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REMARKS ON THE CH_2 OF CUBIC HYPERSURFACES

RENÉ MBORO

ABSTRACT. This paper presents two approaches to reducing problems on 2-cycles on a smooth cubic hypersurface X over an algebraically closed field of characteristic $\neq 2$, to problems on 1-cycles on its variety of lines $F(X)$. The first one relies on bitangent lines of X and Tsen-Lang theorem. It allows to prove that $\text{CH}_2(X)$ is generated, via the action of the universal \mathbb{P}^1 -bundle over $F(X)$, by $\text{CH}_1(F(X))$. When the characteristic of the base field is 0, we use that result to prove that if $\dim(X) \geq 7$, then $\text{CH}_2(X)$ is generated by classes of planes contained in X and if $\dim(X) \geq 9$, then $\text{CH}_2(X) \simeq \mathbb{Z}$. Similar results, with slightly weaker bounds, had already been obtained by Pan ([27]). The second approach consists of an extension to subvarieties of X of higher dimension of an inversion formula developed by Shen ([30], [31]) in the case of curves of X . This inversion formula allows to lift torsion cycles in $\text{CH}_2(X)$ to torsion cycles in $\text{CH}_1(F(X))$. For complex cubic 5-folds, it allows to prove that the birational invariant provided by the group $\text{CH}^3(X)_{\text{tors}, AJ}$ of homologically trivial, torsion codimension 3 cycles annihilated by the Abel-Jacobi morphism is controlled by the group $\text{CH}_1(F(X))_{\text{tors}, AJ}$ which is a birational invariant of $F(X)$, possibly always trivial for Fano varieties.

INTRODUCTION

Let $X \subset \mathbb{P}_{\mathbb{C}}^{n+1}$ be a smooth hypersurface of degree $d \geq 2$. Let $F_r(X) \subset G(r+1, n+2)$ be the variety of \mathbb{P}^r 's contained in X and $P_r = \mathbb{P}(\mathcal{E}_{r+1}|_{F_r(X)}) \subset F_r(X) \times X$ be the universal \mathbb{P}^r -bundle. One has the incidence correspondence

$$p_r : P_r \rightarrow F_r(X), \quad q_r : P_r \rightarrow X.$$

We will be particularly interested in this paper in the case $r = 1$ and $r = 2$, $d = 3$. It is known (see for example [14], [36]) that if X is covered by projective spaces of dimension $1 \leq r < \frac{n}{2}$, that is q_r is surjective, then $\text{CH}_i(X)_{\mathbb{Q}} \simeq \mathbb{Q}$ for $i < r$ and for $\frac{n}{2} > i \geq r$, there is an inversion formula implying that

$$P_{r,*} : \text{CH}_{i-r}(F_r(X))_{\text{hom}, \mathbb{Q}} \rightarrow \text{CH}_i(X)_{\text{hom}, \mathbb{Q}}$$

is surjective. We recall briefly how it works: Up to taking a desingularization and general hyperplane sections of $F_r(X)$, we can assume that $F_r(X)$ is smooth and q_r is generically finite of degree $N > 0$. Let $H_X = c_1(\mathcal{O}_X(1)) \in \text{CH}^1(X)$ and $h = q_r^* H_X \in \text{CH}^1(P_r)$. Given a cycle $\Gamma \in \text{CH}_i(X)_{\text{hom}}$, we have $N\Gamma = q_{r*} q_r^* \Gamma$ and we can write $q_r^* \Gamma = \sum_{j=0}^r h^j \cdot p_r^* \gamma_j$ where $\gamma_j \in \text{CH}_{i+j-r}(F_r(X))_{\text{hom}}$. Now, $dq_{r*}(h^j \cdot p_r^* \gamma_j) = dH_X^j \cdot q_{r*}(p_r^* \gamma_j) = 0$ for $j > 0$ since

$$dH_X \cdot = i_X^* i_{X,*} : \text{CH}_l(X)_{\text{hom}} \rightarrow \text{CH}_{l-1}(X)_{\text{hom}},$$

where i_X is the inclusion of X into \mathbb{P}^{n+1} , factors through $\text{CH}_l(\mathbb{P}_{\mathbb{C}}^{n+1})_{\text{hom}}$ hence is zero. So we get

$$dN\Gamma = q_{r*} p_r^*(d\gamma_0)$$

which gives $\text{CH}_i(X)_{\text{hom}, \mathbb{Q}} = 0$ for $i < r$ since in this range $\text{CH}_{i-r}(F_r(X)) = 0$, and more generally the desired surjectivity. Working a little more, this method gives, in the case of 2-cycles on cubic fivefolds, the following result (which is a precision of [14], [26]):

Proposition 0.1. *Let X be a smooth cubic fivefold. Then the kernel of the Abel-Jacobi map $\text{CH}_2(X)_{AJ} := \text{Ker}(\Phi_X : \text{CH}_2(X)_{\text{hom}} \rightarrow J^5(X))$ is of 18-torsion.*

Proof. For cubic hypersurfaces of dimension ≥ 3 , after taking hyperplane sections of $F_1(X)$, the degree of the generically finite morphism $P_1 \rightarrow X$ is 6. If $\Gamma \in \text{CH}_2(X)_{AJ}$, we can use

the fact that $3H_X \cdot \Gamma = 0$ in $\text{CH}^4(X)$ and thus (using the notation P, p, q in this case)

$$(1) \quad 3h \cdot q^* \Gamma = 3(h \cdot p^* \gamma_0 + h^2 \cdot p^* \gamma_1) = 0$$

in $\text{CH}^4(P)$. As $h^2 = h \cdot p^* l - p^* c_2$ in $\text{CH}^2(P)$, where l and c_2 are the natural Chern classes on $F_1(X)$ restricted from the Grassmannian, we deduce from (1):

$$3\gamma_0 = -3l \cdot \gamma_1 \text{ in } \text{CH}^3(F_1(X)).$$

Combining this with the previous argument then gives $18\Gamma = q_* p^*(3\gamma_1 \cdot l^3)$ where $3\gamma_1$ is a codimension 2-cycle homologous to 0 and Abel-Jacobi equivalent to 0 on $F_1(X)$. Finally we conclude using [7, Theorem 1 (i)] and the fact that $F_1(X)$ is rationally connected, which implies that $\text{CH}^2(F_1(X))_{AJ} = 0$. \square

The denominators appearing in the above argument do not allow to understand 2-torsion cycles. On the other hand, as smooth cubic hypersurfaces admit a degree 2 unirational parametrization ([8]), all birational invariants are 2-torsion so that, for functorial birational invariant constructed using torsion cycles, the above method gives no interesting information. Our aim in this paper is to give inversion formulas with integral coefficients, allowing in some cases to also control the torsion of the group of cycles, which is especially important for those hypersurfaces in view of rationality problems.

In this paper, we present two approaches to study the surjectivity of the map P_{1*} on cycles with integral coefficient for cubic hypersurfaces. The first one is presented in the first section and uses the bitangent lines of X ; it gives the following result:

Theorem 0.2. *Let $X \subset \mathbb{P}_k^{n+1}$, with $n \geq 2^i + 1$ be a smooth cubic hypersurface over an algebraically closed field k of characteristic not equal to 2, containing a linear subspace of dimension $i < \frac{n}{2}$. Assuming resolution of singularities in dimension $\leq i$, $P_{1,*} : \text{CH}_{i-1}(F_1(X)) \rightarrow \text{CH}_i(X)/\mathbb{Z} \cdot H^{n-i}$ is surjective.*

In the case where $i = 2$, the theorem associates to any 2-cycle a 1-cycle on $F_1(X)$. As, for $i = 2$, the condition to apply the theorem is $\dim_k(X) \geq 5$, $F_1(X)$ is a smooth Fano variety hence separably rationally connected in characteristic 0. By work of Tian and Zong ([34]), $\text{CH}_1(F_1(X))$ is then generated by classes of rational curves. A direct consequence is the following:

Corollary 0.3. *Let $X \subset \mathbb{P}_k^{n+1}$ be a smooth cubic hypersurface over an algebraically closed field k of characteristic 0. If $n \geq 5$, then $\text{CH}_2(X)$ is generated by cycle classes of rational surfaces.*

Remark 0.4. This result is true for a different reason also in dimension 4, see Proposition 2.3.

In the second section, we study 1-cycles on $F_1(X)$ and refine Corollary 0.3 to obtain:

Proposition 0.5. *Let $X \subset \mathbb{P}_k^{n+1}$ be a smooth cubic hypersurface over an algebraically closed field k of characteristic 0. If $n \geq 7$, then $\text{CH}_2(X)$ is generated by classes of planes $\mathbb{P}^2 \subset X$ and therefore $\text{CH}_2(X)_{\text{hom}} = \text{CH}_2(X)_{\text{alg}}$. If $n \geq 9$, then $\text{CH}_2(X) \simeq \mathbb{Z}$.*

Remark 0.6. Some of the results of the first two sections had already been obtained by Pan ([27]) in characteristic 0 namely the surjectivity of $P_{1,*} : \text{CH}_1(F_1(X)) \rightarrow \text{CH}_2(X)$ for $n \geq 17$, the fact that $\text{CH}_1(F_1(X))_{\text{hom}} = \text{CH}_1(F_1(X))_{\text{alg}}$ for $n \geq 13$ and that $\text{CH}_2(X) = \mathbb{Z}$ for $n \geq 18$. He has, more generally, obtained, for hypersurfaces of degree d , bounds to get similar results on the CH_2 of the hypersurfaces and CH_1 of their variety of lines.

The last section is devoted to a second approach to the integral coefficient problem; it consists of a generalization of a formula developed by Shen ([30], see also [31]) in the case of 1-cycles of cubic hypersurfaces. Concretely, we prove the following:

Theorem 0.7. *Let $X \subset \mathbb{P}_k^{n+1}$ be a smooth cubic hypersurface over a field k and Σ a “sufficiently general” smooth subvariety of X of dimension r . Then, denoting $\gamma_\Sigma \in \text{CH}_{r-1}(F_1(X))$*

the cycle class of a hyperplane section of dimension $r - 1$ of the subscheme of $F_1(X)$ parametrizing the bisecant lines to Σ , we have

$$(2\text{deg}(\Sigma) - 3)\Sigma + P_{1,*}\gamma_\Sigma = m_\Sigma H_X^{n-r}$$

for an integer m_Σ .

This inversion formula is more powerful as it will allow us to lift, modulo $\mathbb{Z} \cdot H_X^{n-2}$, torsion 2-cycles on X to torsion 1-cycles on $F_1(X)$. The application we have in mind is the study of certain birational invariants of X . When $k = \mathbb{C}$, it was observed in [39] that the group $\text{CH}_{\text{tors},AJ}^3$ of homologically trivial torsion codimension 3 cycles annihilated by the Abel-Jacobi map is a birational invariant of smooth projective varieties which is trivial for stably rational varieties and more generally for varieties admitting a Chow-theoretic decomposition of the diagonal. This is a consequence of the deep result due to Bloch ([5], [11]) that the group $\text{CH}^2(Y)_{\text{tors},AJ}$ of homologically trivial torsion codimension 2 cycles annihilated by the Abel-Jacobi map is 0 for any smooth projective variety. For cubic hypersurfaces, as already mentioned, it follows from the existence of a unirational parametrization of degree 2 that $\text{CH}^3(X)_{\text{tors},AJ}$ is a 2-torsion group. Although we have not been able to compute this group, we obtain the following:

Theorem 0.8. *Let $X \subset \mathbb{P}_{\mathbb{C}}^{n+1}$ be a smooth cubic hypersurface, with $n \geq 5$. Then for any $\Gamma \in \text{CH}_2(X)_{\text{tors},AJ}$, there are a homologically trivial cycle $\gamma \in \text{CH}_1(F_1(X))_{\text{tors},AJ}$ and an odd integer m such that $P_*(\gamma) = m\Gamma$. In particular, when $n = 5$, if the 2-torsion part of $\text{CH}_1(F_1(X))_{\text{tors},AJ}$ is 0 then $\text{CH}^3(X)_{\text{tors},AJ} = 0$.*

As we will explain in Section 3.3, the group $\text{CH}_1(F_1(X))_{\text{tors},AJ}$ is a stable birational invariant of the variety $F_1(X)$, which is likely to be trivial for rationally connected varieties.

The group $\text{CH}^3(X)_{\text{tors},AJ}$ has a quotient which has an interpretation in terms of unramified cohomology. We recall that, for a smooth complex projective variety Y and an abelian group A , the degree i unramified cohomology group $H_{nr}^i(Y, A)$ of Y with coefficients in A can be defined (see [6]) as the group of global sections $H^0(Y, \mathcal{H}^i(A))$, $\mathcal{H}^i(A)$ being the sheaf associated to the presheaf $U \mapsto H^i(U(\mathbb{C}), A)$, where this last group is the Betti cohomology of the complex variety $U(\mathbb{C})$. The groups $H_{nr}^i(Y, A)$ provide stable birational invariants (see [10]) of Y , which vanish for projective space i.e. these groups are invariants under the relation:

$$Y \sim Z \text{ if } X \times \mathbb{P}^r \text{ is birationally equivalent to } Z \times \mathbb{P}^s \text{ for some } r, s.$$

Unramified cohomology group with coefficients in $\mathbb{Z}/m\mathbb{Z}$ or \mathbb{Q}/\mathbb{Z} has been used in the study of Lüroth problem, that is the study of unirational varieties which are not rational, to provide examples of unirational varieties which are not stably rational (see [2],[10]). In the case of smooth cubic hypersurfaces $X \subset \mathbb{P}_{\mathbb{C}}^{n+1}$, since there is a unirational parametrization of degree 2 of X (see [8]) and there is an action of correspondences on unramified cohomology groups compatible with composition of correspondences (see [12, Appendice]), the groups $H_{nr}^i(X, \mathbb{Q}/\mathbb{Z})$, $i \geq 1$, are 2-torsion groups. It is known that $H_{nr}^1(X, \mathbb{Q}/\mathbb{Z}) = 0$ for $n \geq 2$ since this group is isomorphic to the torsion in the Picard group of X (see [9, Proposition 4.2.1]).

Since for cubic hypersurfaces of dimension at least 2, $H_{nr}^2(X, \mathbb{Q}/\mathbb{Z})$ is equal to the Brauer group $Br(X)$ (see [9, Proposition 4.2.3]), we have $H_{nr}^2(X, \mathbb{Q}/\mathbb{Z}) = 0$.

