Nevanlinna classes for non radial weights in the unit disc. Applications

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Abstract

We introduce Nevanlinna classes associated to non radial weights in the unit disc in the complex plane and we get Blaschke type theorems relative to these classes by use of several complex variables methods. This gives alternative proofs and improve some results of Boritchev, Golinski and Kupin useful, in particular, for the study of eigenvalues of non self adjoint Schrödinger operators.

Contents

1 Introduction.

We shall work with classes of holomorphic functions whose zeroes may appear as eigenvalues of Schrödinger operators with complex valued potential. So having information on these zeroes gives information on the operator.

Boritchev, Golinski and Kupin, in [3], got information on the distribution of the zeroes of functions in these classes to get interesting counterparts of famous Lieb-Thirring inequalities for complex Jacobi matrices. To get other motivations for this, see the nice introduction in [3].

This work is directly inspired by their papers and gives alternative proofs and improve some of their results.

The only measures we shall deal with are the Lebesgue measures: of the plane when we integrate in $\mathbb{D}$ or of the torus when we integrate on $\mathbb{T}$. So usually I shall not write it.

Let $\mathbb{D}$ be the unit disc in the complex plane $\mathbb{C}$. For $\varphi(z) = |R(z)|$ with $R(z) = \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ \eta_j \in \mathbb{T},\ q_j \in \mathbb{R}$, or $\varphi(z) = d(z,E)^q$ where E is a any closed set in $\mathbb{T} = \partial\mathbb{D}$ and $d(z,E)$ is the euclidean distance from z to E, or a mix $\varphi(z) = |R(z)|\, d(z,E)^q$ provided that $\forall j = 1, ..., n,\ \eta_j \notin E$, we set:

Definition 1.1 *We say that an holomorphic function f in the unit disc $\mathbb{D}$, such that $|f(0)| = 1$ is in the generalised Nevanlinna class $\mathcal{N}_{\varphi,p}(\mathbb{D})$ for $p > 0$, if $\exists \delta > 0,\ \delta < 1$ such that*

$$\|f\|_{\mathcal{N}_{\varphi,p}} := \sup_{1-\delta<s<1} \int_{\mathbb{D}} (1-|z|^2)^{p-1} \varphi(sz) \log^+ |f(sz)| < \infty.$$

To explicit the results we get, we set:

- for $\varphi(z) = |R(z)|$, if $q_j > -p/2$, $\tilde{q}_j := q_j$ else choose any $\tilde{q}_j > -p/2$ and set $\tilde{\varphi}(z) := \left| \prod_{j=1}^{n} (z-\eta_j)^{\tilde{q}_j} \right|$;
- for $\varphi(z) = d(z,E)^q$, we set $\forall \epsilon > 0,\ \tilde{\varphi}(z) := d(a,E)^{(q)_+ (1+\epsilon)}$, where $(q)_+ := \max(q, 0)$;
- for $\varphi(z) = |R(z)|\, d(z,E)^q$, we set $\forall \epsilon > 0,\ \tilde{\varphi}(z) := \left| \prod_{j=1}^{n} (z-\eta_j)^{\tilde{q}_j} \right| d(a,E)^{(q)_+ (1+\epsilon)}$.

With these definitions we get a Blaschke type condition, with $Z(f)$ the set of zeroes of f :

Theorem 1.2 *Suppose $p > 0$, and $f \in \mathcal{N}_{\varphi,p}(\mathbb{D})$ with $|f(0)| = 1$, then*

$$\sum_{a \in Z(f)} (1-|a|^2)^{1+p} \tilde{\varphi}(a) \leq c(\varphi) \|f\|_{\mathcal{N}_{\varphi,p}},$$

the constant $c(\varphi)$ depending only on φ.

For the case $p = 0$ we have to define:

- for $\varphi(z) = |R(z)|$, we set $\gamma(z) := \sum_{j=1}^{n} |q_j|\, |z-\eta_j|^{-1}$;
- for $\varphi(z) = d(z,E)^q$, we set $\gamma(z) := d(z,E)^{-1}$;
- for $\varphi(z) = |R(z)|\, d(z,E)^q$, we set $\gamma(z) := d(z,E)^{-1} \sum_{j=1}^{n} |q_j|\, |z-\eta_j|^{-1}$.

Definition 1.3 *We say that an holomorphic function f such that $|f(0)| = 1$ is in the generalised Nevanlinna class $\mathcal{N}_{\varphi,0}(\mathbb{D})$ if $\exists \delta > 0,\ \delta < 1$ such that*

$$\|f\|_{\mathcal{N}_{\varphi,0}} := \sup_{1-\delta<s<1} \int_{\mathbb{T}} \varphi(se^{i\theta}) \log^+ \left|f(se^{i\theta})\right| + \sup_{1-\delta<s<1} \int_{\mathbb{D}} \gamma(sz)\varphi(sz) \log^+ |f(sz)| < \infty,$$

with $\gamma(z)$ depending on the choice of φ as above.

Again we have to define the $\tilde{\varphi}$ associated to φ.

- for $\varphi(z) = |R(z)|$, we set $\tilde{\varphi}(z) := \left|\prod_{j=1}^{n} (z-\eta_j)^{(q_j)_+}\right|$;
- for $\varphi(z) = d(z,E)^q$, we set $\forall \epsilon > 0,\ \tilde{\varphi}(z) := d(z,E)^{(q)_+(1+\epsilon)}$;
- for $\varphi(z) = |R(z)|\, d(z,E)^q$, we set $\forall \epsilon > 0,\ \tilde{\varphi}(z) := \left|\prod_{j=1}^{n} (z-\eta_j)^{(q_j)_+}\right| d(z,E)^{(q)_+(1+\epsilon)}$.

With these definitions we get a Blaschke type condition:

Theorem 1.4 *Suppose $f \in \mathcal{N}_{\varphi,0}(\mathbb{D})$ with $|f(0)| = 1$, then there exists a constant $c(\varphi)$ depending only on φ such that*

$$\sum_{a\in Z(f)} (1-|a|^2)\tilde{\varphi}(a) \le c(\varphi)\|f\|_{\mathcal{N}_{\varphi,0}}.$$

All these results are new and generalise the usual Nevanlinna classes associated to radial weights.

We can apply these theorems to the case of L^∞ estimates.

With $R(z) := \prod_{j=1}^{n} (z-\eta_j)^{q_j}$, $\eta_j \in \mathbb{T}$, $q_j \in \mathbb{R}$, we set $\forall \epsilon > 0$, $R_\epsilon(z) := \prod_{j=1}^{n} (z-\eta_j)^{(q_j-1+\epsilon)_+}$. We define, $\forall j = 1, ..., n$, if $q_j - 1 > -p/2$, $\tilde{q}_j = q_j$ else we choose $\tilde{q}_j > 1 - p/2$, and we set $\tilde{R}_0(z) := \prod_{j=1}^{n} (z-\eta_j)^{\tilde{q}_j - 1}$.

Then we have as a first application (see the next section for other ones):

Theorem 1.5 *Suppose the holomorphic function f in $\mathbb{D}$ verifies $|f(z)| \le \exp \dfrac{D}{(1-|z|^2)^p\,|R(z)|}$*

with $R(z) := \prod_{j=1}^{n} (z-\eta_j)^{q_j}$, $\eta_j \in \mathbb{T}$, $q_j \in \mathbb{R}$, then we have:

for $p = 0$,

$$\sum_{a\in Z(f)} (1-|a|)\,|R_\epsilon(a)| \le Dc(R).$$

For $p > 0$

$$\forall \epsilon > 0,\ \sum_{a\in Z(f)} (1-|a|)^{1+p+\epsilon} \left|\tilde{R}_0(a)\right| \le Dc(\epsilon, R).$$

Remark 1.6 *When we compare the results of Boritchev, Golinski and Kupin [3] we find that, for $p > 0$ and $q_j \le -p/2$ their result is better and for $q_j > -p/2$, theorem 1.5 is better, because their threshold is $-p$ and mine is $-p/2$.*

As we shall see our results are based only on:
• the green formula ;
• the "zeroes" formula (see the next section) ;
which are the tools we use in several complex variables when dealing with problems on zeroes of holomorphic functions.

The methods used in several complex variables already proved their usefulness in the one variable case. For instance:
• the corona theorem of Carleson [5] is easier to prove and to understand thanks to the proof of T. Wolff based on L. Hörmander [7] ;
• the characterization of interpolating sequences by Carleson for H^∞ and by Shapiro & Shields for H^p are also easier to prove by these methods (see [1], last section, where they allow to get the bounded linear extension property for the case H^p ; the H^∞ case being done by Pehr Beurling [2]).

So it is not surprising that in the case of zero set, they can also be useful.

In this paper all the computations are *completely elementary:* derivations of usual functions and straightforward estimates. Because we have to be careful with signs and constants, I explicit the computations, so the paper is (too !) long.

This work was already presented in an international workshop in November 2016 in Toulouse, France.

2 Basic notations and results.

Let f be an holomorphic function in the unit disk $\mathbb{D}$ of the complex plane, $\mathcal{C}^\infty(\bar{\mathbb{D}})$, and g a $\mathcal{C}^\infty$ smooth function in the closed unit disk $\bar{\mathbb{D}}$ such that $g = 0$ on $\mathbb{T}$.

The Green formula gives:

$$\int_{\mathbb{D}} (g\triangle \log|f| - \log|f| \triangle g) = \int_{\mathbb{T}} (g\partial_n \log|f| - \log|f| \partial_n g) \tag{2.1}$$

where ∂_n is the normal derivative. With the "zero" formula: $\Delta \log|f| = \sum_{a\in Z(f)} \delta_a$ we get

$$\sum_{a\in Z(f)} g(a) = \int_{\mathbb{D}} \log|f| \triangle g + \int_{\mathbb{T}} (g\partial_n \log|f| - \log|f| \partial_n g).$$

Because $g = 0$ on $\mathbb{T}$,

$$\sum_{a\in Z(f)} g(a) = \int_{\mathbb{D}} \log|f| \triangle g - \int_{\mathbb{T}} \log|f| \partial_n g. \tag{2.2}$$

So, in order to get estimates on $\sum_{a\in Z(f)} g(a)$, we have to compute $\partial_n g$ and Δg. In this work, g will always be of the form

$$g_s(z) = (1-|z|^2)^{1+p}\varphi(sz),$$

where $\varphi(z)$ will be smooth and positive in $\mathbb{D}$.

We get a Blaschke type theorem if we can control

$$\int_{\mathbb{D}} \log|f| \triangle g - \int_{\mathbb{T}} \log|f| \, \partial_n g \leq c\|f\|$$

because then we get

$$\sum_{a\in Z(f)} (1-|a|^2)^{p+1}\varphi(sa) \leq c\|f\|,$$

where $\|f\|$ is a "norm" linked to the function f. To get an idea of what happens here, suppose first that $p>0$, and we set $f_s(z) := f(sz)$; so the equation (2.2) simplifies to

$$\sum_{a\in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log|f(sz)| \triangle g_s(z) = \int_{\mathbb{D}} \log^+|f(sz)| \triangle g_s(z) - \int_{\mathbb{D}} \log^-|f(sz)| \triangle g_s(z).$$

The strategy is quite obvious: we compute Δg_s and we estimate the two quantities

$$A_+(s) := \int_{\mathbb{D}} \log^+|f(sz)| \triangle g_s(z) \text{ and } A_-(s) := -\int_{\mathbb{D}} \log^-|f(sz)| \triangle g_s(z).$$

Because $\log^+|f(sz)|$ is directly related to the size of f, we just take the sum of the absolute value of the terms in Δg_s to estimate A_+.

For A_- we have to be more careful because we want to control terms containing $\log^-|f(sz)|$ by terms containing only $\log^+|f(sz)|$.

The case of a closed set $E\subset\mathbb{T}$ is more delicate than the case of the rational function and we build and use a smooth function $h(z)=h_\epsilon(z)$ in $\mathbb{D}$ such that:

$\forall\epsilon>0,\ \forall z\in\mathbb{D},\ d(z,E)^{1+\epsilon} \lesssim h_\epsilon(z) \lesssim d(z,E)$.

With this function we get precise results which give also results for the function $d(z,E)$.

Remark 2.1 *In fact the only property we use about $u=\log|f(sz)|$ is that u is a sub harmonic function such that $u(0)=0$. So we can replace $\log|f|$ by a sub harmonic function u and the "zero" formula: $\Delta\log|f| = \sum_{a\in Z(f)}\delta_a$ by the Riesz measure μ associated to u, $\mu:=\Delta u$ which is a positive measure in $\mathbb{D}$. This way we can retrieve some results of Favorov & Golinskii [6] by a different approach.*

To keep the fact that we are mainly interested in zeroes of holomorphic functions, we continue with $\log|f|$.

Let $E=\bar{E}\subset\mathbb{T}$; recall its Ahern-Clark type $\alpha(E)$:

$\alpha(E) := \sup\{\alpha\in\mathbb{R} :: |\{t\in\mathbb{T} :: d(t,E)<x\}| = \mathcal{O}(x^\alpha),\ x\to+0\}$,

where $|A|$ denotes the Lebesgue measure of the set A.

We get, denoting by $\mathcal{H}(\mathbb{D})$ the space of holomorphic functions in the disc:

Theorem 2.2 *Suppose that $f\in\mathcal{H}(\mathbb{D}),\ |f(0)|=1,\ q\geq 0$ and*

$$\forall z\in\mathbb{D},\ \log^+|f(z)| \leq \frac{K}{(1-|z|^2)^p}\frac{1}{d(z,E)^q},$$

then we have, $\forall\epsilon>0$,

$$\sum_{a\in Z(f)} (1-|a|^2)^{1+p} d(a,E)^{(q-\alpha(E)+\epsilon)_+} \leq c(p,q,E,\epsilon)K.$$

We get also the mixed case:

Theorem 2.3 *Suppose that $f\in\mathcal{H}(\mathbb{D}),\ |f(0)|=1$ and*

$$\forall z \in \mathbb{D},\ \log^+ |f(z)| \leq \frac{K}{(1-|z|^2)^p} \frac{1}{|R(z)|\, d(z,E)^q},$$

with $p > 0$ *and* $R(z) := \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ q_j > 1 - p/2,\ \eta_j \notin E,$

then we have, with $\epsilon > 0$,

$$\sum_{a \in Z(f)} (1-|a|^2)^{1+p+\epsilon} |R_0(a)|\, d(a,E)^{(q-\alpha(E)+\epsilon)_+} \leq c(p,q,R,E,\epsilon)K.$$

And

Theorem 2.4 *Suppose that* $f \in \mathcal{H}(\mathbb{D}),\ |f(0)| = 1$ *with* $R(z) := \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ \eta_j \notin E$, *and*

$$\forall z \in \mathbb{D},\ \log^+ |f(z)| \leq K \frac{1}{|R(z)|\, d(z,E)^q},$$

then, with $\epsilon > 0$,

$$\sum_{a \in Z(f)} (1-|a|^2) |R_\epsilon(a)|\, d(a,E)^{(q-\alpha(E)+\epsilon)_+} \leq c(q,R,E,\epsilon)K.$$

We use the same method for the two cases : the case of the rational function $R(z)$ and the case of the distance to a closed set E in the torus $\mathbb{T}$, in contrast to the works of Boritchev, Golinski & Kupin [3] for the rational function $R(z)$ and of Favorov & Golinskii [6] for the distance to a closed set E in the torus $\mathbb{T}$.

This work is presented the following way.

• In the next section we study the case of $\varphi(z) = |R(z)|^2$ with $R(z) = \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ \eta_j \in \mathbb{T},\ q_j \in \mathbb{R}$ and $p > 0$. This is the easiest case but the problematic is already here.

