity (2.30) by $\chi(x_n)$, we obtain that

$$\left\{\begin{aligned}&\partial_t^2 u_\chi^{(k)} - \nabla\cdot(\tilde{c}_1\nabla u_\chi^{(k)}) + R_k(\partial_t u_\chi^{(1)},\partial_t u_\chi^{(2)},\cdots,\partial_t u_\chi^{(k)},\nabla u_\chi^{(0)},\nabla u_\chi^{(1)},\cdots,\nabla u_\chi^{(k-1)})\\ &\qquad = P_k(f_{c\chi}^{(0)},f_{c\chi}^{(1)},\cdots,f_{c\chi}^{(k)}) - g_k\ \text{ in } Q,\\ &u^{(k)} = 0\ \text{ on } \Sigma,\end{aligned}\right. \tag{2.31}$$

with

$$f_{c\chi}^{(j)} := \nabla\cdot(c_\chi\nabla u_2^{(j)}),\qquad j = 0,1,2,\dots,k, \tag{2.32}$$

and g_k is supported in $\tilde{Q}_\epsilon := \{x = (x', x_n, t),\ \ x'\in\omega,\ \ |x_n| \in (L-2\epsilon, L-\epsilon),\text{ and } t \in (-T,T)\}$.

Having said that we may now upper bound, up to suitable additive and multiplicative constants, the $e^{s\varphi(\cdot,0)}$-weighted first Sobolev norm of the conductivity c_χ in Ω_L, by the corresponding norm of the initial condition $u_\chi^{(2)}(\cdot,0)$.

Lemma 2.5. *We assume that (2.6)-(2.8) hold. Let u be the solution to the linearized problem (2.29) and let χ be defined by (2.27). Then there exist two constants $s_* > 0$ and $C > 0$, depending only on ω, ε and the constant M_0 defined by (2.8), such that the estimate*

$$\sum_{j=0,1} \| e^{s\varphi(\cdot,0)} \nabla^j c_\chi \|^2_{0,\Omega_L} \leq C s^{-1} \left(\sum_{j=0,1} \| e^{s\varphi(\cdot,0)} \nabla^j u^{(2)}_\chi(\cdot,0) \|^2_{0,\Omega_L} + e^{2s\tilde{d}_\ell} \right),$$

holds for all $s \geq s_*$.

For the proof see Lemma 4.1 in p.419 of [13].

Now, we end the proof of theorem 2.1. We first give an upper bound $u^{(2)}_\chi(\cdot,0)$ in the $e^{s\varphi(\cdot,0)}$-weighted $H^1(\Omega_L)$-norm topology, by the corresponding norms of $u^{(2)}_\chi$ and $u^{(3)}_\chi$ in Q_L.

Lemma 2.6. *There exists a constant $s_* > 0$ depending only on T such that we have*

$$\| z(\cdot,0) \|^2_{0,\Omega_L} \leq 2 \left(s \| z \|^2_{0,Q_L} + s^{-1} \| \partial_t z \|^2_{0,Q_L} \right),$$

for all $s \geq s_*$ and $z \in H^1(-T,T;L^2(\Omega_L))$.

For the proof see Lemma 3.2 in p.13 of [9].

We apply Lemma 2.6 with $z = e^{s\varphi} \partial^j_i u^{(2)}_\chi$ for $i \in \mathbb{N}^*_n$ and $j = 0,1$, getting

$$\| e^{s\varphi(\cdot,0)} \partial^j_i u^{(2)}_\chi(\cdot,0) \|^2_{0,\Omega_L} \leq C \left(s \| e^{s\varphi} \partial^j_i u^{(2)}_\chi \|^2_{0,Q_L} + s^{-1} \| e^{s\varphi} \partial^j_i u^{(3)}_\chi \|^2_{0,Q_L} \right), \; s \geq s_*.$$

Summing up the above estimate over $i \in \mathbb{N}^*_n$ and $j = 0,1$, we obtain for all $s \geq s_*$ that

$$\sum_{j=0,1} \| e^{s\varphi(\cdot,0)} \nabla^j u^{(2)}_\chi(\cdot,0) \|^2_{0,\Omega_L} \leq C \sum_{j=0,1} \left(s \| e^{s\varphi} \nabla^j u^{(2)}_\chi \|^2_{0,Q_L} + s^{-1} \| e^{s\varphi} \nabla^j u^{(3)}_\chi \|^2_{0,Q_L} \right). \tag{2.33}$$

Then we majorize the right hand side of (2.33) with

$$\mathfrak{h}_k(s) := \sum_{j=0,1} s^{2(1-j)} \| e^{s\varphi} \nabla^j_{x,t} u^{(k)}_\chi \|^2_{0,\Sigma_L}, \; k = 0,1,2,3. \tag{2.34}$$

Indeed, since $u^{(k)}_\chi$, for $k = 0,1,2,3$, is solution to (2.26) with $\tilde{c} = \tilde{c}_1$ and $f = P_k(f^{(0)}_{c_\chi}, f^{(1)}_{c_\chi}, \cdots, f^{(k)}_{c_\chi}) - g_k - R_k(\partial_t u^{(1)}_\chi, \partial_t u^{(2)}_\chi, ..., \partial_t u^{(k)}_\chi, \nabla u^{(0)}_\chi, \nabla u^{(1)}_\chi, ..., \nabla u^{(k-1)}_\chi)$, according to (2.31), then Corollary 2.4 yields

$$\begin{aligned} &\sum_{k=0}^{3} \left(s \sum_{j=0,1} s^{2(1-j)} \| e^{s\varphi} \nabla^j_{x,t} u^{(k)}_\chi \|^2_{0,Q_L} \right) \\ &\leq C \sum_{k=0}^{3} \left(\| e^{s\varphi} f^{(k)}_{c_\chi} \|^2_{0,Q_L} + \| e^{s\varphi} g_k \|^2_{0,\tilde{Q}_\epsilon} + s^3 e^{2s\tilde{d}_\ell} \| u^{(k)}_\chi \|^2_{1,Q_L} + s\mathfrak{h}_k(s) \right), \end{aligned}$$

for s large enough, because the terms coming from the operators R_k are absorbed by the terms in the left hand side.

