Nevanlinna classes for non radial weights in the unit disc. Applications

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Abstract

We introduce Nevanlinna classes associated to non radial weights in the unit disc in the complex plane and we get Blaschke type theorems relative to these classes by use of several complex variables methods. This gives alternative proofs and improve some results of Boritchev, Golinskii and Kupin useful, in particular, for the study of eigenvalues of non self adjoint Schrödinger operators.

Contents

1 Introduction.

We shall work with classes of holomorphic functions whose zeroes may appear as eigenvalues of Schrödinger operators with complex valued potential. So having information on these zeroes gives information on the operator.

Let $F := \{\eta_j,\ j = 1, ..., n\} \subset \mathbb{T}$; we associate to F the rational function with $q_j \in \mathbb{R},\ R(z) := \prod_{j=1}^{n} (z - \eta_j)^{q_j}$ and we set, as a clearly non radial weight, $\varphi(z) = |R(z)|^2$; we also need to set $\gamma(z) := \left| \sum_{j=1}^{n} q_j (z - \eta_j)^{-1} \right|$.

Definition 1.1 *We shall say that the holomorphic function f is in the generalised Nevanlinna class with weight φ, $\mathcal{N}_{\varphi,p}(\mathbb{D})$, if there is $0 < \delta < 1$ such that, for $p > 0$:*

$$\|f\|_{\mathcal{N}_{\varphi,p}} := \sup_{1-\delta \le s < 1} \int_{\mathbb{D}} (1 - |z|^2)^{p-1} \varphi(sz) \log^+ |f(sz)| < \infty.$$

For $p = 0$:

$$\|f\|_{\mathcal{N}_{\varphi,0}} := \sup_{1-\delta \le s < 1} \int_{\mathbb{T}} \varphi(se^{i\theta}) \log^+ \left| f(se^{i\theta}) \right| d\theta + \\ + \sup_{1-\delta \le s < 1} \int_{\mathbb{D}} \varphi(sz)\gamma(sz) \log^+ |f(sz)| < \infty.$$

In order to state the results we get, we set, for $p > 0$:
if $q_j > -p/2,\ \tilde{q}_j := q_j$; else we choose any $\tilde{q}_j > -p/2$; for $p = 0 :\ \tilde{q}_j := (q_j)_+$; then we set

$$\tilde{\varphi}(z) := \left| \prod_{j=1}^{n} (z - \eta_j)^{\tilde{q}_j} \right|.$$

We get the following Blaschke type theorem:

Theorem 1.2 *Suppose $f \in \mathcal{N}_{\varphi,p}(\mathbb{D})$ is such that $|f(0)| = 1$, then we have:*

$$\sum_{a \in Z(f)} (1 - |a|^2)^{p+1} \tilde{\varphi}(a) \le c(\tilde{\varphi}) \|f\|_{\mathcal{N}_{\varphi,p}},$$

the constant $c(\tilde{\varphi})$ depending only on $\tilde{\varphi}$.

We can apply these theorems to the case of L^∞ bounds.
With $R(z) := \prod_{j=1}^{n} (z - \eta_j)^{q_j},\ \eta_j \in \mathbb{T},\ q_j \in \mathbb{R}$, we set $\forall \epsilon > 0,\ R_\epsilon(z) := \prod_{j=1}^{n} (z - \eta_j)^{(q_j - 1 + \epsilon)_+}$. We define, $\forall j = 1, ..., n$, if $q_j - 1 > -p/2,\ \tilde{q}_j = q_j$ else we choose $\tilde{q}_j > 1 - p/2$, and we set $\tilde{R}_0(z) := \prod_{j=1}^{n} (z - \eta_j)^{\tilde{q}_j - 1}$.
We get as a corollary of our results:

Theorem 1.3 *Suppose the holomorphic function f in $\mathbb{D}$ verifies $|f(0)| = 1$ and $|f(z)| \le \exp \frac{D}{(1 - |z|^2)^p |R(z)|}$ with $R(z) := \prod_{j=1}^{n} (z - \eta_j)^{q_j},\ \eta_j \in \mathbb{T},\ q_j \in \mathbb{R}$, then we have:*

for $p = 0$,

$$\sum_{a \in Z(f)} (1 - |a|) |R_\epsilon(a)| \le D c(R).$$

For $p > 0$

$$\forall \epsilon > 0, \ \sum_{a \in Z(f)} (1-|a|)^{1+p+\epsilon} \left| \tilde{R}_0(a) \right| \leq Dc(\epsilon, R).$$

Now recall that Boritchev, Golinskii and Kupin [4] proved, in particular:

Theorem 1.4 *Let $f \in \mathcal{H}(\mathbb{D})$, $|f(0)| = 1$ and ζ_j, $\xi_k \in \mathbb{T}$, satisfy the growth condition :*

$$\log^+ |f(z)| \leq \frac{K}{(1-|z|)^p} \frac{\prod_{j=1}^n |z-\zeta_j|^{r_j}}{\prod_{k=1}^m |z-\xi_k|^{q_k}}, \quad z \in \mathbb{D}, \ p, \ q_k, \ r_j \geq 0.$$

Then for every $\epsilon > 0$, there is a positive number $C_3 = C_3(E, F, p, \{q_k\}, \{r_j\}, \epsilon)$ such that the following Blaschke condition holds:

$$\sum_{\zeta \in Z(f)} (1-|\zeta|)^{p+1+\epsilon} \frac{\prod_{k=1}^m |\zeta - \xi_k|^{(q_k-1+\epsilon)_+}}{\prod_{j=1}^n |\zeta - \zeta_j|^{\min(p, r_j)}} \leq C_3 \cdot K.$$

If $p = 0$, the factor $(1-|\zeta|)^{1+\epsilon}$ can be replaced by $(1-|\zeta|)$.

Comparing our result with the previous one, we get:
- for $p > 0$ and $q \leq -p/2$ their result is better ;
- for $p > 0$ and $q > -p/2$ our is better ;
- for $p = 0$ the two results are identical.

The reason is that they have a threshold of $-p$ and our is $-p/2$.

As we shall see our results are based only on:
- the green formula ;
- the "zeroes" formula (see the next section) ;

which are the tools we use in several complex variables when dealing with problems on zeroes of holomorphic functions.

The methods used in several complex variables already proved their usefulness in the one variable case. For instance:
- the corona theorem of Carleson [6] is easier to prove and to understand thanks to the proof of T. Wolff based on L. Hörmander [7] ;
- the characterization of interpolating sequences by Carleson for H^∞ and by Shapiro & Shields for H^p are also easier to prove by these methods (see [1], last section, where they allow me to get the bounded linear extension property for the case H^p ; the H^∞ case being done by Pehr Beurling [3]).

So it is not surprising that in the case of zero set, they can also be useful.

In this paper all the computations are *completely elementary:* derivations of usual functions and straightforward estimates.

This work was already presented in an international workshop in November 2016 in Toulouse, France and in May 2017 in Bedlewo, Poland, during the conference on : "Hilbert spaces of entire functions and their applications".

2 Basic notations and results.

Let f be an holomorphic function in the unit disk $\mathbb{D}$ of the complex plane, $\mathcal{C}^\infty(\bar{\mathbb{D}})$, and g a $\mathcal{C}^\infty$ smooth function in the closed unit disk $\bar{\mathbb{D}}$ such that $g = 0$ on $\mathbb{T}$.

The only measures we shall deal with are the Lebesgue measures: of the plane when we integrate in the unit disc $\mathbb{D}$ or of the torus when we integrate on $\mathbb{T} := \partial\mathbb{D}$. So usually I shall not write explicitly the measure.

The Green formula gives:

$$\int_{\mathbb{D}} (g\triangle \log|f| - \log|f| \triangle g) = \int_{\mathbb{T}} (g\partial_n \log|f| - \log|f| \partial_n g) \tag{2.1}$$

where ∂_n is the normal derivative. With the "zero" formula: $\Delta \log|f| = \sum_{a\in Z(f)} \delta_a$ we get

$$\sum_{a\in Z(f)} g(a) = \int_{\mathbb{D}} \log|f| \triangle g + \int_{\mathbb{T}} (g\partial_n \log|f| - \log|f| \partial_n g).$$

Because $g = 0$ on $\mathbb{T}$,

$$\sum_{a\in Z(f)} g(a) = \int_{\mathbb{D}} \log|f| \triangle g - \int_{\mathbb{T}} \log|f| \partial_n g. \tag{2.2}$$

So, in order to get estimates on $\sum_{a\in Z(f)} g(a)$, we have to compute $\partial_n g$ and Δg. In this work, g will always be of the form

$$g_s(z) = (1-|z|^2)^{1+p}\varphi(sz),$$

where $\varphi(z)$ will be smooth and positive in $\mathbb{D}$.

We get a Blaschke type theorem if we can control

$$\int_{\mathbb{D}} \log|f| \triangle g - \int_{\mathbb{T}} \log|f| \partial_n g \leq c\|f\|$$

because then we get

$$\sum_{a\in Z(f)} (1-|a|^2)^{p+1}\varphi(sa) \leq c\|f\|,$$

where $\|f\|$ is a "norm" linked to the function f. To get an idea of what happens here, suppose first that $p > 0$, and we set $f_s(z) := f(sz)$; so the equation (2.2) simplifies to

$$\sum_{a\in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log|f(sz)| \triangle g_s(z) = \int_{\mathbb{D}} \log^+|f(sz)| \triangle g_s(z) - \int_{\mathbb{D}} \log^-|f(sz)| \triangle g_s(z).$$

The strategy is quite obvious: we compute Δg_s and we estimate the two quantities

$$A_+(s) := \int_{\mathbb{D}} \log^+|f(sz)| \triangle g_s(z) \text{ and } A_-(s) := -\int_{\mathbb{D}} \log^-|f(sz)| \triangle g_s(z).$$

Because $\log^+|f(sz)|$ is directly related to the size of f, we just take the sum of the absolute value of the terms in Δg_s to estimate A_+.

For A_- we have to be more careful because we want to control terms containing $\log^-|f(sz)|$ by terms containing only $\log^+|f(sz)|$.

This work is presented the following way.

- In the next section we study the case of $\varphi(z) = |R(z)|^2$ with $R(z) = \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ \eta_j \in \mathbb{T},\ q_j \in \mathbb{R}$ and $p > 0$. This is the easiest case but the problematic is already here.
- In section 4 we study, with the same φ, the case $p = 0$.

• In section 5 we get the L^∞ bounds and we retrieve some results of Boritchev, Golinskii and Kupin [4].

• In section 6 we recall the case of a weight which is a power of the distance to a closed set E in $\mathbb{T}$.

• in section 7 we study the mixed case associated to a closed set E in $\mathbb{T}$ and a finite set F.

• Finally in the appendix we prove technical, but important, lemmas.

3 Case $p > 0$.

Let $F := \{\eta_1, ..., \eta_n\} \subset \mathbb{T}$ be a finite sequence of points on $\mathbb{T}$. We shall work with the rational function $R(z) = \prod_{j=1}^{n} (z-\eta_j)^{q_j}$, $q_j \in \mathbb{R}$ and we set $\varphi(z) := |R(z)|^2$. In order to have a smooth function in the disc we set $g_s(z) := (1-|z|^2)^{1+p} |R(sz)|^2$, with $0 \leq s < 1$, and:

$$\Delta g_s = 4\partial\bar{\partial} g_s = 4\partial\bar{\partial}[(1-|z|^2)^{1+p} |R(sz)|^2] = \Delta[(1-|z|^2)^{p+1}] |R(sz)|^2 + (1-|z|^2)^{p+1}\Delta[|R(sz)|^2] + \\ +8\Re[\partial((1-|z|^2)^{p+1})\bar{\partial}(|R(sz)|^2)].$$

Straightforward computations give the following lemma, which separates the positive terms, the negative terms and the terms with no fixed sign:

Lemma 3.1 *We have*

$$\Delta g_s(z) = \Delta_+ - \Delta_- + \Delta_\mp$$

with

$$\Delta_+ := 4(1-|z|^2)^{p-1}[p(p+1)|z|^2 + s^2(1-|z|^2)^2 \left|\sum_{j=1}^{n} q_j (sz-\eta_j)^{-1}\right|^2] |R(sz)|^2$$

$$\Delta_- := 4(p+1)(1-|z|^2)^p |R(sz)|^2$$

$$\Delta_\mp := 8s\Re[(-(r+1)(1-|z|^2)^r \bar{z}\ )(\sum_{j=1}^{n} q_j (s\bar{z}-\bar{\eta}_j)^{-1})] |R(sz)|^2 .$$

Because $p > 0 \Rightarrow \partial_n g_s = 0$ on $\mathbb{T}$, and formula (2.2), with $f_s(z) := f(sz)$, reduces to:

$$\sum_{a \in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log|f(sz)| \triangle g_s(z).$$

We have to estimate $\int_{\mathbb{D}} \log|f(sz)| \triangle g_s(z)$ and for it, we decompose:

$$\log|f(sz)| \triangle g_s(z) = \log^+|f(sz)| \triangle g_s(z) - \log^-|f(sz)| \triangle g_s(z).$$

We shall first group the terms containing $\log^+|f(sz)|$. We set

$A_+(s) := \Delta_+ \log^+|f(sz)| - \Delta_- \log^+|f(sz)| + \Delta_\mp \log^+|f(sz)|$.