• In section 4 we study, with the same φ, the case $p = 0$.

• In section 5 we study, for any closed set $E \subset \mathbb{T}$, the case $p > 0$.

• In section 6 we study the same case with $p = 0$.

• In section 7 we study the mixed case for the Nevanlinna classes.

• In section 8 we apply what precede to L^∞ estimates in the case of rational function and in the case of the distance to a closed set in $\mathbb{T}$.

• In section 9 we get the mixed case for the L^∞ estimates and we retrieve some results of Boritchev, Golinski and Kupin [3].

• Finally in the appendix we prove technical, but important, lemmas.

3 Case E finite with $p > 0$.

Let $E := \{\eta_1, ..., \eta_n\} \subset \mathbb{T}$ be a finite sequence of points on $\mathbb{T}$. We shall work with the rational function $R(z) = \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ q_j \in \mathbb{R}$ and we set $\varphi(z) := |R(z)|^2$. In order to have a smooth

function in the disc we set $g_s(z) := (1-|z|^2)^{1+p}\,|R(sz)|^2$, with $0 < 1-\delta < s < 1$, where $\delta > 0$ is fixed, and:

$$\Delta g_s = 4\partial\bar\partial g_s = 4\partial\bar\partial[(1-|z|^2)^{1+p}\,|R(sz)|^2] = \Delta[(1-|z|^2)^{p+1}]\,|R(sz)|^2 + (1-|z|^2)^{p+1}\Delta[|R(sz)|^2] + \\ +8\Re[\partial((1-|z|^2)^{p+1})\bar\partial(|R(sz)|^2)].$$

Straightforward computations give the lemma, which separate the positive term, the negative term and the term with no fixed sign:

Lemma 3.1 *We have*

$$\Delta g_s(z) = \Delta_+ - \Delta_- + \Delta_\mp$$

with

$$\Delta_+ := 4(1-|z|^2)^{p-1}[p(p+1)\,|z|^2 + s^2(1-|z|^2)^2\left|\sum_{j=1}^n q_j(sz-\eta_j)^{-1}\right|^2]\,|R(sz)|^2$$

$$\Delta_- := 4(p+1)(1-|z|^2)^p\,|R(sz)|^2$$

$$\Delta_\mp : += 8s\Re[(-(r+1)(1-|z|^2)^r\bar z\ )(\sum_{j=1}^n q_j(s\bar z-\bar\eta_j)^{-1})]\,|R(sz)|^2\,.$$

Because $p > 0 \Rightarrow \partial_n g_s = 0$ on $\mathbb{T}$, formula (2.2), with $f_s(z) := f(sz)$, reduces to:

$$\sum_{a\in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log|f(sz)|\,\triangle g_s(z).$$

We have to estimate $\displaystyle\int_{\mathbb{D}} \log|f(sz)|\,\triangle g_s(z)$ and for it, we decompose:

$$\log|f(sz)|\,\triangle g_s(z) = \log^+|f(sz)|\,\triangle g_s(z) - \log^-|f(sz)|\,\triangle g_s(z).$$

We shall first group the terms containing $\log^+|f(sz)|$. We set

$A_+(s) := \Delta_+\log^+|f(sz)| - \Delta_-\log^+|f(sz)| + \Delta_\mp\log^+|f(sz)|$.

And $T_+(s) := \displaystyle\int_{\mathbb{D}} A_+(s)dm(z)$. We set also $P_{\mathbb{D},+}(s) := \displaystyle\int_{\mathbb{D}} (1-|z|^2)^{p-1}\,|R(sz)|^2\log^+|f(sz)|$.

Proposition 3.2 *We have, with* $|q| := \displaystyle\sum_{j=1}^n |q_j|$, $T_+(s) \le 4[p(p+1)\,|z|^2 + 4\,|q|^2 + 2\,|q|]P_{\mathbb{D},+}(s)$.

Proof.

We have $A_+ \le \Delta_+\log^+|f(sz)| + \Delta_\mp\log^+|f(sz)|$ because $-\Delta_-$ is negative. We use that $(1-|z|^2) \le 2\,|sz-\eta_j|$ then elementary estimates on the modulus of the reminding terms end the proof. ■

We shall now group the terms containing $\log^-|f(sz)|$. We set

$A_-(s,z) := -\Delta_+\log^-|f(sz)| + \Delta_-\log^-|f(sz)| - \Delta_\mp\log^-|f(sz)|$

and $P_{\mathbb{D},-}(s) := \displaystyle\int_{\mathbb{D}} (1-|z|^2)^{p-1}\,|z|^2\,|R(sz)|^2\log^-|f(sz)|$ and $T_-(s) := \displaystyle\int_{\mathbb{D}} A_-(s,z)$.

Proposition 3.3 *Suppose that* $\forall j = 1,...,n,\ q_j \ge 0$, *then*

$$T_-(s) \le (p+1)[4c(1,u) + s\,|q|\,c(1/2,u)]P_{\mathbb{D},+}(s).$$

Proof.
Set

$$A_2 := \Delta_- \log^- |f(sz)| = 4(p+1)(1-|z|^2)^p |R(sz)|^2 \log^- |f(sz)| .$$

We apply the "substitution" lemma 10.1 from the appendix with $\delta = 1$, to get

$$\int_{\mathbb{D}} A_2 \leq 4(p+1)(1-u^2)\frac{1}{u^2} P_{\mathbb{D},-}(s) + 4(p+1)c(1,u)P_{\mathbb{D},+}(s).$$

Now set

$$B_j := 8q_j(p+1)(1-|z|^2)^p \Re[\bar{z}\,(\bar{z}-\bar{\eta}_j)^{-1}] \, |R(sz)|^2 \log^- |f(sz)| ,$$

and

$$A_3 := -\Delta_+ \log^- |f(sz)| = -8\Re[(-(p+1)(1-|z|^2)^p \bar{z}\,)(\sum_{j=1}^{n} q_j(\bar{z}-\bar{\eta}_j)^{-1})] \, |R(sz)|^2 \log^- |f(sz)| \; ;$$

we get $A_3 = \sum_{j=1}^{n} B_j$. But $\Re[\bar{z}(s\bar{z}-\bar{\eta}_j)^{-1}] = \dfrac{1}{|sz-\eta_j|^2}\Re[\bar{z}(sz-\eta_j)]$ hence by lemma 10.2 from the appendix, we have $\Re(\bar{z}(z-\eta)) \leq 0$ iff $z \in \mathbb{D} \cap D(\dfrac{\eta_j}{2},\ \dfrac{1}{2})$. So, with $q_j \geq 0$, the part in $\mathbb{D} \cap D(\dfrac{\eta_j}{2},\ \dfrac{1}{2})$ is negative and can be ignored. So it remains

$$B_j \leq (p+1)s(1-|z|^2)^p \, |R(sz)|^2 \mathbb{1}_{D(\frac{\eta_j}{2},\frac{1}{2})^c}(z)\Re[q_j \Re[\bar{z}(\bar{z}-\bar{\eta}_j)^{-1}] \log^- |f(sz)| .$$

But for $z \in D(\dfrac{\eta_j}{2},\dfrac{1}{2})^c,\ (1-|z|^2) \leq 2\,|z-\eta_j|^2$ hence,

$$\mathbb{1}_{D(\frac{\eta_j}{2},\frac{1}{2})^c}(z)\Re[\bar{z}(\bar{z}-\bar{\eta}_j)^{-1}] \leq 2(1-|z|^2)^{-1/2}\mathbb{1}_{D(\frac{\eta_j}{2},\frac{1}{2})^c}(z) \leq 2(1-|z|^2)^{-1/2}.$$

So we get

$$B_j \leq sq_j(p+1)(1-|z|^2)^{p-1/2} \, |R(sz)|^2 \log^- |f(sz)|$$

and, provided that $q_j \geq 0$,

$$A_3 = \sum_{j=1}^{n} B_j \leq s\,|q|\,(p+1)(1-|z|^2)^{p-1/2} \, |R(sz)|^2 \log^- |f(sz)| . \tag{3.3}$$

We can again apply the "substitution" lemma 10.1 with $\delta = 1/2$, this time and we get

$$\int_{\mathbb{D}} (1-|z|^2)^{p-1/2} \, |R(sz)|^2 \log^- |f(z)| \leq (1-u^2)^{1/2}\frac{1}{u^2} P_{\mathbb{D},-}(s) + c(1/2,u)P_{\mathbb{D},+}(s).$$

So finally

$$\int_{\mathbb{D}} A_3 \leq s\,|q|\,(p+1)(1-u^2)^{1/2}\frac{1}{u^2} P_{\mathbb{D},-}(s) + s\,|q|\,(p+1)c(1/2,u)P_{\mathbb{D},+}(s).$$

Integrating $A_-(s,z)$ over $\mathbb{D}$ and adding, we get, with $A_1 := -\Delta_+ \log^- |f(sz)|$,

$$\begin{aligned} T_-(s) \leq \int_{\mathbb{D}} (A_1+A_2+A_3) \leq & -4p(p+1)P_{\mathbb{D},-}(s) + 4(p+1)(1-u^2)\frac{1}{u^2}P_{\mathbb{D},-}(s) + \\ & +4(p+1)c(1,u)P_{\mathbb{D},+}(s) + s\,|q|\,(p+1)(1-u^2)^{1/2}\frac{1}{u^2}P_{\mathbb{D},-}(s) + \\ & +s\,|q|\,(p+1)c(1/2,u)P_{\mathbb{D},+}(s). \end{aligned}$$

The key point here is that the "bad terms" in $\log^- |f(z)|$ can be controlled by the "good" one: $A_1 := -\Delta_+ \log^- |f(sz)|$.

We can choose $0 < u < 1$ such that

$$-4p(p+1) + 4(p+1)(1-u^2)\frac{1}{u^2} + s\,|q|\,(p+1)(1-u^2)^{1/2}\frac{1}{u^2} \leq 0$$

just taking, because $p > 0$, $\sqrt{1-u^2} \leq \dfrac{4p}{4+s\,|q|}$. Hence we get, provided that

$$\forall j=1,...,n\ q_j\geq 0,\ T_-(s)\leq (p+1)[4c(1,u)+s\,|q|\,c(1/2,u)]P_{\mathbb{D},+}(s).\ \blacksquare$$

We can also get results for $q_j<0$ the following way.

Proposition 3.4 *Set* $|q|_\infty:=\max\limits_{k=1,...,n}|q_k|$ *and suppose* $|q|_\infty<p/4$, *then there exist* $u<1,\ \gamma<1$ *such that:*

$$T_-(s)\leq 4(p+1)[c(1,u)+2\frac{|q|}{\delta}c(1,u)+2\,|q|_\infty\,(1-\gamma)^{-1}c(1,\gamma)]P_{\mathbb{D},+}(s).$$

Proof.
We have

$$|-\Delta_\mp|=\left|-8s\Re[(-(p+1)(1-|z|^2)^p\bar{z}\ )(\sum_{j=1}^n q_j(s\bar{z}-\bar{\eta}_j)^{-1})]\right||R(sz)|^2\leq$$
$$\leq 8(p+1)(1-|z|^2)^p\sum_{j=1}^n|q_j|\,|sz-\eta_j|^{-1}\,|R(sz)|^2\,.$$

We cut the disc in disjoint sectors around the points $\eta_j:\ \mathbb{D}=\Gamma_0\cup\bigcup\limits_{j=1}^n\Gamma_j$ with

$$\forall j=1,...,n,\ \Gamma_j:=\{z\in\mathbb{D}::\left|\frac{z}{|z|}-\eta_j\right|<\delta\},\ \Gamma_0:=\mathbb{D}\backslash\bigcup_{j=1}^n\Gamma_j.$$

This is possible because the points η_j are disjoint and in finite number so $\delta>0$ exists.
Now we set

$$A_3':=\left|-\Delta_\mp\log^-|f(sz)|\right|\leq 8(p+1)(1-|z|^2)^p\sum_{j=1}^n|q_j|\,|sz-\eta_j|^{-1}\,|R(sz)|^2\log^-|f(sz)|$$

and

$$\forall k=0,1,...,n,\ f_k(z):=8(p+1)(1-|z|^2)^p\sum_{j=1,j\neq k}^n|q_j|\,|sz-\eta_j|^{-1}\,|R(sz)|^2\log^-|f(sz)|$$

and on Γ_k, including $k=0$, we get

$$\forall z\in\Gamma_k,\ f_k(z)\leq 8(p+1)\frac{|q|}{\delta}(1-|z|^2)^p\,|R(sz)|^2\log^-|f(sz)|\,.$$

Hence we have

$$\forall k=0,...,n,\ \forall z\in\Gamma_k,\ A_3'\leq 8(p+1)\frac{|q|}{\delta}(1-|z|^2)^p+8(p+1)(1-|z|^2)^p\,|q_k|\,|sz-\eta_k|^{-1}\,|R(sz)|^2\log^-|f(sz)|\,.$$

Now we integrate in the disc and we get

$$\int_{\mathbb{D}}A_3'\leq 8(p+1)\frac{|q|}{\delta}\sum_{k=0}^n\int_{\Gamma_k}(1-|z|^2)^p\,|R(sz)|^2\log^-|f(sz)|+$$
$$+8(p+1)\sum_{k=0}^n|q_k|\int_{\Gamma_k}(1-|z|^2)^p\,|sz-\eta_k|^{-1}\,|R(sz)|^2\log^-|f(sz)|=:B_1+B_2.$$

But

$$\int_{\Gamma_k}(1-|z|^2)^p\,|R(sz)|^2\log^-|f(sz)|\leq\int_{\mathbb{D}}(1-|z|^2)^p\,|R(sz)|^2\log^-|f(sz)|$$

and we can apply the "substitution" lemma 10.1, with $\delta=1$, to get

$$\int_{\mathbb{D}}(1-|z|^2)^p\,|R(sz)|^2\log^-|f(sz)|\leq(1-u^2)\frac{1}{u^2}P_{\mathbb{D},-}(s)+c(1,u)P_{\mathbb{D},+}(s).$$

So the first term in $\int_{\mathbb{D}} A_3'$ is controlled by

$$B_1 \leq 8(p+1)\frac{|q|}{\delta}(1-u^2)\frac{1}{u^2}P_{\mathbb{D},-}(s) + 8(p+1)\frac{|q|}{\delta}c(1,u)P_{\mathbb{D},+}(s).$$

For the second one we first localise near the boundary:

$$\begin{aligned} B_2 &:= 8(p+1)\sum_{k=0}^{n}|q_k|\int_{\Gamma_k}(1-|z|^2)^p\,|sz-\eta_k|^{-1}\,|R(sz)|^2\log^-|f(sz)| = \\ &= 8(p+1)\sum_{k=0}^{n}|q_k|\int_{D(0,\gamma)\cap\Gamma_k}(1-|z|^2)^p\,|sz-\eta_k|^{-1}\,|R(sz)|^2\log^-|f(sz)| + \\ &\quad +8(p+1)\sum_{k=0}^{n}|q_k|\int_{\Gamma_k\setminus D(0,\gamma)}(1-|z|^2)^p\,|sz-\eta_k|^{-1}\,|R(sz)|^2\log^-|f(sz)| =: \\ &=: C_1 + C_2. \end{aligned}$$

For the first term we get

$$C_1 \leq 8(p+1)\,|q|_\infty\,(1-\gamma)^{-1}\int_{D(0,\gamma)}(1-|z|^2)^p\,|R(sz)|^2\log^-|f(sz)|.$$