The time dependence of $\tilde{c}_i$ implies to manage more terms at this part of the proof than in the inequality (4.13) in p.421 of [13]. More precisely, we have also to deal in the right hand side with the terms $\mathfrak{h}_k(s)$ for $k = 0$ and $k = 1$. In light of (2.33) this entails that

$$\begin{aligned} &\sum_{j=0,1} \| e^{s\varphi(\cdot,0)} \nabla^j u^{(2)}_\chi(\cdot,0) \|^2_{0,\Omega_L} \\ \leq \; & C \sum_{k=0}^{3} \left(\| e^{s\varphi} f^{(k)}_{c_\chi} \|^2_{0,Q_L} + \| e^{s\varphi} g_k \|^2_{0,\tilde{Q}_\epsilon} + s^3 e^{2s\tilde{d}_\ell} \| u^{(k)}_\chi \|^2_{1,Q_L} + s\mathfrak{h}_k(s) \right). \end{aligned} \tag{2.35}$$

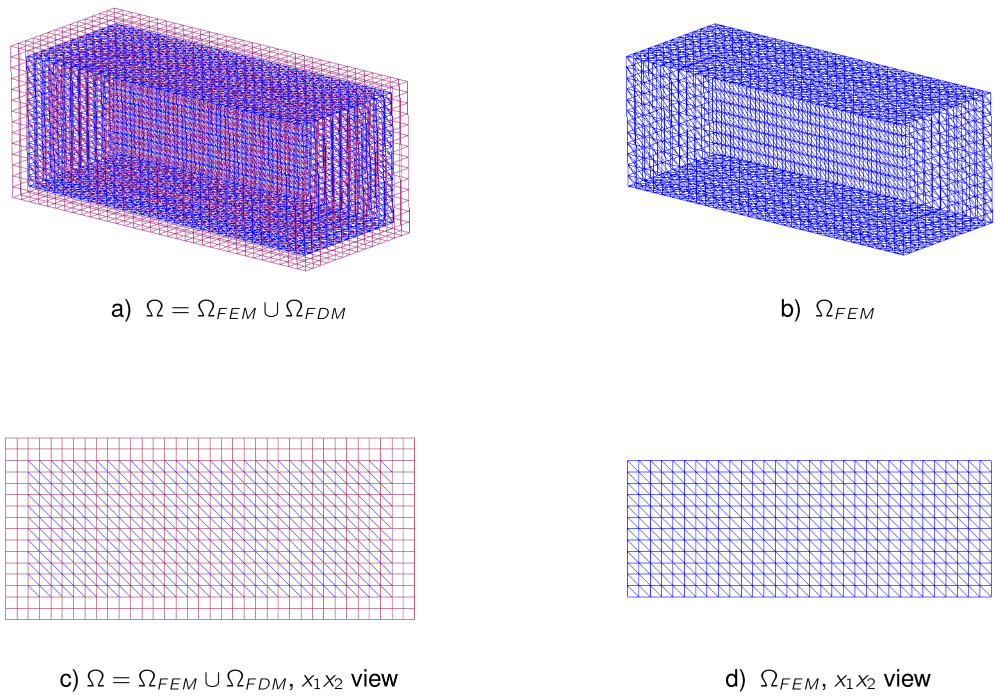
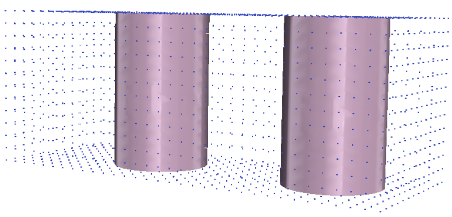


Figure 1. *a) Wireframe of the hybrid finite element/finite difference mesh used in the domain decomposition method of the domain $\Omega = \Omega_{FEM} \cup \Omega_{FDM}$. b) Wireframe of the finite element mesh of the domain Ω_{FEM}.*

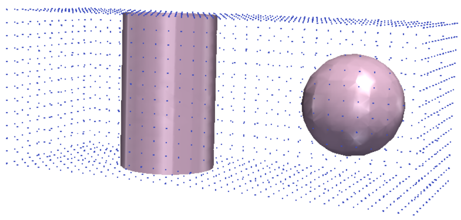
Further, we see that the first (resp., second) term of the sum in the right hand side of (2.35) is upper bounded up to some multiplicative constant, by $\sum_{j=0,1} \|e^{s\varphi} \nabla^j c_\chi\|^2_{0,Q_L}$ (resp., $e^{2s\tilde{d}_\ell}(\|u^{(k)}\|^2_{1,Q_L} + 1)$), as in (4.14) in p.421 of [13]. Using this upper bound and Lemma 2.5 it follows that for sufficiently large s we get

$$
\begin{aligned}
& C\, s \sum_{j=0,1} \|e^{s\varphi(\cdot,0)} \nabla^j c_\chi\|^2_{0,\Omega_L} \qquad (2.36)\\
\le & \sum_{j=0,1} \|e^{s\varphi} \nabla^j c_\chi\|^2_{0,Q_L} + e^{2s\tilde{d}_\ell} + \sum_{k=0}^{3} \left(s^3 e^{2s\tilde{d}_\ell} \|u^{(k)}\|^2_{1,Q_L} + s\mathfrak{h}_k(s) \right).
\end{aligned}
$$