As for $H_{nr}^3(Y, \mathbb{Q}/\mathbb{Z})$, it was reinterpreted in [12, Theorem 1.1] for rationally connected varieties Y as the torsion in the group $Z^4 := \text{Hdg}^4(Y)/H^4(Y, \mathbb{Z})_{\text{alg}}$, quotient of degree 4 Hodge classes by the subgroup of $H^4(Y, \mathbb{Z})$ generated by classes of codimension 2 algebraic cycles, i.e. $H_{nr}^3(Y, \mathbb{Q}/\mathbb{Z})$ measures the failure of the integral Hodge conjecture in degree 4. For cubic hypersurfaces $X \subset \mathbb{P}_{\mathbb{C}}^{n+1}$, by Lefschetz hyperplane theorem, the only non trivial case where the integral Hodge conjecture could fail in degree 4 is for cubic 4-folds but it was proved to hold by Voisin in [37].

The group $H_{nr}^4(Y, \mathbb{Q}/\mathbb{Z})$ was reinterpreted in [39, Corollary 0.3] for rationally connected varieties Y as the group $\text{CH}^3(Y)_{\text{tors},AJ}/\text{alg}$ of homologically trivial torsion codimension 3 cycles annihilated by Abel-Jacobi map (or torsion codimension 3 cycles annihilated by

Deligne cycle map) modulo algebraic equivalence. As $H_{nr}^4(X, \mathbb{Q}/\mathbb{Z}) \simeq \text{CH}^3(X)_{tors, AJ}/alg$, we have $H_{nr}^4(X, \mathbb{Q}/\mathbb{Z}) = 0$ for cubic hypersurfaces of dimension 2, 3 (because cubic three-folds are rationally connected) and 4 (because the 1-cycles on cubic 4-folds are generated by lines by [30, Theorem 1.1]). So the first non trivial case is the case of codimension 3 cycles on cubic 5-folds.

As suggested by work of Tian and Zong ([34]), who proved that the Griffiths group of homologically trivial 1-cycles on Fano complete intersection of index 2, modulo algebraic equivalence is 0, and by a question at the end of Section 2 in [40], it is likely that $\text{CH}_1(Y)_{tors, AJ}/alg$, which is also a birational invariant of smooth projective varieties Y vanishing for stably rational varieties, vanishes for Fano varieties, such as $F(X)$ when $\dim(X) \geq 5$, or even for rationally connected varieties. So we would expect the group $H_{nr}^4(X, \mathbb{Q}/\mathbb{Z})$ to be trivial for cubic 5-folds. We have the partial result:

Theorem 0.9. *Let $X \subset \mathbb{P}_{\mathbb{C}}^6$ be a smooth cubic hypersurface. Then $H_{nr}^4(X, \mathbb{Q}/\mathbb{Z})$ is finite.*

1. FIRST FORMULA

Let $X \subset \mathbb{P}_k^{n+1}$ be a smooth hypersurface of degree $d \geq 2$ and dimension $n \geq 3$ over an algebraically closed field k . Let us denote $F(X) \subset G(2, n+2)$ its variety of lines and $P \subset F(X) \times X$ the correspondence given by the universal \mathbb{P}^1 -bundle, and

$$p : P \rightarrow F(X), \quad q : P \rightarrow X$$

the two projections. For a general hypersurface of degree $d \leq 2n - 2$, $F(X)$ is a smooth connected variety ([21, Theorem 4.3]).

Let us denote $Q = \{([l], x) \in \mathbb{P}(\mathcal{E}_2), l \subset X \text{ or } l \cap X = \{x\}\}$ the correspondence associated to the family of osculating lines of X , and

$$\pi : Q \rightarrow X, \quad \varphi : Q \rightarrow G(2, n+2)$$

the two projections. We have $P \subset Q$.

The following lemma is obvious:

Lemma 1.1. *The fiber of $\pi : Q \rightarrow X$ (resp. $q : P \rightarrow X$) over any point x in the image of π (resp. of q) is isomorphic to an intersection of hypersurfaces of type $(2, 3, \dots, d-1)$ in $P(T_{X,x})$ (resp. of type $(2, 3, \dots, d)$). Moreover, for X general, Q is a local complete intersection subscheme of $\mathbb{P}(\mathcal{E}_2)$ of dimension $2n - d + 1$. If $\text{char}(k) = 0$, then Q is smooth for X general.*

Proof. By definition Q is the set of $([l], x)$ in $\mathbb{P}(\mathcal{E}_2)$ over $G(2, n+2)$ where the restriction of the equation defining X is 0 or proportional to λ_x^d , where λ_x is the linear form defining x in l . Let $x \in X$ and \mathcal{P} a hyperplane not containing x . There is an isomorphism $P(T_{\mathbb{P}^{n+1}, x}) \rightarrow \mathcal{P}$ given by $[v] \mapsto l_{(x,v)} \cap \mathcal{P}$, where $l_{(x,v)}$ is the line of \mathbb{P}^{n+1} determined by (x, v) . We can assume that $x = [1, 0, \dots, 0]$ and $\mathcal{P} = \{X_0 = 0\}$. Let l be a line through x and $[0, Y_1, \dots, Y_{n+1}] \in \mathcal{P}$ the point associated to l . Then, denoting f an equation defining X , since $x \in X$, we can write $f(X_0, \dots, X_{n+1}) = \sum_{i=0}^{d-1} X_0^i f_{d-i}(X_1, \dots, X_{n+1})$, where f_i is a homogeneous polynomial of degree i . The general point of l has coordinates $(\mu, \lambda Y_1, \dots, \lambda Y_{n+1})$ where $\lambda = \lambda_x$ and μ form a basis of linear forms on l . The restriction of f to l thus writes $\sum_{i=0}^{d-1} \mu^i \lambda^{d-i} f_{d-i}(Y_1, \dots, Y_{n+1})$. Thus the line l is osculating if and only if $f_j(Y_1, \dots, Y_{n+1}) = 0, \forall j < d$. The first equation f_1 is the differential of f at x and its vanishing hyperplane is $P(T_{X,x})$, so we proved that $\pi^{-1}(x)$ is isomorphic to an intersection of hypersurfaces of type $(2, 3, \dots, d-1)$ in $P(T_{X,x})$. We show likewise that the fiber $q^{-1}(x)$ is isomorphic to an intersection of hypersurfaces of type $(2, 3, \dots, d)$.

On the projective bundle $p_G : \mathbb{P}(\mathcal{E}_2) \rightarrow G(2, n+2)$, we have the exact sequence:

$$(2) \quad 0 \rightarrow \Omega_{\mathbb{P}(\mathcal{E}_2)/G(2, n+2)}(1) \rightarrow p_G^* \mathcal{E}_2 \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E}_2)}(1) \rightarrow 0$$

The last morphism being the evaluation morphism, we see that $\Omega_{\mathbb{P}(\mathcal{E}_2)/G(2,n+2)}(1)_{([l],x)}$ is the ideal sheaf of x in l . Taking the symmetric power of the dual of (2) yields the exact sequence:

$$0 \rightarrow \mathrm{Sym}^d(\Omega_{\mathbb{P}(\mathcal{E}_2)/G(2,n+2)}(1)) \rightarrow p_G^* \mathrm{Sym}^d \mathcal{E}_2 \rightarrow p_G^* \mathrm{Sym}^{d-1} \mathcal{E}_2 \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_2)}(1) \rightarrow 0$$

where the first morphism is the d -th symmetric power of the (first) inclusion in (2).

Now, let f be an equation defining X ; it gives rise to a section σ_f of $p_G^* \mathrm{Sym}^d \mathcal{E}_2$. Let $\bar{\sigma}_f$ be the section of $p_G^* \mathrm{Sym}^{d-1} \mathcal{E}_2 \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_2)}(1)$ induced by σ_f . Then the zero locus of $\bar{\sigma}_f$ is exactly the locus of $([l], x)$ where the restriction to l of the equation defining X is 0 or equal to the linear form induced by x on l to the power d . So Q is the zero locus in $\mathbb{P}(\mathcal{E}_2)$ of a section of the vector bundle $p_G^* \mathrm{Sym}^{d-1} \mathcal{E}_2 \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}_2)}(1)$. As this vector bundle is globally generated, the zero locus of a general section is a local complete intersection (even regular if $\mathrm{char}(k) = 0$) subscheme of $\mathbb{P}(\mathcal{E}_2)$ of dimension $2n - d + 1$. \square

Theorem 1.2. *Let $X \subset \mathbb{P}_k^{n+1}$ be a smooth hypersurface of degree d and let $P \subset F(X) \times X$ be the incidence correspondence. Assume $\sum_{i=1}^{d-1} i^r \leq n$ with $r > 0$ and, if $r > 3$ and $\mathrm{char}(k) > 0$, assume resolution of singularities of varieties of dimension r . Then for any cycle $\Gamma \in \mathrm{CH}_r(X)$ there is a $\gamma \in \mathrm{CH}_{r-1}(F(X))$ such that*

$$d\Gamma + P_*(\gamma) \in \mathbb{Z} \cdot H_X^{n-r}$$

where $H_X = c_1(\mathcal{O}_X(1))$.

Proof. Let $\Sigma \subset X$ be an integral subvariety of dimension $r > 0$. By Tsen-Lang theorem ([22], [33, Theorem 2.10]), the function field $k(\Sigma)$ of Σ is C_r . As the fibers of $\pi : Q \rightarrow X$ are isomorphic to intersection of hypersurfaces of type $(2, 3, \dots, d-1)$ and $\sum_{i=1}^{d-1} i^r \leq n$, the restriction $\pi_\Sigma : Q|_\Sigma \rightarrow \Sigma$ admits a rational section $\sigma : \Sigma \dashrightarrow Q$.

Case 1: The rational section σ is actually a rational section of $P|_\Sigma \rightarrow \Sigma$. This means that for any $x \in \Sigma$, the line $p \circ \sigma(x)$ is contained in X . We have the following diagram of resolution of indeterminacies:

$$\begin{array}{ccc} \tilde{\Sigma} & & \\ \downarrow \tau & \searrow \tilde{\sigma} & \\ \Sigma & \dashrightarrow & P \xrightarrow{p} F(X) \end{array}$$

Let us denote $\mathbb{P}_{\tilde{\Sigma}}$ the pull-back via $p \circ \tilde{\sigma}$ of the \mathbb{P}^1 -bundle on $F(X)$, $f : \mathbb{P}_{\tilde{\Sigma}} \rightarrow X$ the projection on X (which is the restriction of q) and $p_\Sigma : \mathbb{P}_{\tilde{\Sigma}} \rightarrow \tilde{\Sigma}$ the projective bundle. The line bundle $\tau^* \mathcal{O}_X(1)|_\Sigma$ gives rise to a section $\eta : \tilde{\Sigma} \rightarrow \mathbb{P}_{\tilde{\Sigma}}$ (given by $s \mapsto (p \circ \tilde{\sigma}(s), \tau(s))$) of p_Σ . We have the decomposition $\mathrm{Pic}(\mathbb{P}_{\tilde{\Sigma}}) \simeq \mathbb{Z} \cdot f^* H_X \oplus p_\Sigma^* \mathrm{Pic}(\tilde{\Sigma})$ so that we can write

$$(3) \quad \eta(\tilde{\Sigma}) = f^* H_X + p_\Sigma^* D$$

for D a divisor on $\tilde{\Sigma}$. We apply f_* to that equality: we have $f_* \eta(\tilde{\Sigma}) = \tau_*(\tilde{\Sigma}) = \Sigma$ in $\mathrm{CH}_r(X)$. Projection formula yields $f_* f^* H_X = H_X \cdot f_*(1)$. Finally, we see that $f_* p_\Sigma^* D = P_*(p_* \tilde{\sigma}_*(D))$. So, we get

$$\Sigma = H_X \cdot f_*(1) + P_*(p_* \tilde{\sigma}_*(D)).$$

Remembering that $dH_X \cdot f_*(1) = i_X^* i_{X,*} f_*(1) \in \mathbb{Z} \cdot H_X^{n-r}$, we are done for this case.

Case 2: The rational section σ is not a rational section of $P|_\Sigma \rightarrow \Sigma$. This means that for the general point $x \in \Sigma$, the line $\varphi \circ \sigma(x)$ is not contained in X , hence intersects X at x

with multiplicity d . We have the following diagram of resolution of indeterminacies:

$$\begin{array}{ccc} \widetilde{\Sigma} & & \\ \downarrow \tau & \searrow \tilde{\sigma} & \\ \Sigma & \dashrightarrow Q & \xrightarrow{\varphi} G(2, n+2) \end{array}$$

Let again $\mathbb{P}_{\widetilde{\Sigma}}$ be the pull-back via $\varphi \circ \tilde{\sigma}$ of the \mathbb{P}^1 -bundle on $G(2, n+2)$ and let $f : \mathbb{P}_{\widetilde{\Sigma}} \rightarrow \mathbb{P}^{n+1}$ be the natural morphism. Let $\widetilde{\Sigma}_1$ be the locus in $\widetilde{\Sigma}$ consisting of $x \in \widetilde{\Sigma}$ such that the line $\varphi \circ \tilde{\sigma}(x)$ is contained in X . We have an equality of r -cycles

$$(4) \quad (f_* \mathbb{P}_{\widetilde{\Sigma}})|_X = d\Sigma + R$$

in $\text{CH}_r(X)$, where the residual cycle R is supported on the r -dimensional locus $\mathbb{P}_{\widetilde{\Sigma}_1}$, or rather its image in X . It is clear that R is a cycle in the image of P_* so that (4) proves the result in this case. \square

In the case of smooth cubic hypersurfaces of dimension ≥ 3 , $F(X)$ is always smooth and connected ([1, Corollary 1.11, Theorem 1.16]). We have the following result which is essentially Theorem 0.2 of the introduction:

Theorem 1.3. *Let $X \subset \mathbb{P}_k^{n+1}$, with $n \geq 3$ and $\text{char}(k) > 2$, be a smooth cubic hypersurface containing a linear space of dimension $d \geq 1$. Then, for $1 \leq i \leq d$ and $2i \neq n$,*

$$P_* : \text{CH}_{i-1}(F(X)) \rightarrow \text{CH}_i(X)/\mathbb{Z} \cdot H_X^{n-i}$$

is surjective on $2\text{CH}_i(X)/\mathbb{Z} \cdot H_X^{n-i}$.