The proof of the "substitution" lemma 10.1, gives with γ in place of u,

$$C_1 \leq 8(p+1)\,|q|_\infty\,(1-\gamma)^{-1}c(1,\gamma)P_{\mathbb{D},+}(s).$$

Now for the second term we have

$$\begin{aligned} C_2 &:= 8(p+1)\sum_{k=0}^{n}|q_k|\int_{\Gamma_k\setminus D(0,\gamma)}(1-|z|^2)^p\,|sz-\eta_k|^{-1}\,|R(sz)|^2\log^-|f(sz)| \leq \\ &\leq 8(p+1)\sum_{k=0}^{n}|q_k|\,\frac{1}{\gamma^2}\int_{\Gamma_k\setminus D(0,\gamma)}(1-|z|^2)^p\,|z|^2\,|sz-\eta_k|^{-1}\,|R(sz)|^2\log^-|f(sz)|. \end{aligned}$$

We use $(1-|z|^2)\leq 2\,|sz-\eta_k|$ to get

$$C_2 \leq 16(p+1)\frac{1}{\gamma^2}\sum_{k=0}^{n}|q_k|\int_{\Gamma_k}(1-|z|^2)^{p-1}\,|z|^2\,|R(sz)|^2\log^-|f(sz)| \leq 16(p+1)\,|q|_\infty\,\frac{1}{\gamma^2}P_{\mathbb{D},-}(s).$$

We have, with the notations of proposition 3.3, replacing A_3 by A_3',

$$\begin{aligned} T_-(s) &\leq \int_{\mathbb{D}}(A_1+A_2+A_3') \leq \\ &\quad -4p(p+1)P_{\mathbb{D},-}(s) + 4(p+1)(1-u^2)\frac{1}{u^2}P_{\mathbb{D},-}(s) + 4(p+1)c_3(1,u)P_{\mathbb{D},+}(s) + \\ &\quad +8(p+1)\frac{|q|}{\delta}(1-u^2)\frac{1}{u^2}P_{\mathbb{D},-}(s) + 8(p+1)\frac{|q|}{\delta}c_3(1,u)P_{\mathbb{D},+}(s) + \\ &\quad +8(p+1)\,|q|_\infty\,(1-\gamma)^{-1}c(1,\gamma)P_{\mathbb{D},+}(s) + 16(p+1)\,|q|_\infty\,\frac{1}{\gamma^2}P_{\mathbb{D},-}(s). \end{aligned}$$

Let us see the terms containing $\log^-|f(sz)|$, we set:

$$D(s,\gamma,u) := [-4p(p+1) + 8(p+1)\frac{|q|}{\delta}(1-u^2)\frac{1}{u^2} + 16(p+1)\,|q|_\infty\,\frac{1}{\gamma^2}]P_{\mathbb{D},-}(s).$$

So

$$D(s,\gamma,u) = 16(-\frac{p}{4} + \frac{|q|_\infty}{\gamma^2} + \frac{|q|}{2\delta}\frac{1-u^2}{u^2})(p+1)P_{\mathbb{D},-}(s).$$

Now suppose that $|q|_\infty < p/4$ and first choose $\gamma < 1$ big enough to have $-\frac{p}{4} + \frac{|q|_\infty}{\gamma^2} =: -\epsilon < 0$ which is clearly possible, then choose $u < 1$ such that $\frac{|q|}{2\delta}\frac{1-u^2}{u^2} - \epsilon \leq 0$ which is also clearly

possible because $\epsilon > 0$. So we get with these choices of u and γ,

$$T_-(s) \leq [4(p+1)c(1,u) + 8(p+1)\frac{|q|}{\delta}c(1,u) + 8(p+1)\,|q|_\infty\,(1-\gamma)^{-1}c(1,\gamma)]P_{\mathbb{D},+}(s). \blacksquare$$

As a corollary of these two propositions, we get

Corollary 3.5 *Suppose* $\forall j,\ q_j > -p/4$, *then there is a constant* $c(p,R)$ *such that:*
$T_-(s) \leq c(p,R)P_{\mathbb{D},+}(s)$.

Proof.
As above we can separate the points η_j where $-p/4 < q_j < 0$ from the points η_j with $q_j \geq 0$. Then we apply the relevant proof to each case. $\blacksquare$

We are lead to the following definition:

Definition 3.6 *Let* $R(z) = \prod_{j=1}^{n}(z-\eta_j)^{q_j},\ q_j \in \mathbb{R}$. *We say that an holomorphic function* f *such that* $|f(0)| = 1$ *is in the generalised Nevanlinna class* $\mathcal{N}_{|R|^2,p}(\mathbb{D})$ *for* $p > 0$, *if* $\exists \delta > 0,\ \delta < 1$ *such that*

$$\|f\|_{\mathcal{N}_{|R|^2,p}} := \sup_{1-\delta<s<1}\int_{\mathbb{D}}(1-|z|^2)^{p-1}\,|R(sz)|^2\log^+|f(sz)| < \infty.$$

And we get the Blaschke type condition:

Theorem 3.7 *Let* $R(z) = \prod_{j=1}^{n}(z-\eta_j)^{q_j},\ q_j \in \mathbb{R}$. *Suppose* $p > 0,\ j = 1,...,n,\ q_j > -p/4$ *and* $f \in \mathcal{N}_{|R|^2,p}(\mathbb{D})$ *with* $|f(0)| = 1$, *then*

$$\sum_{a\in Z(f)}(1-|a|^2)^{1+p}\,|R(a)|^2 \leq c(p,R)\|f\|_{\mathcal{N}_{|R|^2,p}}.$$

Proof.
We apply the formula (2.2), to get, with $g_s(z) = (1-|z|^2)^{1+p}\,|R(sz)|^2$,

$$\forall s<1,\quad \sum_{a\in Z(f_s)}(1-|a|^2)^{1+p}\,|R(sa)|^2 = \int_{\mathbb{D}}\log|f(sz)|\,\triangle g_s(z)$$

because with $p > 0,\ \partial_n g_s = 0$ on $\mathbb{T}$.
Now we use Proposition 3.2 to get that

$$\int_{\mathbb{D}}\log^+|f(sz)|\,\triangle g_s(z) \leq 4[p(p+1)\,|z|^2 + 4\,|q|^2 + 2\,|q|]P_{\mathbb{D},+}(s),$$

and corollary 3.5 to get

$$-\int_{\mathbb{D}}\log^-|f(sz)|\,\triangle g_s(z) \leq c(p,R)P_{\mathbb{D},+}(s).$$

So adding we get

$$\forall s<1,\quad \sum_{a\in Z(f_s)}(1-|a|^2)^{1+p}\,|R(sa)|^2 \leq c(p,R)P_{\mathbb{D},+}(s).$$

We are in position to apply lemma 10.6 from the appendix, with $\varphi(z) = |R(z)|^2$, to get

$$\sum_{a\in Z(f)}(1-|a|^2)^{1+p}\,|R(a)|^2 \leq c(p,R)\sup_{1-\delta<s<1}P_{\mathbb{D},+}(s),$$

because $|R(z)|^2$ is positive. $\blacksquare$

Corollary 3.8 *Let* $R(z)=\prod_{j=1}^{n}(z-\eta_j)^{q_j}$, $q_j\in\mathbb{R}$. *Suppose* $p>0$ *and* $f\in\mathcal{N}_{|R|,p}(\mathbb{D})$ *with* $|f(0)|=1$, *and let* $\forall j=1,...,n$, *if* $q_j>-p/2$, $\tilde{q}_j=q_j$ *else choose* $\tilde{q}_j>-p/2$, *and set* $\tilde{R}(z):=\prod_{j=1}^{n}(z-\eta_j)^{\tilde{q}_j}$, *then*

$$\sum_{a\in Z(f)}(1-|a|^2)^{1+p}\left|\tilde{R}(a)\right|\le c(p,\tilde{q},R)\|f\|_{\mathcal{N}_{|R|,p}}.$$

Proof.
In order to apply theorem 3.7 to $\tilde{R}$ we have to show that $f\in\mathcal{N}_{|R|,p}(\mathbb{D})\Rightarrow f\in\mathcal{N}_{\left|\tilde{R}\right|,p}(\mathbb{D})$.
But

$$\tilde{R}(sz):=\prod_{j=1}^{n}(sz-\eta_j)^{\tilde{q}_j}=\prod_{j=1}^{n}(sz-\eta_j)^{q_j}\times\prod_{j=1}^{n}(sz-\eta_j)^{\tilde{q}_j-q_j},$$

and the only point is for the j such that $q_j\le -p/2$. So set $r_j:=\tilde{q}_j-q_j\ge 0$, we have $|sz-\eta_j|\le 2$ hence $|sz-\eta_j|^{r_j}\le 2^{r_j}$ so $\left|\tilde{R}(sz)\right|\le 2^{|r|}\,|R(sz)|$ with $|r|:=\sum_{j=1}^{n}r_j$.

Putting it $\|f\|_{\mathcal{N}_{|\tilde{R}|,p}}$ we get

$$\|f\|_{\mathcal{N}_{|\tilde{R}|,p}}:=\sup_{1-\delta<s<1}\int_{\mathbb{D}}(1-|z|^2)^{p-1}\left|\tilde{R}(sz)\right|\log^+|f(sz)|\le$$
$$\le 2^{|r|}\sup_{1-\delta<s<1}\int_{\mathbb{D}}(1-|z|^2)^{p-1}\,|R(sz)|\log^+|f(sz)|=2^{|r|}\|f\|_{\mathcal{N}_{|R|,p}}.$$

So we are done. ■

4 Case E finite with $p=0$.

Now we set: $g_s(z)=(1-|z|^2)\,|R(sz)|^2$ and we have that
$\partial_n g_s(z)=-2\,|z|\,|R(sz)|^2+(1-|z|^2)\partial_n(|R(sz)|^2)$
which is not 0 on $\mathbb{T}$, so we have to add the boundary term:

$$B(s):=-\int_{\mathbb{T}}\log|f(sz)|\,\partial_n g_s=2\int_{\mathbb{T}}|R(sz)|^2\log^+|f(sz)|-2\int_{\mathbb{T}}|R(sz)|^2\log^-|f(sz)|=:$$
$$=:B_+(s)-B_-(s).$$

This time we shall use, for $t_0\in[1-\delta,1[$,

$$P_{\mathbb{T},-}(t_0):=\sup_{1-\delta\le s\le t_0}\int_{\mathbb{T}}\left|R(se^{i\theta})\right|^2\log^-\left|f(se^{i\theta})\right|$$

and

$$P_{\mathbb{T},+}(t_0):=\sup_{1-\delta\le s\le t_0}\int_{\mathbb{T}}\left|R(se^{i\theta})\right|^2\log^+\left|f(se^{i\theta})\right|.$$

Now we set

$$A_+(s):=4s^2(1-|z|^2)[\left|\sum_{j=1}^{n}q_j(sz-\eta_j)^{-1}\right|^2]\,|R(sz)|^2\log^+|f(sz)|-4\,|R(sz)|^2\log^+|f(sz)|+$$
$$+8s\Re[(-\bar{z}\,)(\sum_{j=1}^{n}q_j(s\bar{z}-\bar{\eta}_j)^{-1})]\,|R(sz)|^2\log^+|f(sz)|+B_+(s).$$

Set also $T_+(s) := \int_{\mathbb{D}} A_+(s)$, and with $\gamma(z) := \sum_{j=1}^{n} |q_j| \, |z - \eta_j|^{-1}$,

$$P_{\gamma,+}(s) := \int_{\mathbb{D}} \gamma(sz) \, |R(sz)|^2 \log^+ |f(sz)|.$$

Proposition 4.1 *We have*

$$T_+(s) \leq 8(|q| + 1)P_{\gamma,+}(s) + B_+(s).$$

Proof.

Set

$$A_1 := 4s^2 \int_{\mathbb{D}} (1 - |z|^2)[\left| \sum_{j=1}^{n} q_j (sz - \eta_j)^{-1} \right|^2] \, |R(sz)|^2 \log^+ |f(sz)|.$$

Using $(1 - |z|^2) \leq 2 \, |sz - \eta_j|$, we get $A_1 \leq 8 \, |q| \, P_{\gamma,+}(s)$.

Set $A_2 := -\int_{\mathbb{D}} 4 \, |R(sz)|^2 \log^+ |f(sz)|$. Then $A_2 \leq 0$ and it can be forgotten.

Finally set

$$A_3 := \int_{\mathbb{D}} 8s\Re[(-\bar{z}\,)(\sum_{j=1}^{n} q_j (s\bar{z} - \bar{\eta}_j)^{-1})] \, |R(sz)|^2 \log^+ |f(sz)| \, .$$

Again we get $A_3 \leq 8sP_{\gamma,+}(s)$.

Summing the A_j we get

$$T_+(s) \leq 8(|q| + 1)P_{\gamma,+}(s) + B_+(s). \blacksquare$$

We shall now group the terms containing $\log^- |f(sz)|$. We set

$$-A_-(s,z) := -4 \, |R(sz)|^2 \log^- |f(sz)| + (1 - |z|^2)\Delta(|R(sz)|^2)(sz) \log^- |f(sz)| +$$
$$+8s\Re[(-\bar{z}\,)(\sum_{j=1}^{n} q_j (s\bar{z} - \bar{\eta}_j)^{-1})] \, |R(sz)|^2 \log^- |f(sz)| + B_-(s).$$

and $T_-(s) := \int_{\mathbb{D}} A(s,z)$.

Proposition 4.2 *We have*

$$T_-(s) \leq 2[2c_3'(1,u) + 2 \, |q| \, c_3'(1/2,u)]P_{\mathbb{T},+}(t_0) +$$
$$+2(1 - \sqrt{u})^{1/2}[2(1 - \sqrt{u})^{1/2} + 2 \, |q|]P_{\mathbb{T},-}(t_0) - B_-(s).$$

Proof.

We have $\Delta[(1 - |z|^2)] = -4$ so

$$A_1(s) := -\int_{\mathbb{D}} \Delta((1 - |z|^2)) \, |R(sz)|^2 \log^- |f(sz)| = 4 \int_{\mathbb{D}} |R(sz)|^2 \log^- |f(sz)|.$$

We can apply the second part of the substitution lemma 10.1 with $\delta = 1$, we get for any $u < 1$,

$$\forall t \leq t_0, \ \int_{\mathbb{D}} |R(sz)|^2 \log^- |f(tz)| \leq c(0,u)P_{\mathbb{T},+}(t_0) + \frac{1}{2}(1 - \sqrt{u})P_{\mathbb{T},-}(t_0).$$

So we get

$$A_1(s) \leq 4c(0,u)P_{\mathbb{T},+}(t_0) + 2(1 - \sqrt{u})P_{\mathbb{T},-}(t_0).$$

For

$$A_2 := -\int_{\mathbb{D}} (1 - |z|^2)\Delta(|R(sz)|^2)(sz) \log^- |f(sz)| = -4s^2 \int_{\mathbb{D}} (1 - |z|^2) \, |R'(sz)|^2 \, (sz) \log^- |f(sz)| \leq$$

0

so we can forget it.