Finally, the end of the proof is similar to Step 3 in p.422 of [13], adding in the right hand side the terms in the sum for $k = 0$ and $k = 1$. But, these additive terms are already embedded in the norm used in our final result, and we get the stability inequality of theorem 2.1.



a) Test 1



b) Test 2

Figure 2. *Slices of the exact function $c(x)$ in Ω_{FEM} given by: a) (3.40) in Test 1 and b) (3.41) in Test 2.*

3. Numerical Studies

In this section, we present numerical studies for the determination of the unknown function $c(x)$ of the equation (1.1). To solve problem (1.1) we consider a convex bounded domain $\Omega \subset \mathbb{R}^3$. For its numerical solution, we apply overlapping domain decomposition finite element/finite difference method of [3]. To do this, we divide Ω into two subregions, Ω_{FEM} and Ω_{FDM} such that $\Omega = \Omega_{\text{FEM}} \cup \Omega_{\text{FDM}}$ with $\Omega_{\text{FEM}} \subset \Omega_{\text{FDM}}$, see Figure 1. The communication between these domains is arranged using a mesh overlapping through a two layers of structured nodes around Ω_{FEM}, see Figure 1-c), d) and Figure 2 in [2] for details. The key idea with such a domain decomposition is to apply different numerical methods in different computational domains. For the numerical solution of (1.1) in Ω_{FDM} we will use the finite difference method on a structured mesh. In Ω_{FEM}, we will use finite elements on a sequence of unstructured meshes $K_h = \{K\}$, with elements K consisting of tetrahedra in $\mathbb{R}^3$ satisfying maximal angle condition. This approach combines the flexibility of the finite elements and the efficiency of the finite differences in terms of speed and memory usage and is implemented in the high performance software package WavES [30] using C++ and PETSc [29]. Moreover, the efficient implementation of the absorbing boundary conditions at the boundary of Ω_{FDM} in WavES [30] allows us to solve our inverse problem more precisely.

We introduce dimensionless spatial variables $x' = x/(1m)$ and define Ω_{FEM} and Ω_{FDM} as the following dimensionless computational domains:

$$\Omega_{FEM} = \{x = (x_1, x_2, x_3);\ x_1 \in (-1.6, 1.6),\ x_2 \in (-0.6, 0.6),\ x_3 \in (-0.6, 0.6)\},$$

$$\Omega = \{x = (x_1, x_2, x_3);\ x_1 \in (-1.8, 1.8),\ x_2 \in (-0.8, 0.8),\ x_3 \in (-0.8, 0.8)\}.$$

The boundary $\partial\Omega$ of the domain Ω is such that $\partial\Omega = \partial_1\Omega \cup \partial_2\Omega \cup \partial_3\Omega$ where $\partial_1\Omega$ and $\partial_2\Omega$ are, respectively, front and back sides of Ω, and $\partial_3\Omega$ is the union of left, right, top and bottom sides of this domain. The boundary $\partial_1\Omega$ represents backscattering side of the domain Ω where we collect our time-dependent observations. We also define $S_{1,1} := \partial_1\Omega \times (0, t_1]$, $S_{1,2} := \partial_1\Omega \times (t_1, T)$, $S_2 := \partial_2\Omega \times (0, T)$, $S_3 := \partial_3\Omega \times (0, T)$, and $S_T := \partial_1\Omega \times (0, T)$. Note, that we have introduced different notions for time intervals $(0, t_1]$ and $(t_1, T]$ since we initialize the plane wave only in time $(0, t_1]$ at $\partial_1\Omega$, and at all times $(t_1, T]$ the plane wave travels through the computational domain Ω. Thus, additional boundary conditions are needed to be imposed at $S_{1,2}$, which will be first order absorbing boundary conditions [17].

Our model problem used in computations is the following:

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - \nabla \cdot (\tilde{c} \nabla u) &= 0 \text{ in } \Omega_T, \\
u(x,0) = \theta_0(x), \quad u_t(x,0) &= 0 \text{ in } \Omega, \\
\partial_n u &= f(t) \text{ on } S_{1,1}, \\
\partial_n u &= -\partial_t u \text{ on } S_{1,2}, \\
\partial_n u &= -\partial_t u \text{ on } S_2, \\
\partial_n u &= 0 \text{ on } S_3,
\end{aligned} \tag{3.37}$$

with $\tilde{c}$ is given by (1.2).

In (3.37) the function $f(t)$ is the single direction of a plane wave which is initialized at $\partial_1 \Omega$ in time $T = [0, 3.0]$ and is defined as

$$f(t) = \begin{cases} \sin(\omega_f t), & \text{if } t \in \left(0, \frac{2\pi}{\omega_f}\right), \\ 0, & \text{if } t > \frac{2\pi}{\omega_f}. \end{cases} \tag{3.38}$$

We take $\omega_f = 30, 40, 50$ in all our tests, see Tables 1, 2 for results.