If moreover, $n \geq 2r + 1$ for a $r > 0$ and resolution of singularities holds of k -varieties of dimension r , then $P_ : \text{CH}_{i-1}(F(X)) \rightarrow \text{CH}_i(X)/\mathbb{Z} \cdot H_X^{n-i}$ is surjective for any $i \neq \frac{n}{2}$, $1 \leq i \leq r$.*

Proof. According to [8, Appendix B], X admits a unirational parametrization of degree 2 constructed as follows: for a general line Δ in X , consider the projective bundle $P(T_{X|\Delta})$ over Δ and the rational map $\varphi : P(T_{X|\Delta}) \dashrightarrow X$ which to a point $x \in \Delta$ and a nonzero vector $v \in T_{X,x}$ associated the residual point to x (x has multiplicity 2) in the intersection $X \cap l_{(x,v)}$ of X with the line of \mathbb{P}^{n+1} determined by (x, v) . The indeterminacy locus Z corresponds to the (x, v) such that $l_{(x,v)} \subset X$. It has codimension 2 for general lines. Indeed, if Δ is general, it is generally contained in the locus where the fibers of the projection $q : P \rightarrow X$ are complete (since P has dimension $2n - d$) intersection of type $(2, 3)$ in the projectivized tangent spaces so that the general fiber of $Z \rightarrow \Delta$ has dimension $n - 3$. Choosing a sufficiently general Δ , we can also assume that Z is smooth. Then, blowing-up $P(T_{X|\Delta})$ along Z yields the resolution of indeterminacies; let us denote $\tau : \widetilde{P(T_{X|\Delta})} \rightarrow P(T_{X|\Delta})$ that blow-up, E the exceptional divisor and $\tilde{\varphi} : \widetilde{P(T_{X|\Delta})} \rightarrow X$ the resulting degree 2 morphism. For $1 \leq i \leq d$, by the formulas for blowing-up, we have the decomposition

$$\text{CH}_i(\widetilde{P(T_{X|\Delta})}) = \tau^* \text{CH}_i(P(T_{X|\Delta})) \oplus j_{E,*} \tau_{|E}^* \text{CH}_{i-1}(Z) \oplus j_{E,*} (j_E^* \tilde{\varphi}^* H_X) \cdot \tau_{|E}^* \text{CH}_i(Z).$$

As $\tau_{|E}$ is flat, we can see that $\tilde{\varphi}_* j_{E,*} \tau_{|E}^*$ identifies with the composition of the morphism $\text{CH}_*(Z) \rightarrow \text{CH}_*(F(X))$ (induced by the restriction of natural morphism $P(T_X) \rightarrow G(2, n+2)$) followed by the action P_* .

So let $\Gamma \in \text{CH}_i(X)$, with $2i \neq n$, be a cycle on X . As X contains a linear space of dimension i and $H_{\text{ét}}^{2(n-i)}(X, \mathbb{Z}_\ell) = \mathbb{Z}_\ell$ ($\forall \ell \neq \text{char}(k)$) by Lefschetz hyperplane section theorem ($n \neq 2i$), for any $\mathcal{P} \simeq \mathbb{P}^i \subset X$, $\Gamma - \text{deg}(\Gamma)[\mathcal{P}]$ is homologically trivial and $\tilde{\varphi}_* \tilde{\varphi}^*(\Gamma - \text{deg}(\Gamma)[\mathcal{P}]) = 2(\Gamma - \text{deg}(\Gamma)[\mathcal{P}])$. As $P(T_{X|\Delta})$ is a projective bundle over \mathbb{P}^1 , $\text{CH}_*(P(T_{X|\Delta}))_{\text{hom}} = 0$ so, from the above discussion, we conclude that there are a $(i-1)$ -cycle $\gamma \in \text{CH}_{i-1}(F(X))_{\text{hom}}$ and a i -cycle $D_\Gamma \in \text{CH}_i(F(X))_{\text{hom}}$ such that

$$2(\Gamma - \text{deg}(\Gamma)[\mathcal{P}]) = P_* \gamma + H_X \cdot P_* D_\Gamma.$$

It remains to deal with the term $H_X \cdot P_* D_\Gamma$. For this, let $j : Y \hookrightarrow X$ be a hyperplane section with one ordinary double point p_0 as singularity. Then $H_X \cdot P_* D_\Gamma = j_* j^* P_* D_\Gamma$.

We have $Y \subset \mathbb{P}^n$ and if we choose coordinates in which $p_0 = [0 : \dots : 0 : 1]$, the equation of Y has the following form: $F(X_0, \dots, X_n) = X_n Q(X_0, \dots, X_{n-1}) + T(X_0, \dots, X_{n-1})$ where $Q(X_0, \dots, X_{n-1})$ is a quadratic homogeneous polynomial and $T(X_0, \dots, X_{n-1})$ is a degree 3 homogeneous polynomial. The linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ centered at p_0 induces a birational map $Y \dashrightarrow \mathbb{P}^{n-1} \simeq [p_0]$ where $[p_0]$ denotes the scheme parametrizing lines of \mathbb{P}^n passing through p_0 . The indeterminacies of the inverse map $\mathbb{P}^{n-1} \dashrightarrow Y$ are resolved by blowing-up \mathbb{P}^{n-1} along the complete intersection $F_{p_0}(Y) = \{Q = 0\} \cap \{T = 0\}$ of type $(2, 3)$. The variety of lines of Y passing through p_0 is isomorphic to $F_{p_0}(Y)$ and we have the following diagram:

$$\begin{array}{ccc} \widetilde{\mathbb{P}^{n-1}}^{F_{p_0}(Y)} & & \\ \downarrow \chi & \searrow q & \\ \mathbb{P}^{n-1} & \dashrightarrow & Y \end{array}$$

By projection formula, $(j \circ q)_* (j \circ q)^* P_* D_\Gamma = P_* D_\Gamma \cdot j_* q_* 1 = P_* D_\Gamma \cdot [Y] = P_* D_\Gamma \cdot H_X$ and $(j \circ q)^* P_* D_\Gamma$ is a homologically trivial cycle on $\widetilde{\mathbb{P}^{n-1}}^{F_{p_0}(Y)}$. But since the ideal $\text{CH}_*(\mathbb{P}^{n-1})_{\text{hom}}$ of homologically trivial cycles on \mathbb{P}^{n-1} is 0, from the decomposition of the Chow groups of a blow-up, we get that $(j \circ q)^* P_* D_\Gamma$ can be written $j_{E_{F_{p_0}(Y), *}} \chi_{|E_{F_{p_0}(Y), *}}^* w$ for a cycle w on $F_{p_0}(Y)$ so that $H_X \cdot P_* D_\Gamma = j_* q_* j_{E_{F_{p_0}(Y), *}} \chi_{|E_{F_{p_0}(Y), *}}^* w$ which can be written $P_* i_{F_{p_0}(Y), *} w$ where $i_{F_{p_0}(Y), *} : F_{p_0}(Y) \hookrightarrow F(X)$ is the inclusion. Finally, \mathcal{P} is in $\text{Im}(P_*)$ so we have: $2\Gamma = 2\mathcal{P} + P_*(\gamma + i_{F_{p_0}(Y), *} w)$ which proves that $2\text{CH}_i(X)$ is in the image of P_* .

When $n \geq 2^r + 1$, we can also apply Theorem 1.2; we get, for any cycle $\Gamma \in \text{CH}_i(X)$, a cycle $\gamma' \in \text{CH}_{i-1}(F(X))$ such that $3\Gamma + P_* \gamma' \in \mathbb{Z} \cdot H_X^{n-i}$ in $\text{CH}_i(X)$. \square

Proposition 1.4. *Let $X \subset \mathbb{P}_k^{n+1}$, with $n \geq 4$ and $\text{char}(k) > 2$, be a smooth cubic hypersurface. Then $H_X^{n-2} \in \text{Im}(P_* : \text{CH}_1(F(X)) \rightarrow \text{CH}_2(X))$. In particular, by Theorem 1.3, for $n \geq 5$, $P_* : \text{CH}_1(F(X)) \rightarrow \text{CH}_2(X)$ is surjective.*

Proof. Since, according to [25, Lemma 1.4], any smooth cubic threefold contains some lines of second type (lines whose normal bundle contains a copy of $\mathcal{O}_{\mathbb{P}^1}(-1)$), X contains lines of second type. Let $l_0 \subset X$ be a line of second type. According to [8, Lemma 6.7], there is a (unique) $\mathbb{P}^{n-1} \subset \mathbb{P}^{n+1}$ tangent to X along l_0 . So, when $n \geq 4$, we can choose a $P_0 \simeq \mathbb{P}^3 \subset \mathbb{P}^{n+1}$ tangent to X along l_0 . Then $S := P_0 \cap X$ is a cubic surface singular along l_0 which is ruled by lines of X . Indeed, for any $x \in S \setminus l_0$, $\text{span}(x, l_0) \cap S$ is a plane cubic containing l_0 with multiplicity 2; so that the residual curve is a line passing through x . So, we can write $S = q(p^{-1}(D))$ for a closed subscheme of pure dimension 1, $D \subset F(X)$ so, in $\text{CH}_2(X)$, we have $H_X^{n-2} = [S] = P_*([D])$. \square

Here is one consequence of this proposition:

Corollary 1.5. *Let $\pi : \mathcal{X} \rightarrow B$ be a family of complex cubic hypersurfaces of dimension $n \geq 5$ i.e. π is a smooth projective morphism of connected quasi-projective complex varieties with n -dimensional cubic hypersurfaces as fibers. Then, the specialization map*

$$\text{CH}_2(X_{\bar{\eta}})/\text{alg} \rightarrow \text{CH}_2(X_t)/\text{alg}$$

where $X_{\bar{\eta}}$ is the geometric generic fiber and $X_t := \pi^{-1}(t)$ for $t \in B(k)$ any closed point, is surjective.

Proof. The statement follows from Proposition 1.4 and the following property, essentially written in the proof of [38, Lemma 2.1]

Proposition 1.6. ([38, Lemma 2.1]). *Let $\pi : \mathcal{Y} \rightarrow B$ be a smooth projective morphism with rationally connected fibers. Then for any $t \in B(k)$, the specialization map $\text{CH}_1(Y_{\bar{\eta}})/\text{alg} \rightarrow \text{CH}_1(Y_t)/\text{alg}$ is surjective.*

Proof. We just recall briefly the proof: by attaching sufficiently very free rational curves to it (so that the resulting curve is smoothable), any curve $C \subset Y_t$ is algebraically equivalent to a (non effective) sum of curves $C_i \subset Y_t$ such that $H^1(C_i, N_{C_i/Y_t}) = 0$. Then the morphism of deformation of each (C_i, Y_t) to B is smooth. So we have a curve $C_{i,\eta} \subset Y_{K_i}$ where K_i is a finite extension of the function field of B , which is sent by specialization in the fiber Y_t , to C_i . \square

Applying this proposition to the relative variety of lines $F(\mathcal{X}) \rightarrow B$, yields a surjective map: $\mathrm{CH}_1(F(X_{\bar{\eta}}))/\mathrm{alg} \rightarrow \mathrm{CH}_1(F(X_t))/\mathrm{alg}$. The universal \mathbb{P}^1 -bundle $\mathcal{P} \subset F(\mathcal{X}) \times_B \mathcal{X}$ gives the surjective maps $\mathcal{P}_{t,*} : \mathrm{CH}_1(F(X_t))/\mathrm{alg} \rightarrow \mathrm{CH}_2(X_t)/\mathrm{alg}$ and $\mathcal{P}_{\bar{\eta},*} : \mathrm{CH}_1(F(X_{\bar{\eta}}))/\mathrm{alg} \rightarrow \mathrm{CH}_2(X_{\bar{\eta}})/\mathrm{alg}$ and they commute ([15, 20.3]). \square

2. ONE-CYCLES ON THE VARIETY OF LINES OF A FANO HYPERSURFACE IN \mathbb{P}^n

Throughout this section, k will designate an algebraically closed field. According to [21, Theorem 4.3], for a general hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \leq 2n - 2$, the variety of lines $F(X)$ is smooth, connected of dimension $2n - d - 1$. In the case of cubic hypersurfaces of dimension $n \geq 3$, we even know, by work of Altman and Kleiman ([1, Corollary 1.11, Theorem 1.16], see also [3]) that for any smooth hypersurface X , $F(X)$ is smooth and connected.

We recall that, for a smooth hypersurface $X \subset \mathbb{P}_k^{n+1}$ of degree d , when $F(X)$ has the expected dimension $2n - d - 1$, it is the zero-locus in $G(2, n+2)$ of a regular section of $\mathrm{Sym}^d(\mathcal{E}_2)$, where \mathcal{E}_2 is the rank 2 quotient bundle on $G(2, n+2)$ and its dualizing sheaf, given by adjunction formula ([17, Theorem III 7.11]), is $-((n+2) - \frac{d(d+1)}{2})$ times the Plücker line bundle on $G(2, n+2)$ restricted to $F(X)$. In particular, when $F(X)$ is smooth, connected and $\frac{d(d+1)}{2} < (n+2)$, $F(X)$ is Fano so rationally connected.

From now, we assume that the condition $d(d+1) < 2(n+2)$ holds and that $X \subset \mathbb{P}_k^{n+1}$ is a smooth hypersurface such that $F(X)$ is smooth and connected. Then the following theorem applies to $F(X)$ if $\mathrm{char}(k) = 0$ or, when $\mathrm{char}(k) > 0$, if $F(X)$ is, moreover separably rationally connected:

Theorem 2.1. ([34, Theorem 1.3]). *Let Y be a smooth proper and separably rationally connected variety over an algebraically closed field. Then every 1-cycle is rationally equivalent to a \mathbb{Z} -linear combination of cycle classes of rational curves. That is, the Chow group $\mathrm{CH}_1(Y)$ is generated by rational curves.*

Corollary 2.2. (i) *When $\mathrm{char}(k) = 0$ and X is a smooth cubic hypersurface of dimension ≥ 5 , $F(X)$ is separably rationally connected; then Proposition 1.3 together with Theorem 2.1 yields that $\mathrm{CH}_2(X)$ is generated by classes of rational surfaces. In positive characteristic, the same is true for smooth cubic hypersurfaces X whose variety of lines $F(X)$ is separably rationally connected.*

(ii) *When $k = \mathbb{C}$ and X is a smooth cubic hypersurface of dimension 5, the group of 1-cycles modulo algebraic equivalence, $\mathrm{CH}_1(F(X))/\mathrm{alg}$ is finitely generated. Thus $\mathrm{CH}_2(X)/\mathrm{alg}$ is finitely generated. In particular, the group $H_{nr}^4(X, \mathbb{Q}/\mathbb{Z})$ is finite.*

Proof. (ii) According to [21, Theorem 5.7], any rational curve is algebraically equivalent to a sum of rational curves of anticanonical degree at most $\dim_k(F(X)) + 1$. As there are finitely many irreducible varieties parametrizing rational curves of bounded degree, $\mathrm{CH}_1(F(X))/\mathrm{alg}$, is finitely generated. Proposition 1.4 tells us that $P_* : \mathrm{CH}_1(F(X))/\mathrm{alg} \rightarrow \mathrm{CH}_2(X)/\mathrm{alg}$ is surjective so that $\mathrm{CH}_2(X)/\mathrm{alg}$ is finitely generated. Now, by work of Voisin ([39]) $H_{nr}^4(X, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{CH}_2(X)_{\mathrm{tors}, AJ}/\mathrm{alg} \subset \mathrm{CH}_2(X)/\mathrm{alg}$ so it is finitely generated. On the other hand, we know that, since X admits a unirational parametrization of degree 2, $H_{nr}^4(X, \mathbb{Q}/\mathbb{Z})$ is a 2-torsion group so it is a finitely generated $\mathbb{Z}/2\mathbb{Z}$ -vector space hence $H_{nr}^4(X, \mathbb{Q}/\mathbb{Z})$ is finite. \square

Actually, by completely different methods using a variant of [37, Theorem 18], the first item of Corollary 2.2 turns out to be true for cubic 4-folds also in characteristic 0.