Now we arrive at the "bad term"

$$A_3 := -\int_{\mathbb{D}} 8\Re[\partial((1-|z|^2))\bar{\partial}(|R(sz)|^2)(sz)]\log^- |f(sz)|.$$

Copying the proof done in the case $p>0$, we use again lemma 10.2 and we integrate inequality (3.3) with $p=0$:

$$A_3 \leq s\,|q| \int_{\mathbb{D}} (1-|z|^2)^{-1/2}\,|R(sz)|^2.$$

Now we are in position to apply the second part of lemma 10.1 with $\delta = 1/2$, so we get

$$\forall t \leq t_0,\ \int_{\mathbb{D}} (1-|z|^2)^{-1/2}\,|R(sz)|^2 \log^- |f(tz)| \leq 2c(0,u)P_{\mathbb{T},+}(t_0) + (1-\sqrt{u})^{1/2}P_{\mathbb{T},-}(t_0),$$

and

$$A_3 \leq 2s\,|q|\,c(0,u)P_{\mathbb{T},+}(t_0) + 2s\,|q|\,(1-\sqrt{u})^{1/2}P_{\mathbb{T},-}(t_0).$$

Summing all, we get

$$\begin{aligned} T_-(s) \leq\ & 4c(0,u)P_{\mathbb{T},+}(t_0) + 2(1-\sqrt{u})P_{\mathbb{T},-}(t_0) + 2s\,|q|\,c(0,u)P_{\mathbb{T},+}(t_0) + \\ & + 2s\,|q|\,(1-\sqrt{u})^{1/2}P_{\mathbb{T},-}(t_0) - B_-(s). \end{aligned}$$

Hence

$$T_-(s) \leq 2[2c(0,u) + 2\,|q|\,c(0,u)]P_{\mathbb{T},+}(t_0) + 2(1-\sqrt{u})^{1/2}[2(1-\sqrt{u})^{1/2} + 2\,|q|]P_{\mathbb{T},-}(t_0) - B_-(s).$$

■

Definition 4.3 *Let* $R(z) = \prod_{j=1}^{n}(z-\eta_j)^{q_j},\ q_j \in \mathbb{R}$. *We say that an holomorphic function* f *such that* $|f(0)| = 1$ *is in the generalised Nevanlinna class* $\mathcal{N}_{|R|^2,0}(\mathbb{D})$ *if* $\exists \delta > 0,\ \delta < 1$ *such that*

$$\begin{aligned} \|f\|_{\mathcal{N}_{|R|^2,0}} := & \sup_{1-\delta<s<1} \int_{\mathbb{T}} \left|R(se^{i\theta})\right|^2 \log^+ \left|f(se^{i\theta})\right| + \\ & + \sup_{1-\delta<s<1} \int_{\mathbb{D}} \gamma(sz)\,|R(sz)|^2 \log^+ |f(sz)| < \infty, \end{aligned}$$

with $\gamma(z) := \sum_{j=1}^{n} |q_j|\,|z-\eta_j|^{-1}$.

We get the Blaschke type condition:

Theorem 4.4 *Let* $R(z) = \prod_{j=1}^{n}(z-\eta_j)^{q_j},\ q_j \in \mathbb{R}$. *Suppose* $\forall j = 1,...,n,\ q_j \geq 0$ *and* $f \in \mathcal{N}_{|R|^2,0}(\mathbb{D})$ *with* $|f(0)| = 1$, *then there exists a constant* $c(R)$ *depending only on* R *such that*

$$\sum_{a \in Z(f)} (1-|a|^2)\,|R(a)|^2 \leq c(R)\|f\|_{\mathcal{N}_{|R|^2,0}}.$$

Proof.

Fix $t_0 \in [1-\delta, 1[$ and recall that

$$P_{\mathbb{T},-}(t_0) := \sup_{1-\delta\leq s\leq t_0} \int_{\mathbb{T}} \left|R(se^{i\theta})\right|^2 \log^- \left|f(se^{i\theta})\right|\ ;$$

and

$$P_{\mathbb{T},+}(t_0) := \sup_{1-\delta\leq s\leq t_0} \int_{\mathbb{T}} \left|R(se^{i\theta})\right|^2 \log^+ \left|f(se^{i\theta})\right|\ ;$$

by lemma 10.3 in the appendix, we have that

$$h(s) := \int_{\mathbb{T}} \left|R(se^{i\theta})\right|^2 \log^- \left|f(se^{i\theta})\right|$$

is a continuous function of $s \in [1-\delta, t_0]$ hence its supremum is achieved at $s_0 = s(t_0) \in [1-\delta, t_0]$, i.e.

$$P_{\mathbb{T},-}(t_0) = -B_-(s_0) := \int_{\mathbb{T}} \left|R(s_0e^{i\theta})\right|^2 \log^- \left|f(s_0e^{i\theta})\right|.$$

Let us consider, for $t \in [t_0, 1[$,

$$\Sigma(t, s_0) := \sum_{a\in Z(f_t)} g_t(a) + \sum_{a\in Z(f_{s_0})} g_{s_0}(a).$$

We have, by (2.2),

$$\Sigma(t, s_0) \le T_+(t) + T_+(s_0) + T_-(t) + T_-(s_0).$$

By use of proposition 4.1 we get

$$T_+(s) \le 8(|q|+1) \int_{\mathbb{D}} \gamma(z) \left|R(sz)\right|^2 \log^+ |f(sz)| + B_+(s),$$

and by use of proposition 4.2 we get

$$T_-(s) \le 2[2c(0,u) + 2|q|\, c(0,u)]P_{\mathbb{T},+}(t_0) + 2(1-\sqrt{u})^{1/2}[2(1-\sqrt{u})^{1/2} + 2|q|]P_{\mathbb{T},-}(t_0) - B_-(s).$$

Hence

$$\begin{aligned}\Sigma(t, s_0) \le T_+(t) + T_+(s_0) + T_-(t) + T_-(s_0) \le 8(|q|+1) \int_{\mathbb{D}} \gamma(z) \left|R(tz)\right|^2 \log^+ |f(tz)| + B_+(t) + \\ +8(|q|+1) \int_{\mathbb{D}} \gamma(z) \left|R(s_0z)\right|^2 \log^+ |f(s_0z)| + B_+(s_0) + \\ +4[2c(0,u) + 2s|q|\, c(0,u)]P_{\mathbb{T},+}(t_0) + \\ +4(1-\sqrt{u})^{1/2}[2(1-\sqrt{u})^{1/2} + 2|q|]P_{\mathbb{T},-}(t_0) - B_-(t) - B_-(s_0).\end{aligned}$$

We forget the negative term $-B_-(t) := -\int_{\mathbb{T}} 2\left|R(tz)\right|^2 \log^- |f| \le 0$ and we recall that

$$P_{\mathbb{T},-}(t_0) = B_-(s_0) := \int_{\mathbb{T}} 2\left|R(s_0z)\right|^2 \log^- |f|.$$

Now choose $u < 1$ such that $4(1-\sqrt{u})^{1/2}[2(1-\sqrt{u})^{1/2} + 2|q|] - 1 \le 0$ i.e. $(1-\sqrt{u})^{1/2} \le \dfrac{1}{8(|q|+1)}$, which is independent of t_0. It remains

$$\begin{aligned}\Sigma(t, s_0) \le 8(|q|+1) \int_{\mathbb{D}} \gamma(z) \left|R(tz)\right|^2 \log^+ |f(tz)| + B_+(t) + \\ +8(|q|+1) \int_{\mathbb{D}} \gamma(z) \left|R(s_0z)\right|^2 \log^+ |f(s_0z)| + B_+(s_0) + \\ +4[2c(0,u) + 2s|q|\, c(0,u)]P_{\mathbb{T},+}(t_0).\end{aligned}$$

Then, because $t \in [1-\delta, t_0]$, $s_0 \in [1-\delta, t_0]$, we get $B_+(t) \le P_{\mathbb{T},+}(t_0)$; $B_+(s_0) \le P_{\mathbb{T},+}(t_0)$; hence

$$\Sigma(t, s_0) \le 16(|q|+1)P_{\gamma,+}(t_0) + 2P_{\mathbb{T},+}(t_0) + 4[2c(0,u) + 2|q|\, c(0,u)]P_{\mathbb{T},+}(t_0).$$

So finally

$$\Sigma(t, s_0) \le 16(|q|+1)P_{\gamma,+}(t_0) + 2[1 + 2(2c(0,u) + 2|q|\, c(0,u))]P_{\mathbb{T},+}(t_0).$$

We get, taking $t = t_0 < 1$,

$$\sum_{a\in Z(f_{t_0})} g_{t_0}(a) \le \Sigma(t, s_0) \le 16(|q|+1)P_{\gamma,+}(t_0) + 2[1 + 2(2c(0,u) + 2|q|\, c(0,u))]P_{\mathbb{T},+}(t_0).$$

Setting

$$c(R) := \max(16(|q|+1),\ 2[1 + 2(2c(0,u) + 2|q|\, c(0,u))]),$$

which is independent of t_0, we get

$$\forall t_0 \in [1-\delta, 1[, \quad \sum_{a \in Z(f_{t_0})} (1-|a|^2)\,|R(t_0 a)|^2 \leq c(R)\|f\|_{\mathcal{N}_{|R|^2,0}}$$

hence using the second part of lemma 10.6 from the appendix, with $\varphi(z) = \gamma(z)\,|R(z)|^2$, $\psi(z) = |R(z)|^2$, we get

$$\sum_{a \in Z(f)} (1-|a|^2)\,|R(a)|^2 \leq c(R)\|f\|_{\mathcal{N}_{|R|^2,0}}. \blacksquare$$

Corollary 4.5 *Let* $R(z) = \prod_{j=1}^{n} (z-\eta_j)^{q_j}$, $q_j \in \mathbb{R}$. *Suppose* $f \in \mathcal{N}_{|R|,0}(\mathbb{D})$ *with* $|f(0)| = 1$, *and set* $\tilde{R}(z) := \prod_{j=1}^{n} (z-\eta_j)^{(q_j)_+}$, *then there exists a constant* $c(R)$ *depending only on* R *such that*

$$\sum_{a \in Z(f)} (1-|a|^2)\left|\tilde{R}(a)\right| \leq c(R)\|f\|_{\mathcal{N}_{|R|,0}}.$$

Proof.
We have to prove that $f \in \mathcal{N}_{|R|,0} \Rightarrow f \in \mathcal{N}_{\left|\tilde{R}\right|,0}$. But if $q < 0$ then:

$$|z-\eta| \leq 2 \Rightarrow |z-\eta|^q \geq 2^q \Rightarrow 1 = |z-\eta|^{(q)_+} \leq 2^{-q}\,|z-\eta|^q.$$

Putting it the definition of $\|f\|_{\mathcal{N}_{|R|,0}}$ we are done. $\blacksquare$

5 Case E infinite and $p > 0$.

Let $E = \bar{E} \subset \mathbb{T}$ be a closed set ; we have $\mathbb{T} = \bigcup_{j \in \mathbb{N}} (\alpha_j, \beta_j) \cup E$ where the (α_j, β_j) are the contiguous intervals to E.

Let $\Gamma_j := \{z = re^{i\psi} \in \mathbb{D} :: \psi \in (\alpha_j, \beta_j)\}$ the conical set based on $F_j := (\alpha_j, \beta_j)$ and $\Gamma_E := \{z = re^{i\psi} \in \mathbb{D} :: \psi \in E\}$.

We fix an arc $F_j := (\alpha_j, \beta_j) = (\theta \in \mathbb{T} :: |\theta - \gamma_j| < \delta_j)$, with γ_j the midpoint of (α_j, β_j) on $\mathbb{T}$ and $2\delta_j$ is the length of the arc (α_j, β_j).

By rotation we shall suppose that $\gamma_j = 0$ and we set $\delta := \delta_j$. So we start with the arc $(-\delta, \delta)$ we fix an $\epsilon > 0$ and we associate to it the function

$$h(z) = h_\epsilon(\rho e^{i\theta}) := 1 - \rho + \chi(\frac{1-\rho}{\delta})\delta^{1+\epsilon} H(\theta/\delta),$$

with

$$\chi(t) \geq 0,\ \chi(t) = 1 \text{ for } t \leq 1 \text{ and } \chi(t) = 0 \text{ for } t \geq k+1,\ \chi \in \mathcal{C}^\infty(\mathbb{R}^+)\ ;$$

we can manage χ to have $\chi'(t) \leq 0$, $|\chi'(t)| \leq 1/k$. And

$$H(t) \in \mathcal{C}^\infty([-1,1]),\ H(t) \geq 0,\ H \text{ is zero up to infinite order at } t = -1 \text{ and } t = 1,$$

with $H(0) = 1$, $H'(0) = H''(0) = 0$, and we can manage to have $|H'| \leq 1 + \gamma$ for any $\gamma > 0$.

The values k and $\gamma > 0$ are fixed the following way: if $\delta_0 := \sup_{j \in \mathbb{N}} \delta_j$ then $\delta_0^{2\epsilon} \leq (\frac{1-\gamma}{1+\gamma})^2(1-\frac{1}{k})$, which is possible because, for E with more than 2 points, we have $\delta_0^{2\epsilon} < 1$. (For E a singleton this was done with $R(z) = (z-\eta)^q$, $\eta \in \mathbb{T}$, in the previous section.) This implies

$$\forall j \in \mathbb{N},\ \delta_j^{2\epsilon} \leq (\frac{1-\gamma}{1+\gamma})^2(1-\frac{1}{k}). \tag{5.4}$$

Moreover we set $\chi = \chi' = 0$ for $\rho \leq 1/2$.

We shall also use the notations:

$$I(\rho) := \{j \in \mathbb{N} :: \delta_j \geq (1-\rho)/(k+1)\} \text{ and } F(s,\rho) := \bigcup_{j \in I(s\rho)} F_j.$$

This means that $\forall i \notin I(\rho),\ h(\rho e^{i\theta}) := 1-\rho$, and also $\forall \rho \leq 1/2,\ h(\rho e^{i\theta}) := 1-\rho$.

Lemma 5.1 *We have:*

$\forall z \in \Gamma_\delta := \{z = re^{i\psi} \in \mathbb{D} :: \psi \in (e^{-i\delta},\ e^{i\delta})\},\ cd(z, \{e^{-i\delta},\ e^{i\delta}\}) \geq h_\epsilon(z) \geq c'd(z, \{e^{-i\delta},\ e^{i\delta}\})^{1+\epsilon}$, *with c and c' are 2 absolute constants, and $h(z)$ coincides up to infinite order with $(1-\rho)$ on the boundary $z = re^{-i\delta},\ z = re^{i\delta}$.*

Proof.

If $|z| \leq 1/2$, we have $h(\rho e^{i\theta}) := 1-\rho$ and $d(z, e^{i\delta}) = \left|\rho e^{i\theta} - e^{i\delta}\right| \leq 2$ so

$$h(\rho e^{i\theta}) \geq \frac{1}{2} \geq \frac{1}{4} \times 2 \geq \frac{1}{4} d(z, e^{i\delta}) \Rightarrow \frac{1}{4} d(z, e^{i\delta})^{1+\epsilon} \leq \frac{1}{4} d(z, e^{i\delta}) \leq h(z) \leq d(z, e^{i\delta}).$$

We have

$$d(z, e^{i\delta})^2 = \left|\rho e^{i\theta} - e^{i\delta}\right|^2 \simeq (1-\rho)^2 + |\delta - \theta|^2 \Rightarrow d(z, e^{i\delta}) \simeq (1-\rho) + (\delta - \theta),$$

the $\simeq$ meaning that there are 2 absolute constants c, c' such that

$$c[(1-\rho) + (\delta - \theta)] \leq d(z, e^{i\delta}) \leq c'[(1-\rho) + (\delta - \theta)].$$

Now for $|z| > 1/2$ we always have, by definition of H,

$$\forall \theta \in (0, \delta),\ h(\rho e^{i\theta}) = 1 - \rho + \chi(\frac{1-\rho}{\delta})\delta^{1+\epsilon} H(\theta/\delta) \simeq$$
$$\simeq (1-\rho) + \chi\delta^\epsilon(\delta - \theta) \leq (1-\rho) + (\delta - \theta) \lesssim d(z, e^{i\delta}).$$

For the other direction, we have

$$\forall \theta \in (0, \delta),\ h(\rho e^{i\theta}) = 1 - \rho + \chi(\frac{1-\rho}{\delta})\delta^{1+\epsilon} H(\theta/\delta) \simeq (1-\rho) + \chi\delta^\epsilon(\delta - \theta).$$

So two cases:

• $1 - \rho \geq \delta$, hence $(\delta - \theta) \leq \delta \leq (1-\rho) \Rightarrow h(\rho e^{i\theta}) \geq (1-\rho) \geq \frac{1}{2}(1-\rho) + \frac{1}{2}((\delta - \theta)) \gtrsim d(z, e^{i\delta}) \geq d(z, e^{i\delta})^{1+\epsilon}$.