We initialize initial condition $\theta_0(x)$ which satisfies inequality (2.6) at the backscattered side $\partial_1 \Omega$ as

$$u(x,0) = \theta_0(x) = e^{-(x_1^2 + x_2 + x_3^3)}. \tag{3.39}$$

To be able to use the method given in [3] we assume that the functions $c(x) = 1$ and $c_0(x,t) = 0$ inside Ω_{FDM}. The goal of our numerical tests is to reconstruct a smooth function $c(x)$ only inside Ω_{FEM}, which satisfies the conditions (2.3) and (3.43). In Test 1 we define the function $c(x)$ as

$$\begin{aligned} c(x) = 1.0 &+ 5.0 \cdot e^{-((x_1-0.5)^2/0.2 + x_2/200 + x_3{}^2/0.2)} \\ &+ 5.0 \cdot e^{-((x_1+1)^2/0.2 + x_2/200 + x_3{}^2/0.2)}, \end{aligned} \tag{3.40}$$

and in Test 2 the function $c(x)$ is chosen as

$$\begin{aligned} c(x) = 1.0 &+ 7.0 \cdot e^{-((x_1-0.5)^2/0.2 + x_2/200 + x_3{}^2/0.2)} \\ &+ 7.0 \cdot e^{-((x_1+1)^2/0.2 + x_2{}^2/0.2 + x_3{}^2/0.2)}. \end{aligned} \tag{3.41}$$

We also assume that the function $c_0(x,t)$ is known inside Ω_{FEM}, and we define this function as

$$c_0(x,t) = 0.01 \ \cos t \cdot e^{-(x_1{}^2/0.2 + x_2/200 + x_3{}^2/0.2)}. \tag{3.42}$$

Figure 2 presents slices of the exact function $c(x)$ for $c(x) = 5.5$ given by (3.40) and (3.41) for Tests 1, 2, correspondingly. Figure 3 presents isosurfaces of the exact function $\tilde{c}$ in Test 1, and Figure 4 presents isosurfaces of the exact function $\tilde{c}$ in Test 2 at different times. These figures also show that though the coefficient $c_0(x,t)$ given by (3.42) is small, we observe clear dependence on time of the time-dependent function $\tilde{c}$: first, at times $t = 1.2, 1.8$ we see how two functions tend to separate. Next, at further times $t = 2.4$ and $t = 2.7$ functions start drifting away from each other. Thus, time-dependence in all our computational tests is relevant and takes place.

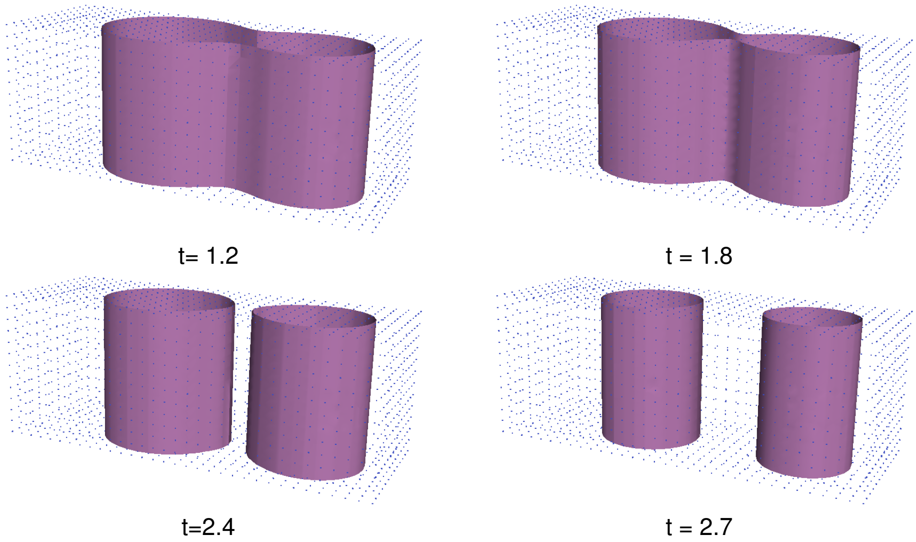


Figure 3. *Test 1. Slices of the exact space and time-dependent function $\tilde{c}$ at different times.*

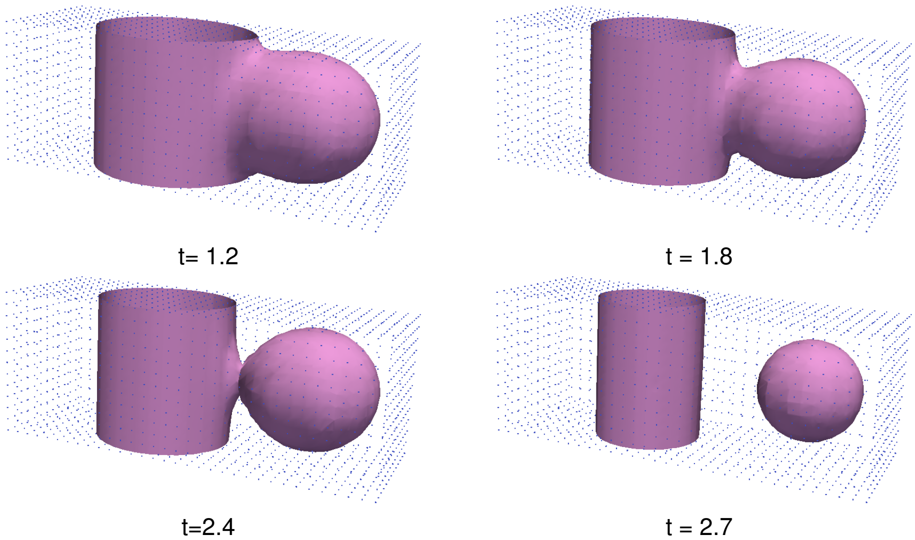


Figure 4. *Test 2. Slices of the exact space and time-dependent function $\tilde{c}$ at different times.*