Proposition 2.3. *Let $X \subset \mathbb{P}_{\mathbb{C}}^5$ be a smooth cubic hypersurface. Then $\text{CH}_2(X)$ is generated by classes of rational surfaces.*

Proof. In the proof by Voisin of the integral Hodge conjecture of cubic 4-folds ([37, Theorem 18]), one can replace the parametrization of the family of intermediate jacobians associated to a Lefschetz pencil of X , with rationally connected fibers given by [23] and [20] (the family of elliptic curves of degree 5) by the one given by [16, Theorem 9.2] (the family of rational curves of degree 4); her proof then shows that any degree 4 Hodge class is homologically equivalent to the class of a combination of rational surfaces swept-out by a family of rational curves of degree 4 in X parameterized by a rational curve. Finally, since X is rationally connected and the intermediate jacobian $J^3(X)$ is trivial, Bloch-Srinivas [7, Theorem 1] applies and says that codimension 2 cycles homologically trivial on X are rationally trivial so that we have proved that any 2-cycle on X is rationally equivalent to a combination of rational surfaces. \square

2.1. One-cycles modulo algebraic equivalence. In this section, we apply the methods of [34, Theorem 6.2], using a coarse parametrization of rational curves lying on $F(X)$, to study 1-cycles on varieties $F(X)$. Our goal is to prove:

Theorem 2.4. *Let $X \subset \mathbb{P}_k^{n+1}$ be a smooth hypersurface of degree d , with $\frac{d(d+1)}{2} < n$, such that $F(X)$ is smooth, connected. Then every rational curve on $F(X)$ is algebraically equivalent to an integral sum of lines. In particular (assuming moreover that $F(X)$ is separably rationally connected if $\text{char}(k) > 0$), any 1-cycle on $F(X)$ is algebraically equivalent to an integral sum of lines and thus $\text{CH}_1(F(X))_{\text{hom}} = \text{CH}_1(F(X))_{\text{alg}}$.*

We start with some preparation. Let V be a $(n+2)$ -dimensional k -vector space and $X \subset \mathbb{P}(V) \simeq \mathbb{P}_k^{n+1}$ a smooth hypersurface of degree d . A morphism $r : \mathbb{P}^1 \rightarrow G(2, V)$ such that $r^* \mathcal{O}_{G(2, V)}(1) \simeq \mathcal{O}_{\mathbb{P}^1}(e)$, with $e \geq 1$, is associated to the datum of a globally generated rank 2 vector bundle on \mathbb{P}^1 , which is a quotient of the trivial bundle $V \otimes \mathcal{O}_{\mathbb{P}^1}$ i.e. to an exact sequence

$$V \otimes \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \rightarrow 0$$

with $a, b \geq 0$ and $a + b = e$. So a natural parameter space for those morphisms is

$$\mathbb{P} := \mathbb{P}(\text{Hom}(V^*, H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b)))).$$

Given $[P_0, \dots, P_{n+1}, Q_0, \dots, Q_{n+1}] \in \mathbb{P}$, where the P_i 's are in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ and the Q_i 's are in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b))$, the points in the image in $\mathbb{P}(V)$ of $\text{Im}(\mathbb{P}^1 \rightarrow G(2, n+2))$ under the correspondence given by the universal \mathbb{P}^1 -bundle are of the form $[P_0(Y_0, Y_1)\lambda + Q_0(Y_0, Y_1)\mu, \dots, P_{n+1}(Y_0, Y_1)\lambda + Q_{n+1}(Y_0, Y_1)\mu]$ where $\text{Span}(Y_0, Y_1) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. Let $\prod_{i=0}^{n+1} X_i^{\alpha_i} \in H^0(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(d))$ be a monomial with $\sum_{i=0}^{n+1} \alpha_i = d$. Then the induced equation on the image in \mathbb{P}^{n+1} of the morphism $\mathbb{P}^1 \rightarrow G(2, n+2)$ associated to $[P_0, \dots, P_{n+1}, Q_0, \dots, Q_{n+1}]$ has the following form:

$$\sum_{k=0}^d \left(\sum_{\substack{0 \leq l_0 \leq \alpha_0, \dots, 0 \leq l_{n+1} \leq \alpha_{n+1} \\ \sum_i l_i = k}} \prod_{i=0}^{n+1} \binom{\alpha_i}{l_i} P_i^{\alpha_i - l_i} Q_i^{l_i} \right) \lambda^{d-k} \mu^k$$

so that, denoting \mathbb{P}_X , the closed subset of \mathbb{P} parametrizing the $[P_0, \dots, P_{n+1}, Q_0, \dots, Q_{n+1}]$ whose image in \mathbb{P}^{n+1} is contained in the hypersurface X of degree d , is defined by $\sum_{k=0}^d (a(d-k) + bk + 1) = 0$ homogeneous polynomials of degree d on \mathbb{P} .

The closed subset $B \subset \mathbb{P}$ parametrizing the $M \in \mathbb{P}(\text{Hom}(V^*, H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b))))$ whose rank is ≤ 2 has dimension $2(e+n)+3$. Now, we have the following lemma:

Lemma 2.5. ([18]). *Let Y be a subscheme of a projective space \mathbb{P}^N defined by M homogeneous polynomials. Let Z be a closed subset of Y with dimension $< N - M - 1$. Then $Y \setminus Z$ is connected.*

The closed subset $B \cap \mathbb{P}_X$ of \mathbb{P}_X has dimension

$$\dim_k(F(X)) + 2(a+1) + 2(b+1) - 1 = 2n - d - 1 + 2e + 3 = 2(e+n) - d + 2$$

since it parametrizes (generically) a point of $F(X)$ and over that point 2 polynomials in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a))$ and 2 polynomials in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b))$. Applying Lemma 2.5 with $Y = \mathbb{P}_X$ and $Z = B \cap \mathbb{P}_X$, so that $N = (n+2)(e+2) - 1$ and $M = \sum_{k=0}^d (a(d-k) + bk + 1) = d+1 + e \frac{d(d+1)}{2}$, yields the following condition for the connectedness of $\mathbb{P}_X \setminus (B \cap \mathbb{P}_X)$:

$$(5) \quad e\left(n - \frac{d(d+1)}{2}\right) > 1$$

Proof of Theorem 2.4. We proceed by induction on the degree of the considered rational curve, following the arguments of [34, Theorem 6.2].

Let $D \subset \mathbb{P}$ be the closed subset parametrizing $2(n+2)$ -tuples $[P_0, \dots, P_{n+1}, Q_0, \dots, Q_{n+1}]$ that have a common non constant factor. Assume $e \geq 2$. Let $p \in \mathbb{P}_X \setminus (\mathbb{P}_X \cap (B \cup D))$ be a point parametrizing a degree e morphism $\mathbb{P}^1 \rightarrow F(X)$ generically injective. As $e \geq 2$, $\mathbb{P}_X \setminus (\mathbb{P}_X \cap B)$ is connected; so there is a connected curve γ in $\mathbb{P}_X \setminus (\mathbb{P}_X \cap B)$ connecting p to a point $q = [P_{0,q}, \dots, P_{n+1,q}, Q_{0,q}, \dots, Q_{n+1,q}]$ of $\mathbb{P}_X \cap D \setminus (\mathbb{P}_X \cap B)$. Factorizing out the common factor of $(P_{i,q}, Q_{i,q})_{i=0 \dots n+1}$, we get a $(P'_{i,q}, Q'_{i,q})_{i=0 \dots n+1}$ which parametrizes a morphism $\mathbb{P}^1 \rightarrow F(X)$ of degree $< e$ (finite onto its image), since $q \notin B$. So, approaching q from points of γ outside D and using standard bend-and-break construction, we get from q a morphism from a connected curve whose components are isomorphic to \mathbb{P}^1 to $F(X)$ such that the restriction to each component yields a rational curve of degree $< e$ (or a contraction). So the rational curve parametrized by p is algebraically equivalent to a sum of rational curve each of which has degree $< e$. We conclude by induction on e that the rational curve parametrized by p is algebraically equivalent to a sum of lines. \square

2.2. One-cycles modulo rational equivalence. From now on, we will assume that $X \subset \mathbb{P}_k^{n+1}$ is a smooth hypersurface of degree $d > 2$, with $d(d+1)/2 < n$, and that $\text{char}(k) = 0$. The following is proved in [13, Proposition 6.2]:

Proposition 2.6. *Assume $\text{char}(k) = 0$ and $X \subset \mathbb{P}_k^{n+1}$ is a smooth hypersurface of degree $d > 2$, with $d(d+1)/2 < n$. Then, $F(X)$ is chain connected by lines.*

Proceeding as in [34], we get the following result:

Theorem 2.7. *Let $X \subset \mathbb{P}_k^{n+1}$ be a smooth hypersurface of degree $d > 2$ over an algebraically closed field of characteristic 0, with $\frac{d(d+1)}{2} < n$, such that $F(X)$ is smooth and connected. Then $\text{CH}_1(F(X))$ is generated by lines i.e. any 1-cycle is rationally equivalent to a \mathbb{Z} -linear combination of lines.*

Proof. Let γ be a 1-cycle on $F(X)$. According to Theorem 2.4, there is a \mathbb{Z} -linear combination of lines $\sum_i m_i l_i$ such that $\gamma - \sum_i m_i l_i$ is algebraically equivalent to 0. Then, using [34, Proposition 3.1], we know there is a positive integer N such that for every 1-cycle C on $F(X)$, NC is rationally equivalent to a \mathbb{Z} -linear combination of lines. As the group $\text{CH}_1(X)_{\text{alg}}$ of 1-cycles algebraically equivalent to 0 is divisible ([4, Lemme 0.1.1]), we conclude that $\gamma - \sum_i m_i l_i$ is rationally equivalent to a \mathbb{Z} -linear combination of lines. \square

This provides us the following results for cubic hypersurfaces (cf. Proposition 0.5):

Corollary 2.8. *Let k be an algebraically closed field of characteristic 0 and X a smooth cubic hypersurface. We have the following properties:*

- (i) *if $\dim_k(X) \geq 7$, then, $\text{CH}_2(X)$ is generated (over \mathbb{Z}) by cycle classes of planes contained in X and $\text{CH}_2(X)_{\text{hom}} = \text{CH}_2(X)_{\text{alg}}$;*
- (ii) *if $\dim_k(X) \geq 9$, then, $\text{CH}_2(X) \simeq \mathbb{Z}$*

Proof. Item (ii) is an application of Proposition 1.4 and Theorem 2.7.

(iii) The variety of lines $F(F(X))$ of $F(X)$ is isomorphic to the projective bundle $\mathbb{P}(\mathcal{E}_3|_{F_2(X)})$ over $F_2(X) \subset G(3, n+2)$, where \mathcal{E}_3 is the rank 3 quotient bundle on $G(3, n+2)$ and $F_2(X)$ is the variety on planes of X , since a line in $F(X)$ correspond to the lines of \mathbb{P}^{n+1} contained

in a plane $\mathbb{P}^2 \simeq P \subset \mathbb{P}^{n+1}$ passing through a given point of P . Now, when $n \geq 9$, according to [13, Proposition 6.1], $\text{CH}_0(F_2(X)) \simeq \mathbb{Z}$ so that $\text{CH}_0(F(F(X))) \simeq \mathbb{Z}$. \square

3. INVERSION FORMULA

Let $X \subset \mathbb{P}_k^{n+1}$, where $n \geq 3$, be a smooth cubic hypersurface over a field k . Let as before $F(X) \subset G(2, n+2)$ be the variety of lines of X and $P \subset F(X) \times X$ the correspondence given by the universal \mathbb{P}^1 -bundle over $F(X)$. The variety $F(X)$ is smooth, connected of dimension $2n-4$ ([1, Corollary 1.11, Theorem 1.16]).

3.1. Inversion formula. In this section, adapting constructions and arguments developed in [30], we establish an inversion formula for a smooth subvariety Σ of X using the scheme of the lines of X which are bisecant to Σ . The lines of \mathbb{P}_k^{n+1} bisecant to Σ are naturally in relation with the punctual Hilbert scheme $\text{Hilb}_2(\Sigma)$, that we shall denote $\Sigma^{[2]}$, via the morphism $\varphi : \Sigma^{[2]} \rightarrow G(2, n+2)$ which associates to a length 2 subscheme of Σ the line it determines. There is a natural inclusion $\Sigma^{[2]} \subset X^{[2]}$.

We recall that $\Sigma^{[2]}$ is smooth and is obtained as the quotient of the blow-up $\widetilde{\Sigma \times \Sigma}$ of $\Sigma \times \Sigma$ along the diagonal Δ_Σ , by its natural involution. Let us denote $q : \widetilde{\Sigma \times \Sigma} \rightarrow \Sigma^{[2]}$ the quotient morphism, $\tau : \widetilde{\Sigma \times \Sigma} \rightarrow \Sigma \times \Sigma$ the blow-up and $j_E : E \hookrightarrow \widetilde{\Sigma \times \Sigma}$ the exceptional divisor of τ . As the involution acts trivially on E , $q|_E$ is an isomorphism onto its image and q is a double cover of $\Sigma^{[2]}$ ramified along $q(E)$. So let us denote $\delta \in \text{CH}^1(\Sigma^{[2]})$ a divisor satisfying $[q(E)] = 2\delta$.