And $T_+(s) := \int_{\mathbb{D}} A_+(s) dm(z)$. We set also $P_{\mathbb{D},+}(s) := \int_{\mathbb{D}} (1-|z|^2)^{p-1} |R(sz)|^2 \log^+|f(sz)|$.

Proposition 3.2 *We have, with* $|q| := \sum_{j=1}^{n} |q_j|$, $T_+(s) \leq 4[p(p+1)|z|^2 + 4|q|^2 + 2|q|] P_{\mathbb{D},+}(s)$.

Proof.
We have $A_+ \leq \Delta_+ \log^+ |f(sz)| + \Delta_\mp \log^+ |f(sz)|$ because $-\Delta_-$ is negative. We use that $(1-|z|^2) \leq 2\,|sz-\eta_j|$ then elementary estimates on the modulus of the reminding terms end the proof. ■

We shall now group the terms containing $\log^- |f(sz)|$. We set

$A_-(s,z) := -\Delta_+ \log^- |f(sz)| + \Delta_- \log^- |f(sz)| - \Delta_\mp \log^- |f(sz)|$

and $P_{\mathbb{D},-}(s) := \int_{\mathbb{D}} (1-|z|^2)^{p-1} |z|^2 |R(sz)|^2 \log^- |f(sz)|$ and $T_-(s) := \int_{\mathbb{D}} A_-(s,z)$.

Proposition 3.3 *Suppose that* $\forall j = 1,...,n,\ q_j \geq 0$, *then*

$T_-(s) \leq (p+1)[4c(1,u) + s\,|q|\,c(1/2,u)]P_{\mathbb{D},+}(s)$.

Proof.
Set

$A_2 := \Delta_- \log^- |f(sz)| = 4(p+1)(1-|z|^2)^p |R(sz)|^2 \log^- |f(sz)|$.

We apply the "substitution" lemma 9.1 from the appendix with $\delta = 1$, to get

$$\int_{\mathbb{D}} A_2 \leq 4(p+1)(1-u^2)\frac{1}{u^2} P_{\mathbb{D},-}(s) + 4(p+1)c(1,u)P_{\mathbb{D},+}(s).$$

Now set

$B_j := 8q_j(p+1)(1-|z|^2)^p \Re[\bar{z}(\bar{z}-\bar{\eta}_j)^{-1}] |R(sz)|^2 \log^- |f(sz)|$,

and

$A_3 := -\Delta_\mp \log^- |f(sz)| =$

$= -8\Re[(-(p+1)(1-|z|^2)^p \bar{z}\ )(\sum_{j=1}^n q_j(\bar{z}-\bar{\eta}_j)^{-1})] |R(sz)|^2 \log^- |f(sz)|$;

we get $A_3 = \sum_{j=1}^n B_j$. But

$$\Re[\bar{z}(s\bar{z}-\bar{\eta}_j)^{-1}] = \frac{1}{|sz-\eta_j|^2}\Re[\bar{z}(sz-\eta_j)],$$

hence by lemma 9.2 from the appendix, we have $\Re(\bar{z}(z-\eta)) \leq 0$ iff $z \in \mathbb{D} \cap D(\frac{\eta_j}{2},\ \frac{1}{2})$. So, with $q_j \geq 0$, the part in $\mathbb{D} \cap D(\frac{\eta_j}{2},\ \frac{1}{2})$ is negative and can be ignored. It remains

$B_j \leq (p+1)s(1-|z|^2)^p |R(sz)|^2 \mathbb{1}_{D(\frac{\eta_j}{2},\frac{1}{2})^c}(z)\Re[q_j\bar{z}(\bar{z}-\bar{\eta}_j)^{-1}] \log^- |f(sz)|$.

But for $z \in D(\frac{\eta_j}{2},\frac{1}{2})^c,\ (1-|z|^2) \leq 2\,|z-\eta_j|^2$ hence,

$\mathbb{1}_{D(\frac{\eta_j}{2},\frac{1}{2})^c}(z)\Re[\bar{z}(\bar{z}-\bar{\eta}_j)^{-1}] \leq 2(1-|z|^2)^{-1/2}\mathbb{1}_{D(\frac{\eta_j}{2},\frac{1}{2})^c}(z) \leq 2(1-|z|^2)^{-1/2}$.

So we get

$B_j \leq sq_j(p+1)(1-|z|^2)^{p-1/2} |R(sz)|^2 \log^- |f(sz)|$

and, provided that $q_j \geq 0$,

$$A_3 = \sum_{j=1}^n B_j \leq s\,|q|\,(p+1)(1-|z|^2)^{p-1/2} |R(sz)|^2 \log^- |f(sz)|. \tag{3.3}$$

We can again apply the "substitution" lemma 9.1 with $\delta = 1/2$, this time and we get

$$\int_{\mathbb{D}}(1-|z|^2)^{p-1/2}\left|R(sz)\right|^2\log^-|f(z)|\leq(1-u^2)^{1/2}\frac{1}{u^2}P_{\mathbb{D},-}(s)+c(1/2,u)P_{\mathbb{D},+}(s).$$

So finally

$$\int_{\mathbb{D}}A_3\leq s\left|q\right|(p+1)(1-u^2)^{1/2}\frac{1}{u^2}P_{\mathbb{D},-}(s)+s\left|q\right|(p+1)c(1/2,u)P_{\mathbb{D},+}(s).$$

Integrating $A_-(s,z)$ over $\mathbb{D}$ and adding, we get, with $A_1:=-\Delta_+\log^-|f(sz)|$,

$$\begin{aligned}T_-(s)\leq\int_{\mathbb{D}}(A_1+A_2+A_3)\leq&-4p(p+1)P_{\mathbb{D},-}(s)+4(p+1)(1-u^2)\frac{1}{u^2}P_{\mathbb{D},-}(s)+\\&+4(p+1)c(1,u)P_{\mathbb{D},+}(s)+s\left|q\right|(p+1)(1-u^2)^{1/2}\frac{1}{u^2}P_{\mathbb{D},-}(s)+\\&+s\left|q\right|(p+1)c(1/2,u)P_{\mathbb{D},+}(s).\end{aligned}$$

The key point here is that the "bad terms" in $\log^-|f(z)|$ can be controlled by the "good" one: $A_1:=-\Delta_+\log^-|f(sz)|$.

We can choose $0<u<1$ such that

$$-4p(p+1)+4(p+1)(1-u^2)\frac{1}{u^2}+s\left|q\right|(p+1)(1-u^2)^{1/2}\frac{1}{u^2}\leq 0$$

just taking, because $p>0$, $\sqrt{1-u^2}\leq\dfrac{4p}{4+s\left|q\right|}$. Hence we get, provided that $\forall j=1,...,n,\ q_j\geq 0$,

$$T_-(s)\leq(p+1)[4c(1,u)+s\left|q\right|c(1/2,u)]P_{\mathbb{D},+}(s).$$ ■

We can also get results for $q_j<0$ the following way. We cut the disc in disjoint sectors around the points $\eta_j:\ \mathbb{D}=\Gamma_0\cup\bigcup_{j=1}^n\Gamma_j$ with

$$\forall j=1,...,n,\ \Gamma_j:=\{z\in\mathbb{D}::\left|\frac{z}{|z|}-\eta_j\right|<\alpha\},\ \Gamma_0:=\mathbb{D}\backslash\bigcup_{j=1}^n\Gamma_j.$$

This is possible because the points η_j are in finite number so $\alpha>0$ exists.

Proposition 3.4 *Set* $|q|_\infty:=\max_{k=1,...,n}|q_k|$ *and suppose* $|q|_\infty<p/4$, *then there exist* $u<1,\ \gamma<1$ *such that:*

$$T_-(s)\leq 4(p+1)[c(1,u)+2\frac{|q|}{\alpha}c(1,u)+2\left|q\right|_\infty(1-\gamma)^{-1}c(1,\gamma)]P_{\mathbb{D},+}(s).$$

Proof.

We have

$$\begin{aligned}|-\Delta_\mp|=&\left|-8s\Re[(-(p+1)(1-|z|^2)^p\bar{z}\ )(\sum_{j=1}^n q_j(s\bar{z}-\bar{\eta}_j)^{-1})]\right|\left|R(sz)\right|^2\leq\\&\leq 8(p+1)(1-|z|^2)^p\sum_{j=1}^n\left|q_j\right|\left|sz-\eta_j\right|^{-1}\left|R(sz)\right|^2.\end{aligned}$$

Now we set

$$A_3':=\left|-\Delta_\mp\log^-|f(sz)|\right|\leq 8(p+1)(1-|z|^2)^p\sum_{j=1}^n\left|q_j\right|\left|sz-\eta_j\right|^{-1}\left|R(sz)\right|^2\log^-|f(sz)|$$

and

$$\forall k=0,1,...,n,\ f_k(z):=8(p+1)(1-|z|^2)^p\sum_{j=1,j\neq k}^n\left|q_j\right|\left|sz-\eta_j\right|^{-1}\left|R(sz)\right|^2\log^-|f(sz)|$$

and on Γ_k, including $k=0$, we get

$$\forall z\in\Gamma_k,\ f_k(z)\le 8(p+1)\frac{|q|}{\alpha}(1-|z|^2)^p\left|R(sz)\right|^2\log^-|f(sz)|\,.$$

Hence we have

$$\forall k=0,...,n,\ \forall z\in\Gamma_k,\ A_3'\le$$
$$\le 8(p+1)\frac{|q|}{\alpha}(1-|z|^2)^p+8(p+1)(1-|z|^2)^p\,|q_k|\,|sz-\eta_k|^{-1}\left|R(sz)\right|^2\log^-|f(sz)|\,.$$

Now we integrate in the disc and we get

$$\int_{\mathbb{D}}A_3'\le 8(p+1)\frac{|q|}{\alpha}\sum_{k=0}^{n}\int_{\Gamma_k}(1-|z|^2)^p\left|R(sz)\right|^2\log^-|f(sz)|+$$
$$+8(p+1)\sum_{k=0}^{n}|q_k|\int_{\Gamma_k}(1-|z|^2)^p\,|sz-\eta_k|^{-1}\left|R(sz)\right|^2\log^-|f(sz)|=:B_1+B_2.$$

But

$$\int_{\Gamma_k}(1-|z|^2)^p\left|R(sz)\right|^2\log^-|f(sz)|\le\int_{\mathbb{D}}(1-|z|^2)^p\left|R(sz)\right|^2\log^-|f(sz)|$$

and we can apply the "substitution" lemma 9.1, with $\delta=1$, to get

$$\int_{\mathbb{D}}(1-|z|^2)^p\left|R(sz)\right|^2\log^-|f(sz)|\le(1-u^2)\frac{1}{u^2}P_{\mathbb{D},-}(s)+c(1,u)P_{\mathbb{D},+}(s).$$

So the first term in $\int_{\mathbb{D}}A_3'$ is controlled by

$$B_1\le 8(p+1)\frac{|q|}{\alpha}(1-u^2)\frac{1}{u^2}P_{\mathbb{D},-}(s)+8(p+1)\frac{|q|}{\delta}c(1,u)P_{\mathbb{D},+}(s).$$