We choose the mesh size $h = 0.1$ in the overlapping layers between Ω_{FEM} and Ω_{FDM} as well as in the computations of the inverse problem. However, we have generated our backscattered data using the several times locally refined mesh Ω_{FEM}, and in a such way we avoid problem with variational crime. To generate backscattered data we solve the model problem (3.37) in time $T = [0, 3.0]$ with the time step $\tau = 0.003$ which satisfies the CFL condition [15], and supply simulated backscattered data by additive noise $\sigma = 3\%, 10\%$ at $\partial_1 \Omega$.

We also assume that the reconstructed function $c(x)$ belongs to the set of admissible parameters

$$M_c \in \{ 1 \leq c(x) \leq 10 \ \forall x \in \Omega_{FEM}, c(x) = 1 \ on \ \partial\Omega_{FEM} \}, \tag{3.43}$$

where $\partial\Omega_{FEM}$ is the boundary of Ω_{FEM}.

3.1. Optimization method

Our coefficient inverse problem which we use in computations, is the following.

Inverse Problem (IP) *Assume that the function $c(x)$ of the model problem (3.37) is unknown while the function $c_0(x,t)$ is known. Let $c(x)$ satisfies conditions (3.43) and $c(x) = 1, c_0(x,t) = 0$ in the domain $\Omega \backslash \Omega_{\mathrm{FEM}}$. Determine the function $c(x)$ for $x \in \Omega \backslash \Omega_{\mathrm{FDM}}$, assuming that the following function $\tilde{u}(x,t)$ is known*

$$u(x,t) = \tilde{u}(x,t), \forall (x,t) \in S_T. \tag{3.44}$$

The function $\tilde{u}(x,t)$ in (3.44) represents the time-dependent measurements at the boundary $\partial_1\Omega$. Let us introduce the following spaces of real valued functions

$$\begin{aligned} H_u^1(\Omega_T) &:= \{u \in H^1(\Omega_T) : u(\cdot, 0) = 0\}, \\ H_\lambda^1(\Omega_T) &:= \{\lambda \in H^1(\Omega_T) : \lambda(\cdot, T) = 0\}, \\ U^1 &= H_u^1(\Omega_T) \times H_\lambda^1(\Omega_T) \times C\left(\overline{\Omega}\right). \end{aligned} \tag{3.45}$$

For solution of the **IP** for the model problem (3.37) we minimize the following Tikhonov functional

$$J(u,c) = \frac{1}{2} \int_{S_T} (u - \tilde{u})^2 z_\delta(t) ds dt + \frac{1}{2} \gamma \int_\Omega (c - c^0)^2 \ dx, \tag{3.46}$$

where u satisfies (3.37) and $z_\delta(t)$ is a cut-off function ensuring the compatibility conditions at $\overline{\Omega}_T \cap \{t = T\}$ for the adjoint problem and can be chosen as in [3]. We denote by c^0 the initial guess for c, and by $\gamma > 0$ the regularization parameter. Similarly with [3] in all our computations we choose constant regularization parameter $\gamma = 0.01$ because it gives smallest relative error in the reconstruction of the function $c(x)$. Different techniques for the computation of a regularization parameter are presented in works [1, 16, 28], and checking of performance of these techniques for the solution of our inverse problem can be a challenge for our future research. To find a minimum of the functional (3.46) we introduce the Lagrangian

$$L(v) = J(u,c) + \int_\Omega \int_0^T \lambda \Big(\frac{\partial^2 u}{\partial t^2} - \nabla \cdot (\tilde{c} \nabla u) \Big) \ dx dt, \tag{3.47}$$

where $v = (u, \lambda, c) \in U^1$. Our goal is to find a stationary point of the Lagrangian with respect to v satisfying $\forall \bar{v} = (\bar{u}, \bar{\lambda}, \bar{c}) \in U^1$

$$L'(v; \bar{v}) = 0, \tag{3.48}$$

where $L'(v; \cdot)$ is the Jacobian of L at v.