Let us denote P_2 the \mathbb{P}^2 -bundle over $F(X)$ defined as the symmetric product of P over $F(X)$. There is a natural embedding $P_2 \subset X^{[2]}$ under which P_2 parametrizes length 2 subschemes contained in a line of X . Let us make the following definition:

Definition 3.1. For a smooth subvariety $\Sigma \subset X$, let us denote $\text{Bisec}'(\Sigma) = \Sigma^{[2]} \cap P_2$ the locus in $\Sigma^{[2]}$ parametrizing the lines of X bisecant to Σ . The morphism φ induces a morphism φ' from $\text{Bisec}'(\Sigma)$ to the variety of lines $F(X)$ of X . Set $\text{Bisec}(\Sigma) = \varphi'(\text{Bisec}'(\Sigma))$.

A smooth subvariety $\Sigma \subset X$ is well-positionned if $\text{Bisec}'(\Sigma)$ is a smooth closed subscheme of $\Sigma^{[2]}$ of codimension 2.

We have the following inversion formula for well-positionned subvarieties:

Theorem 3.2. Let $X \subset \mathbb{P}_k^{n+1}$ be a smooth cubic hypersurface and $\Sigma \subset X$ a well-positionned subvariety of dimension $1 \leq d \leq n$. Then, the following equality holds in $\text{CH}_d(X)$:

$$(6) \quad (2\text{deg}(\Sigma) - 3)[\Sigma] + P_*(\gamma_\Sigma) = m(\Sigma)H_X^{n-d}$$

where $m(\Sigma)$ is an integer, $H_X = c_1(\mathcal{O}_X(1))$, and γ_Σ is the $(d-1)$ -cycle of $F(X)$ given by the intersection of $\text{Bisec}(\Sigma)$ with $(d-1)$ general hyperplanes i.e. $\text{Bisec}(\Sigma) \cdot c_1(\mathcal{E}_{2|F(X)})^{d-1}$, \mathcal{E}_2 being the rank 2 quotient bundle on $G(2, n+2)$.

Proof. We have a natural residual construction $\Sigma^{[2]} \dashrightarrow X$ which associates to a length 2 subscheme of Σ , $x+y$, the point $z \in X$ residual to $x+y$ in the intersection $l_{(x,y)} \cap X$, $l_{(x,y)}$ being the line determined by $x+y$. We can resolve the indeterminacies of this map by blowing up the smooth codimension 2 subscheme $\text{Bisec}'(\Sigma) = P_2 \cap \Sigma^{[2]}$:

$$\begin{array}{ccc} \widetilde{\Sigma^{[2]}} & & \\ \downarrow & \searrow r & \\ \Sigma^{[2]} & \dashrightarrow & X \end{array}$$

Let us denote $E_{\text{Bisec}'(\Sigma)}$ the exceptional divisor of that blow-up. There is a divisor $\delta \in \text{CH}^1(\Sigma^{[2]})$ which is especially related to the geometry of Σ since $q^*\delta = [q^{-1}(\delta)] = E$ is a projective bundle over Σ so that $q^*\delta^d$ is supported on a subvariety isomorphic to Σ . So the computation of $\left[\widetilde{\Sigma^{[2]}} \right]_* (\delta^d) \in \text{CH}_d(X)$ is of special interest. To do this computation, we introduce the following construction:

Using the morphism $\varphi : \Sigma^{[2]} \rightarrow G(2, n+2)$, we can see that $\widetilde{\Sigma^{[2]}}$ admits a natural embedding in $\mathbb{P}(\varphi^*\mathcal{E}_2)$. Denoting f the projection $\mathbb{P}(\varphi^*\mathcal{E}_2) \rightarrow \mathbb{P}_k^{n+1}$, we have the natural inclusion of codimension 1 subschemes $\widetilde{\Sigma^{[2]}} \subset f^{-1}(X)$ and $r : \widetilde{\Sigma^{[2]}} \rightarrow X$ factorizes through the restriction $f' : f^{-1}(X) \rightarrow X$ of f . We following proposition proves that the residual divisor to $\widetilde{\Sigma^{[2]}}$ in $f^{-1}(X)$ is isomorphic to $\widetilde{\Sigma \times \Sigma}$:

Proposition 3.3. *There is an embedding $\sigma : \widetilde{\Sigma \times \Sigma} \rightarrow \mathbb{P}(\varphi^*\mathcal{E}_2)$ under which $\sigma(\widetilde{\Sigma \times \Sigma})$ is the residual scheme to $\widetilde{\Sigma^{[2]}}$ in $f^{-1}(X)$. Moreover, the following equalities hold in $\text{CH}^1(\mathbb{P}(\varphi^*\mathcal{E}_2))$:*

$$[\sigma(\widetilde{\Sigma \times \Sigma})] = 2f^*H - \pi^*(q_*\tau^*pr_1^*H_\Sigma - 2\delta)$$

and

$$[\widetilde{\Sigma^{[2]}}] = f^*H + \pi^*(q_*\tau^*pr_1^*H_\Sigma - 2\delta)$$

where $H = c_1(\mathcal{O}_{\mathbb{P}_k^{n+1}}(1))$, H_Σ is the restriction of H to Σ , $pr_1 : \Sigma \times \Sigma \rightarrow \Sigma$ the first projection and $\pi : \mathbb{P}(\varphi^*\mathcal{E}_2) \rightarrow \Sigma^{[2]}$ is the projective bundle. Moreover, $\text{Bisec}'(\Sigma) = c_2(\pi_*(\mathcal{O}_{\mathbb{P}(\varphi^*\mathcal{E}_2)}(\widetilde{\Sigma^{[2]}})))$ in $\text{CH}^2(\Sigma^{[2]})$.

Proof. As for any point $p \in \widetilde{\Sigma \times \Sigma}$, the point $pr_1(\tau(p))$ lies on the line $\varphi(q(p))$ (defined over $k(p)$), the evaluation morphism $q^*\varphi^*\mathcal{E}_2 \rightarrow \tau^*pr_1^*\mathcal{O}_\Sigma(1)$, where $\mathcal{O}_\Sigma(1) \simeq \mathcal{O}_{\mathbb{P}_k^{n+1}}(1)|_\Sigma$, is surjective i.e. gives rise to a section σ' of the projective bundle $\pi' : \mathbb{P}(q^*\varphi^*\mathcal{E}_2) \rightarrow \widetilde{\Sigma \times \Sigma}$. Let us denote $q' : \mathbb{P}(q^*\varphi^*\mathcal{E}_2) \rightarrow \mathbb{P}(\varphi^*\mathcal{E}_2)$ the base change of q ; it is also a ramified double cover. The composition $\sigma := q' \circ \sigma' : \widetilde{\Sigma \times \Sigma} \rightarrow \mathbb{P}(\varphi^*\mathcal{E}_2)$ is an isomorphism onto its image and we have the inclusion of divisors $\sigma(\widetilde{\Sigma \times \Sigma}) \subset f^{-1}(X)$. Let us denote R the residual scheme to $\sigma(\widetilde{\Sigma \times \Sigma})$ in $f^{-1}(X)$. We need to prove that $\pi|_R$ is the blow-up of $\Sigma^{[2]}$ along $\text{Bisec}'(\Sigma)$ i.e. $R \simeq \widetilde{\Sigma^{[2]}}$.

As σ' is the section of the projective bundle $\mathbb{P}(q^*\varphi^*\mathcal{E}_2)$ given by $\tau^*pr_1^*\mathcal{O}_\Sigma(1)$, its class in $\text{CH}^1(\mathbb{P}(q^*\varphi^*\mathcal{E}_2))$ is given by $c_1(\pi'^*\mathcal{K}^\vee \otimes \mathcal{O}_{\mathbb{P}(q^*\varphi^*\mathcal{E}_2)}(1))$, where \mathcal{K} is defined by the exact sequence:

$$(7) \quad 0 \rightarrow \mathcal{K} \rightarrow q^*\varphi^*\mathcal{E}_2 \rightarrow \tau^*pr_1^*\mathcal{O}_\Sigma(1) \rightarrow 0$$

and $\mathcal{O}_{\mathbb{P}(q^*\varphi^*\mathcal{E}_2)}(1) \simeq q'^*f^*\mathcal{O}_{\mathbb{P}_k^{n+1}}(1)$. As q' is a double cover ramified along $\pi^{-1}(q(E))$, the push-forward $q'_*(\pi'^*\mathcal{K}^\vee \otimes q'^*f^*\mathcal{O}_{\mathbb{P}_k^{n+1}}(1)) \simeq q'_*(\pi'^*\mathcal{K}^\vee) \otimes f^*\mathcal{O}_{\mathbb{P}_k^{n+1}}(1)$ is a rank 2 vector bundle on $\mathbb{P}(\varphi^*\mathcal{E}_2)$ and Grothendieck-Riemann-Roch theorem gives

$$c_1(q'_*(\pi'^*\mathcal{K}^\vee \otimes q'^*f^*\mathcal{O}_{\mathbb{P}_k^{n+1}}(1))) = q'_*c_1(\pi'^*\mathcal{K}^\vee \otimes q'^*f^*\mathcal{O}_{\mathbb{P}_k^{n+1}}(1)) - \pi^*\delta = q'_*\sigma'(\widetilde{\Sigma \times \Sigma}) - \pi^*\delta.$$

On the other hand, we have:

$$\begin{aligned} c_1(q'_*(\pi'^*\mathcal{K}^\vee) \otimes f^*\mathcal{O}_{\mathbb{P}_k^{n+1}}(1))) &= 2f^*c_1(\mathcal{O}_{\mathbb{P}_k^{n+1}}(1)) + c_1(q'_*\pi'^*\mathcal{K}^\vee) \\ &= 2f^*H + \pi^*c_1(q_*\mathcal{K}^\vee) \text{ since } q \text{ and } \pi \text{ are proper and flat} \\ &= 2f^*H + \pi^*(q_*c_1(\mathcal{K}^\vee) - \delta) \text{ applying GRR to } q \\ &= 2f^*H + \pi^*[q_*(\tau^*pr_1^*c_1(\mathcal{O}_\Sigma(1)) - q^*c_1(\varphi^*\mathcal{E}_2)) - \delta] \text{ using (7)} \end{aligned}$$

As a linear form on \mathbb{P}^1 is determined by its value on a length 2 subscheme, the evaluation morphism yields an isomorphism of sheaves:

$$(8) \quad \varphi^*\mathcal{E}_2 \simeq q_*\tau^*pr_1^*\mathcal{O}_\Sigma(1),$$

so that, using Grothendieck-Riemann-Roch theorem for q , we have the equality $c_1(\varphi^*\mathcal{E}_2) = q_*c_1(\tau^*pr_1^*\mathcal{O}_\Sigma(1)) - \delta$. We end the computation of $[\sigma(\widetilde{\Sigma \times \Sigma})]$ as follows:

$$\begin{aligned} c_1(q'_*(\pi'^*\mathcal{K}^\vee \otimes q'^*f^*\mathcal{O}_{\mathbb{P}_k^{n+1}}(1))) &= 2f^*H + \pi^*[q_*\tau^*pr_1^*c_1(\mathcal{O}_\Sigma(1)) - 2c_1(\varphi^*\mathcal{E}_2) - \delta] \\ &= 2f^*H + \pi^*[(1-2)q_*\tau^*pr_1^*c_1(\mathcal{O}_\Sigma(1)) + (2-1)\delta] \end{aligned}$$

and $q'_*\sigma'(\widetilde{\Sigma \times \Sigma}) = c_1(q'_*(\pi'^*\mathcal{K}^\vee \otimes q'^*f^*\mathcal{O}_{\mathbb{P}_k^{n+1}}(1))) + \pi^*\delta$

Now, we have $R = [f^{-1}(X)] - [\sigma(\widetilde{\Sigma \times \Sigma})] = 3f^*H - [\sigma(\widetilde{\Sigma \times \Sigma})] = f^*H + \pi^*(q_*\tau^*pr_1^*H_\Sigma - 2\delta)$ so that by projection formula $\pi_*\mathcal{O}_{\mathbb{P}(\varphi^*\mathcal{E}_2)}(R) \simeq \varphi^*\mathcal{E}_2 \otimes \mathcal{O}_{\Sigma^{[2]}}(q_*\tau^*pr_1^*H_\Sigma - 2\delta)$. Letting $s_R \in |\mathcal{O}_{\mathbb{P}(\varphi^*\mathcal{E}_2)}(R)|$ be a section whose zero locus is equal to R , we can consider s_R as a section of the rank 2-vector bundle $\pi_*\mathcal{O}_{\mathbb{P}(\varphi^*\mathcal{E}_2)}(R)$. Then the zero locus of this section correspond to length 2 subschemes whose associated line is contained in X that is to $Bisec'(\Sigma)$. As Σ is assumed to be well-positionned, $Bisec'(\Sigma)$ has codimension 2 in $\Sigma^{[2]}$ so that its class in $\text{CH}^2(\Sigma^{[2]})$ is $c_2(\pi_*\mathcal{O}_{\mathbb{P}(\varphi^*\mathcal{E}_2)}(R))$.