For the second one we first localise near the boundary:

$$B_2:=8(p+1)\sum_{k=0}^{n}|q_k|\int_{\Gamma_k}(1-|z|^2)^p\,|sz-\eta_k|^{-1}\left|R(sz)\right|^2\log^-|f(sz)|=$$
$$=8(p+1)\sum_{k=0}^{n}|q_k|\int_{D(0,\gamma)\cap\Gamma_k}(1-|z|^2)^p\,|sz-\eta_k|^{-1}\left|R(sz)\right|^2\log^-|f(sz)|+$$
$$+8(p+1)\sum_{k=0}^{n}|q_k|\int_{\Gamma_k\setminus D(0,\gamma)}(1-|z|^2)^p\,|sz-\eta_k|^{-1}\left|R(sz)\right|^2\log^-|f(sz)|=:$$
$$=:C_1+C_2.$$

We get

$$C_1\le 8(p+1)\,|q|_\infty\,(1-\gamma)^{-1}\int_{D(0,\gamma)}(1-|z|^2)^p\left|R(sz)\right|^2\log^-|f(sz)|.$$

The proof of the "substitution" lemma 9.1, gives with γ in place of u,

$$C_1\le 8(p+1)\,|q|_\infty\,(1-\gamma)^{-1}c(1,\gamma)P_{\mathbb{D},+}(s).$$

Now for C_2 we have

$$C_2:=8(p+1)\sum_{k=0}^{n}|q_k|\int_{\Gamma_k\setminus D(0,\gamma)}(1-|z|^2)^p\,|sz-\eta_k|^{-1}\left|R(sz)\right|^2\log^-|f(sz)|\le$$
$$\le 8(p+1)\sum_{k=0}^{n}|q_k|\,\frac{1}{\gamma^2}\int_{\Gamma_k\setminus D(0,\gamma)}(1-|z|^2)^p\,|z|^2\,|sz-\eta_k|^{-1}\left|R(sz)\right|^2\log^-|f(sz)|.$$

We use $(1-|z|^2)\le 2\,|sz-\eta_k|$ to get

$$C_2\le 16(p+1)\frac{1}{\gamma^2}\sum_{k=0}^{n}|q_k|\int_{\Gamma_k}(1-|z|^2)^{p-1}\,|z|^2\left|R(sz)\right|^2\log^-|f(sz)|\le 16(p+1)\,|q|_\infty\,\frac{1}{\gamma^2}P_{\mathbb{D},-}(s).$$

We have, with the notations of proposition 3.3, replacing A_3 by A_3',

$$T_-(s) \leq \int_{\mathbb{D}} (A_1 + A_2 + A_3') \leq$$
$$-4p(p+1)P_{\mathbb{D},-}(s) + 4(p+1)(1-u^2)\frac{1}{u^2}P_{\mathbb{D},-}(s) + 4(p+1)c_3(1,u)P_{\mathbb{D},+}(s) +$$
$$+8(p+1)\frac{|q|}{\alpha}(1-u^2)\frac{1}{u^2}P_{\mathbb{D},-}(s) + 8(p+1)\frac{|q|}{\alpha}c_3(1,u)P_{\mathbb{D},+}(s) +$$
$$+8(p+1)\,|q|_\infty\,(1-\gamma)^{-1}c(1,\gamma)P_{\mathbb{D},+}(s) + 16(p+1)\,|q|_\infty\,\frac{1}{\gamma^2}P_{\mathbb{D},-}(s).$$

Let us see the terms containing $\log^-|f(sz)|$, we set:

$$D(s,\gamma,u) := [-4p(p+1) + 8(p+1)\frac{|q|}{\alpha}(1-u^2)\frac{1}{u^2} + 16(p+1)\,|q|_\infty\,\frac{1}{\gamma^2}]P_{\mathbb{D},-}(s).$$

So

$$D(s,\gamma,u) = 16(-\frac{p}{4} + \frac{|q|_\infty}{\gamma^2} + \frac{|q|}{2\alpha}\frac{1-u^2}{u^2})(p+1)P_{\mathbb{D},-}(s).$$

Now suppose that $|q|_\infty < p/4$ and first choose $\gamma < 1$ big enough to have $-\frac{p}{4} + \frac{|q|_\infty}{\gamma^2} =: -\epsilon < 0$ which is clearly possible, then choose $u < 1$ such that $\frac{|q|}{2\alpha}\frac{1-u^2}{u^2} - \epsilon \leq 0$ which is also clearly possible because $\epsilon > 0$. So we get with these choices of u and γ,

$$T_-(s) \leq [4(p+1)c(1,u) + 8(p+1)\frac{|q|}{\alpha}c(1,u) + 8(p+1)\,|q|_\infty\,(1-\gamma)^{-1}c(1,\gamma)]P_{\mathbb{D},+}(s). \quad \blacksquare$$

As a corollary of these two propositions, we get

Corollary 3.5 *Suppose* $\forall j,\ q_j > -p/4$, *then there is a constant* $c(p,R)$ *such that:*
$T_-(s) \leq c(p,R)P_{\mathbb{D},+}(s)$.

Proof.
As above we can separate the points η_j where $-p/4 < q_j < 0$ from the points η_j with $q_j \geq 0$. Then we apply the relevant proof to each case. $\blacksquare$

We are lead to the following definition:

Definition 3.6 *Let* $R(z) = \prod_{j=1}^{n}(z-\eta_j)^{q_j},\ q_j \in \mathbb{R}$. *We say that an holomorphic function* f *is in the generalised Nevanlinna class* $\mathcal{N}_{|R|^2,p}(\mathbb{D})$ *for* $p > 0$, *if* $\exists \delta > 0,\ \delta < 1$ *such that*

$$\|f\|_{\mathcal{N}_{|R|^2,p}} := \sup_{1-\delta<s<1}\int_{\mathbb{D}}(1-|z|^2)^{p-1}\,|R(sz)|^2\log^+|f(sz)| < \infty.$$

And we get the Blaschke type condition:

Theorem 3.7 *Let* $R(z) = \prod_{j=1}^{n}(z-\eta_j)^{q_j},\ q_j \in \mathbb{R}$. *Suppose* $p > 0,\ j = 1,...,n,\ q_j > -p/4$ *and* $f \in \mathcal{N}_{|R|^2,p}(\mathbb{D})$ *with* $|f(0)| = 1$, *then*

$$\sum_{a\in Z(f)}(1-|a|^2)^{1+p}\,|R(a)|^2 \leq c(p,R)\|f\|_{\mathcal{N}_{|R|^2,p}}.$$

Proof.
We apply the formula (2.2), to get, with $g_s(z) = (1-|z|^2)^{1+p}\,|R(sz)|^2$,

$$\forall s<1,\quad \sum_{a\in Z(f_s)} (1-|a|^2)^{1+p}\,|R(sa)|^2 = \int_{\mathbb{D}} \log|f(sz)|\,\triangle g_s(z)$$

because with $p>0$, $\partial_n g_s = 0$ on $\mathbb{T}$.
Now we use Proposition 3.2 to get that

$$\int_{\mathbb{D}} \log^+|f(sz)|\,\triangle g_s(z) \le 4[p(p+1)\,|z|^2 + 4\,|q|^2 + 2\,|q|]P_{\mathbb{D},+}(s),$$

and corollary 3.5 to get

$$-\int_{\mathbb{D}} \log^-|f(sz)|\,\triangle g_s(z) \le c(p,R)P_{\mathbb{D},+}(s).$$

So adding we get

$$\forall s<1,\quad \sum_{a\in Z(f_s)} (1-|a|^2)^{1+p}\,|R(sa)|^2 \le c(p,R)P_{\mathbb{D},+}(s).$$

We are in position to apply lemma 9.5 from the appendix, with $\varphi(z) = |R(z)|^2$, to get

$$\sum_{a\in Z(f)} (1-|a|^2)^{1+p}\,|R(a)|^2 \le c(p,R)\sup_{1-\delta<s<1} P_{\mathbb{D},+}(s),$$

because $|R(z)|^2$ is positive. ■

Corollary 3.8 *Let* $R(z) = \prod_{j=1}^{n}(z-\eta_j)^{q_j}$, $q_j\in\mathbb{R}$. *Suppose* $p>0$ *and* $f\in\mathcal{N}_{|R|,p}(\mathbb{D})$ *with* $|f(0)|=1$, *and let* $\forall j=1,...,n$, *if* $q_j>-p/2$, $\tilde{q}_j = q_j$ *else choose* $\tilde{q}_j > -p/2$, *and set* $\tilde{R}(z) := \prod_{j=1}^{n}(z-\eta_j)^{\tilde{q}_j}$, *then*

$$\sum_{a\in Z(f)} (1-|a|^2)^{1+p}\left|\tilde{R}(a)\right| \le c(p,\tilde{q},R)\|f\|_{\mathcal{N}_{|R|,p}}.$$

Proof.
In order to apply theorem 3.7 to $\tilde{R}$ we have to show that $f\in\mathcal{N}_{|R|,p}(\mathbb{D}) \Rightarrow f\in\mathcal{N}_{|\tilde{R}|,p}(\mathbb{D})$.
But

$$\tilde{R}(sz) := \prod_{j=1}^{n}(sz-\eta_j)^{\tilde{q}_j} = \prod_{j=1}^{n}(sz-\eta_j)^{q_j}\times\prod_{j=1}^{n}(sz-\eta_j)^{\tilde{q}_j-q_j},$$

and the only point is for the j such that $q_j\le -p/2$. So set $r_j := \tilde{q}_j - q_j \ge 0$, we have $|sz-\eta_j|\le 2$ hence $|sz-\eta_j|^{r_j}\le 2^{r_j}$ so $\left|\tilde{R}(sz)\right| \le 2^{|r|}\,|R(sz)|$ with $|r| := \sum_{j=1}^{n} r_j$.

Putting it in $\|f\|_{\mathcal{N}_{|\tilde{R}|,p}}$ we get

$$\|f\|_{\mathcal{N}_{|\tilde{R}|,p}} := \sup_{1-\delta<s<1}\int_{\mathbb{D}} (1-|z|^2)^{p-1}\left|\tilde{R}(sz)\right|\log^+|f(sz)| \le$$
$$\le 2^{|r|}\sup_{1-\delta<s<1}\int_{\mathbb{D}} (1-|z|^2)^{p-1}\,|R(sz)|\log^+|f(sz)| = 2^{|r|}\|f\|_{\mathcal{N}_{|R|,p}}.$$

So we are done. ■

4 Case $p = 0$.

Now we set: $g_s(z) = (1-|z|^2)\,|R(sz)|^2$ and we have that

$\partial_n g_s(z) = -2\,|z|\,|R(sz)|^2 + (1-|z|^2)\partial_n(|R(sz)|^2)$

which is not 0 on $\mathbb{T}$, so we have to add the boundary term:

$$B(s) := -\int_{\mathbb{T}} \log|f(sz)|\,\partial_n g_s = 2\int_{\mathbb{T}} |R(sz)|^2 \log^+|f(sz)| - 2\int_{\mathbb{T}} |R(sz)|^2 \log^-|f(sz)| =: \\ =: B_+(s) - B_-(s).$$

We shall use as above, for $t_0 \in [0,1[$,

$$P_{\mathbb{T},-}(t_0) := \sup_{0\le s\le t_0} \int_{\mathbb{T}} \left|R(se^{i\theta})\right|^2 \log^-\left|f(se^{i\theta})\right|$$

and

$$P_{\mathbb{T},+}(t_0) := \sup_{0\le s\le t_0} \int_{\mathbb{T}} \left|R(se^{i\theta})\right|^2 \log^+\left|f(se^{i\theta})\right|.$$

Now we set

$$A_+(s) := 4s^2(1-|z|^2)[\left|\sum_{j=1}^n q_j(sz-\eta_j)^{-1}\right|^2]\,|R(sz)|^2\log^+|f(sz)| - 4\,|R(sz)|^2\log^+|f(sz)| + \\ +8s\Re[(-\bar z\,)(\sum_{j=1}^n q_j(s\bar z - \bar\eta_j)^{-1})]\,|R(sz)|^2\log^+|f(sz)| + B_+(s).$$

Set also $T_+(s) := \int_{\mathbb{D}} A_+(s)$, and with $\gamma(z) := \sum_{j=1}^n |q_j|\,|z-\eta_j|^{-1}$,

$$P_{\gamma,+}(s) := \int_{\mathbb{D}} \gamma(sz)\,|R(sz)|^2\log^+|f(sz)|.$$

Proposition 4.1 *We have*

$T_+(s) \le 8(|q|+1)P_{\gamma,+}(s) + B_+(s)$.