We assume that $\lambda(x,T) = \partial_t \lambda(x,T) = 0$ and impose such conditions on the function λ that $L(v) = F(u,c)$. We use facts that $\lambda(x,T) = \frac{\partial \lambda}{\partial t}(x,T) = 0$ as well as that by our assumption we have $c = 1, c_0 = 0$ on $\partial\Omega$. Then the equation (3.48) together with initial and boundary conditions of (3.37) expresses that for all $\bar{v} \in U^1$,

$$\begin{aligned} 0 = \frac{\partial L}{\partial \lambda}(v)(\bar{\lambda}) = &- \int_{\Omega_T} \frac{\partial \bar{\lambda}}{\partial t} \frac{\partial u}{\partial t} \ dx dt + \int_{\Omega_T} (\tilde{c} \nabla u)(\nabla \bar{\lambda}) \ dx dt \\ &- \int_{S_{1,1}} \bar{\lambda} f(t) \ ds dt + \int_{S_{1,2} \cup S_2} \bar{\lambda} \partial_t u \ ds dt, \quad \forall \bar{\lambda} \in H_\lambda^1(\Omega_T), \end{aligned} \tag{3.49}$$

$$0 = \frac{\partial L}{\partial u}(v)(\bar{u}) = \int_{S_T} (u - \widetilde{u})\ \bar{u}\ z_\delta\ dsdt - \int_\Omega \frac{\partial \lambda}{\partial t}(x,0)\bar{u}(x,0)\ dx - \int_{S_{1,2 \cup S_2}} \frac{\partial \lambda}{\partial t}\bar{u}\ dsdt$$
$$- \int_{\Omega_T} \frac{\partial \lambda}{\partial t}\frac{\partial \bar{u}}{\partial t}\ dxdt + \int_{\Omega_T} (\tilde{c}\nabla\lambda)(\nabla\bar{u})\ dxdt,\ \forall \bar{u} \in H_u^1(\Omega_T). \tag{3.50}$$

The last optimality equation expresses that the gradient with respect to c vanishes:

$$0 = \frac{\partial L}{\partial c}(v)(\bar{c}) = \int_{\Omega_T} (\nabla u)(\nabla\lambda)\bar{c}\ dxdt + \gamma\int_\Omega (c - c^0)\bar{c}\ dx,\ x \in \Omega. \tag{3.51}$$

The equation (3.49) is the weak formulation of the state equation (3.37) and the equation (3.50) is the weak formulation of the following adjoint problem

$$\begin{aligned}
\frac{\partial^2 \lambda}{\partial t^2} - \nabla\cdot(\tilde{c}\nabla\lambda) &= -(u - \widetilde{u})z_\delta \text{ in } \Omega_T,\\
\lambda(\cdot, T) = \frac{\partial \lambda}{\partial t}(\cdot, T) &= 0 \text{ in } \Omega,\\
\partial_n\lambda &= \partial_t\lambda \text{ on } S_{1,2} \cup S_2,\\
\partial_n\lambda &= 0 \text{ on } S_3 \cup S_{1,1},
\end{aligned} \tag{3.52}$$

with $\tilde{c}$ is given by (1.2).

3.2. *The conjugate gradient algorithm for the solution of the inverse problem*

For the numerical solution of inverse problem **IP** we use the conjugate gradient method (CGM) to find the approximation c^m of the function c at the optimization iteration m. Using the optimality condition for the function c given by (3.51) we denote

$$g^m(x) = \int_0^T \nabla u^m \nabla\lambda^m dt + \gamma(c^m - c^0), \tag{3.53}$$

where functions u^m, λ^m are the computed solutions of the forward (3.37) and the adjoint (3.52) problems at the iteration m, respectively.

Algorithm

Step 0. Start with the initial approximation c^0 and compute the approximations c^m as follows:

Step 1. Compute solutions $u^m := u\,(x, t, c^m)$ and $\lambda^m := \lambda\,(x, t, c^m)$ of the forward and the adjoint problems using the domain decomposition method of [3].

Step 2. Update function $c := c^{m+1}$ using the CGM method

$$c^{m+1} = c^m + \alpha d^m(x),$$

where α is the step-size in the gradient update (see details in [27]) and

$$d^m(x) = -g^m(x) + \beta^m d^{m-1}(x),$$

where

$$\beta^m = \frac{\|g^m(x)\|^2}{\|g^{m-1}(x)\|^2},$$

with $d^0(x) = -g^0(x)$.

Table 1. *Results of reconstruction in Test 1 together with computational errors* $|\max_{\Omega_{FEM}} c^N - \max_{\Omega_{FEM}} c|$ *in percents. Here,* N *is the final number of iteration in the conjugate gradient method.*

$\sigma = 3\%$				$\sigma = 10\%$			
ω_f	$\max_{\Omega_{FEM}} c^N$	error, %	N	ω_f	$\max_{\Omega_{FEM}} c^N$	error, %	N
$\omega_f = 30$	5.5	8	11	$\omega_f = 30$	5	17	11
$\omega_f = 40$	5.5	8	12	$\omega_f = 40$	5.5	8	12
$\omega_f = 50$	5.5	8	12	$\omega_f = 50$	6.5	8	13

Table 2. *Results of reconstruction in Test 2 together with computational errors* $|\max_{\Omega_{FEM}} c^N - \max_{\Omega_{FEM}} c|$ *in percents. Here,* N *is the final number of iteration in the conjugate gradient method.*

$\sigma = 3\%$				$\sigma = 10\%$			
ω_f	$\max_{\Omega_{FEM}} c^N$	error, %	N	ω_f	$\max_{\Omega_{FEM}} c^N$	error, %	N
$\omega_f = 30$	7	13	11	$\omega_f = 30$	7.5	6	11
$\omega_f = 40$	7	13	11	$\omega_f = 40$	7.5	6	12
$\omega_f = 50$	7.5	6	11	$\omega_f = 50$	9.8	23	11

Step 3. Stop computations and obtain the function $c^N = c^m$, $N = m$, if either $\|g^m\|_{L_2(\Omega)} \leq tol$ or norms $\|g^m\|_{L_2(\Omega)}$ are stabilized. Here, tol is the tolerance chosen by the user in CGM. Otherwise set $m := m + 1$ and go to step 1.