Let $U \simeq \text{Spec}(A)$ be an affine open subset of $\Sigma^{[2]}$ such that $\mathbb{P}(\mathcal{E})|_U \simeq \mathbb{P}_A^1$. Denoting $[Y_0 : Y_1]$ the homogeneous (relative) coordinates on \mathbb{P}_A^1 , the equation s_R of $R|_U \subset \mathbb{P}_A^1$, is of the form $f_0Y_0 + f_1Y_1 = 0$, where $f_0, f_1 \in A$, since $R \in |\mathcal{O}_{\mathbb{P}(\varphi^*\mathcal{E}_2)}(1) \otimes \pi_*\mathcal{O}_{\Sigma^{[2]}}(q_*\tau^*pr_1^*H_\Sigma - 2\delta)|$. Then the section s_R of $(\pi_*\mathcal{O}_{\mathbb{P}(\varphi^*\mathcal{E}_2)}(R))|_U$ is (f_0, f_1) so that, since Σ is well-positionned, the ideal of $Bisec'(\Sigma) \cap U$ in U is generated by (f_0, f_1) and it is a regular sequence in A . As (f_0, f_1) is a regular sequence, the equation $f_0Y_0 + f_1Y_1 = 0$ tells exactly that R is the blow-up of $\Sigma^{[2]}$ along $Bisec'(\Sigma)$ i.e. $R \simeq \widetilde{\Sigma^{[2]}}$. \square

The divisors $\widetilde{\Sigma^{[2]}}$, $\sigma(\widetilde{\Sigma \times \Sigma})$ and $f^{-1}(X)$ can be considered as correspondences from $\Sigma^{[2]}$ to X . The following fiber square:

$$\begin{array}{ccc} f^{-1}(X) & \xrightarrow{f'} & X \\ \downarrow i'_X & & \downarrow i_X \\ \mathbb{P}(\varphi^*\mathcal{E}_2) & \xrightarrow{f} & \mathbb{P}^{n+1} \\ \downarrow \pi & & \\ \widetilde{\Sigma^{[2]}} & & \end{array}$$

yields the following easy lemma:

Lemma 3.4. *The action $[f^*(X)]_* : \text{CH}^*(\Sigma^{[2]}) \rightarrow \text{CH}^*(X)$ factors through $\text{CH}^*(\mathbb{P}^{n+1})$ i.e. for any $z \in \text{CH}^i(\Sigma^{[2]})$, there is an integer $m_z \in \mathbb{Z}$ such that $[f^{-1}(X)]_*z = m_z H_X^{n+i-2d}$.*

By Lemma 3.4, $[\widetilde{\Sigma^{[2]}}]_* + [\sigma(\widetilde{\Sigma \times \Sigma})]_* : \text{CH}^*(\Sigma^{[2]}) \rightarrow \text{CH}^*(X)$ factors through $\text{CH}^*(\mathbb{P}_k^{n+1})$. As $[\sigma(\widetilde{\Sigma \times \Sigma})]$ is tautological, we can compute the action of $[\widetilde{\Sigma^{[2]}}$ modulo cycles coming from \mathbb{P}_k^{n+1} . In particular, since $[\sigma(\widetilde{\Sigma \times \Sigma})]_*(\delta^d) = f'_{|\sigma(\widetilde{\Sigma \times \Sigma}),*} q^* \delta^d = (-1)^{d-1} f'_{|\sigma(\widetilde{\Sigma \times \Sigma}),*} (j_E(E_{|E}^{d-1})) = [\Sigma]$ in $\text{CH}_d(X)$, we have $[\widetilde{\Sigma^{[2]}}]_*(\delta^d) = -[\Sigma] \bmod \mathbb{Z} \cdot H_X^{n-d}$.

Another relation we have is that $f'_{|\widetilde{\Sigma^{[2]},*} E_{Bisec'(\sigma)} = P_*(Bisec(\Sigma))$ so that, in order to obtain the inversion formula, we just have to find a suitable relation between $\pi^*_{|\Sigma^{[2]}} \delta$ and $E_{Bisec'(\sigma)}$.

Lemma 3.5. *We have the following equality in $\text{CH}^1(\widetilde{\Sigma^{[2]})$:*

$$(9) \quad (f^*H)_{|\widetilde{\Sigma^{[2]}}} = 2\pi^*_{|\widetilde{\Sigma^{[2]}}} q_*\tau^*pr_1^*H_\Sigma - 3\pi^*_{|\widetilde{\Sigma^{[2]}}} \delta - E_{Bisec'(\Sigma)}.$$

Proof. Using that $\pi_{|\widetilde{\Sigma^{[2]}}$ is a blow-up, we have $K_{\widetilde{\Sigma^{[2]}}} = \pi^*_{|\Sigma^{[2]}}} K_{\Sigma^{[2]}} + E_{Bisec'(\Sigma)}$. Adjunction formula gives $K_{\widetilde{\Sigma^{[2]}}} = (K_{\mathbb{P}(\varphi^*\mathcal{E}_2)} + \widetilde{\Sigma^{[2]}})_{|\widetilde{\Sigma^{[2]}}}$. As $K_{\mathbb{P}(\varphi^*\mathcal{E}_2)} = \pi^* K_{\Sigma^{[2]}} + K_{\mathbb{P}(\varphi^*\mathcal{E}_2)/\Sigma^{[2]}}$, we get

$$(10) \quad E_{Bisec'(\Sigma)} = (K_{\mathbb{P}(\varphi^*\mathcal{E}_2)/\Sigma^{[2]}} + \widetilde{\Sigma^{[2]}})_{|\widetilde{\Sigma^{[2]}}}$$

Using formulas for projective bundle, we have

$$K_{\mathbb{P}(\varphi^*\mathcal{E}_2)/\Sigma^{[2]}} = -2c_1(\mathcal{O}_{\mathbb{P}(\varphi^*\mathcal{E}_2)}(1)) + \pi^* c_1(\varphi^*\mathcal{E}_2) = -2f^*H + \pi^*(q_*\tau^*pr_1^*c_1(\mathcal{O}_\Sigma(1)) - \delta).$$

Then, (10) yields

$$E_{Bisec'(\Sigma)} = (-f^*H + 2\pi^*q_*\tau^*pr_1^*H_\Sigma - 3\pi^*\delta)_{|\widetilde{\Sigma^{[2]}}}$$

□

We recall from (8) that $c_1(\varphi^*\mathcal{E}_2) = q_*\tau^*pr_1^*H_\Sigma - \delta$. Intersecting (9) with $\pi_{\widetilde{\Sigma^{[2]}}}^*c_1(\varphi^*\mathcal{E}_2)^{d-1}$ and projecting to X , we get:

$$(11) \quad H_X \cdot \left[\widetilde{\Sigma^{[2]}} \right]_* (c_1(\varphi^*\mathcal{E}_2)^{d-1}) = \left[\widetilde{\Sigma^{[2]}} \right]_* [(2q_*\tau^*pr_1^*H_\Sigma - 3\delta) \cdot c_1(\varphi^*\mathcal{E}_2)^{d-1}] - f'_{|\widetilde{\Sigma^{[2]}}} (E_{Bisec'(\Sigma)} \cdot \pi_{\widetilde{\Sigma^{[2]}}}^*c_1(\varphi^*\mathcal{E}_2)^{d-1})$$

Lemma 3.6. *We have the following formulas:*

- (i) for $k \geq 1$, $(q_*\tau^*pr_1^*H_\Sigma)^k = \sum_{j=0}^{k-1} \binom{k-1}{j} q_*\tau^*(pr_1^*H_\Sigma^{k-j} \cdot pr_2^*H_\Sigma^j)$;
- (ii) for $k, k' \geq 0$ and $m \geq 1$,

$$q_*\tau^*(pr_1^*H_\Sigma^k \cdot pr_2^*H_\Sigma^{k'}) \cdot \delta^m = q_*j_{E,*}[\tau_{|E}^*i_{\Delta_\Sigma}^*(pr_1^*H_\Sigma^k \cdot pr_2^*H_\Sigma^{k'}) \cdot (E_{|E})^{m-1}]$$

where $j_E : E \hookrightarrow \widetilde{\Sigma \times \Sigma}$ is the inclusion of the exceptional divisor, $i_{\Delta_\Sigma} : \Delta_\Sigma \hookrightarrow \Sigma \times \Sigma$ is the inclusion of the diagonal (so that $i_{\Delta_\Sigma}^*(pr_1^*H_\Sigma^k \cdot pr_2^*H_\Sigma^{k'})$ is the hyperplane section $H_\Sigma^{k+k'}$ on $\Sigma \simeq \Delta_\Sigma$) and $c_1(\mathcal{O}_E(-1)) \simeq E_{|E}$ is the tautological line bundle of the projective bundle $\tau_{|E} : E \rightarrow \Delta_\Sigma$.

(iii) it follows that for $m \geq 2$,

$$\begin{aligned} c_1(\varphi^*\mathcal{E}_2)^m &= \sum_{l=0}^{m-1} \binom{m-1}{l} q_*\tau^*(pr_1^*H_\Sigma^{m-l} \cdot pr_2^*H_\Sigma^l) \\ &+ \sum_{k=1}^m \sum_{l=0}^{m-1-k} (-1)^k \binom{m}{k} \binom{m-1-k}{l} q_*j_{E,*}(\tau_{|E}^*i_{\Delta_\Sigma}^*(pr_1^*H_\Sigma^{m-k-l} \cdot pr_2^*H_\Sigma^l) \cdot E_{|E}^{k-1}) \end{aligned}$$

In order to establish (6), let us now compute the different terms of (11).

(1) We have $[\sigma(\widetilde{\Sigma \times \Sigma})]_*(q_*\tau^*(pr_1^*H_\Sigma^{d-1-l} \cdot pr_2^*H_\Sigma^l)) = 0$ since the support in X of this cycle is $H_\Sigma^{d-1-l} \cup H_\Sigma^l$ which has dimension $\leq d$ whereas $\pi_{|\sigma(\widetilde{\Sigma \times \Sigma})}^*(q_*\tau^*(pr_1^*H_\Sigma^{d-1-l} \cdot pr_2^*H_\Sigma^l))$ has dimension $2d - (d-1) = d+1$.

Likewise, for $d-1 \geq k \geq 1$, $[\sigma(\widetilde{\Sigma \times \Sigma})]_*(q_*j_{E,*}(\tau_{|E}^*i_{\Delta_\Sigma}^*(pr_1^*H_\Sigma^{d-1-k-l} \cdot pr_2^*H_\Sigma^l) \cdot E_{|E}^{k-1}))$ is 0 in $\text{CH}^*(X)$ since $\pi_{|\sigma(\widetilde{\Sigma \times \Sigma})}^*(q_*j_{E,*}(\tau_{|E}^*i_{\Delta_\Sigma}^*(pr_1^*H_\Sigma^{d-1-k-l} \cdot pr_2^*H_\Sigma^l) \cdot E_{|E}^{k-1}))$ has dimension $d+1 > d \geq k+1$ whereas the support of $f_{|\sigma(\widetilde{\Sigma \times \Sigma})} (q_*j_{E,*}(\tau_{|E}^*i_{\Delta_\Sigma}^*(pr_1^*H_\Sigma^{d-1-k-l} \cdot pr_2^*H_\Sigma^l) \cdot E_{|E}^{k-1}))$ in X is H_Σ^{d-1-k} which has dimension $k+1$. So $\left[\widetilde{\Sigma^{[2]}} \right]_* (c_1(\varphi^*\mathcal{E}_2)^{d-1}) \in \mathbb{Z} \cdot H_X^{n-d-1}$.

(2) We have $[\sigma(\widetilde{\Sigma \times \Sigma})]_*(q_*\tau^*pr_1^*H_\Sigma^d) = \text{deg}(\Sigma)[\Sigma]$ in $\text{CH}_d(X)$ and for $d-1 \geq l \geq 1$, $[\sigma(\widetilde{\Sigma \times \Sigma})]_*(q_*\tau^*(pr_1^*H_\Sigma^{d-l} \cdot pr_2^*H_\Sigma^l)) = 0$ since the support of this cycle is $H_\Sigma^{d-l} \cup H_\Sigma^l$ which has dimension at most $d-1$ whereas $pr_1^*H_\Sigma^{d-l} \cdot pr_2^*H_\Sigma^l$ has dimension d . For $d-1 \geq k \geq 1$, we have also $[\sigma(\widetilde{\Sigma \times \Sigma})]_*(q_*j_{E,*}(\tau_{|E}^*i_{\Delta_\Sigma}^*(pr_1^*H_\Sigma^{d-k-l} \cdot pr_2^*H_\Sigma^l) \cdot E_{|E}^{k-1})) = 0$ since its support is H_Σ^{d-k} which has dimension $k \leq d-1$ whereas $(\tau_{|E}^*i_{\Delta_\Sigma}^*(pr_1^*H_\Sigma^{d-k-l} \cdot pr_2^*H_\Sigma^l) \cdot E_{|E}^{k-1})$ has dimension d so that $\left[\widetilde{\Sigma^{[2]}} \right]_* (q_*\tau^*pr_1^*H_\Sigma \cdot c_1(\varphi^*\mathcal{E}_2)^{d-1}) = -\text{deg}(\Sigma)[\Sigma] \bmod \mathbb{Z} \cdot H_X^{n-d}$

(3) Likewise, we get $[\sigma(\widetilde{\Sigma \times \Sigma})]_*(\delta \cdot c_1(\varphi^*\mathcal{E}_2)^{d-1}) = [\sigma(\widetilde{\Sigma \times \Sigma})]_*(\delta^d) = [\Sigma]$ in $\text{CH}_d(X)$ so that $\left[\widetilde{\Sigma^{[2]}} \right]_* (\delta \cdot c_1(\varphi^*\mathcal{E}_2)^{d-1}) = -[\Sigma] \bmod \mathbb{Z} \cdot H_X^{n-d}$.

(4) For the last term, we have

$$f'_{|R,*}(E_{\text{Bisec}'(\Sigma)} \cdot \pi_{|\Sigma^{[2]}}^* c_1(\varphi^* \mathcal{E}_2)^{d-1}) = P_*(\varphi'_*(i_{\text{Bisec}'(\Sigma)}^* c_1(\varphi^* \mathcal{E}_2)^{d-1})).$$

So, we see, from the computation, that (11) yields the inversion formula (6) of the Theorem. \square

Remark 3.7. *In the proof of Theorem 3.2, we could have intersected (9) with $\pi_{|\Sigma^{[2]}}^* \delta^{d-1}$ instead of $\pi_{|\Sigma^{[2]}}^* c_1(\varphi^* \mathcal{E}_2)^{d-1}$, we would have got a formula of the form*

$$3[\Sigma] + P_*\gamma \in \mathbb{Z} \cdot H_X^{n-d}$$

with γ a $(d-1)$ -cycle on $F(X)$. But when we cut $\text{Bisec}(\Sigma)$ by powers of $c_1(\varphi^* \mathcal{E}_2)$ we just look at hyperplane sections of $\text{Bisec}(\Sigma)$ whereas cutting $\text{Bisec}'(\Sigma)$ by powers of $\pi_{|\Sigma^{[2]}}^* \delta$ gives a cycle γ which depends even more on the geometry of Σ . As we will see in Section 3.3, $\text{Bisec}(\Sigma)$ seems to be the natural object to associate to Σ .

3.2. A digression on the Hilbert square of subvarieties. Assume $k = \mathbb{C}$. Recall that P_2 is defined as the symmetric product of P over $F(X)$. It admits an imbedding $i_{P_2} : P_2 \hookrightarrow X^{[2]}$ and a projection $p_{P_2} : P_2 \rightarrow F(X)$ which is a \mathbb{P}^2 -bundle. As P_2 is smooth and has codimension 2 in $X^{[2]}$, X is well-positioned and, with the notations of 3.1, the class of P_2 in $\text{CH}^2(X^{[2]})$ is $c_2(\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\widehat{X}^{[2]}))$. Let $\Sigma \subset X$ be a smooth well-positioned d -dimensional subvariety of a cubic hypersurface of dimension $n \geq 5$, with $1 \leq d \leq n-2$. Then from Proposition 3.3, we get that $[\text{Bisec}'(\Sigma)] = i_{\Sigma^{[2]}}^* P_2$; since it comes from a cycle constructed in $P_2 \cap \Sigma^{[2]}$, we see that $[\text{Bisec}(\Sigma)] = p_{P_2,*} i_{P_2}^* \Sigma^{[2]}$ in $\text{CH}_{2d-2}(F(X))$. So the inversion formula of Theorem 3.2, relies on the operation $\Sigma \mapsto \Sigma^{[2]}$.