Proof.

Set

$$A_1 := 4s^2\int_{\mathbb{D}} (1-|z|^2)[\left|\sum_{j=1}^n q_j(sz-\eta_j)^{-1}\right|^2]\,|R(sz)|^2\log^+|f(sz)|.$$

Using $(1-|z|^2) \le 2\,|sz-\eta_j|$, we get $A_1 \le 8\,|q|\,P_{\gamma,+}(s)$.

Set $A_2 := -\int_{\mathbb{D}} 4\,|R(sz)|^2\log^+|f(sz)|$. Then $A_2 \le 0$ and it can be forgotten.

Finally set

$$A_3 := \int_{\mathbb{D}} 8s\Re[(-\bar z\,)(\sum_{j=1}^n q_j(s\bar z - \bar\eta_j)^{-1})]\,|R(sz)|^2\log^+|f(sz)|.$$

Again we get $A_3 \le 8sP_{\gamma,+}(s)$.

Summing the A_j we get

$T_+(s) \le 8(|q|+1)P_{\gamma,+}(s) + B_+(s)$. ■

We shall now group the terms containing $\log^-|f(sz)|$. We set

$-A_-(s,z) := -4\,|R(sz)|^2\log^-|f(sz)| + (1-|z|^2)\Delta(|R(sz)|^2)(sz)\log^-|f(sz)| +$

$$+8s\Re[(-\bar{z}\,)(\sum_{j=1}^{n} q_j(s\bar{z}-\bar{\eta}_j)^{-1})]\,|R(sz)|^2\log^-|f(sz)| + B_-(s).$$

and $T_-(s) := \int_{\mathbb{D}} A(s,z)$.

Proposition 4.2 *We have*

$$T_-(s) \le 2[2c_3'(1,u) + 2|q|\,c_3'(1/2,u)]P_{\mathbb{T},+}(t_0)+ \\ +2(1-u^2)^{1/2}[2(1-u^2)^{1/2}+2|q|]P_{\mathbb{T},-}(t_0) - B_-(s).$$

Proof.

We have $\Delta[(1-|z|^2)] = -4$ so

$$A_1(s) := -\int_{\mathbb{D}} \Delta((1-|z|^2))\,|R(sz)|^2\log^-|f(sz)| = 4\int_{\mathbb{D}} |R(sz)|^2\log^-|f(sz)|.$$

We can apply the second part of the substitution lemma 9.1 with $\delta = 1$, we get for any $u < 1$,

$$\forall s \le t_0,\ \int_{\mathbb{D}} |R(sz)|^2\log^-|f(sz)| \le c(1,u)P_{\mathbb{T},+}(t_0) + \frac{1}{2}(1-u^2)P_{\mathbb{T},-}(t_0).$$

So we get

$$A_1(s) \le 4c(1,u)P_{\mathbb{T},+}(t_0) + 2(1-u^2)P_{\mathbb{T},-}(t_0).$$

For

$$A_2 := -\textstyle\int_{\mathbb{D}} (1-|z|^2)\Delta(|R(sz)|^2)(sz)\log^-|f(sz)| = \\ = -4s^2\int_{\mathbb{D}} (1-|z|^2)\,|R'(sz)|^2\,(sz)\log^-|f(sz)| \le 0,$$

so we can forget it.

Now we arrive at the "bad term"

$$A_3 := -\int_{\mathbb{D}} 8\Re[\partial((1-|z|^2))\bar{\partial}(|R(sz)|^2)]\log^-|f(sz)|.$$

Copying the proof done in the case $p > 0$, we use again lemma 9.2 and we integrate inequality (3.3) with $p = 0$:

$$A_3 \le s|q|\int_{\mathbb{D}} (1-|z|^2)^{-1/2}\,|R(sz)|^2.$$

Now we are in position to apply the second part of lemma 9.1 with $\delta = 1/2$, so we get

$$\forall s \le t_0,\ \int_{\mathbb{D}} (1-|z|^2)^{-1/2}\,|R(sz)|^2\log^-|f(sz)| \le 2c(1/2,u)P_{\mathbb{T},+}(t_0) + (1-u^2)^{1/2}P_{\mathbb{T},-}(t_0),$$

and

$$A_3 \le 2s|q|\,c(1/2,u)P_{\mathbb{T},+}(t_0) + 2s|q|\,(1-u^2)^{1/2}P_{\mathbb{T},-}(t_0).$$

Summing all, we get

$$T_-(s) \le 4c(1,u)P_{\mathbb{T},+}(t_0) + 2(1-u^2)P_{\mathbb{T},-}(t_0) + 2s|q|\,c(1/2,u)P_{\mathbb{T},+}(t_0)+ \\ +2s|q|\,(1-u^2)^{1/2}P_{\mathbb{T},-}(t_0) - B_-(s).$$

Hence

$$T_-(s) \le 2[2c(1,u)+2|q|\,c(1/2,u)]P_{\mathbb{T},+}(t_0) + 2(1-u^2)^{1/2}[2(1-u^2)^{1/2}+2|q|]P_{\mathbb{T},-}(t_0) - B_-(s).$$

■

Definition 4.3 *Let* $R(z) = \prod_{j=1}^{n}(z-\eta_j)^{q_j},\ q_j \in \mathbb{R}$. *We say that an holomorphic function* f *is in the generalised Nevanlinna class* $\mathcal{N}_{|R|^2,0}(\mathbb{D})$ *if* $\exists\delta > 0,\ \delta < 1$ *such that*

$$\|f\|_{\mathcal{N}_{|R|^2,0}} := \sup_{1-\delta<s<1} \int_{\mathbb{T}} \left|R(se^{i\theta})\right|^2 \log^+ \left|f(se^{i\theta})\right| + \\ + \sup_{1-\delta<s<1} \int_{\mathbb{D}} \gamma(sz) \left|R(sz)\right|^2 \log^+ \left|f(sz)\right| < \infty,$$

with $\gamma(z) := \sum_{j=1}^{n} |q_j| \, |z-\eta_j|^{-1}$.

We get the Blaschke type condition:

Theorem 4.4 *Let* $R(z) = \prod_{j=1}^{n} (z-\eta_j)^{q_j}$, $q_j \in \mathbb{R}$. *Suppose* $\forall j = 1, ..., n,\ q_j \geq 0$ *and* $f \in \mathcal{N}_{|R|^2,0}(\mathbb{D})$ *with* $|f(0)| = 1$, *then there exists a constant* $c(R)$ *depending only on* R *such that*

$$\sum_{a \in Z(f)} (1-|a|^2) \left|R(a)\right|^2 \leq c(R) \|f\|_{\mathcal{N}_{|R|^2,0}}.$$

Proof.

Fix $t_0 \in [0,1[$, by lemma 9.3 in the appendix, we have that

$$h(s) := \int_{\mathbb{T}} \left|R(se^{i\theta})\right|^2 \log^- \left|f(se^{i\theta})\right|$$

is a continuous function of $s \in [0, t_0]$ hence its supremum is achieved at $s_0 = s(t_0) \in [0, t_0]$, i.e.

$$P_{\mathbb{T},-}(t_0) = B_-(s_0) := \int_{\mathbb{T}} \left|R(s_0 e^{i\theta})\right|^2 \log^- \left|f(s_0 e^{i\theta})\right|.$$

Let us consider, for any $t \in [0, t_0]$,

$$\Sigma(t, s_0) := \sum_{a \in Z(f_t)} g_t(a) + \sum_{a \in Z(f_{s_0})} g_{s_0}(a).$$

We have, by (2.2),

$$\Sigma(t, s_0) \leq T_+(t) + T_+(s_0) + T_-(t) + T_-(s_0).$$

By use of proposition 4.1 we get

$$T_+(s) \leq 8(|q|+1) \int_{\mathbb{D}} \gamma(z) \left|R(sz)\right|^2 \log^+ \left|f(sz)\right| + B_+(s),$$

and by use of proposition 4.2 we get for $s \in [0, t_0]$,

$$T_-(s) \leq 2[2c(1,u) + 2|q| \, c(1/2,u)] P_{\mathbb{T},+}(t_0) + 2(1-u^2)^{1/2}[2(1-u^2)^{1/2} + 2|q|] P_{\mathbb{T},-}(t_0) - B_-(s).$$

Hence

$$\Sigma(t, s_0) \leq T_+(t) + T_+(s_0) + T_-(t) + T_-(s_0) \leq 8(|q|+1) \int_{\mathbb{D}} \gamma(z) \left|R(tz)\right|^2 \log^+ \left|f(tz)\right| + B_+(t) + \\ + 8(|q|+1) \int_{\mathbb{D}} \gamma(z) \left|R(s_0 z)\right|^2 \log^+ \left|f(s_0 z)\right| + B_+(s_0) + \\ + 4[2c(1,u) + 2s|q| \, c(1/2,u)] P_{\mathbb{T},+}(t_0) + \\ + 4(1-u^2)^{1/2}[2(1-u^2)^{1/2} + 2|q|] P_{\mathbb{T},-}(t_0) - B_-(t) - B_-(s_0).$$

We forget the negative term $-B_-(t) := -\int_{\mathbb{T}} 2 \left|R(tz)\right|^2 \log^- |f| \leq 0$ and we recall that

$$P_{\mathbb{T},-}(t_0) = B_-(s_0) := \int_{\mathbb{T}} \left|R(s_0 z)\right|^2 \log^- |f|.$$

Now choose a suitable $u < 1$ such that

$$4(1-u^2)^{1/2}[2(1-u^2)^{1/2} + 2|q|] \ - 1 \leq 0$$

i.e. $(1-u^2)^{1/2} \leq \dfrac{1}{8(|q|+1)}$, which is independent of t_0. It remains

$$\Sigma(t,s_0) \leq 8(|q|+1)\int_{\mathbb{D}} \gamma(z)\,|R(tz)|^2 \log^+|f(tz)| + B_+(t)+$$
$$+8(|q|+1)\int_{\mathbb{D}} \gamma(z)\,|R(s_0z)|^2 \log^+|f(s_0z)| + B_+(s_0)+$$
$$+4[2c(1,u)+2s\,|q|\,c(1/2,u)]P_{\mathbb{T},+}(t_0).$$

Then, because $t \in [0,t_0],\ s_0 \in [0,t_0]$, we get $B_+(t) \leq P_{\mathbb{T},+}(t_0)\ ;\ B_+(s_0) \leq P_{\mathbb{T},+}(t_0)$; hence

$$\Sigma(t,s_0) \leq 16(|q|+1)P_{\gamma,+}(t_0) + 2P_{\mathbb{T},+}(t_0) + 4[2c(1,u)+2\,|q|\,c(1/2,u)]P_{\mathbb{T},+}(t_0).$$

So finally

$$\Sigma(t,s_0) \leq 16(|q|+1)P_{\gamma,+}(t_0) + 2[1+2(2c(1,u)+2\,|q|\,c(1/2,u))]P_{\mathbb{T},+}(t_0).$$

We get, taking $t = t_0 < 1$ and the suitable u, *independent of* t_0,

$$\sum_{a\in Z(f_{t_0})} g_{t_0}(a) \leq \Sigma(t,s_0) \leq 16(|q|+1)P_{\gamma,+}(t_0) + 2[1+2(2c(1,u)+2\,|q|\,c(1/2,u))]P_{\mathbb{T},+}(t_0).$$

Setting

$$c(R) := \max(16(|q|+1),\ 2[1+2(2c(1,u)+2\,|q|\,c(1/2,u))]),$$

which is still independent of t_0, we get

$$\forall t_0 \in [0,1[,\quad \sum_{a\in Z(f_{t_0})} (1-|a|^2)\,|R(t_0a)|^2 \leq c(R)\|f\|_{\mathcal{N}_{|R|^2,0}}$$

hence using the second part of lemma 9.5 from the appendix, with $\varphi(z) = \gamma(z)\,|R(z)|^2$, $\psi(z) = |R(z)|^2$, we get

$$\sum_{a\in Z(f)} (1-|a|^2)\,|R(a)|^2 \leq c(R)\|f\|_{\mathcal{N}_{|R|^2,0}}. \qquad \blacksquare$$

Corollary 4.5 *Let* $R(z) = \prod_{j=1}^{n}(z-\eta_j)^{q_j}$, $q_j \in \mathbb{R}$. *Suppose* $f \in \mathcal{N}_{|R|,0}(\mathbb{D})$ *with* $|f(0)| = 1$, *and set* $\tilde{R}(z) := \prod_{j=1}^{n}(z-\eta_j)^{(q_j)_+}$, *then there exists a constant* $c(R)$ *depending only on* R *such that*

$$\sum_{a\in Z(f)} (1-|a|^2)\left|\tilde{R}(a)\right| \leq c(R)\|f\|_{\mathcal{N}_{|R|,0}}.$$

Proof.