We note that in all our tests we start the conjugate gradient algorithm with $c^0(x) = 1.0$ at all points in Ω. This choice corresponds to the starting of the algorithm from the homogeneous domain and is similar as was used in our previous computational works [2]-[6].

3.3. *Test 1*

In this numerical test we reconstructed the function $c(x)$ given by (3.40), see Figure 2-a), with $c_0(x, t)$ given by (3.42) since by our assumption this function is known. To get reasonable reconstruction in this test we runned the conjugate gradient algorithm of section 3.2 in time $T = [0, 1.5]$ with the time step $\tau = 0.003$.

Figure 5 displays results of the reconstruction of the function given by (3.40) with additive noise $\sigma = 3\%$. Quite similar results are obtained for $\sigma = 10\%$, see Figure 6. We observe that the location of the maximal value of the function (3.40) is imaged correctly. We also observe the influence of the time-dependent function $c_0(x, t)$ given by (3.42) on the determined function $c(x)$: compare reconstructions obtained in Figures 5, 6 with slices of time-dependent function $\tilde{c}$ given in Figure 3.

It follows from Figures 5 and Table 1 that the imaged contrast in this function is $5.5 : 1 = \max_{\Omega_{FEM}} c_{11} : 1$, where $m := N = 11$ is the final iteration in the conjugate gradient method of section 3.2. Similar observation we made using the Figure 6 and Table 1 where the imaged contrast is $5.5 : 1 = \max_{\Omega_{FEM}} c_{12} : 1$, $m := N = 12$. However, from these figures we also observe that because of the data post-processing procedure [3] the values of the background of function (3.40) are not reconstructed but are smoothed out. Thus, we are able to reconstruct only maximal values of the function (3.40). Comparison of Figures 5 with Figure 2-a) reveals that it is desirable to improve the shape of the reconstructed function $c(x)$ in x_3 direction.

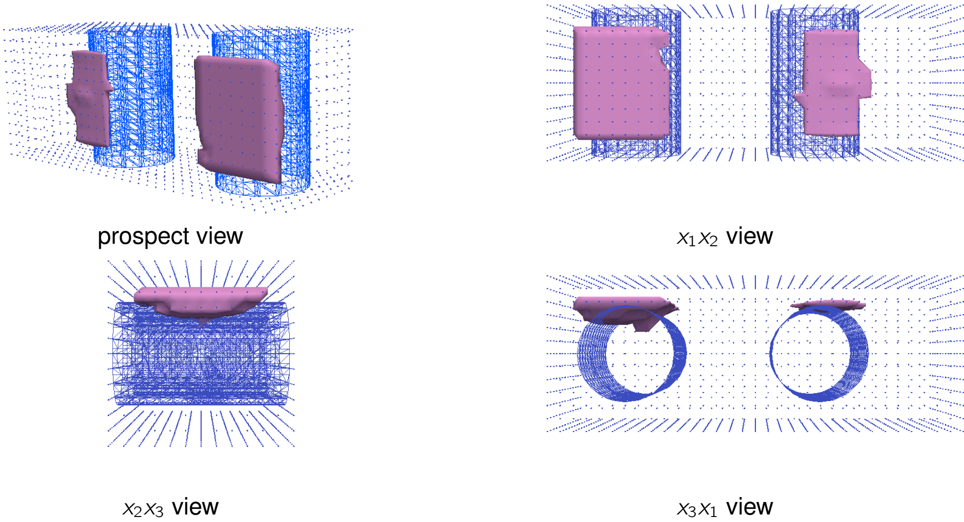


Figure 5. *Test 1. Isosurface of the reconstructed function $c(x)$ with $\max_{\Omega_{FEM}} c^{11}(x) = 5.5$ (in red). Computations are done for $\omega_f = 30$ in (3.38) and with noise level $\sigma = 3\%$ in data $\tilde{u}$. The cylindrical wireframes present the isosurfaces for the exact function $c(x)$ given by (3.40), which correspond to the value of the reconstructed c^{11}.*