• $1 - \rho < \delta \Rightarrow \chi = 1$ and $h(\rho e^{i\theta}) \gtrsim (1-\rho) + \delta^\epsilon(\delta - \theta) \geq (1-\rho)^{1+\epsilon} + (\delta - \theta)^{1+\epsilon} \gtrsim d(z, e^{i\delta})^{1+\epsilon}$, because $1 - \rho \leq 1$ and $|\delta - \theta| \leq \delta$.

The fact that $h(z)$ coincides up to infinite order with $(1-\rho)$ on the boundary $z = re^{-i\delta},\ z = re^{i\delta}$ is because $H(t)$ is zero up to infinite order for $t = \mp 1$. ■

Back to the Γ_j we set, still with $z = \rho e^{i\theta}$,

$$\forall z \in \Gamma_j,\ h_j(z) := 1 - \rho + \chi(\frac{1-\rho}{\delta_j})\delta_j^{1+\epsilon} H(\frac{\theta - \gamma_j}{\delta_j}), \text{ and } \forall z \in \Gamma_E,\ h_E(z) := 1 - \rho.$$

Now we consider

$$\forall z \in \Gamma_j,\ g_s(z) := (1-\rho)^{p+1} h_j(sz)^q$$

and

$$\forall z \in \Gamma_E,\ g_s(z) := (1-\rho)^{p+1} h_E(sz)^q = (1-\rho)^{p+1}(1 - s\rho)^q,$$

because these functions coincide up to infinite order on the boundary of their domain of definition, they define a unique function in $\mathbb{D}$, $g_s(z)$, and $\forall s < 1,\ g_s(z) \in \mathcal{C}^\infty(\bar{\mathbb{D}})$, so we can apply the Green formula to it. Recall that $f_s(z) := f(sz)$.

With the "zero" formula: $\Delta \log |f_s| = \sum_{a \in Z(f_s)} \delta_a$ we get

$$\sum_{a\in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log|f(sz)|\,\triangle g_s(z) + \int_{\mathbb{T}} (g_s\partial_n \log|f(sz)| - \log|f(sz)|\,\partial_n g_s).$$

So, because $g_s = 0$ on $\mathbb{T}$, $\displaystyle\sum_{a\in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log|f(sz)|\,\triangle g_s(z) - \int_{\mathbb{T}} \log\left|f(se^{i\theta})\right| \partial_n g_s(e^{i\theta})$.

If, moreover $p > 0$, $\partial_n g_s = 0$ on $\mathbb{T}$, hence $\displaystyle\sum_{a\in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log|f(sz)|\,\triangle g_s(z)$.

So we have to compute $\triangle g_s(z)$. We set

$$\forall z \in \Gamma_j,\ \varphi(z) := h_j(z)^q = [1-\rho+\chi(\frac{1-\rho}{\delta_j})\delta_j^{1+\epsilon} H(\frac{\theta-\gamma_j}{\delta_j})]^q.$$

Then we have, setting $\chi' := \chi'(\frac{1-s\rho}{\delta_j})$ and $H' := H(\frac{\theta-\gamma_j}{\delta_j})$,

$$\forall z \in \Gamma_j,\ \frac{\partial}{\partial\rho}\varphi(sz) = -sqh(sz)^{q-1}[1+\chi'\delta_j^\epsilon H]\ ;$$

and

$$\forall z \in \Gamma_j,\ \frac{\partial^2}{\partial\rho^2}\varphi(sz) = s^2q(q-1)h(sz)^{q-2}[1+\chi'\delta_j^\epsilon H]^2 + s^2qh(sz)^{q-1}[\chi''\frac{1}{\delta_j^{1-\epsilon}}H].$$

The same way

$$\forall z \in \Gamma_j,\ \frac{\partial}{\partial\theta}\varphi(z) = qh(z)^{q-1}\chi\delta_j^\epsilon H'\ ;$$

and

$$\forall z \in \Gamma_j,\ \frac{\partial^2}{\partial\theta^2}\varphi(z) = q(q-1)h(z)^{q-2}(\chi\delta_j^\epsilon H')^2 + qh(z)^{q-1}\chi\frac{1}{\delta_j^{1-\epsilon}}H''.$$

Proposition 5.2 *We have*

$$\begin{aligned}\Delta g_s(z) = {} & q(q-1)(1-\rho)^{p+1}h(sz)^{q-2}\{s^2(1+\chi'\delta_j^\epsilon H)^2 + \frac{1}{\rho^2}(\chi\delta_j^\epsilon H')^2\}+\\ & +q(1-\rho)^p h(sz)^{q-1}\{-s^2(1-\rho)\frac{1}{\delta_j^{1-\epsilon}}\chi''H + 2s(p+1)(1+\chi'\delta_j^\epsilon H)-\\ & -s\frac{1}{\rho}(1-\rho)(1+\chi'\delta_j^\epsilon H) + \frac{1}{\rho^2}(1-\rho)\frac{1}{\delta_j^{1-\epsilon}}(\chi H'')\}+\\ & +(p+1)(1-\rho)^{p-1}h(sz)^q\{p-\frac{1}{\rho}(1-\rho)\}.\end{aligned}$$

Proof.
We shall use the expression of the laplacian in polar coordinates:

$$\Delta f = \frac{\partial^2 f}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial f}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\theta^2}$$

and a straightforward computation gives the proposition. ■

Now we set, taking the terms with $\log^+|f(sz)|$, $A_+(s,z) := \Delta g_s(z)\log^+|f(sz)|$ and

$$T_+(s,\rho) := \int_{\mathbb{T}} A_+(s,\rho e^{i\theta}) \text{ and } T_+(s) := \int_0^1 T_+(s,\rho)\rho d\rho.$$

Set also

$$P_+(s,z) := (1-\rho)^{p-1}\varphi(s\rho e^{i\theta})^q \log^+\left|f(s\rho e^{i\theta})\right|, \text{ and } P_{\mathbb{D},+}(s) := \int_{\mathbb{D}} P_+(s,z).$$

Proposition 5.3 *We have*
$T_+(s) \le cP_{\mathbb{D},+}(s)$.

Proof.
We fix $\rho < 1$; for $\forall z = \rho e^{i\theta} \in \Gamma_j$ and $\delta_j < (1-s\rho)/k+1$, then $\chi = \chi' = 0$ hence $h(sz) = 1-s\rho$ and we set

$$T_+^1(s,\rho) := \sum_{j::\delta_j<(1-s\rho)/k+1,} \int_{|\theta-\gamma_j|<\delta_j} A_+(s,\rho e^{i\theta}) + \int_E A_+(s,\rho e^{i\theta}) \ ;$$

and

$$T_+^2(s,\rho) := \sum_{j::\delta_j\ge(1-s\rho)/k+1,} \int_{|\theta-\gamma_j|<\delta_j} A_+(s,\rho e^{i\theta}).$$

Set, for $\delta_j \ge (1-s\rho) \iff sz \in \Gamma_j$,

$$\forall z = \rho e^{i\theta} \in \Gamma_j,\ A_j := q(q-1)(1-\rho)^{p+1} h(sz)^{q-2} \{ s^2(1+\chi' H)^2 + \frac{1}{\rho^2}(\chi H')^2 \} \log^+ |f(sz)| \, .$$

This can be written

$$A_j := \mathbb{1}_{\Gamma_j}(sz) q(q-1)(1-\rho)^{p+1} h(sz)^q h(sz)^{-2} \times \{ s^2(1+\chi' H)^2 + \frac{1}{\rho^2}(\chi H')^2 \} \log^+ |f(sz)| \, .$$

Because

$$h(sz) := 1 - s\rho + \chi_{\delta_j}(s\rho)\delta_j H((\theta-\gamma_j)/\delta_j) \ge (1-s\rho)$$

we have $h(sz)^{-2} \le (1-s\rho)^{-2}$ and

$$|A_j| \le \mathbb{1}_{\Gamma_j}(sz) q \, |q-1| \, (1-\rho)^{p+1} (1-s\rho)^{-2} \varphi(sz)^q \times \left| \{ s^2(1+\chi' H)^2 + \frac{1}{\rho^2}(\chi H')^2 \} \right| \log^+ |f(sz)| \, .$$

But, for $\rho \le 1/2$, we have set $\chi = \chi' = 0$, so

$$\left| \{ s^2(1+\chi' H)^2 + \frac{1}{\rho^2}(\chi H')^2 \} \right| \le \left| \{ s^2(1+\chi' H)^2 + 4(\chi H')^2 \} \right| \le c_1,$$

and

$$|A_j| \le q \, |q-1| \, c_1 (1-\rho)^{p+1} (1-s\rho)^{-2} \varphi(sz)^q \log^+ |f(sz)| \, .$$

Now because $s \le 1$ we have $(1-\rho) \le (1-s\rho)$ hence

$$A_j \le |A_j| \le q \, |q-1| \, c_1 (1-\rho)^{p-1} \varphi(sz)^q \log^+ |f(sz)| = \mathbb{1}_{\Gamma_j}(sz) q \, |q-1| \, c_1 P_+(s,z).$$

Adding over all j's because this is also trivially true in case $h(sz) = (1-s\rho)$, we get

$$T_+(s,\rho) := \int_{\mathbb{T}} A_+(s,\rho e^{i\theta}) \le q \, |q-1| \, c_1 \int_{\mathbb{T}} P_+(s,\rho e^{i\theta}).$$

Set for $\forall z = \rho e^{i\theta} \in \Gamma_j$, and $\delta_j \ge (1-s\rho)$,

$$B_j := q(1-\rho)^p h(sz)^{q-1} \{ -s^2(1-\rho)\frac{1}{\delta_j}\chi'' H + 2s(p+1)(1+\chi' H) -$$
$$-s\frac{1}{\rho}(1-\rho)(1+\chi' H) + \frac{1}{\rho^2}(1-\rho)\frac{1}{\delta_j}(\chi H'') \} \log^+ |f(sz)| \, .$$

Because $\delta_j \ge (1-s\rho) \Rightarrow \dfrac{(1-\rho)}{\delta_j} \le \dfrac{(1-s\rho)}{\delta_j} \le 1$ hence, because here $\rho \ge 1/2$,

$$|B_j| \le q(1-\rho)^p h(sz)^{q-1} \{ s^2 \, |\chi'' H| + 2s(p+1) \, |(1+\chi' H)| + 2s(1-\rho) \, |1+\chi' H| + 4 \, |\chi H''| \} \log^+ |f(sz)| \, .$$

So

$$B_j \le |B_j| \le q(1-\rho)^{p-1} \varphi(sz) \{ s^2 c_2 + 2s(p+1)c_1 + 2s(1-\rho)c_1 + 4c_2 \} \log^+ |f(sz)| \, ,$$

i.e.

$$B_j \le q c_3 (1-\rho)^{p-1} \varphi(sz) \log^+ |f(sz)| \le q c_3 1_{\Gamma_j}(sz) P_+(s,z),$$

with

$$c_3 := \{ s^2 c_2 + 2s(p+1)c_1 + 2s(1-\rho)c_1 + 4c_2 \}.$$

Finally set

$$C(s,z) := (p+1)(1-\rho)^{p-1}h(sz)^q\{p-\frac{1}{\rho}(1-\rho)\}\log^+|f(sz)|\,,$$

then

$$C \le \left|p-\frac{1}{\rho}(1-\rho)\right|(p+1)P_+(s,z).$$

Integrating all these terms in $\mathbb{D}$ we get

$T_+(s) \le c(s)P_{\mathbb{D},+}(s)$. ■

We set, taking the terms with $\log^-|f(sz)|$,

$$A_-(s,z) := \Delta g_s(z)\log^-|f(sz)| \text{ and } T_-(s,\rho) := \int_{\mathbb{T}} A_-(s,\rho e^{i\theta}) \text{ and } T_-(s) := \int_0^1 T_-(s,\rho)\rho d\rho.$$

We set also

$$F(\rho,s) := \bigcup_{j\in I(s\rho)} F_j \subset \mathbb{T}.$$

We fix $\rho<1$; for $\forall z=\rho e^{i\theta}\in\Gamma_j$ and $\delta_j < \dfrac{1-\rho s}{k+1}$, then $\chi=\chi'=0$ hence $h(sz)=1-s\rho$ and we set

$$T_-^1(s,\rho) := \int_{\mathbb{T}\setminus F(\rho,s)} A_-(s,\rho e^{i\theta}), \text{ and } T_-^2(s,\rho) := \int_{F(\rho,s)} A_-(s,\rho e^{i\theta}).$$

Recall that $I(\rho) := \{j\in\mathbb{N} :: \delta_j \ge (1-s\rho)/(k+1)\}$ and for $j\in I(\rho)$,

$$\forall z=\rho e^{i\theta}\in\Gamma_j,\ A_j := -q(q-1)(1-\rho)^{p+1}h(sz)^{q-2}\{s^2(1+\chi'\delta_j^\epsilon H)^2+\frac{1}{\rho^2}(\chi\delta_j^\epsilon H')^2\}\log^-|f(sz)|\,.$$

This can be written

$$A_j := -q(q-1)(1-\rho)^{p+1}h(sz)^{q-1}h(sz)^{-1}\times\delta_j^{2\epsilon}\{s^2(1+\chi'H)^2+\frac{1}{\rho^2}(\chi H')^2\}\log^-|f(sz)|\,.$$

Now we have two cases.