We have to prove that $f \in \mathcal{N}_{|R|,0} \Rightarrow f \in \mathcal{N}_{\left|\tilde{R}\right|,0}$. But if $q < 0$ then:

$$|z-\eta| \leq 2 \Rightarrow |z-\eta|^q \geq 2^q \Rightarrow 1 = |z-\eta|^{(q)_+} \leq 2^{-q}\,|z-\eta|^q.$$

Putting it in the definition of $\|f\|_{\mathcal{N}_{|R|,0}}$ we are done. $\blacksquare$

5 Application : L^∞ bounds.

We shall retrieve some of the results of Boritchev, Golinskii and Kupin [4], [5].

Suppose the function f verifies $|f(z)| \leq \exp\dfrac{D}{|R(z)|}$ with $R(z) := \prod_{j=1}^{n}(z-\eta_j)^{q_j}$.

We deduce that $|R(z)|\log|f(z)|$ is in $L^1(\mathbb{T})$ with a better exponent of almost 1 over the rational function R. Precisely set

$$\forall \epsilon \geq 0,\ R_\epsilon(z) := \prod_{j=1}^{n} (z-\eta_j)^{q_j-1+\epsilon},$$

we have:

Lemma 5.1 *If the function f verifies $|f(z)| \leq \exp \frac{D}{|R(z)|}$ with $R(z) := \prod_{j=1}^{n} (z-\eta_j)^{q_j}$, we have*

$$\forall \epsilon > 0,\ \int_{\mathbb{T}} \left|R_\epsilon(e^{i\theta})\right| \log^+ \left|f(e^{i\theta})\right| \leq DC(\delta,\epsilon).$$

Proof.
The hypothesis gives $|R(z)| \log^+ |f(z)| \leq D$ and

$$\frac{R_\epsilon(z)}{R(z)} = \prod_{j=1}^{n} \frac{(z-\eta_j)^{q_j-1+\epsilon}}{(z-\eta_j)^{q_j}} = \prod_{j=1}^{n} (z-\eta_j)^{-1+\epsilon},$$

so

$$|R_\epsilon(z)| \log^+ |f(z)| \leq \frac{|R_\epsilon(z)|}{|R(z)|} D \leq D \prod_{j=1}^{n} (z-\eta_j)^{-1+\epsilon}.$$

Because the points $\{\eta_k\}$ are separated on the torus $\mathbb{T}$ by $\alpha > 0$ say and $|z-\eta_j|^{-1+\epsilon}$ is integrable for the Lebesgue measure on the torus $\mathbb{T}$ because $\epsilon > 0$, we get:

$$\int_{\mathbb{T}} \frac{\left|R_\epsilon(e^{i\theta})\right|}{|R(e^{i\theta})|} \left|R(e^{i\theta})\right| \log^+ \left|f(e^{i\theta})\right| \leq D \int_{\mathbb{T}} \prod_{j=1}^{n} \left|e^{i\theta}-\eta_j\right|^{-1+\epsilon} \leq DC(\alpha,\epsilon). \qquad \blacksquare$$

Theorem 5.2 *Suppose the holomorphic function f verifies $|f(0)| = 1$ and $|f(z)| \leq \exp \frac{D}{(1-|z|^2)^p |R(z)|}$ with $R(z) := \prod_{j=1}^{n} (z-\eta_j)^{q_j}$, $q_j \in \mathbb{R}$. For $p = 0$, we set $\tilde{R}_\epsilon(z) := \prod_{j=1}^{n} (z-\eta_j)^{(q_j-1+\epsilon)_+}$ and we get:*

$$\sum_{a \in Z(f)} (1-|a|) \left|\tilde{R}_\epsilon(a)\right| \leq Dc(\epsilon,p,R).$$

For $p > 0$, $\forall j = 1,...,n$, if $q_j - 1 > -p/2$ set $\tilde{q}_j = q_j$ else choose $\tilde{q}_j > 1 - p/2$, and set $\tilde{R}_0(z) := \prod_{j=1}^{n} (z-\eta_j)^{\tilde{q}_j - 1}$, then:

$$\forall \epsilon > 0,\ \sum_{a \in Z(f)} (1-|a|)^{1+p+\epsilon} \left|\tilde{R}_0(a)\right| \leq Dc(\epsilon,R).$$

Proof.
• **Case** $p = 0$.

We shall apply the corollary 4.5 with R_ϵ instead of R.
To apply corollary 4.5 we have to show that

$$\sup_{s<1} \int_{\mathbb{D}} |R_\epsilon(sz)|\, s \left|\sum_{j=1}^{n} q_j (z-\eta_j)^{-1}\right| \log^+ |f(sz)| < \infty$$

and

$$\sup_{s<1}\int_{\mathbb{T}}\left|R_\epsilon(se^{i\theta})\right|\log^+\left|f(se^{i\theta})\right|<\infty.$$

The hypothesis gives $|R(z)|\log^+|f(z)|\le D$ so we get

$$|R_\epsilon(sz)|\log^+|f(sz)|\le D\prod_{j=1}^n|1-s\bar\eta_j z|^{-1+\epsilon},$$

because, as already seen, $\dfrac{R_\epsilon(sz)}{R(sz)}=\prod_{j=1}^n(1-s\bar\eta_j z)^{-1+\epsilon}$,

so we get:

$$|R_\epsilon(sz)|\sum_{k=1}^n|1-s\bar\eta_k z|^{-1+\epsilon}\log^+|f(z)|\le 2D\,|q|\sum_{k=1}^n\prod_{j\neq k}\left(|1-s\bar\eta_j z|^{-1+\epsilon}\right)|1-s\bar\eta_k z|^{-2+\epsilon}.$$

Because the points $\{\eta_k\}$ are separated by an $\alpha>0$ and $|1-\bar\eta_j z|^{-2+\epsilon}$ is integrable for the Lebesgue measure on the disc $\mathbb{D}$ because $\epsilon>0$, we get:

$$\sup_{s<1}\int_{\mathbb{D}}|R_\epsilon(sz)|\,s\sum_{j=1}^n|q_j|\,|1-s\bar\eta_j z|^{-1}\log^+|f_s|\,dm(z)\le 2D\,|q|\,c(\alpha,\epsilon).$$

Now to apply corollary 4.5 we need also to compute

$$\int_{\mathbb{T}}\left|R_\epsilon(se^{i\theta})\right|\log^+\left|f(se^{i\theta})\right|\le\int_{\mathbb{T}}\frac{\left|R_\epsilon(se^{i\theta})\right|}{|R(e^{i\theta})|}\left|R(se^{i\theta})\right|\log^+\left|f(se^{i\theta})\right|\le$$
$$\le D\int_{\mathbb{T}}\left|\prod_{j=1}^n(1-s\bar\eta_j e^{i\theta})^{-1+\epsilon}\right|.$$

Again the points $\{\eta_k\}$ are separated by α and $\left|1-\bar\eta_j e^{i\theta}\right|^{-1+\epsilon}$ is integrable for the Lebesgue measure on the torus $\mathbb{T}$ because $\epsilon>0$. So we get:

$$\sup_{s<1}\int_{\mathbb{T}}\left|R_\epsilon(se^{i\theta})\right|\log^+\left|f(se^{i\theta})\right|\le c(\alpha,\epsilon),$$

which ends the proof of the case $p=0$.

- **Case** $p>0$.

We shall show that $\forall\epsilon>0,\ f\in\mathcal{N}_{R_0,p+\epsilon}(\mathbb{D})$. For this we have to prove:

$$\|f\|_{R_0,p+\epsilon}:=\sup_{s<1}\Big(\int_{\mathbb{D}}(1-|z|^2)^{p+\epsilon-1}\,|R_0(sz)|\log^+|f(sz)|\Big)<\infty.$$

Because $|f(sz)|\le\exp\dfrac{D}{(1-|sz|^2)^p\,|R(sz)|}$ we get

$$I(s,\epsilon):=\int_{\mathbb{D}}(1-|z|^2)^{p+\epsilon-1}\,|R_0(sz)|\log^+|f(sz)|)\le\int_{\mathbb{D}}(1-|z|^2)^{p+\epsilon-1}\frac{|R_0(sz)|}{|R(sz)|}\,|R(sz)|\log^+|f|\le$$
$$\le\int_{\mathbb{D}}(1-|z|^2)^{p+\epsilon-1}\frac{|R_0(sz)|}{|R(sz)|}\frac{D}{(1-|sz|^2)^p}.$$

Now, as already seen, $\dfrac{R_0(sz)}{R(sz)}=\prod_{j=1}^n(1-s\bar\eta_j z)^{-1}$, so we get, because $\forall s\le 1,\ (1-|z|^2)\le(1-|sz|^2)$,

$$I(s,\epsilon)\le D\int_{\mathbb{D}}(1-|z|^2)^{\epsilon-1}\prod_{j=1}^n(1-s\bar\eta_j z)^{-1}.$$

Now we apply lemma 9.4 with $p=\epsilon$ to get

$$\sup_{s<1}\int_{\mathbb{D}}(1-|sz|^2)^{-1+\epsilon}\prod_{j=1}^{n}(1-s\bar{\eta}_j z)^{-1}\leq c(\epsilon,\alpha).$$

Hence

$$\|f\|_{R_0,p+\epsilon}\leq Dc(\epsilon,\delta)\Rightarrow f\in\mathcal{N}_{R_0,p+\epsilon}(\mathbb{D}).$$

But then corollary 3.8 gives that

$$\sum_{a\in Z(f)}(1-|a|)^{1+p+\epsilon}\left|\tilde{R}_0(a)\right|\leq C\|f\|_{R_0,p+\epsilon}\leq CDc(\epsilon,\alpha),$$

which ends the proof of the theorem. ■

6 Case of a closed set in $\mathbb{T}$.

Let $E=\bar{E}\subset\mathbb{T}$ be a closed set in $\mathbb{T}$; in [2], we associate to it a $\mathcal{C}^\infty(\mathbb{D})$ function $h(z)$ (called $\varphi(z)$ in [2]) such that $h(z)\simeq d(z,E)$ and setting $g_s(z):=(1-|z|^2)^{p+1}h(sz)^q\in\mathcal{C}^\infty(\bar{\mathbb{D}})$, with $0<s<1$ and $q>0$, we proved there:

Theorem 6.1 *We have:*

$$\int_{\mathbb{D}}\triangle g_s(z)\log|f(sz)|\lesssim\int_{\mathbb{D}}(1-|z|^2)^{p-1}h(sz)^q\log^+|fsz|.$$

This lead to the definition:

Definition 6.2 *Let $E=\bar{E}\subset\mathbb{T}$. We say that an holomorphic function f is in the generalised Nevanlinna class $\mathcal{N}_{h^q,p}(\mathbb{D})$ for $p>0$ if $\exists\delta>0,\ \delta<1$ such that*

$$\|f\|_{\mathcal{N}_{h^q,p}}:=\sup_{1-\delta<s<1}\int_{\mathbb{D}}(1-|z|)^{p-1}h(sz)^q\log^+|f(sz)|<\infty.$$

And we proved the Blaschke type condition:

Theorem 6.3 *Let $E=\bar{E}\subset\mathbb{T}$. Suppose $q>0$ and $f\in\mathcal{N}_{h^q,p}(\mathbb{D})$ with $|f(0)|=1$, then*

$$\sum_{a\in Z(f)}(1-|a|^2)^{1+p}h(a)^q\leq c\|f\|_{\mathcal{N}_{h^q,p}}.$$

Corollary 6.4 *Let $E=\bar{E}\subset\mathbb{T}$. Suppose $q\in\mathbb{R}$ and $f\in\mathcal{N}_{d(\cdot,E)^q,p}(\mathbb{D})$ with $|f(0)|=1$, then*

$$\sum_{a\in Z(f)}(1-|a|^2)^{1+p}d(a,E)^q\leq c\|f\|_{\mathcal{N}_{d(\cdot,E)^q,p}}.$$

7 The mixed case.

We shall combine the case of the rational function $R(z)=\prod_{j=1}^{n}(z-\eta_j)^{q_j},\ q_j\in\mathbb{R}$ with the case of the closed set $E\subset\mathbb{T}$ treated in [2]. For this we shall consider $\varphi(z):=|R(sz)|^2h(sz)^q$ and $g_s(z):=(1-|z|^2)^{1+p}\varphi(sz)$.