On the other hand, since the coefficient appearing with Σ in the formula is odd and X admits a degree 2 unirational parametrisation, by Theorem 3.2, the operation $\Sigma \mapsto \Sigma^{[2]}$ become central in the study of birational invariants constructed from torsion algebraic cycles for cubic hypersurfaces. The operation $\Sigma \mapsto \Sigma^{[2]}$ have the following general properties:

Proposition 3.8. *Let Y be a smooth projective k -variety. Let V, V' be smooth subvarieties of Y of dimension $d < \dim(Y)$ and $N > 0$ an integer such that $N(i_{V,*}(c(V)^{-1}) - i_{V',*}(c(V')^{-1})) = 0$ in $\text{CH}_*(Y)$ (resp. $\text{CH}_*(Y)/\text{alg}$), where i_V (resp. $i_{V'}$) is the inclusion of V (resp. of V') in Y .*

(i) *Then $2N(V^{[2]} - V'^{[2]}) = 0$ in $\text{CH}_{2d}(Y^{[2]})$ (resp. $\text{CH}_{2d}(Y^{[2]})/\text{alg}$).*

(ii) *Moreover if the groups $\text{CH}_i(Y)$ are torsion-free for $i \leq 2d$, then $V^{[2]} = V'^{[2]}$ in $\text{CH}_{2d}(Y^{[2]})$.*

Proof. (i) Let us denote $\tau : \widetilde{Y \times Y} \rightarrow Y \times Y$ the blow-up of $Y \times Y$ along the diagonal Δ_Y and $q : \widetilde{Y \times Y} \rightarrow Y^{[2]}$ the quotient by the involution. For a smooth subvariety $V \subset Y$, we have $q_*(\widetilde{V \times V}^{\Delta_V}) = 2V^{[2]}$ in $\text{CH}_{2d}(Y^{[2]})$, where $\widetilde{V \times V}^{\Delta_V}$ is the blow-up of $V \times V$ along its diagonal Δ_V , i.e. the proper transform of $V \times V$ under τ . When V is smooth, using [15, Theorem 6.7 and Corollary 4.2.1], we have

$$(12) \quad \widetilde{V \times V}^{\Delta_V} = \tau^*(V \times V) - j_{E_Y,*} \{c(\tau_{|E_Y}^* T_Y) c(E_{Y|E_Y})^{-1} \cdot \tau_{|E_Y}^* i_{V,*}(c(T_V)^{-1})\}_{2d} \text{ in } \text{CH}_{2d}(\widetilde{Y \times Y})$$

where $j_{E_Y} : E_Y \hookrightarrow \widetilde{Y \times Y}$ is the exceptional divisor of τ and for an element $z \in \text{CH}_*(\widetilde{Y \times Y})$, $\{z\}_k$ is the part of dimension k of z .

Now, if $N(V - V') = 0$ in $\text{CH}_d(Y)$ (resp. $\text{CH}_d(Y)/\text{alg}$), V and V' being smooth subvarieties of Y , then

$$N(V \times V) = N pr_1^* V \cdot pr_2^* V = pr_1^*(NV) \cdot pr_2^* V = pr_1^*(NV') \cdot pr_2^* V = pr_1^* V' \cdot pr_2^*(NV) = N(V' \times V')$$

in $\mathrm{CH}_{2d}(Y \times Y)$ (resp. $\mathrm{CH}_{2d}(Y \times Y)/\mathrm{alg}$). So we see that the hypothesis yields

$$\begin{aligned} 2N(V^{[2]} - V'^{[2]}) &= Nq_*(\widetilde{V \times V}^{\Delta_V} - \widetilde{V' \times V'}^{\Delta_{V'}}) \\ &= \tau^*N[(V \times V) - (V' \times V')] \\ &\quad - j_{E_Y,*}\{c(\tau_{|E_Y}^*T_Y)c(E_Y|E_Y)^{-1} \cdot \tau_{|E_Y}^*N((i_{V,*}(c(T_V)^{-1}) - i_{V',*}(c(T_{V'})^{-1})))\}_{2d} \\ &= 0 \text{ in } \mathrm{CH}_{2d}(Y^{[2]}) \text{ (resp. } \mathrm{CH}_{2d}(Y^{[2]})/\mathrm{alg}). \end{aligned}$$

(ii) As $\mathrm{CH}_{*\leq d}(Y)$ is assumed to be torsion-free, $V = V'$ in $\mathrm{CH}_d(Y)$. Then, by [24, Proposition 1.4], $V^{(2)} = V'^{(2)}$ in $\mathrm{CH}_{2d}(Y^{(2)})$, where, for a variety Z , $Z^{(2)}$ is the symmetric product of Z . We have the localisation exact sequence

$$\mathrm{CH}_{2d}(E_Y) \rightarrow \mathrm{CH}_{2d}(Y^{[2]}) \rightarrow \mathrm{CH}_{2d}(Y^{[2]}\setminus E_Y) \rightarrow 0$$

and since $Y^{[2]}\setminus E_Y \simeq Y^{(2)}\setminus \Delta_Y$, $V^{[2]} - V'^{[2]}$ can be written $q_*j_{E_Y,*}\gamma$ for a $2d$ -cycle $\gamma \in \mathrm{CH}_{2d}(E_Y)$. According to item (i), $2(V^{[2]} - V'^{[2]}) = 0$ so that $q_*j_{E,*}(2\gamma) = 0$. As, q is flat, $q^*q_*j_{E,*}(2\gamma) = [q^{-1}q_*j_{E,*}(2\gamma)] = j_{E,*}(2\gamma)$ and by the decomposition of the Chow groups of the blow-up $Y \times Y$, $2\gamma = 0$ in $\mathrm{CH}_{2d}(E_Y)$. So, by the decomposition of the Chow groups of projective bundle and torsion-freeness of $\mathrm{CH}_{*\leq d}(Y)$, $\gamma = 0$ i.e. $V^{[2]} - V'^{[2]} = 0$ in $\mathrm{CH}_{2d}(Y^{[2]})$. \square

Unfortunately, in general, for a smooth subvariety V of a smooth projective variety Y , one cannot expect the class of $V^{[2]}$ in $\mathrm{CH}_*(Y^{[2]})$ to be determined by $(i_{V,*}(c(T_V)^{-1}))$ as the following example, which was communicated to the author by Voisin, shows.

Let S be an abelian surface and $x, y \in S$ be two distinct 2-torsion points. For any sufficiently ample linear system \mathcal{L} on S , there exists a curve $C_x \in |\mathcal{L}|$ not containing y , resp. $C_y \in |\mathcal{L}|$ not containing x , which is smooth away from x , resp. y , and has an ordinary double point at x , resp. y . Let $\tau: \widetilde{S} \rightarrow S$ be the blow-up of S at x and y and E_x, E_y the corresponding exceptional divisors. The normalization \widetilde{C}_x (resp. \widetilde{C}_y) of C_x (resp. C_y) is the strict transform of C_x (resp. C_y) under τ and its class in $Pic(\widetilde{S})$ is $\tau^*c_1(\mathcal{L}) - 2E_x$ (resp. $\tau^*c_1(\mathcal{L}) - 2E_y$).

Let $\mathcal{N} \in Pic(S)$ be sufficiently ample on S so that the line bundle $\tau_{|\widetilde{C}_x}^*\mathcal{N}|_{C_x}$ is very ample (once its degree on \widetilde{C}_x is large enough) on \widetilde{C}_x and $\tau_{|\widetilde{C}_y}^*\mathcal{N}|_{C_y}$ is very ample on \widetilde{C}_y . We can pick a meromorphic function $f_x: \widetilde{C}_x \rightarrow \mathbb{P}^1$ in $|\tau_{|\widetilde{C}_x}^*\mathcal{N}|_{C_x}|$ such that, denoting x_1 and x_2 the points lying over the node x , $f_x(x_1) \neq f_x(x_2)$. Likewise, we can pick a meromorphic function $f_y: \widetilde{C}_y \rightarrow \mathbb{P}^1$ in $|\tau_{|\widetilde{C}_y}^*\mathcal{N}|_{C_y}|$ such that $f_y(y_1) \neq f_y(y_2)$, where y_1, y_2 are the points lying over the node y .

Let $X = S \times \mathbb{P}^1$ be the trivial projective bundle over S . By construction the morphisms $(\tau_{|\widetilde{C}_x}, f_x): \widetilde{C}_x \rightarrow X$ and $(\tau_{|\widetilde{C}_y}, f_y): \widetilde{C}_y \rightarrow X$ are embeddings so $D_x = (\tau_{|\widetilde{C}_x}, f_x)(\widetilde{C}_x)$ and $D_y = (\tau_{|\widetilde{C}_y}, f_y)(\widetilde{C}_y)$ are smooth curves on X .

Proposition 3.9. *In this situation, we have $i_{D_x,*}(c(T_{D_x})^{-1}) = i_{D_y,*}(c(T_{D_y})^{-1})$ in $\mathrm{CH}_*(X)$ but $D_x^{[2]} \neq D_y^{[2]}$ in $\mathrm{CH}_2(X^{[2]})$.*

Proof. We have the decomposition $\mathrm{CH}_1(X) \simeq pr_1^*\mathrm{CH}_0(S) \oplus pr_1^*\mathrm{CH}_1(S)$. The projection on $\mathrm{CH}_1(S)$ is given by $pr_{1,*}$; we have $pr_{1,*}D_x = \tau_{|\widetilde{C}_x,*}\widetilde{C}_x = C_x$ and $pr_{1,*}D_y = \tau_{|\widetilde{C}_y,*}\widetilde{C}_y = C_y$ and $C_x, C_y \in |\mathcal{L}|$. As the Chern classes of the trivial bundle are trivial the projection on $\mathrm{CH}_0(S)$ is given by the composition of the intersection with $pr_2^*c_1(\mathcal{O}_{\mathbb{P}^1}(1))$ followed by $pr_{1,*}$. We have $f_x^*\mathcal{O}_{\mathbb{P}^1}(1) \simeq \tau_{|\widetilde{C}_x}^*\mathcal{N}|_{C_x}$ and using projection formula and $C_x \in |\mathcal{L}|$, we get $pr_{1,*}(D_x \cdot pr_2^*c_1(\mathcal{O}_{\mathbb{P}^1}(1))) = c_1(\mathcal{L}) \cdot c_1(\mathcal{N})$ in $\mathrm{CH}_0(S)$. Likewise, we have $pr_{1,*}(D_y \cdot pr_2^*c_1(\mathcal{O}_{\mathbb{P}^1}(1))) = c_1(\mathcal{L}) \cdot c_1(\mathcal{N})$. So $D_x = D_y$ in $\mathrm{CH}_1(X)$.

By adjunction, we have $K_{\widetilde{C}_x} = (K_{\widetilde{S}} + \widetilde{C}_x)|_{\widetilde{C}_x} = (\tau^*(c_1(\mathcal{L}) + K_S) + E_y - E_x)|_{\widetilde{C}_x}$ so that

in $\text{CH}_0(X) \simeq \text{CH}_0(S)$,

$$\begin{aligned} i_{D_x,*}K_{D_x} &= (\tau \circ i_{\widetilde{C}_x})_* i_{\widetilde{C}_x}^*(\tau^*c_1(\mathcal{L}) + E_y - E_x) \\ &= \tau_* i_{\widetilde{C}_x,*} i_{\widetilde{C}_x}^*(\tau^*c_1(\mathcal{L}) + E_y - E_x) \\ &= \tau_*((\tau^*c_1(\mathcal{L}) + E_y - E_x) \cdot \widetilde{C}_x) \\ &= \tau_*(\tau^*(c_1(\mathcal{L})^2 + 2E_x^2)) \\ &= c_1(\mathcal{L})^2 - 2x \end{aligned}$$

Likewise $i_{D_y,*}K_{D_y} = c_1(\mathcal{L})^2 - 2y$. As $2x = 2y$ in $\text{CH}_0(S)$, $i_{D_x,*}K_{D_x} = i_{D_y,*}K_{D_y}$ in $\text{CH}_0(X)$. So $i_{D_x,*}(c(T_{D_x})^{-1}) = i_{D_y,*}(c(T_{D_y})^{-1})$.

The variety of lines of X , with respect to a very ample line bundle of the form $pr_1^*\mathcal{L}' \otimes pr_2^*\mathcal{O}_{\mathbb{P}^1}(1)$, is isomorphic to S since any morphism from a projective space to an abelian variety is constant. Let us denote $P_2 = \mathbb{P}(\text{Sym}^2\mathcal{E}) \simeq S \times \mathbb{P}^2$; it parametrizes the length 2 subschemes of X contained in a line of X . So $D_x^{[2]} \cap P_2$ parametrizes length 2 subschemes of D_x such that the associated line is contained in X . But by construction, since $pr_{1|D_x} : D_x \rightarrow C_x$ is an isomorphism above $C_x \setminus \{x\}$, the only length 2 subscheme whose associated line is contained in X is $\{x_1 + x_2\}$ whose associated line is $\mathbb{P}(\mathcal{E}_x)$. So, denoting $i_{P_2} : P_2 \hookrightarrow X^{[2]}$ the natural inclusion and $\pi_1 : P_2 \rightarrow S$ the first projection, we have $\pi_{1,*}(i_{P_2}^*D_x^{[2]}) = x$ in $\text{CH}_0(S)$.