5.1 Case $q \le 1$.

Then, because $h(sz) := 1-s\rho+\chi(\dfrac{1-s\rho}{\delta_j})\delta_j^{1+\epsilon}H \ge (1-s\rho)$ we have $h(sz)^{-1}\le(1-s\rho)^{-1}$ and

$$A_j = q(1-q)(1-\rho)^{p+1}(1-s\rho)^{-1}h(sz)^{q-1}\times\delta_j^{2\epsilon}\{s^2(1+\chi'H)^2+\frac{1}{\rho^2}(\chi H')^2\}\log^-|f(sz)|\,.$$

Set

$$B_j := q(1-\rho)^p h(sz)^{q-1}\{s^2(1-\rho)\frac{1}{\delta_j^{1-\epsilon}}\chi''H - 2s(p+1)(1+\chi'\delta_j^\epsilon H)+$$
$$+s\frac{1}{\rho}(1-\rho)(1+\chi'\delta_j^\epsilon H)-\frac{1}{\rho^2}(1-\rho)\frac{1}{\delta_j^{1-\epsilon}}(\chi H'')\}\log^-|f(sz)|\,.$$

This can be written:

$$B_j := q(1-\rho)^p h(sz)^{q-1}\{s^2(1-\rho)\frac{1}{\delta_j^{1-\epsilon}}\chi''H+s\frac{1}{\rho}(1-\rho)(1+\chi'\delta_j^\epsilon H)\}\log^-|f(sz)|-$$
$$-q(1-\rho)^p h(sz)^{q-1}\{2s(p+1)(1+\chi'\delta_j^\epsilon H)+\frac{1}{\rho^2}(1-\rho)\frac{1}{\delta_j^{1-\epsilon}}(\chi H'')\}\log^-|f(sz)|\,.$$

Group A_j with a negative term in B_j :

$$C_j := q(1-q)(1-\rho)^{p+1}(1-s\rho)^{-1}h(sz)^{q-1}\times\delta_j^{2\epsilon}\{s^2(1+\chi'H)^2+\frac{1}{\rho^2}(\chi H')^2\}\log^-|f(sz)|-$$
$$-2sq(p+1)(1-\rho)^p h(sz)^{q-1}(1+\chi'\delta_j^\epsilon H)\log^-|f(sz)|\,.$$

The aim is to show that, for $\rho > 1-\gamma$ we have that $C_j \le 0$, i.e.

$$(1-q)(1-\rho)^{p+1}(1-s\rho)^{-1}h(sz)^{q-1}\times\delta_j^{2\epsilon}\{s^2(1+\chi' H)^2+\frac{1}{\rho^2}(\chi H')^2\}\le$$
$$\le 2s(p+1)(1-\rho)^p h(sz)^{q-1}(1+\chi'\delta_j^\epsilon H)).$$

Because everything is continuous in s, we can make directly $s=1$. So it suffices to show

$$(1-\rho)^p h(sz)^{q-1}\times\delta_j^{2\epsilon}\{(1+\chi' H)^2+\frac{1}{\rho^2}(\chi H')^2\} < 2(1-\rho)^p h(sz)^{q-1}(1+\chi'\delta_j^\epsilon H)).$$

We have to prove

$$\delta_j^{2\epsilon}\{(1+\chi' H)^2+\frac{1}{\rho^2}(\chi H')^2\} < 2(1+\chi'\delta_j^\epsilon H).$$

But recall that $1\ge(1+\chi'\delta_j^\epsilon H)\ge(1-\frac{1}{k})$ then $(1+\chi'\delta_j^\epsilon H)^2\le(1+\chi'\delta_j^\epsilon H)$ so it remains to show

$$\delta_j^{2\epsilon}\frac{1}{\rho^2}(\chi H')^2 < (1+\chi'\delta_j^\epsilon H).$$

If $\rho > 1-\gamma$ and because $|H'|\le 1+\gamma$ we get

$$\delta_j^{2\epsilon}\frac{1}{\rho^2}(\chi H')^2 < (\frac{1-\gamma}{1+\gamma})^2\delta_j^{2\epsilon}.$$

Now we use inequality (5.4): $\delta_j^{2\epsilon}\le(\frac{1-\gamma}{1+\gamma})^2(1-\frac{1}{k})$ to get $(\frac{1-\gamma}{1+\gamma})^2\delta_j^{2\epsilon}\le(1+\chi'\delta_j^\epsilon H)$, and we are done.

So we proved

Lemma 5.4 *We have, for $\rho > 1-\gamma$ and $q\le 1$,*

$$A_j+B_j\le D_j(z) := q(1-\rho)^p h(sz)^{q-1}\{s^2(1-\rho)\frac{1}{\delta_j^{1-\epsilon}}\chi'' H+s\frac{1}{\rho}(1-\rho)(1+\chi'\delta_j^\epsilon H)\}\log^-|f(sz)|-$$
$$-q(1-\rho)^p h(sz)^{q-1}\{\frac{1}{\rho^2}(1-\rho)\frac{1}{\delta_j^{1-\epsilon}}(\chi H'')\}\log^-|f(sz)|.$$

Now we have

$$|D_j|\le q(1-\rho)^p h(sz)^{q-1}\{s^2(1-\rho)\frac{1}{\delta_j^{1-\epsilon}}|\chi'' H|+s\frac{1}{\rho}(1-\rho)(1+\chi'\delta_j^\epsilon H)\}\log^-|f(sz)|+$$
$$+q(1-\rho)^p h(sz)^{q-1}\{\frac{1}{\rho^2}(1-\rho)\frac{1}{\delta_j^{1-\epsilon}}|\chi H''|\}\log^-|f(sz)|.$$

and, because $\frac{1-\rho}{\delta_j}\le k+1$, we have $(\frac{1-\rho}{\delta_j})^{1-\epsilon}\le(k+1)^{1-\epsilon}$, and $(1+\chi'\delta_j^\epsilon H)\le 1$, so

$$|D_j|\le q(1-\rho)^p h(sz)^{q-1}\{s^2(1-\rho)^\epsilon(k+1)\|\chi''\|_\infty+s\frac{1}{\rho}(1-\rho)\}\log^-|f(sz)|+$$
$$+q(1-\rho)^p h(sz)^{q-1}\{\frac{1}{\rho^2}(1-\rho)^\epsilon(k+1)\|H''\|_\infty\log^-|f(sz)|.$$

Recall we impose that for $\rho\le 1/2$, $\chi\equiv 0$, so,

$$|D_j|\le q(1-\rho)^p h(sz)^{q-1}\{(1-\rho)^\epsilon(k+1)\|\chi''\|_\infty+2(1-\rho)+4(1-\rho)^\epsilon(k+1)\|H''\|_\infty\}\log^-|f(sz)|.$$

Which gives

$$\forall j\in I(\rho),\ |D_j|\le q(1-\rho)^p h(sz)^{q-1}(1-\rho)^\epsilon(k+1)\{\|\chi''\|_\infty+2+4\|H''\|_\infty\}\log^-|f(sz)|\le$$
$$\le c_6 q(1-\rho)^{p+\epsilon}h(sz)^{q-1}\log^-|f(sz)|.$$

We use again that $(1-\rho)\le h(sz)\Rightarrow h(sz)^{-1}\le(1-\rho)^{-1}$ to get

$$|D_j|\le c_6 q(1-\rho)^{p-1+\epsilon}h(sz)^q\log^-|f(sz)|.$$

So we can apply the "substitution" lemma 10.4 in the appendix, and we get

$$\forall \rho \in (0,1),\ \int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^- \left|f(s\rho e^{i\theta})\right| \leq (\frac{2}{(1-\rho)})^q \int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^+ \left|f(s\rho e^{i\theta})\right|.$$

Now we group the D_j hence

$$\sum_{j\in I(\rho)} \int_{F_j} \left|D_j(\rho e^{i\theta})\right| \leq c_6 q(1-\rho)^{p-1+\epsilon} \int_{F(s,\rho)} h(s\rho e^{i\theta})^q \log^- \left|f(s\rho e^{i\theta})\right| \leq$$

$$c_6 q(1-\rho)^{p-1+\epsilon} \int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^- \left|f(s\rho e^{i\theta})\right|.$$

So we proved

Lemma 5.5 *We have for $\rho \geq 1-\gamma$ and $q \leq 1$,*

$$\sum_{j\in I(\rho)} \int_{F_j} \left|D_j(\rho e^{i\theta})\right| \leq c_6 q(1-\rho)^{p-1+\epsilon} \int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^- \left|f(s\rho e^{i\theta})\right|.$$

It remains to control when $\rho < 1-\gamma$, but we have

$$|A_j| \leq c_7 \mathbb{1}_{F_j}(e^{i\theta}) q(1-q)(1-\rho)^{p+1} h(s\rho e^{i\theta})^{q-2} \log^- \left|f(s\rho e^{i\theta})\right|,$$

and from $(1-\rho) \leq h(sz)$, we get $(1-\rho)^{p+1} h(sz)^{q-2} \leq (1-\rho)^{p-1} h(sz)^q$, so

$$|A_j| \leq c_7 \mathbb{1}_{F_j}(e^{i\theta}) q(1-q)(1-\rho)^{p-1} h(s\rho e^{i\theta})^q \log^- \left|f(s\rho e^{i\theta})\right|,$$

and, for $\rho \leq 1-\gamma$,

$$\sum_{j\in I(\rho)} \int_{F_j} |A_j| \leq c_7 q(1-q)(1-\rho)^{p-1} \int_{F(s,\rho)} h(s\rho e^{i\theta})^q \log^- \left|f(s\rho e^{i\theta})\right|,$$

where $h(sz)$ is the (smooth) extension of $h = h_j$ to the disc.

Now we use the "substitution" lemma 10.4 and we get

$$\sum_{j\in I(\rho)} \int_{F_j} |A_j| \leq c_7 2^q q(1-q)(1-\rho)^{p-q-1} \int_{F(s,\rho)} h(s\rho e^{i\theta})^q \log^+ \left|f(s\rho e^{i\theta})\right|,$$

We proceed exactly the same way with

$$B_j := q(1-\rho)^p h(sz)^{q-1} \{ s^2(1-\rho) \frac{1}{\delta_j^{1-\epsilon}} \chi'' H - 2s(p+1)(1+\chi' \delta_j^\epsilon H) +$$

$$+ s\frac{1}{\rho}(1-\rho)(1+\chi'\delta_j^\epsilon H) - \frac{1}{\rho^2}(1-\rho)\frac{1}{\delta_j^{1-\epsilon}}(\chi H'')\} \log^- |f(sz)|.$$

So because $(1+\chi'\delta_j^\epsilon H) \geq 0$, we ignore the negative term:

$$q(1-\rho)^p h(sz)^{q-1} \{-2s(p+1)(1+\chi'\delta_j^\epsilon H)\},$$

and we get

$$B_j \leq B_j' := q(1-\rho)^p h(sz)^{q-1} \{ s^2(1-\rho) \frac{1}{\delta_j^{1-\epsilon}} \chi'' H +$$

$$+ s\frac{1}{\rho}(1-\rho)(1+\chi'\delta_j^\epsilon H) - \frac{1}{\rho^2}(1-\rho)\frac{1}{\delta_j^{1-\epsilon}}(\chi H'')\} \log^- |f(sz)|.$$

So still with $(1-\rho) \leq h(sz)$, we have

$$\left|B_j'\right| \leq c_8 q(1-\rho)^{p-1+\epsilon} h(sz)^q \log^- |f(sz)|.$$

And we get

$$\sum_{j\in I(\rho)} \int_{F_j} B_j \leq \sum_{j\in I(\rho)} \int_{F_j} \left|B_j'\right| \leq (1-\rho)^{p-1+\epsilon} \int_{F(s,\rho)} h(s\rho e^{i\theta})^q \log^- |f(sz)|. \tag{5.5}$$

So we have

Proposition 5.6 *We have, for $q \leq 1$, for any $\gamma \leq u < 1$, and $\rho < u$,*

$$\sum_{j\in I(\rho)} \int_{F_j} (A_j + B_j) \leq c_9 c_3(0,\gamma)(1-\rho)^{p-q-1} \int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^+ \left|f(s\rho e^{i\theta})\right|.$$

And for $q \leq 1$ and $\rho \geq u \geq \gamma$,

$$\sum_{j\in I(\rho)} \int_{F_j} (A_j + B_j) \leq c_6 q(1-\rho)^{p-1+\epsilon} \int_{F(s,\rho)} h(s\rho e^{i\theta})^q \log^- \left|f(s\rho e^{i\theta})\right|.$$

Proof.
We get the second assertion directly from (5.5). For the first one we use the "substitution" lemma 10.4. ■

5.2 Case $q > 1$.

Then $A_j \leq 0$ and can be ignored and it remains

$$B_j := q(1-\rho)^p h(sz)^{q-1}\{s^2(1-\rho)\frac{1}{\delta_j^{1-\epsilon}}\chi'' H + s\frac{1}{\rho}(1-\rho)(1+\chi' H)\}\log^- |f(sz)| -$$
$$-q(1-\rho)^p h(sz)^{q-1}\{2s(p+1)(1+\chi' H) + \frac{1}{\rho^2}(1-\rho)\frac{1}{\delta_j^{1-\epsilon}}(\chi H'')\}\log^- |f(sz)|\,,$$

which, ignoring the negative term:

$$-q(1-\rho)^p h(sz)^{q-1} 2s(p+1)(1+\chi' H)\log^- |f(sz)|\,,$$

because $(1+\chi' H) \geq 0$, leads to

$$B_j \leq q(k+1)c_5(1-\rho)^{p+\epsilon} h(sz)^{q-1}\log^- |f(sz)|\,,$$

with again $c_5 := \sup_t(|\chi''(t)| + \frac{1}{\rho} + \frac{1}{\rho^2}|H''(t)|)$.
Using again $(1-\rho) \leq h(sz)$ we get

$$B_j \leq q(k+1)c_5(1-\rho)^{p-1+\epsilon} h(sz)^q \log^- |f(sz)|\,.$$

Proposition 5.7 *We have, for $q > 1$, for any $u < 1$, and for any $\rho \geq u$,*

$$\sum_{j\in I(\rho)} \int_{F_j} B_j \leq c_9(1-\rho)^{p-1+\epsilon} \int_{F(s,\rho)} h(s\rho e^{i\theta})^q \log^- \left|f(s\rho e^{i\theta})\right|.$$

And for $q > 1$ and $\rho < u$,

$$\sum_{j\in I(\rho)} \int_{F_j} B_j \leq c_5 2^q q(k+1)(1-\rho)^{p-q-1+\epsilon} \int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^+ \left|f(s\rho e^{i\theta})\right|,$$

Proof.
We have for any $\rho < 1$,

$$\sum_{j\in I(\rho)} \int_{F_j} B_j \leq c_5 q(k+1)(1-\rho)^{p-1+\epsilon} \int_{F(s,\rho)} h(s\rho e^{i\theta})^q \log^- \left|f(s\rho e^{i\theta})\right|,$$

which gives the first assertion.
We use the "substitution" lemma 10.4 to get

$$\int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^- \left|f(s\rho e^{i\theta})\right| \leq (\frac{2}{(1-\rho)})^q \int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^+ \left|f(s\rho e^{i\theta})\right|,$$

hence

$$\sum_{j\in I(\rho)} \int_{F_j} B_j \leq c_5 q(k+1)(1-\rho)^{p-1+\epsilon}(\frac{2}{(1-\rho)})^q \int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^+ \left|f(s\rho e^{i\theta})\right|,$$

so

$$\sum_{j\in I(\rho)}\int_{F_j} B_j \le c_5 2^q q(k+1)(1-\rho)^{p-q-1+\epsilon}\int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^+\left|f(s\rho e^{i\theta})\right|,$$

which proves the proposition. ■

Finally set

$$C(s,z) := -(p+1)(1-\rho)^{p-1}h(sz)^q\{p-\frac{1}{\rho}(1-\rho)\}\log^-|f(sz)|\,.$$

Proposition 5.8 *We set* $\rho(p) := \dfrac{2}{2+p} \iff \dfrac{(1-\rho)}{\rho} = p/2$. *Now fix* $u :: \rho(p) \le u < 1$, *then for any* $\rho \ge u \ge \rho(p)$,

$$\int_{F(s,\rho)} C(s,\rho e^{i\theta}) \le -\frac{p}{2}(p+1)\int_{F(s,\rho)} (1-\rho)^{p-1}h(s\rho e^{i\theta})^q \log^-\left|f(s\rho e^{i\theta})\right|.$$

And for $\rho < u$,

$$\int_{F(s,\rho)} C(s,\rho e^{i\theta}) \le (p+1)2^q(1-\rho)^{p-1-q}\int_{\mathbb{T}} h(sz)^q \log^+\left|f(s\rho e^{i\theta})\right|.$$

Proof.

Because $p > 0$, the condition is valid and we have the first assertion.

For the second one we apply the "substitution" lemma 10.4 as we already did. ■

Proposition 5.9 *We have with* $T^2_-(s,\rho) := \displaystyle\int_{F(\rho,s)} A_-(s,\rho e^{i\theta})$ *that it exists* $u = u(p,q) < 1$ *such that:*

- *For* $\rho < u$
$$T^2_-(s,\rho) \le c_{12}(1-\rho)^{p-q-1}\int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^+\left|f(s\rho e^{i\theta})\right|.$$
- *For* $\rho \ge u$
$$T^2_-(s,\rho) \le 0.$$

Proof.