We make the hypothesis that $\forall j=1,...,n,\ \eta_j\notin E$. We set $2\mu:=\min_{j=1,...,n}d(\eta_j,E)$ then we have that $\mu>0$.

Because

$$\Delta g_s(z) = \Delta[(1-|z|^2)^{p+1}]\varphi(sz) + (1-|z|^2)^{p+1}\Delta[\varphi(sz)] + 8\Re[\partial((1-|z|^2)^{p+1})\bar{\partial}(\varphi(sz))],$$

and

$$\Delta[\varphi(sz)] = s^2 h(sz)^q \Delta[|R(sz)|^2] h(sz)^q + s^2 |R(sz)|^2 \Delta[h(sz)^q] + 8s^2\Re[\bar{\partial}|R(sz)|^2 \times \partial(h(sz)^q)],$$

we are lead to set:

$$A_1 := \frac{1}{2}|R(sz)|^2 \Delta[(1-|z|^2)^{p+1}]h(sz)^q,\ A_2 := \frac{1}{2}h(sz)^q\Delta[(1-|z|^2)^{p+1}]\,|R(sz)|^2$$

so

$$\Delta[(1-|z|^2)^{p+1}]\varphi(sz) = A_1 + A_2.$$

And

$$A_3 := (1-|z|^2)^{p+1}s^2 h(sz)^q \Delta[|R(sz)|^2]h(sz)^q$$
$$A_4 := s^2(1-|z|^2)^{p+1}|R(sz)|^2 \Delta[h(sz)^q]$$
$$A_5 := 8s^2(1-|z|^2)^{p+1}\Re[\bar{\partial}|R(sz)|^2 \times \partial(h(sz)^q)]$$
$$A_6 := 8h(sz)^q\Re[\partial((1-|z|^2)^{p+1})\bar{\partial}(|R(sz)|^2)]$$
$$A_7 := 8|R(sz)|^2\Re[\partial((1-|z|^2)^{p+1})\bar{\partial}(h(sz)^q)]\ ;$$

and we get

$$\Delta g_s(z) = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7.$$

It remains to see that grouping these terms in the right way, this was already treated by the F case or by the E one.

Theorem 7.1 *We have, for $p > 0$:*

$$\int_{\mathbb{D}} \triangle g_s(z)\log|f(sz)| \lesssim \int_{\mathbb{D}} (1-|z|^2)^{p-1}|R(sz)|^2 h(sz)^q \log^+|fsz|.$$

Proof.

We first group the terms

$$B_1 := A_1\log|f(sz)| + A_4\log|f(sz)| + A_7\log|f(sz)|,$$

these terms contain no derivatives of $|R(sz)|^2$ and so verify theorem 6.1 with h^q replaced by $|R(sz)|^2 h(sz)^q$ i.e.

$$\int_{\mathbb{D}} B_1(s,z) \lesssim \int_{\mathbb{D}} (1-|z|^2)^{p-1}\ |R(sz)|^2 h(sz)^q \log^+|fsz|.$$

Now we group the terms

$$B_2 := A_2\log|f(sz)| + A_3\log|f(sz)| + A_6\log|f(sz)|,$$

these terms contain no derivatives of $h(sz)$ and so verify also

$$\int_{\mathbb{D}} B_2(s,z) \lesssim \int_{\mathbb{D}} (1-|z|^2)^{p-1}\ |R(sz)|^2 h(sz)^q \log^+|fsz|.$$

It remains $A_5\log|f(sz)|$ but again the homogeneity is the right one and we get

$$\int_{\mathbb{D}} A_5(s,z)\log^+|fsz| \lesssim \int_{\mathbb{D}} (1-|z|^2)^{p-1}\ |R(sz)|^2 h(sz)^q \log^+|fsz|.$$

So it remains $A_5\log^-|f(sz)|$, and, in order to separate the points, we consider:

$$\forall j = 1,...,n,\ G_j := \{z\in\bar{\mathbb{D}} :: \left|\frac{z}{|z|} - \eta_j\right| < \delta\}\ ;\ G := \bigcup_{j=1}^{n} G_j.$$

Then we need:

Lemma 7.2 *There are two constants $a(\mu)$, $b(\mu)$, just depending on μ, such that:*

$$\forall z\in G,\ \partial h(sz) \simeq a(\mu).$$

And

$$\forall z \notin G, \bar{\partial}\ |R(sz)|^2 \simeq b(\mu).$$

Proof.

Recall that we have $\mathbb{T}\backslash E = \bigcup_{j\in\mathbb{N}} (\alpha_j, \beta_j)$ where the $F_j := (\alpha_j, \beta_j)$ are the contiguous intervals to E and $\Gamma_j := \{z = re^{i\psi} \in \mathbb{D} :: \psi \in (\alpha_j, \beta_j)\}$. We set:

$$\forall z \in \Gamma_j,\ h(z) := \eta_j(z)\psi_j(z)^q + (1-|z|^2)^{2q},\ \forall z \in \Gamma_E,\ h_E(z) := (1-|z|^2)^{2q}$$

with $\chi \in \mathcal{C}^\infty(\mathbb{R}),\ t \leq 2 \Rightarrow \chi(t) = 0,\ t \geq 3 \Rightarrow \chi(t) = 1$ and

$$\forall z \in \Gamma_j,\ \psi_j(z) := \frac{|z-\alpha_j|^2\,|z-\beta_j|^2}{\delta_j^2},\ \eta_j(z) := \chi(\frac{|z-\alpha_j|^2}{(1-|z|^2)^2})\chi(\frac{|z-\beta_j|^2}{(1-|z|^2)^2}).$$

An easy computation using the first lemma in the appendix of [2] gives $\forall z \in G,\ \partial h(sz) \simeq a(\mu)$ because z is far from E.

And with $R(z) = \prod_{j=1}^{n} (z-\eta_j)^{q_j}$, again an easy computation gives $\forall z \notin G, \bar{\partial}\ |R(sz)|^2 \simeq b(\mu)$ because z is far from $\bigcup_{j=1}^{n} \{\eta_j\}$. ■

We can treat the $A_5 \log^- |f(sz)|$ term easily now ; recall

$$A_5 \log^- |f(sz)| := 8s^2(1-|z|^2)^{p+1}\Re[\bar{\partial}\,|R(sz)|^2 \times \partial(h(sz)^q)]\log^- |f(sz)|\ ;$$

cut the disc $\mathbb{D} = G \cup (\mathbb{D}\backslash G)$, so

$$\int_{\mathbb{D}} A_5 \log^- |f(sz)| = \int_G A_5 \log^- |f(sz)| + \int_{\mathbb{D}\backslash G} A_5 \log^- |f(sz)|.$$

On G we have, by lemma 7.2, $\partial h(sz) \simeq a(\mu)$ and we win a $(1-|z|^2)$ so we can apply the substitution lemma 9.1 to get

$$\int_G A_5 \log^- |f(sz)| \leq c_5 P_{\mathbb{D},+}(s).$$

On $\mathbb{D}\backslash G$ we have, by lemma 7.2, $\bar{\partial}\ |R(sz)|^2 \simeq b(\mu)$ and we win again a $(1-|z|^2)$ so we can apply the substitution lemma 9.1 to get

$$\int_{\mathbb{D}\backslash G} A_5 \log^- |f(sz)| \leq c_5' P_{\mathbb{D},+}(s),$$

so finally we get

$$\int_{\mathbb{D}} A_-(s,z) \leq c_6 P_{\mathbb{D},+}(s),$$

which ends the proof of the theorem. ■

So we are lead to

Definition 7.3 *Let $E = \bar{E} \subset \mathbb{T}$ and $R(z) = \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ q_j \in \mathbb{R}$ with $\forall j = 1, ..., n,\ \eta_j \notin E$. Set $\varphi(z) = |R(z)|^2\, h(z)^q$. We say that an holomorphic function f is in the generalised Nevanlinna class $\mathcal{N}_{\varphi,p}(\mathbb{D})$ if $\exists \delta > 0,\ \delta < 1$ such that*

$$\|f\|_{\mathcal{N}_{\varphi,p}} := \sup_{1-\delta<s<1} \int_{\mathbb{D}} (1-|z|)^{p-1} \varphi(sz) \log^+ |f(sz)|.$$

And we have the Blaschke type condition, still using lemma 9.5 from the appendix, with $\varphi(z) = |R(z)|^2\, h(z)^q$:

Theorem 7.4 *Let* $E = \bar{E} \subset \mathbb{T}$ *and* $R(z) = \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ q_j \in \mathbb{R},\ q_j > p/4$, *with* $\forall j = 1, ..., n,\ \eta_j \notin E$. *Suppose* $q > 0$ *and* $f \in \mathcal{N}_{\varphi,p}(\mathbb{D})$ *with* $|f(0)| = 1$, *then*

$$\sum_{a \in Z(f)} (1-|a|^2)^{1+p} \varphi(a) \, |R(a)|^2 \leq c \|f\|_{\mathcal{N}_{\varphi,p}}.$$

As for the case of the rational function R only, we get the

Corollary 7.5 *Let* $E = \bar{E} \subset \mathbb{T}$ *and* $R(z) = \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ q_j \in \mathbb{R}$, *with* $\forall j = 1, ..., n,\ \eta_j \notin E$. *Let* $\forall j = 1, ..., n$, *if* $q_j > -p/2,\ \tilde{q}_j = q_j$ *else choose* $\tilde{q}_j > -p/2$ *and set* $\tilde{R}(z) := \prod_{j=1}^{n} (z-\eta_j)^{\tilde{q}_j}$, *and* $\varphi(z) = |R(z)| \, h(z)^q,\ \tilde{\varphi}(z) = \left|\tilde{R}(z)\right| h(z)^q$. *Suppose* $q > 0$ *and* $f \in \mathcal{N}_{\varphi,p}(\mathbb{D})$ *with* $|f(0)| = 1$, *then*

$$\sum_{a \in Z(f)} (1-|a|^2)^{1+p} \tilde{\varphi}(a) \leq c(\varphi) \|f\|_{\mathcal{N}_{\varphi,p}}.$$

Corollary 7.6 *Let* $E = \bar{E} \subset \mathbb{T}$ *and* $R(z) = \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ q_j \in \mathbb{R}$, *with* $\forall j = 1, ..., n,\ \eta_j \notin E$. *Let* $\forall j = 1, ..., n$, *if* $q_j > -p/2,\ \tilde{q}_j = q_j$ *else choose* $\tilde{q}_j > -p/2$ *and set* $\tilde{R}(z) := \prod_{j=1}^{n} (z-\eta_j)^{\tilde{q}_j}$, *and* $\varphi(z) = |R(z)| \, d(z,E)^q,\ \tilde{\varphi}(z) = \left|\tilde{R}(z)\right| d(z,E)^{(q)_+}$. *Suppose* $f \in \mathcal{N}_{\varphi,p}(\mathbb{D})$ *with* $|f(0)| = 1$, *then*

$$\sum_{a \in Z(f)} (1-|a|^2)^{1+p} \tilde{\varphi}(a) \leq c(\varphi) \|f\|_{\mathcal{N}_{\varphi,p}}.$$