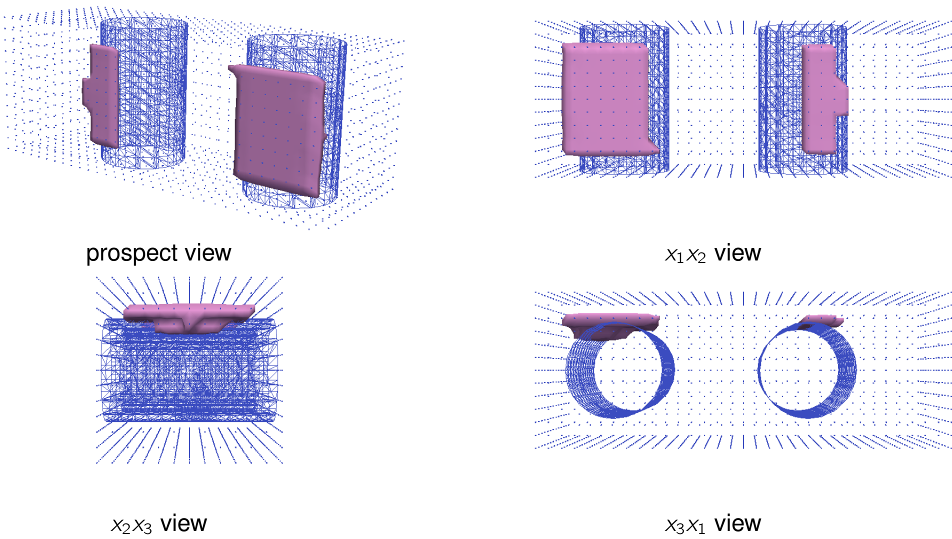


Figure 6. *Test 1. Isosurface of the reconstructed function $c(x)$ with $\max_{\Omega_{FEM}} c^{12}(x) = 5.5$ (in red). Computations are done for $\omega_f = 40$ in (3.38) and with noise level $\sigma = 10\%$ in data $\tilde{u}$. The cylindrical wireframes present the isosurfaces for the exact function $c(x)$ given by (3.40), which correspond to the value of the reconstructed c^{12}.*

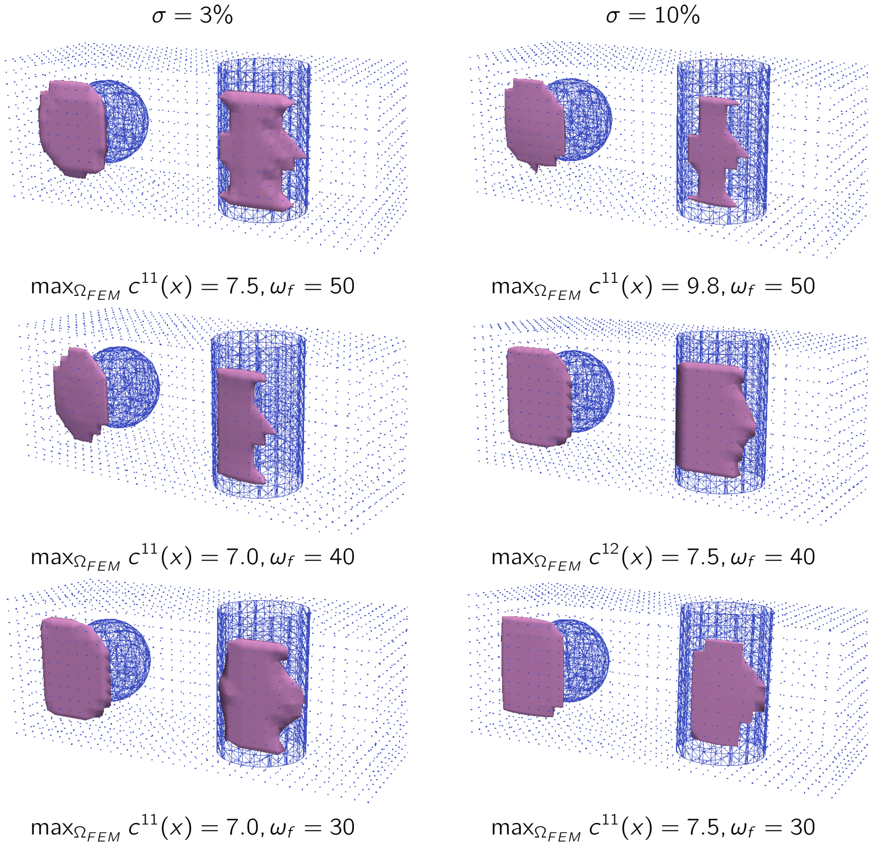


Figure 7. *Test 2. Isosurfaces of the reconstructed function $c(x)$ (in red) for different ω_f in (3.38) and for different noise level σ in data $\tilde{u}$.*

3.4. Test 2

In this numerical test we reconstructed the function $c(x)$ given by (3.41) with the known function $c_0(x, t)$ given by (3.42). Set-up for computations is the same as in the Test 1. Figure 7 and Table 2 show results of the reconstruction of the function $c(x)$ with noise level $\sigma = 3\%$ and $\sigma = 10\%$ in data for different values of ω_f. We observe that the location of the maximal value of the function (3.41) is imaged very well. Again, as in the previous test, the values of the background in (3.41) are smoothed out. Comparing figures with results of reconstruction we conclude that it is desirable improve the shape of the function $c(x)$ in x_3 direction.

4. Conclusions

In this work we present theoretical investigations and numerical studies of the reconstruction of the time and space-dependent coefficient in an infinite cylindrical hyperbolic domain. In the theoretical part of this work we derive a local Carleman estimate which is specially formulated for the infinite cylindrical domain for the case of time-dependent conductivity function.

In the numerical part of the paper we present a computational study of the reconstruction of function $c(x)$ in a hyperbolic problem (3.37) using domain decomposition finite element/difference method of [3]. In our numerical tests we have obtained stable reconstruction of the location and contrasts of the function $c(x)$ in x_1x_2-directions for noise levels $\sigma = 3\%, 10\%$ in backscattered data. However, size and shape on x_3 direction should still be improved in all test cases. Similarly with [2, 5, 6] in our future work we plan to apply an adaptive finite element method in order to get better shapes and sizes of the function $c(x)$ in x_3 direction.

Acknowledgments

The part of the research was done during the sabbatical stay of LB at the Institut de Mathématiques de Marseille, Aix-Marseille University, France, which was supported by the sabbatical programme at the Faculty of Science, University of Gothenburg, Sweden. This paper has been finished during the stay of the second named author at the University of Sciences and Technologie of China, Hefei in April of 2017, and the stay was supported by The USTC.

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