In the case $q \le 1$, we fix $\gamma \le u < 1$ and we have by proposition 5.6 for $\rho < u$,

$$\sum_{j\in I(\rho)}\int_{F_j}(A_j+B_j) \le c_9c_3(0,\gamma)(1-\rho)^{p-q-1}\int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^+\left|f(s\rho e^{i\theta})\right|.$$

By proposition 5.8 we have for $\rho < u$ with $u \ge \rho(p)$

$$\int_{F(s,\rho)} C(s\rho e^{i\theta}) \le (p+1)2^q(1-\rho)^{p-1-q}\int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^+\left|f(s\rho e^{i\theta})\right|.$$

So for $\rho < u$, with $u \ge \max(\rho(p),\ \gamma)$,

$$\begin{aligned} T^2_-(s,\rho) := \int_{F(\rho,s)} A_-(s,\rho e^{i\theta})d\theta &\le c_9c_3(0,\gamma)(1-\rho)^{p-q-1}\int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^+\left|f(s\rho e^{i\theta})\right| + \\ &+(p+1)2^q(1-\rho)^{p-1-q}\int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^+\left|f(s\rho e^{i\theta})\right| \le \\ &\le c_{10}(1-\rho)^{p-q-1}\int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^+\left|f(s\rho e^{i\theta})\right|. \end{aligned}$$

For $\rho > u$, still with $u \ge \max(\rho(p),\ \gamma)$, we have by proposition 5.6

$$\sum_{j\in I(\rho)}\int_{F_j}(A_j+B_j)\le c_6q(1-\rho)^{p-1+\epsilon}\int_{F(s,p)}h(s\rho e^{i\theta})^q\log^-\left|f(s\rho e^{i\theta})\right|,$$

By proposition 5.8 we have

$$\int_{F(s,\rho)}C(s\rho e^{i\theta})\le-\frac{p}{2}(p+1)\int_{F(s,\rho)}(1-\rho)^{p-1}h(s\rho e^{i\theta})^q\log^-\left|f(s\rho e^{i\theta})\right|.$$

Hence adding we get

$$T_-^2(s,\rho)\le c_6q(1-\rho)^{p-1+\epsilon}\int_{F(s,\rho)}h(s\rho e^{i\theta})^q\log^-\left|f(s\rho e^{i\theta})\right|-$$
$$-\frac{p}{2}(p+1)(1-\rho)^{p-1}\int_{F(s,\rho)}h(s\rho e^{i\theta})^q\log^-\left|f(s\rho e^{i\theta})\right|.$$

Now we choose $1-u$ so small than $c_6q(1-u)^\epsilon-\frac{p}{2}(p+1)\le 0$ which is possible because $\epsilon>0,\ p>0$, then we get

$$\forall\rho>u\ge\max(\rho(p),\ \gamma),\ T_-^2(s,\rho)\le 0.$$

Now if $q>1$ we have by proposition 5.7 because there $A_j\le 0$, for $\rho<u$,

$$\sum_{j\in I(\rho)}\int_{F_j}(A_j+B_j)\le c_5 2^q q(k+1)(1-\rho)^{p-q-1+\epsilon}\int_{\mathbb{T}}h(s\rho e^{i\theta})^q\log^+\left|f(s\rho e^{i\theta})\right|,$$

and again by proposition 5.8 for $\rho<u$,

$$\int_{F(s,\rho)}C(s\rho e^{i\theta})\le(p+1)2^q(1-\rho)^{p-1-q}\int_{\mathbb{T}}h(s\rho e^{i\theta})^q\log^+\left|f(s\rho e^{i\theta})\right|.$$

Hence adding for $\rho<u$, with $u\ge\rho(p)$,

$$T_-^2(s,\rho)\le c_{11}(1-\rho)^{p-q-1}\int_{\mathbb{T}}h(s\rho e^{i\theta})^q\log^+\left|f(s\rho e^{i\theta})\right|.$$

For $\rho>u$, by proposition 5.7 because there $A_j\le 0$,

$$\sum_{j\in I(\rho)}\int_{F_j}(A_j+B_j)\le c_9(1-\rho)^{p-1+\epsilon}\int_{F(s,\rho)}h(s\rho e^{i\theta})^q\log^-\left|f(s\rho e^{i\theta})\right|.$$

By proposition 5.8 we have for $\rho\ge u\ge\rho(p)$,

$$\int_{F(s,\rho)}C(s\rho e^{i\theta})\le-\frac{p}{2}(p+1)(1-\rho)^{p-1}\int_{F(s,\rho)}h(s\rho e^{i\theta})^q\log^-\left|f(s\rho e^{i\theta})\right|.$$

Hence adding we get

$$T_-^2(s,\rho)\le c_9(1-\rho)^{p-1+\epsilon}\int_{F(s,\rho)}h(s\rho e^{i\theta})^q\log^-\left|f(s\rho e^{i\theta})\right|-$$
$$-\frac{p}{2}(p+1)(1-\rho)^{p-1}\int_{F(s,\rho)}h(s\rho e^{i\theta})^q\log^-\left|f(s\rho e^{i\theta})\right|.$$

Now we choose $1-u$ so small than $c_9q(1-u)^\epsilon-\frac{p}{2}(p+1)\le 0$ which is possible because $\epsilon>0,\ p>0$, then we get

$$\forall\rho>u\ge\rho(p),\ T_-^2(s,\rho)\le 0.$$

This ends the proof, with suitable constants. ■

Now we have to deal with

$$T_-^1(s,\rho):=\int_{\mathbb{T}\setminus F(\rho,s)}A_-(s,\rho e^{i\theta}).$$

Here we have $g_s(z)=(1-\rho)^{p+1}(1-s\rho)^q$ so

$$\Delta g_s(z) = s^2q(q-1)(1-\rho)^{p+1}(1-s\rho)^{q-2} + q(1-\rho)^p(1-s\rho)^{q-1}\{2s(p+1) - s\frac{1}{\rho}(1-\rho)\}+$$
$$+(p+1)(1-\rho)^{p-1}(1-s\rho)^q\{p - \frac{1}{\rho}(1-\rho)\}.$$

So we have

$$A_-(s,\rho e^{i\theta}) = -B(s,\rho)\log^-|f(sz)| + \{q(1-\rho)^p(1-s\rho)^{q-1}s\frac{1}{\rho}(1-\rho)+$$
$$+(p+1)(1-\rho)^{p-1}(1-s\rho)^q\{\frac{1}{\rho}(1-\rho) - p\}\log^-\left|f(s\rho e^{i\theta})\right|,$$

where $-B(s,\rho)\log^-\left|f(s\rho e^{i\theta})\right|$ is negative so it can be ignored. For the positive ones, because of the $s\frac{1}{\rho}(1-\rho)$ we can proceed as above using the fact that $p>0$ and we have

$$T^1_-(s,\rho) := \int_{\mathbb{T}\setminus F(\rho,s)} A_-(s,\rho e^{i\theta}) \le c_{14}(1-\rho)^{p-1}\int_{\mathbb{T}}\varphi(s\rho e^{i\theta})\log^+\left|f(s\rho e^{i\theta})\right|.$$

So we proved

Proposition 5.10 *We have*

$$T^1_-(s,\rho) \le c_{14}(1-\rho)^{p-1}\int_{\mathbb{T}}\varphi(s\rho e^{i\theta})\log^+\left|f(s\rho e^{i\theta})\right|.$$

And the proposition

Proposition 5.11 *We have:*

$$\int_{\mathbb{D}} A_-(s,z) \le c(s)\int_{\mathbb{D}}(1-|z|)^{p-1}\varphi(sz)\log^+|f(sz)|.$$

Proof.

We have just to add and integrate T^1_- and T^2_-. ■

Now grouping all the terms, we proved

Theorem 5.12 *We have*

$$\forall s<1,\ \sum_{a\in Z(f_s)} g_s(a) = \int_{\mathbb{D}}\log|f(sz)|\,\triangle g_s(z) \le c_{15}\int_{\mathbb{D}}(1-|z|)^{p-1}h(sz)^q\log^+|f(sz)|.$$

so we are lead to

Definition 5.13 *Let $E=\bar{E}\subset\mathbb{T}$. Set $\varphi(z)=h(z)^q$. We say that an holomorphic function f such that $|f(0)|=1$ is in the generalised Nevanlinna class $\mathcal{N}_{\varphi,p}(\mathbb{D})$ for $p>0$ if $\exists\delta>0,\ \delta<1$ such that*

$$\|f\|_{\mathcal{N}_{\varphi,p}} := \sup_{1-\delta<s<1}\int_{\mathbb{D}}(1-|z|)^{p-1}\varphi(sz)\log^+|f(sz)| < \infty.$$

And we proved the Blaschke type condition, using lemma 10.6 with $\varphi(z)=h(z)^q$:

Theorem 5.14 *Let $E=\bar{E}\subset\mathbb{T}$. Suppose $q>0$ and $f\in\mathcal{N}_{\varphi,p}(\mathbb{D})$ with $|f(0)|=1$, then*

$$\sum_{a\in Z(f)}(1-|a|^2)^{1+p}\varphi(a) \le c(\varphi)\|f\|_{\mathcal{N}_{\varphi,p}}.$$

Corollary 5.15 *Let $E = \bar{E} \subset \mathbb{T}$. Suppose $q \in \mathbb{R}$ and $f \in \mathcal{N}_{d(\cdot,E)^q,p}(\mathbb{D})$ with $|f(0)| = 1$, then, $\forall \epsilon > 0$,*

$$\sum_{a\in Z(f)} (1-|a|^2)^{1+p} d(a,E)^{(q)_+(1+\epsilon)} \leq c(\varphi,\epsilon)\|f\|_{\mathcal{N}_{d(\cdot,E)^q,p}}.$$

Proof.
By lemma 5.1 we have $d(z,E)^{1+\epsilon} \lesssim h(z) \lesssim d(z,E)$, so we get for $q \geq 0$, $\|f\|_{\mathcal{N}_{h(\cdot)^q,p}} \lesssim \|f\|_{\mathcal{N}_{d(\cdot,E)^q,p}}$ and $d(a,E)^{1+\epsilon} \lesssim h(a)$.

For $q < 0 \Rightarrow (q)_+ = 0$ and we have to see that $\|f\|_{\mathcal{N}_{1,p}} \lesssim \|f\|_{\mathcal{N}_{d(\cdot,E)^q,p}}$ which is given by the same proof as for corollary 4.5. ■

6 Case E infinite and $p = 0$.

This time we have $g_s(\rho e^{i\theta}) = (1-\rho)h(sz)^q$. So, because $g_s = 0$ on $\mathbb{T}$,

$$\sum_{a\in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log|f(sz)|\, \triangle g_s(z) - \int_{\mathbb{T}} \log\left|f(se^{i\theta})\right| \partial_n g_s(e^{i\theta}).$$

Here we have $\partial_n g_s(z) = -h(sz)^q + (1-\rho)\partial_n(h(sz)^q)$, hence, on $\mathbb{T}$, $\partial_n g_s(z) = -h(sz)^q$. So

$$\forall s < 1,\ \sum_{a\in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log|f(sz)|\, \triangle g_s(z) + \int_{\mathbb{T}} h(se^{i\theta})^q \log\left|f(se^{i\theta})\right|.$$

With the definition of $h(z)$ and with $I(\rho) := \{j \in \mathbb{N} :: \delta_j \geq (1-\rho)/(k+1)\}$, we have, for $z = e^{i\theta} \in \mathbb{T}$,

$$\forall j \in I(s),\ \forall \theta \in F_j,\ h(se^{i\theta}) = 1 - s + \chi(\frac{1-s}{\delta_j})\delta_j^{1+\epsilon} H(\theta/\delta_j)\ ;$$

and

$$\forall j \notin I(s),\ \forall \theta \in F_j,\ h(se^{i\theta}) = 1 - s \text{ and } \forall \theta \in E,\ h(se^{i\theta}) = 1 - s.$$

So we set

$$T(s) := \int_{\mathbb{T}} h(se^{i\theta})^q \log\left|f(se^{i\theta})\right| =$$

$$= \int_{\mathbb{T}} h(se^{i\theta})^q \log^+\left|f(se^{i\theta})\right| - \int_{\mathbb{T}} h(se^{i\theta})^q \log^-\left|f(se^{i\theta})\right| =: T_+(s) - T_-(s). \tag{6.6}$$

We shall proceed the same way as in section 4. We shall set

$$P_{\mathbb{T},+}(t_0) := \sup_{1-\delta\leq s\leq t_0} \int_{\mathbb{T}} h(se^{i\theta})^q \log^+\left|f(se^{i\theta})\right|.$$

and

$$P_{\mathbb{T},-}(t_0) := \sup_{1-\delta\leq s\leq t_0} \int_{\mathbb{T}} h(se^{i\theta})^q \log^-\left|f(se^{i\theta})\right|.$$

Because $\gamma(s) := \int_{\mathbb{T}} h(se^{i\theta})^q \log^-\left|f(se^{i\theta})\right|$ is continuous for $s \in (1-\delta, t_0)$ by lemma 10.3 in the appendix, the sup is achieved for a $s_0 \in [1-\delta, t_0]$ and we have

$$P_{\mathbb{T},-}(t_0) = \int_{\mathbb{T}} h(s_0e^{i\theta})^q \log^-\left|f(s_0e^{i\theta})\right| d\theta.$$

The proposition 5.2 gives, with $p = 0$,

$$\Delta g_s(z) = q(q-1)(1-\rho)h(sz)^{q-2}\{s^2(1+\chi'\delta_j^\epsilon H)^2 + \frac{1}{\rho^2}(\chi\delta_j^\epsilon H')^2\}+$$

$$+qh(sz)^{q-1}\{-s^2(1-\rho)\frac{1}{\delta_j^{1-\epsilon}}\chi''H+2s(1+\chi'\delta_j^\epsilon H)-$$
$$-s\frac{1}{\rho}(1-\rho)(1+\chi'\delta_j^\epsilon H)+\frac{1}{\rho^2}(1-\rho)\frac{1}{\delta_j^{1-\epsilon}}(\chi H'')\}-\frac{1}{\rho}h(sz)^q.$$

We set, taking the terms with $\log^+|f(sz)|$,

$$A_+(s,z):=\Delta g_s(z)\log^+|f(sz)|\leq[c_1q\,|q-1|\,(1-\rho)h(sz)^{q-2}+c_2qh(sz)^{q-1}+2h(sz)^q]\log^+|f(sz)|\,.$$

with

$$c_1:=\{4+4\|H'\|_\infty^2\}\ ;\ c_2:=\|\chi''\|_\infty+2(1+\|\chi'\|_\infty\ +(1+\|\chi'\|_\infty+4\|H''\|_\infty).$$

But using $(1-\rho)\leq(1-s\rho)\leq h(sz)$, we get

$$A_+(s,z)\leq[c_1q\,|q-1|\,h(sz)^{q-1}+c_2qh(sz)^{q-1}+h(sz)^q]\log^+|f(sz)|\leq$$
$$\leq c_3h(sz)^{q-1}\log^+|f(sz)|+h(sz)^q\log^+|f(sz)|\,,$$

with $c_3:=c_1q\,|q-1|+c_2q$. Then we get

$$A_+(s,z)\leq(c_3+h(sz))h(sz)^{q-1}\log^+|f(sz)|\,.$$

We set also

$$P_{\mathbb{D},+}(s):=\int_{\mathbb{D}}h(sz)^{q-1}\log^+|f(sz)|\,dm(z).$$

So we proved

Proposition 6.1 *We have*

$$\int_{\mathbb{D}}\log^+|f(sz)|\,\triangle g_s(z)\leq CP_{\mathbb{D},+}(s)$$

with $C:=c_3+4$.