Proof.
Still using that $h(z) \simeq d(z,E)$ and copying the proof of corollary 4.5 we are done. ■

We proceed exactly the same way for the case $p = 0$ to set, with $\gamma(z) := \sum_{j=1}^{n} |q_j| \, |z-\eta_j|^{-1}$:

Definition 7.7 *Let* $E = \bar{E} \subset \mathbb{T}$ *and* $R(z) = \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ q_j \in \mathbb{R}$ *with* $\forall j = 1, ..., n,\ \eta_j \notin E$. *Set* $\varphi(z) = |R(z)|^2 \, h(z)^q$. *We say that an holomorphic function* f *is in the generalised Nevanlinna class* $\mathcal{N}_{\varphi,0}(\mathbb{D})$ *if* $\exists \delta > 0,\ \delta < 1$ *such that*

$$\|f\|_{\mathcal{N}_{\varphi,0}} := \sup_{1-\delta<s<1} \int_{\mathbb{T}} \varphi(se^{i\theta}) \log^+ \left|f(se^{i\theta})\right| + \sup_{1-\delta<s<1} \int_{\mathbb{D}} \varphi(z)\gamma(z)h(z)^{-1} \log^+ |f(z)|.$$

And we have the Blaschke type condition, still using lemma 9.5 from the appendix,

Theorem 7.8 *Let* $E = \bar{E} \subset \mathbb{T}$ *and* φ *as above. Suppose* $q > 0$ *and* $f \in \mathcal{N}_{\varphi,0}(\mathbb{D})$ *with* $|f(0)| = 1$, *then*

$$\sum_{a \in Z(f)} (1-|a|^2) \varphi(a) \leq c \|f\|_{\mathcal{N}_{\varphi,0}}.$$

Corollary 7.9 *Let* $E = \bar{E} \subset \mathbb{T}$ *and* $R(z) = \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ q_j \in \mathbb{R},$ *with* $\forall j = 1, ..., n,\ \eta_j \notin E.$ *Suppose* $\varphi(z) := |R(z)|\, d(z,E)^q$ *and* $f \in \mathcal{N}_{\varphi,0}(\mathbb{D})$ *with* $|f(0)| = 1,$ *and set* $\tilde{R}(z) := \prod_{j=1}^{n} (z-\eta_j)^{(q_j)_+},$ *then*

$$\sum_{a \in Z(f)} (1-|a|^2) d(a,E)^{(q)_+} \left|\tilde{R}(a)\right|^2 \leq c\|f\|_{\mathcal{N}_{\varphi,0}}.$$

Proof.
Again using that $h(z) \simeq d(z,E)$ and copying the proof of corollary 4.5 we are done. ■

8 Mixed cases with L^∞ bounds.

As in section 7 we can mixed the two previous cases and we get, by a straightforward adaptation of the previous proofs,

Theorem 8.1 *Suppose that* $f \in \mathcal{H}(\mathbb{D}),\ |f(0)| = 1$ *and*

$$\forall z \in \mathbb{D},\ \log^+ |f(z)| \leq \frac{K}{(1-|z|^2)^p} \frac{1}{|R(z)|\, d(z,E)^q},$$

with $p > 0,$ *and* $R(z) := \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ q_j \in \mathbb{R},$ *if* $q_j - 1 > -p/2$ *set* $\tilde{q}_j = q_j$ *else choose* $\tilde{q}_j > 1 - p/2,$ *and set* $\tilde{R}_0(z) := \prod_{j=1}^{n} (z-\eta_j)^{\tilde{q}_j - 1},$ *then we have, with* $\epsilon > 0,$

$$\sum_{a \in Z(f)} (1-|a|^2)^{1+p+\epsilon} \left|\tilde{R}_0(a)\right| d(a,E)^{(q-\alpha(E)+\epsilon)_+} \leq c(p,q,R,E,\epsilon)K.$$

And

Theorem 8.2 *Suppose that* $f \in \mathcal{H}(\mathbb{D}),\ |f(0)| = 1$ *and*

$$\forall z \in \mathbb{D},\ \log^+ |f(z)| \leq K \frac{1}{|R(z)|\, d(z,E)^q},$$

with $p = 0,$ *and* $R(z) := \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ q_j \in \mathbb{R},$ *set* $\tilde{R}_\epsilon(z) := \prod_{j=1}^{n} (z-\eta_j)^{(q_j - 1 + \epsilon)_+}$

then, with $\epsilon > 0,$

$$\sum_{a \in Z(f)} (1-|a|^2) \left|\tilde{R}_\epsilon(a)\right| d(a,E)^{(q-\alpha(E)+\epsilon)_+} \leq c(q,R,E,\epsilon)K.$$

9 Appendix.

Lemma 9.1 *(Substitution) Suppose* $\delta > 0,\ 0 < u < 1$ *and* $|f(0)| = 1$, *then*

$$\int_{\mathbb{D}} (1-|z|^2)^{p-1+\delta} |R(sz)|^2 \log^- |f(sz)| \le (1-u^2)^\delta \frac{1}{u^2} P_{\mathbb{D},-}(s) + c(\delta,u) P_{\mathbb{D},+}(s),$$

with $c(\delta,u) := 2\times 4^{|q|}(1-u)^{\delta-\alpha-\beta},\ \alpha := -2\max_{j=1,\dots,n}(0,-q_j),\ \beta := 2\max_{j=1,\dots,n}(q_j)$,

and $P_{\mathbb{D},-}(s) := \int_{\mathbb{D}} (1-|z|^2)^{p-1}|z|^2 |R(sz)|^2 \log^- |f(sz)|,\ P_{\mathbb{D},+}(s) := \int_{\mathbb{D}} (1-|z|^2)^{p-1} |R(sz)|^2 \log^+ |f(sz)|.$

We also have:

$$\forall s \le t_0,\ \int_{\mathbb{D}} (1-|z|^2)^{\delta-1} \left|R(s\rho e^{i\theta})\right|^2 \log^- |f(sz)| \le c(\delta,u) P_{\mathbb{T},+}(t_0) + \frac{1}{2\delta}(1-u^2)^\delta P_{\mathbb{T},-}(t_0),$$

with

$$P_{\mathbb{T},+}(t_0) := \sup_{0\le s\le t_0} \int_{\mathbb{T}} \left|R(se^{i\theta})\right|^2 \log^+ \left|f(se^{i\theta})\right| d\theta$$

and

$$P_{\mathbb{T},-}(t_0) := \sup_{0\le s\le t_0} \int_{\mathbb{T}} \left|R(se^{i\theta})\right|^2 \log^- \left|f(se^{i\theta})\right| d\theta.$$

Proof.

Because this lemma is a key one for us, we shall give a detailed proof of it. We have

$$A := \int_{\mathbb{D}} (1-|z|^2)^{p-1+\delta} |R(sz)|^2 \log^- |f(sz)| = \int_{D(0,u)} (1-|z|^2)^{p-1+\delta} |R(sz)|^2 \log^- |f(z)| +$$
$$+ \int_{\mathbb{D}\setminus D(0,u)} (1-|z|^2)^{p-1+\delta} |R(sz)|^2 \log^- |f(z)| =: B + C.$$

Clearly for the second term we have

$$C := \int_{\mathbb{D}\setminus D(0,u)} (1-|z|^2)^{p-1+\delta} |R(sz)|^2 \log^- |f(sz)| \le$$
$$(1-u^2)^\delta \frac{1}{u^2} \int_{\mathbb{D}\setminus D(0,u)} (1-|z|^2)^{p-1}|z|^2 |R(sz)|^2 \log^- |f(sz)|.$$

For the first one, we have

$$B := \int_{D(0,u)} (1-|z|^2)^{p-1+\delta} |R(sz)|^2 \log^- |f(sz)|$$

and, changing to polar coordinates,

$$B = \int_0^u (1-\rho^2)^{p-1+\delta} \{ \int_{\mathbb{T}} \left|R(s\rho e^{i\theta})\right|^2 \log^- \left|f(s\rho e^{i\theta})\right| d\theta \} \rho d\rho.$$

We set

$$M(\rho) := \sup_{\theta\in\mathbb{T}} \left|R(\rho e^{i\theta})\right|^2 \le 4^{|q|}(1-\rho)^{-2\max_{j=1,\dots,n}(0,-q_j)},$$

because we have $|z-\eta_j| \le 2$ and $\left|\rho e^{i\theta} - \eta_j\right| \ge (1-\rho)$.

So we get

$$C(s\rho) := \int_{\mathbb{T}} |R(sz)|^2 \log^- |f(sz)| \le M(s\rho) \int_{\mathbb{T}} \log^- \left|f(s\rho e^{i\theta})\right|.$$

Because $\log|f(z)|$ is subharmonic, we get

$$0 = \log|f(0)| \le \int_{\mathbb{T}} \log \left|f(s\rho e^{i\theta})\right| = \int_{\mathbb{T}} \log^+ \left|f(s\rho e^{i\theta})\right| - \int_{\mathbb{T}} \log^- \left|f(s\rho e^{i\theta})\right|.$$

So we have

$$C(s\rho) \leq M(s\rho) \int_{\mathbb{T}} \log^+ \left| f(s\rho e^{i\theta}) \right|. \tag{9.4}$$

Now we set $m(\rho) := \inf_{\theta \in \mathbb{T}} \left| R(\rho e^{i\theta}) \right|^2$ and the same way as for $M(\rho)$, we get $m(\rho) \geq (1-\rho)^{2 \max_{j=1,\ldots,n}(q_j)}$. Putting it in (9.4), we get

$$C(s\rho) \leq M(s\rho) m(s\rho)^{-1} \int_{\mathbb{T}} \left| R(s\rho e^{i\theta}) \right|^2 \log^+ \left| f(s\rho e^{i\theta}) \right|. \tag{9.5}$$

We notice that $\sup_{s<1} \sup_{\rho<u} \dfrac{M(s\rho)}{m(s\rho)} = \sup_{\rho<u} \dfrac{M(\rho)}{m(\rho)}$ hence, setting

$$c(\delta, u) := \sup_{s<1} \sup_{\rho<u} \frac{M(s\rho)}{m(s\rho)} (1-\rho^2)^{\delta},$$

we get

$$c(\delta, u) \leq 2 \times 4^{|q|} (1-u)^{\delta - \alpha - \beta},$$

with

$$\alpha := -2 \max_{j=1,\ldots,n} (0, -q_j), \ \beta := 2 \max_{j=1,\ldots,n} (q_j).$$

Now we have

$$B \leq \int_0^u (1-\rho^2)^{p-1} (1-\rho^2)^{\delta} C(s\rho) \rho d\rho, \tag{9.6}$$

hence $B \leq c(\delta, u) P_{\mathbb{D},+}(s)$.
Adding B and C gives the first part of the lemma.