Likewise $\pi_{1,*}(i_{P_2}^*D_y^{[2]}) = y$ in $\text{CH}_0(S)$. So $\pi_{1,*}i_{P_2}^*(D_x^{[2]} - D_y^{[2]}) = x - y \neq 0$ in $\text{CH}_0(S)$, in particular $D_x^{[2]} - D_y^{[2]}$ is a nonzero 2-torsion element in $\text{CH}_2(X^{[2]})$. \square

3.3. Application to the cycle of bisecant lines. Using the results of the previous sections, we get the following:

Theorem 3.10. *Let $X \subset \mathbb{P}_{\mathbb{C}}^{n+1}$, with $n \geq 5$, be a smooth cubic hypersurface. For any $\Gamma \in \text{CH}_2(X)$ of t -torsion (hence homologically trivial), there is a homologically trivial $2t$ -torsion 1-cycle $\gamma \in \text{CH}_1(F(X))$ and an odd integer m such that $m\Gamma = P_*\gamma$ in $\text{CH}_2(X)$.*

Proof. Let $\Gamma \in \text{CH}_2(X)$ be a cycle annihilated by $t \in \mathbb{Z}_{>0}$. Using Proposition 1.4, we can find a 1-cycle α in $F(X)$ such that $P_*(\alpha) = \Gamma$. As Γ is a torsion cycle and $\text{CH}_0(X) = \mathbb{Z}$, $\Gamma \cdot H^2 = 0$ and since $c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E}_{2|F(X)})}(1)) = q^*H$, we get $\deg(\alpha \cdot c_1(\mathcal{O}_{F(X)}(1))) = \deg(q_*[p^*(\alpha \cdot c_1(\mathcal{O}_{F(X)}(1)) \cdot q^*H]) = 0$, where $\mathcal{O}_{F(X)}(1) = \det(\mathcal{E}_{2|F(X)})$ is the Plücker line bundle, which implies that $\alpha \cdot c_1(\mathcal{O}_{F(X)}(1)) = 0$ in $\text{CH}_0(F(X))$ since $F(X)$ is rationally connected. As $\text{Pic}(F(X)) \simeq \mathbb{Z}$ ([13, Corollaire 3.5]), α is numerically trivial. We have the following lemma:

Lemma 3.11. *Let Y be a smooth projective variety of dimension $d \geq 3$ and D a numerically trivial 1-cycle of Y . Then there are smooth curves $D_1, D_2 \subset Y$ of the same genus such that $D = D_1 - D_2$ in $\text{CH}_1(Y)$.*

Postponing the proof of the lemma, we conclude as follows: let $E_1, E_2 \subset F(X)$ be two smooth curves of genus g such that $\alpha = E_1 - E_2$ in $\text{CH}_1(F(X))$; they have also the same degree (α is numerically trivial) that we shall denote d . Let us denote $S_{E_1} = q(p^{-1}(E_1))$ and $S_{E_2} = q(p^{-1}(E_2))$ the associated ruled surfaces in X , we have $\Gamma = P_*\alpha = S_{E_1} - S_{E_2}$ in $\text{CH}_2(X)$. By transversality arguments, we can arrange that q induces an imbedding of $p^{-1}(E_1)$ (resp. $p^{-1}(E_2)$) in X so that S_{E_1} (resp. S_{E_2}) is smooth and isomorphic to $p^{-1}(E_1)$ (resp. $p^{-1}(E_2)$). An easy computation then gives:

$$\begin{aligned} i_{S_{E_1},*}(c(T_{S_{E_1}})^{-1}) &= S_{E_1} + P_*(i_{E_1,*}K_{E_1} + dc_1(\mathcal{O}_{F(X)}(1)) \cdot E_1) - 2S_{E_1} \cdot H - 2H \cdot P_*(i_{E_1,*}K_{E_1}) \\ &= P_*(E_1) + (2g - 2 + d)P_*[l_0] - 2P_*(E_1) \cdot H - 2(2g - 2)P_*([l_0]) \cdot H \end{aligned}$$

since $\text{CH}_0(F(X)) \simeq \mathbb{Z} \cdot [l_0]$ for a (any) point $[l_0] \in F(X)$. Likewise

$$i_{S_{E_2},*}(c(T_{S_{E_2}})^{-1}) = P_*(E_2) + (2g - 2 + d)P_*[l_0] - 2P_*(E_2) \cdot H - 2(2g - 2)P_*([l_0]) \cdot H$$

so that $i_{S_{E_1},*}(c(T_{S_{E_1}})^{-1}) - i_{S_{E_2},*}(c(T_{S_{E_2}})^{-1}) = (S_{E_1} - S_{E_2}) \cdot (1 - 2H)$ is annihilated by t in $\text{CH}_*(X)$. Using Proposition 3.8, we get that $S_{E_1}^{[2]} - S_{E_2}^{[2]}$ is annihilated by $2t$ in $\text{CH}_4(X^{[2]})$. According to [35, Theorem 2.2], since $H^*(X, \mathbb{Z})$ is torsion-free (by Lefschetz hyperplane and universal coefficient theorems), $H^*(X^{[2]}, \mathbb{Z})$ is torsion-free so that $[S_{E_1}^{[2]} - S_{E_2}^{[2]}] = 0$ in

$H^{4n-8}(X^{[2]}, \mathbb{Z})$.

Now, Theorem 3.2, says that there are integers m_1, m_2 such that

$$(2d-3)S_{E_1} + P_*(p_{P_2,*}i_{P_2}^*S_{E_1}^{[2]} \cdot c_1(\mathcal{O}_{F(X)}(1))) = m_1H^{n-2} \text{ in } \text{CH}_2(X)$$

and

$$(2d-3)S_{E_2} + P_*(p_{P_2,*}i_{P_2}^*S_{E_2}^{[2]} \cdot c_1(\mathcal{O}_{F(X)}(1))) = m_2H^{n-2} \text{ in } \text{CH}_2(X)$$

in particular $(2d-3)\Gamma + P_*(p_{P_2,*}i_{P_2}^*(S_{E_1}^{[2]} - S_{E_2}^{[2]}) \cdot c_1(\mathcal{O}_{F(X)}(1))) \in \mathbb{Z} \cdot H^{n-2}$. But intersecting with H^2 , since Γ and $p_{P_2,*}i_{P_2}^*(S_{E_1}^{[2]} - S_{E_2}^{[2]})$ are torsion cycles, we see that actually:

$$(2d-3)\Gamma + P_*(p_{P_2,*}i_{P_2}^*(S_{E_1}^{[2]} - S_{E_2}^{[2]}) \cdot c_1(\mathcal{O}_{F(X)}(1))) = 0$$

in $\text{CH}_2(X)$. Moreover $p_{P_2,*}i_{P_2}^*(S_{E_1}^{[2]} - S_{E_2}^{[2]})$ is homologically trivial since $S_{E_1}^{[2]} - S_{E_2}^{[2]}$ is. \square

Proof of Lemma 3.11. Using Hironaka's smoothing of cycles ([19]) and moving lemma, we can write $D = \sum_i m_i C_i$ where $(C_i)_{1 \leq i \leq N}$ is a family of smooth pairwise disjoint connected curves. We can always assume that there is a i_0 such that $m_{i_0} = 1$. Indeed, if none of the m_i is equal to 1, then we can pick 2 smooth curves $C_{N+1}, C_{N+2} \subset Y$ which are rationally equivalent such that $(C_i)_{1 \leq i \leq N+2}$ is still a family of pairwise disjoint smooth curves. Then $D = \sum_i m_i C_i + C_{N+1} - C_{N+2}$ in $\text{CH}_1(Y)$.

Let $C \subset Y$ be a smooth curve intersecting C_{i_0} transversally in a unique point and disjoint from the remaining C_i and $Z = (\cup_{i=1}^N C_i) \cup C$. The subscheme Z is purely 1-dimensional and smooth away from the point $C \cap C_{i_0}$ which is an ordinary double point. In particular Z is a local complete intersection subscheme, so that the sheaf I_Z/I_Z^2 on Z is a vector bundle that we shall denote $N_{Z/Y}^\vee$. Let $\mathcal{L} \in \text{Pic}(Y)$ be a very ample line bundle such that $H^1(Y, \mathcal{L} \otimes I_Z^2) = 0$ and $N_{Z/Y}^\vee \otimes \mathcal{L}|_Z$ is globally generated. Then, from the exact sequence

$$0 \rightarrow I_Z^2 \rightarrow I_Z \rightarrow N_{Z/Y}^\vee \rightarrow 0$$

we get a surjective morphism $H^0(Y, \mathcal{L} \otimes I_Z) \xrightarrow{\rho} H^0(Z, N_{Z/Y}^\vee \otimes \mathcal{L}|_Z)$. According to [29, Lemma 1], for any nonzero section s , the zero scheme $V(s) \subset Y$ is singular at a point $x \in Z$ if and only if the section $\rho(s)$ of $N_{Z/Y}^\vee \otimes \mathcal{L}|_Z$ vanishes at x . As, $N_{Z/Y}^\vee \otimes \mathcal{L}|_Z$ is globally generated of rank ≥ 2 , the zero locus of a generic section of $N_{Z/Y}^\vee \otimes \mathcal{L}|_Z$ has codimension $\text{rank}(N_{Z/Y}^\vee \otimes \mathcal{L}|_Z) \geq 2$ i.e. is empty. So we can find a smooth hypersurface in $|\mathcal{L}|$ containing Z . Repeating the process, we can get a smooth surface $S \subset Y$, which is complete intersection of hypersurfaces given by sections of powers of \mathcal{L} , containing Z . Next it is a standard fact (e.g consequence of Riemann-Roch formula) that for any divisor W on a smooth projective surface S , $\text{deg}(W \cdot (W + K_S))$ is even. Applying this fact to the divisor $D = \sum_i m_i C_i$ of S , $\int_S D \cdot (D + K_S)$ is even and since D is numerically trivial on Y and K_S is the restriction of a divisor of Y by adjunction formula (S is complete intersection in Y), $\text{deg}(D^2) \in 2\mathbb{Z}$. Let us write $\text{deg}(D^2) = 2\ell$. Let $H \in \text{Pic}(S)$ be a very ample divisor coming from Y such that the line bundles $\mathcal{O}_S(H - \ell C)$ and $\mathcal{O}_S(H - \ell C + D)$ are ample. We can choose smooth connected curves $E_1 \in |H - \ell C + D|$ and $E_2 \in |H - \ell C|$; we then have $D = E_1 - E_2$ in $\text{Pic}(S)$ (thus, in $\text{CH}_1(Y)$ also). By adjunction formula, we have:

$$\begin{aligned} 2g(E_1) - 2 &= \int_S (H - \ell C + D) \cdot (H - \ell C + D + K_S) \\ &= \int_S (H - \ell C) \cdot (H - \ell C + K_S) + \int_S D \cdot (H - \ell C + D + K_S) + \int_S D \cdot (H - \ell C) \\ &= \int_S (H - \ell C) \cdot (H - \ell C + K_S) + \int_S D^2 - 2\ell D \cdot C \\ &\quad \text{since } D \text{ is numerically trivial on } Y \text{ and } H \text{ and } K_S \text{ come from divisors of } Y \\ &= \int_S (H - \ell C) \cdot (H - \ell C + K_S) \text{ since by construction } \int_S C \cdot D = \int_S C \cdot C_{i_0} = 1 \end{aligned}$$

and

$$2g(E_2) - 2 = \int_S (H - \ell C) \cdot (H - \ell C + K_S).$$

i.e. $g(E_1) = g(E_2)$. \square

Before stating our main corollary, let us prove the following lemma:

Lemma 3.12. *The group $\text{CH}_1(Y)_{\text{tors},AJ}$ is a stable birational invariant for smooth projective varieties Y .*

Proof. As usual, it suffices to prove invariance under taking products with \mathbb{P}^r and under blow-ups. We have

$$\text{CH}_1(Y \times \mathbb{P}^r) = \text{CH}_1(Y) \oplus \text{CH}_0(Y)$$

and this decomposition is compatible with the Deligne cycle class map. As the torsion of $\text{CH}_0(Y)$ injects into $\text{Alb } Y$ by Roitman [28], it follows that $\text{CH}_0(Y)_{\text{tors},AJ} = 0$ which proves the first invariance. Similarly, let $\tilde{Y}_Z \rightarrow Y$ be the blow-up of Y along Z , with Z smooth of codimension ≥ 2 . Then we have

$$\text{CH}_1(\tilde{Y}_Z) = \text{CH}_1(Y) \oplus \text{CH}_0(Z)$$

and we conclude by the same argument invoking [28] that $\text{CH}_1(\tilde{Y}_Z)_{\text{tors},AJ} = \text{CH}_1(Y)_{\text{tors},AJ}$. \square

Remark 3.13. In fact the same arguments show that the group $\text{CH}_1(Y)_{\text{tors},AJ}$ is trivial when Y admits a Chow-theoretic decomposition of the diagonal.

We have the following corollary for cubic 5-folds:

Corollary 3.14. *Let $X \subset \mathbb{P}_{\mathbb{C}}^6$ be smooth cubic hypersurface. Then $P_* : \text{CH}_1(F(X))_{\text{tors},AJ} \rightarrow \text{CH}^3(X)_{\text{tors},AJ}$ is surjective. So the birational invariant $\text{CH}^3(X)_{\text{tors},AJ}$ of X is controlled by the group $\text{CH}_1(F(X))_{\text{tors},AJ}$.*

Proof. Let $\Gamma \in \text{CH}^3(X)_{\text{tors},AJ} \simeq \text{CH}_2(X)_{\text{tors},AJ}$; by Theorem 3.10, there are a homologically trivial torsion cycle $\gamma \in \text{CH}_1(F(X))$ and an integer d such that $(2d-3)\Gamma = P_*\gamma$. Because of the degree 2 unirational parametrization of X , $\text{CH}^3(X)_{\text{tors},AJ}$ is a 2-torsion group; in particular $(2d-3)\Gamma = \Gamma$ in $\text{CH}^3(X)$ and it is equal to $P_*\gamma$. By functoriality of Abel-Jacobi maps (P_* induces morphisms of Hodge structures), denoting, for a complex variety Y , Φ_Y^{2c-1} the Abel-Jacobi map for homologically trivial cycles of codimension c , $\Phi_X^5(\Gamma) = P_*\Phi_{F(X)}^9(\gamma)$. Now, by [32], P_* is an isomorphism of abelian varieties so γ is annihilated by the Abel-Jacobi map. \square

Under the assumption of the corollary, the variety $F(X)$ is Fano, hence rationally connected. Along the lines of the questions asked in [40] for the group $\text{Griff}_1(Y)$, and the results proved in [38] for the group $H_2(Y, \mathbb{Z})/H_2(Y, \mathbb{Z})_{\text{alg}}$ (showing that it should be trivial for rationally connected varieties), it is tempting to believe that the group $\text{CH}_1(Y)_{\text{tors},AJ}$ is always trivial for rationally connected varieties. Thus we can see Corollary 3.14 as an evidence that the group $\text{CH}^3(X)_{\text{tors},AJ}$ should be trivial, hence also the unramified cohomology group $H_{nr}^4(Y, \mathbb{Q}/\mathbb{Z})$ for Y a cubic fivefold.

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