For the second one, from the definition of C with $p = 0$,

$$C := \int_{\mathbb{D} \setminus D(0,u)} (1-|z|^2)^{-1+\delta} \left| R(sz) \right|^2 \log^- \left| f(sz) \right|$$

we get passing in polar coordinates and with $0 \leq s \leq t_0 < 1$,

$$\begin{aligned} C &= \int_u^1 (1-\rho^2)^{\delta-1} \int_{\mathbb{T}} \left| R(s\rho e^{i\theta}) \right|^2 \log^- \left| f(s\rho e^{i\theta}) \right| d\theta \rho d\rho \\ &\leq P_{\mathbb{T},-}(t_0) \int_u^1 (1-\rho^2)^{\delta-1} \rho d\rho \leq \frac{1}{2\delta} (1-u^2)^{\delta} P_{\mathbb{T},-}(t_0). \end{aligned}$$

Now from (9.5) and (9.6) we get

$$B \leq P_{\mathbb{T},+}(t_0) c(\delta, u) \int_0^u (1-\rho^2)^{\delta-1} \rho d\rho \leq P_{\mathbb{T},+}(t_0) c(\delta, u).$$

Adding C with B we get the second part of the lemma. ■

Lemma 9.2 *Let $\eta \in \mathbb{T}$, then we have $\Re(\bar{z}(z-\eta)) \leq 0$ iff $z \in \mathbb{D} \cap D(\frac{\eta}{2}, \frac{1}{2})$.*

Proof.
We set $z = \eta t$, then we have

$$\bar{z}(z-\eta) = \bar{\eta}\bar{t}(\eta t - \eta) = \bar{t}(t-1).$$

Hence

$$\Re(\bar{z}(z-\eta)) = \Re(\bar{t}(t-1)) = \Re(r^2 - re^{i\theta}) = r^2 - r\cos\theta.$$

Hence with $t = x + iy = re^{i\theta,}$ $x = r\cos\theta,\ y = r\sin\theta$, we get

$$\Re(\bar{t}(t-1)) \leq 0 \iff x^2 + y^2 - x \leq 0$$

which means $(x,y) \in D(\frac{1}{2},\ \frac{1}{2})$ hence $z \in \mathbb{D} \cap D(\frac{\eta}{2},\ \frac{1}{2})$. ■

Lemma 9.3 *Let φ be a continuous function in the unit disc $\mathbb{D}$. We have that:*

$$s \leq t \in]0,1[\to \gamma(s) := \int_{\mathbb{T}} \varphi(se^{i\theta}) \log^- \left| f(se^{i\theta}) \right| d\theta$$

is a continuous function of $s \in [0,t]$.

Proof.

Because $s \leq t < 1$, the holomorphic function in the unit disc $f(se^{i\theta})$ has only a finite number of zeroes say $N(t)$. As usual we can factor out the zeros of f to get

$$f(z) = \prod_{j=1}^{N} (z - a_j) g(z)$$

where $g(z)$ has no zeros in the disc $\bar{D}(0,t)$. Hence we get

$$\log|f(z)| = \sum_{j=1}^{N} \log|z - a_j| + \log|g(z)|.$$

Let $a_j = r_j e^{\alpha_j},\ r_j > 0$ because $|f(0)| = 1$, then it suffices to show that

$$\gamma(s) := \int_{\mathbb{T}} \varphi(se^{i\theta}) \log^- \left| se^{i\theta} - re^{i\alpha} \right| d\theta$$

is continuous in s near $s = r$, because $\displaystyle\int_{\mathbb{T}} \varphi(se^{i\theta}) \log^- \left| g(se^{i\theta}) \right| d\theta$ is clearly continuous.

To see that $\gamma(s)$ is continuous at $s = r$, it suffices to show

$$\gamma(s_n) \to \gamma(r) \text{ when } s_n \to r.$$

But

$$\forall \theta \neq 0,\ \varphi(se^{i\theta}) \log \left| se^{i\theta} - r \right| \to \varphi(re^{i\theta}) \log \left| re^{i\theta} - r \right|$$

and $\displaystyle\log \frac{1}{|se^{i\theta} - r|} \leq c_\epsilon \left| se^{i\theta} - r \right|^{-\epsilon}$ with $\epsilon > 0$. So choosing $\epsilon < 1$, we get that $\displaystyle\log \frac{1}{|se^{i\theta} - r|} \in L^1(\mathbb{T})$ uniformly in s. Because $\varphi(se^{i\theta})$ is continuous uniformly in $s \in [0,t]$ we get also $\displaystyle\varphi(se^{i\theta}) \log \frac{1}{|se^{i\theta} - r|} \in L^1(\mathbb{T})$ uniformly in s. So we can apply the dominated convergence theorem of Lebesgue to get the result. ■

Lemma 9.4 *The function $\displaystyle(1 - |z|^2)^{p-1} \prod_{j=1}^{n} |z - \eta_k|^{-1}$, with $p > 0$, is integrable for the Lebesgue measure in the disc $\mathbb{D}$ and we have the estimate*

$$\int_{\mathbb{D}} (1 - |z|^2)^{p-1} \prod_{j=1}^{n} |z - \eta_k|^{-1} \leq c(p,\alpha) < \infty,$$

where the constant α is twice the length of the minimal arc between the points $\{\eta_j\}_{j=1,\ldots,n} \subset \mathbb{T}$.

Proof.

Because the points η_k are separated on the torus $\mathbb{T}$ we can assume that we have disjoint sectors Γ_j based on the arcs $\{\eta_j-\alpha,\eta_j+\alpha\}_{j=1,...,n}\subset\mathbb{T}$ for a $\alpha>0$. Let $\Gamma_0:=\mathbb{D}\backslash\bigcup_{j=1}^{n}\Gamma_j$. We have

$$A:=\int_{\mathbb{D}}(1-|z|^2)^{p-1}\prod_{j=1}^{n}|z-\eta_k|^{-1}dm(z)=\sum_{j=0}^{n}\int_{\Gamma_j}(1-|z|^2)^{p-1}\prod_{k=1}^{n}|z-\eta_k|^{-1}dm(z).$$

We set

$$A_0:=\int_{\Gamma_0}(1-|z|^2)^{p-1}\prod_{k=1}^{n}|z-\eta_k|^{-1}dm(z),$$

and we get

$$\forall z\in\Gamma_0,\ \forall k=1,...,n,\ |z-\eta_k|\geq\alpha\Rightarrow\prod_{k=1}^{n}|z-\eta_k|^{-1}\leq\alpha^{-n}.$$

So

$$A_0\leq\alpha^{-n}\int_{\Gamma_0}(1-|z|^2)^{p-1}dm(z)\leq\alpha^{-n}\int_{\mathbb{D}}(1-|z|^2)^{p-1}dm(z)\leq 2\pi\alpha^{-n}.$$

For computing A_j we can assume that $\eta_j=1$ by rotation and Γ_j based on the arc $(-\alpha,\ \alpha)$; so we have, because $\prod_{k=1}^{n}|z-\eta_k|^{-1}\leq\alpha^{-(n-1)}|1-z|$,

$$A_j:=\int_{\Gamma_j}(1-|z|^2)^{p-1}\prod_{k=1}^{n}|z-\eta_k|^{-1}dm(z)\leq\alpha^{-(n-1)}\int_{\Gamma_j}(1-|z|^2)^{p-1}|1-z|^{-1}dm(z).$$

Set $\beta:=\dfrac{p}{2}>0$, then we have $(1-|z|^2)^{\beta}<2^{\beta}|1-z|^{\beta}$ hence

$$A_j\leq\alpha^{-(n-1)}2^{\beta}\int_{\Gamma_j}(1-|z|^2)^{\beta-1}|1-z|^{\beta-1}dm(z).$$

Changing to polar coordinates, we get

$$A_j\leq\alpha^{-(n-1)}2^{\beta}\int_0^1(1-\rho^2)^{\beta-1}\rho\{\int_{-\delta}^{\delta}\left|1-\rho e^{i\theta}\right|^{\beta-1}d\theta\}d\rho.$$

Because $\beta>0$, we get

$$\forall\rho\leq 1,\ \int_{-\alpha}^{\alpha}\left|1-\rho e^{i\theta}\right|^{\beta-1}d\theta\leq c(\alpha,\beta)$$

and

$$\int_0^1(1-\rho^2)^{\beta-1}\rho d\rho\leq c(\beta).$$

So adding the A_j, we end the proof of the lemma. ■

Lemma 9.5 *Let $\varphi(z)$ be a positive function in $\mathbb{D}$ and $f\in\mathcal{H}(\mathbb{D})$; set $f_s(z):=f(sz)$ and suppose that:*

$$\forall s<1,\ \sum_{a\in Z(f_s)}(1-|a|^2)^{p+1}\varphi(sa)\leq\int_{\mathbb{D}}(1-|z|^2)^{p-1}\varphi(sz)\log^+|f(sz)|,$$

then, for any $1>\delta>0$ we have

$$\sum_{a\in Z(f)}(1-|a|^2)^{p+1}\varphi(a)\leq\sup_{1-\delta<s<1}\int_{\mathbb{D}}(1-|z|^2)^{p-1}\varphi(sz)\log^+|f(sz)|.$$

We have also:
let $\varphi(z),\ \psi(z)$ be positive continuous functions in $\mathbb{D}$ and $f\in\mathcal{H}(\mathbb{D})$ such that:

$$\forall s<1,\ \sum_{a\in Z(f)\cap D(0,s)}(1-|a|^2)\varphi(sa)\le\int_{\mathbb{D}}\varphi(sz)\log^+|f(sz)|+\int_{\mathbb{T}}\psi(se^{i\theta})\log^+\left|f(se^{i\theta})\right|$$

then, for any $1>\delta>0$ *we have*

$$\sum_{a\in Z(f)}(1-|a|^2)\varphi(a)\le\sup_{1-\delta<s<1}\int_{\mathbb{D}}\varphi(sz)\log^+|f(sz)|+\sup_{1-\delta<s<1}\int_{\mathbb{T}}\psi(se^{i\theta})\log^+|f(sz)|.$$

Proof.

We have $a\in Z(f_s)\iff f(sa)=0$, i.e. $b:=sa\in Z(f)\cap D(0,s)$. Hence the hypothesis is

$$\forall s<1,\ \sum_{a\in Z(f)\cap D(0,s)}(1-\left|\frac{a}{s}\right|^2)^{p+1}\varphi(a)\le\int_{\mathbb{D}}(1-|z|^2)^{p-1}\varphi(sz)\log^+|f(sz)|.$$

We fix $1-\delta<r<1,\ r<s<1$, then, because $Z(f)\cap D(0,r)\subset Z(f)\cap D(0,s)$ and $\varphi\ge 0$, we have

$$\sum_{a\in Z(f)\cap D(0,r)}(1-\left|\frac{a}{s}\right|^2)^{p+1}\varphi(a)\le\sum_{a\in Z(f)\cap D(0,s)}(1-\left|\frac{a}{s}\right|^2)^{p+1}\varphi(a)\le$$

$$\le\sup_{1-\delta<s<1}\int_{\mathbb{D}}(1-|z|^2)^{p-1}\varphi(z)\log^+|f(z)|.$$

In $D(0,r)$ we have a finite fixed number of zeroes of f, and, because $(1-\left|\frac{a}{s}\right|^2)^{p+1}$ is continuous in $s\le 1$ for $a\in\mathbb{D}$, we have

$$\forall a\in Z(f)\cap D(0,r),\ \lim_{s\to 1}(1-\left|\frac{a}{s}\right|^2)^{p+1}=(1-|a|^2)^{p+1}.$$

Hence

$$\sum_{a\in Z(f)\cap D(0,r)}(1-|a|^2)^{p+1}\varphi(a)\le\sup_{1-\delta<s<1}\int_{\mathbb{D}}(1-|z|^2)^{p-1}\varphi(sz)\log^+|f(sz)|.$$

Because the right hand side is independent of $r<1$ and φ is positive in $\mathbb{D}$ so the sequence

$$S(r):=\sum_{a\in Z(f)\cap D(0,r)}(1-|a|^2)^{p+1}\varphi(a)$$

is increasing with r, we get

$$\sum_{a\in Z(f)}(1-|a|^2)^{p+1}\varphi(a)\le\sup_{1-\delta<s<1}\int_{\mathbb{D}}(1-|z|^2)^{p-1}\varphi(sz)\log^+|f(sz)|.$$

This proves the first part. The proof of the second one is just identical. ■

Remark 9.6 *(i) As can be easily seen by the change of variables* $u=sz$, *if* $p\ge 1$ *we have:*

$$\sup_{1-\delta<s<1}\int_{\mathbb{D}}(1-|z|^2)^{p-1}\varphi(sz)\log^+|f(sz)|\lesssim\int_{\mathbb{D}}(1-|z|^2)^{p-1}\varphi(z)\log^+|f(z)|.$$

(ii) We also have that if $\varphi(z)\log^+|f(z)|$ *is subharmonic, then:*

$$\sup_{1-\delta<s<1}\int_{\mathbb{D}}(1-|z|^2)^{p-1}\varphi(sz)\log^+|f(sz)|\le\int_{\mathbb{D}}(1-|z|^2)^{p-1}\varphi(z)\log^+|f(z)|.$$

But (ii) is not the case in general in our setting.

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