We set, taking the terms with $\log^-|f(sz)|$, $A_-(s,z):=\Delta g_s(z)\log^-|f(sz)|$ and $T_-(s,\rho):=\int_{\mathbb{T}}A_-(s,\rho e^{i\theta})$ and $T_-(s):=\int_0^1T_-(s,\rho)\rho d\rho$.

We fix $\rho<1$; for $\forall z=\rho e^{i\theta}\in\Gamma_j$ and $\delta_j<\dfrac{1-\rho s}{k+1}$, then $\chi=\chi'=0$ hence $h(sz)=1-s\rho$ and we set

$$T_-^1(s,\rho):=\int_{\mathbb{T}\setminus F(\rho,s)}A_-(s,\rho e^{i\theta}),\text{ and }T_-^2(s,\rho):=\int_{F(\rho,s)}A_-(s,\rho e^{i\theta}).$$

Recall that $I(\rho):=\{j\in\mathbb{N}::\delta_j\geq(1-\rho)/(k+1)\}$ and for $j\in I(s\rho)$,

$$\forall z=\rho e^{i\theta}\in\Gamma_j,\ A_j:=-q(q-1)(1-\rho)h(sz)^{q-2}\{s^2(1+\chi'\delta_j^\epsilon H)^2+\frac{1}{\rho^2}(\chi\delta_j^\epsilon H')^2\}\log^-|f(sz)|\,.$$

This can be written

$$A_j:=-q(q-1)(1-\rho)h(sz)^{q-1}h(sz)^{-1}\times\delta_j^{2\epsilon}\{s^2(1+\chi'H)^2+\frac{1}{\rho^2}(\chi H')^2\}\log^-|f(sz)|\,.$$

Now we have two cases.

• **Case** $q\leq1$.

Using proposition 10.4 with $p=0$, we get, for $q\leq1$, for any $\gamma\leq u<1$, and $\rho<u$,

$$\sum_{j\in I(\rho)}\int_{F_j}(A_j+B_j)\leq c_9c_3(0,\gamma)(1-\rho)^{-q-1}\int_{\mathbb{T}}h(s\rho e^{i\theta})^q\log^+\left|f(s\rho e^{i\theta})\right|.\tag{6.7}$$

And for $q\leq1$ and $\rho\geq u\geq\gamma$,

$$\sum_{j\in I(\rho)}\int_{F_j}(A_j+B_j)\leq c_6q(1-\rho)^{-1+\epsilon}\int_{F(s,\rho)}h(s\rho e^{i\theta})^q\log^-\left|f(s\rho e^{i\theta})\right|.\tag{6.8}$$

• **Case** $q > 1$.

By proposition 5.7 we get for $q > 1$, for any $\gamma \le u < 1$, and $\rho < u$,

$$\sum_{j\in I(\rho)} \int_{F_j} (A_j + B_j) \le c_9 c_3(0,\gamma)(1-\rho)^{-q-1} \int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^+ \left|f(s\rho e^{i\theta})\right|. \tag{6.9}$$

And for $q > 1$ and $\rho \ge u \ge \gamma$,

$$\sum_{j\in I(s\rho)} \int_{F_j} (A_j + B_j) \le c_6 q(1-\rho)^{-1+\epsilon} \int_{F(s,\rho)} h(s\rho e^{i\theta})^q \log^- \left|f(s\rho e^{i\theta})\right|. \tag{6.10}$$

We shall integrate in the disc for $s \le t_0$. We get

• **Case** $q \le 1$.

We have:

$$\int_{\mathbb{D}} A^1_-(s,z) \le \int_{D(0,u)} A^1_-(s,z) + \int_{\mathbb{D}\setminus D(0,u)} A^1_-(s,z),$$

so, by (6.7),

$$\int_{D(0,u)} A^1_-(s,z) \le c_6 q \int_0^u (1-\rho)^{-1-q} \Big(\int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^+ \left|f(s\rho e^{i\theta})\right|\Big)\rho d\rho \le$$
$$\le c_6 q \int_0^u (1-\rho)^{-1-q} \rho d\rho P_{\mathbb{T},+}(t_0) \le c_6 (1-u)^{-q} P_{\mathbb{T},+}(t_0).$$

And, by (6.8),

$$\int_{\mathbb{D}\setminus D(0,u)} A^1_-(s,z) \le c_6 q \int_u^1 (1-\rho)^{-1+\epsilon} \Big(\int_{F(s,\rho)} h(s\rho e^{i\theta})^q \log^- \left|f(s\rho e^{i\theta})\right|\Big)\rho d\rho \le$$
$$\le c_6 q \int_u^1 (1-\rho)^{-1+\epsilon} \rho d\rho P_{\mathbb{T},-}(t_0) \le c_6 \frac{q}{\epsilon}(1-u)^{\epsilon} P_{\mathbb{T},-}(t_0).$$

And with $C(s,z) := h(s\rho e^{i\theta})^q \dfrac{1}{\rho} \log^- \left|f(s\rho e^{i\theta})\right|$ we have $\displaystyle\int_{\mathbb{D}} C(s,z) = \int_{D(0,u)} C(s,z) + \int_{\mathbb{D}\setminus D(0,u)} C(s,z)$,

so

$$\int_{D(0,u)} C(s,z) \le \int_0^u \{\int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^- \left|f(s\rho e^{i\theta})\right|\}\rho d\rho.$$

But by the substitution lemma 10.4, we get

$$\int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^- \left|f(s\rho e^{i\theta})\right| \le c(u) \int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^+ \left|f(s\rho e^{i\theta})\right| \le c(u) P_{\mathbb{T},+}(t_0),$$

hence

$$\int_{D(0,u)} C(s,z) \le c(u) P_{\mathbb{T},+}(t_0) \int_0^u d\rho \le c(u) u P_{\mathbb{T},+}(t_0). \tag{6.11}$$

And

$$\int_{\mathbb{D}\setminus D(0,u)} C(s,z) = \int_{\mathbb{D}\setminus D(0,u)} h(sz)^q \frac{1}{\rho} \log^- |f(sz)| = \int_u^1 d\rho \{\int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^- \left|f(s\rho e^{i\theta})\right|\} \le$$
$$\le (1-u) P_{\mathbb{T},-}(t_0). \tag{6.12}$$

So finally for $q \leq 1$, we get

$$W_-(t_0) \leq (1-u)P_{\mathbb{T},-}(t_0) + c_6\frac{q}{\epsilon}(1-u)^{\epsilon}P_{\mathbb{T},-}(t_0) + c_6(1-u)^{-q}P_{\mathbb{T},+}(t_0) + c(u)P_{\mathbb{T},+}(t_0).$$

• **Case** $q > 1$.

We have:

$$\int_{\mathbb{D}} A^1_-(s,z) \leq \int_{D(0,u)} A^1_-(s,z) + \int_{\mathbb{D}\setminus D(0,u)} A^1_-(s,z),$$

so, by (6.9)

$$\int_{D(0,u)} A^1_-(s,z) \leq c_9c_3(0,\gamma)\int_0^u (1-\rho)^{-q-1}\{\int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^+\left|f(s\rho e^{i\theta})\right|\}\rho d\rho \leq$$

$$\leq c_9c_3(0,\gamma)P_{\mathbb{T},+}(t_0)\int_0^u (1-\rho)^{-q-1}\rho d\rho \leq c_9c_3(0,\gamma)P_{\mathbb{T},+}(t_0)\frac{1}{q}(1-u)^{-q}.$$

And, by (6.10),

$$\int_{\mathbb{D}\setminus D(0,u)} A^1_-(s,z) \leq c_6q\int_u^1 (1-\rho)^{-1+\epsilon}\{\int_{F(s,\rho)} h(s\rho e^{i\theta})^q \log^-\left|f(s\rho e^{i\theta})\right|\}\rho d\rho \leq$$

$$\leq c_6q\int_u^1 (1-\rho)^{-1+\epsilon}\rho d\rho P_{\mathbb{T},-}(t_0) \leq c_6\frac{q}{\epsilon}(1-u)^{\epsilon}P_{\mathbb{T},-}(t_0).$$

And again with

$$C(s,z) := h(s\rho e^{i\theta})^q\frac{1}{\rho}\log^-\left|f(s\rho e^{i\theta})\right|$$

we have by (6.11) and by (6.12)

$$\int_{\mathbb{D}} C(s,z)dm(z) \leq c(u)uP_{\mathbb{T},+}(t_0) + (1-u)P_{\mathbb{T},-}(t_0).$$

So finally for $q > 1$, we get

$$W_-(t_0) \leq (1-u)P_{\mathbb{T},-}(t_0) + c_6\frac{q}{\epsilon}(1-u)^{\epsilon}P_{\mathbb{T},-}(t_0) + c(u)uP_{\mathbb{T},+}(t_0) + c_9c_3(0,\gamma)\frac{1}{q}(1-u)^{-q}P_{\mathbb{T},+}(t_0).$$

Now we have to compensate the terms in $P_{\mathbb{T},-}(t_0)$ and this can be done only by T_- in (6.6). So recall $T_-(s) := \int_{\mathbb{T}} h(se^{i\theta})^q \log^-\left|f(se^{i\theta})\right|$ and there is a $s_0 \in [0,t_0]$ such that:

$$P_{\mathbb{T},-}(t_0) := \sup_{0<s<t_0}\int_{\mathbb{T}} h(sz)^q \log^-\left|f(se^{i\theta})\right| d\theta = \int_{\mathbb{T}} h(s_0z)^q \log^-\left|f(s_0e^{i\theta})\right| d\theta.$$

As in section 4, the idea is to consider, for $t_0 \in [1-\delta, 1[$,

$$\Sigma(s,t_0) := \sum_{a\in Z(f_s)} (1-|a|)h(sa)^q + \sum_{a\in Z(f_{s_0})} (1-|a|)h(s_0a)^q.$$

We get

$$\Sigma(s,t_0) \leq 2CP_{\mathbb{D},+}(s) + 2c_6(1-u)^{-q}P_{\mathbb{T},+}(t_0) + 2c(u)\frac{1}{\rho}P_{\mathbb{T},+}(t_0) +$$

$$+2(1-u)P_{\mathbb{T},-}(t_0) + 2c_6\frac{q}{\epsilon}(1-u)^{\epsilon}P_{\mathbb{T},-}(t_0) + T_+(s) + T_+(s_0) - T_-(s) - T_-(s_0).$$

But

$$T_-(s_0) := \int_{\mathbb{T}} h(s_0e^{i\theta})^q \log^-\left|f(s_0e^{i\theta})\right| = P_{\mathbb{T},-}(t_0),$$

so we take $u < 1$ such that $2(1-u) + 2c_6\frac{q}{\epsilon}(1-u)^{\epsilon} \leq 1$, which is possible because $\epsilon > 0$, so we get

$$\Sigma(s,t_0) \leq C_1P_{\mathbb{T},+}(t_0) + 2CP_{\mathbb{T},+}(t_0),$$

with the constants $C,\ C_1$ *independent* of t_0. So we get with $s = t_0$,

$$\sum_{a\in Z(f_{t_0})} (1-|a|)h(t_0a)^q \le C_1 P_{\mathbb{T},+}(t_0) + CP_{\mathbb{D},+}(t_0),$$

hence, using lemma 10.6 with $\varphi(z) = h(z)^{q-1},\ \psi(z) = h(z)^q$, we get

$$\sum_{a\in Z(f)} (1-|a|)h(a)^q \le$$

$$\le C_1 \sup_{1-\delta<s<1} \int_{\mathbb{T}} h(se^{i\theta})^q \log^+ \left|f(se^{i\theta})\right| + C \sup_{1-\delta<s<1} \int_{\mathbb{D}} h(sz)^{q-1} \log^+ |f(sz)|,$$

and the following definition and theorem.

Definition 6.2 *Let $E = \bar{E} \subset \mathbb{T}$ and $\varphi(z) = h(z)^q$. We say that an holomorphic function f such that $|f(0)| = 1$ is in the generalised Nevanlinna class $\mathcal{N}_{\varphi,0}(\mathbb{D})$ if $\exists \delta > 0,\ \delta < 1$ such that*

$$\|f\|_{\mathcal{N}_{\varphi,0}} := \sup_{1-\delta<s<1} \int_{\mathbb{T}} \varphi(s\rho e^{i\theta}) \log^+ \left|f(s\rho e^{i\theta})\right| + \sup_{1-\delta<s<1} \int_{\mathbb{D}} \varphi(sz) h(sz)^{-1} \log^+ |f(sz)|.$$

We proved the Blaschke type condition.

Theorem 6.3 *Let $E = \bar{E} \subset \mathbb{T}$ and suppose $q > 0$ and $f \in \mathcal{N}_{\varphi,0}(\mathbb{D})$ with $|f(0)| = 1$, then*

$$\sum_{a\in Z(f)} (1-|a|^2)\varphi(a) \le c(\varphi)\|f\|_{\mathcal{N}_{\varphi,0}}.$$

Corollary 6.4 *Let $E = \bar{E} \subset \mathbb{T}$ and suppose $q \in \mathbb{R}$ and $f \in \mathcal{N}_{d(\cdot,E)^q,0}(\mathbb{D})$ with $|f(0)| = 1$, then*

$$\sum_{a\in Z(f)} (1-|a|^2) d(a,E)^{(q)+(1+\epsilon)} \le c(\varphi,\epsilon)\|f\|_{\mathcal{N}_{d(\cdot,E)^q,0}}.$$

Proof.
This is the same proof as for corollary 5.15. ■

7 The mixed case.

We shall combine the case of the rational function $R(z) = \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ q_j \in \mathbb{R}$ with the case of the closed set $E \subset \mathbb{T}$ to which is associated the function $h(z)$.

For this we shall consider $\varphi(z) := |R(sz)|^2 h(sz)^q$ and $g_s(z) := (1-\rho^2)^{1+p}\varphi(sz)$.
We make the hypothesis that $\forall j = 1,...,n,\ \eta_j \notin E$. We set $2\mu := \min_{j=1,...,n} d(\eta_j, E)$ then we have that $\mu > 0$.

Because

$$\Delta g_s(z) = \Delta[(1-|z|^2)^{p+1}]\varphi(sz) + (1-|z|^2)^{p+1}\Delta[\varphi(sz)] + 8\Re[\partial((1-|z|^2)^{p+1})\bar{\partial}(\varphi(sz))],$$

and

$$\Delta[\varphi(sz)] = s^2 h(sz)^q \Delta[|R(sz)|^2] h(sz)^q + s^2 |R(sz)|^2 \Delta[h(sz)^q] + 8s^2\Re[\bar{\partial}|R(sz)|^2 \times \partial(h(sz)^q)],$$

we are lead to set:

$$A_1 := \frac{1}{2}|R(sz)|^2 \Delta[(1-|z|^2)^{p+1}]h(sz)^q,\ A_2 := \frac{1}{2}h(sz)^q \Delta[(1-|z|^2)^{p+1}]\,|R(sz)|^2$$

so

$$\Delta[(1-|z|^2)^{p+1}]\varphi(sz) = A_1 + A_2.$$

And

$$A_3 := (1-|z|^2)^{p+1} s^2 h(sz)^q \Delta[|R(sz)|^2] h(sz)^q$$
$$A_4 := s^2 (1-|z|^2)^{p+1} |R(sz)|^2 \Delta[h(sz)^q]$$
$$A_5 := 8s^2 (1-|z|^2)^{p+1} \Re[\bar{\partial} |R(sz)|^2 \times \partial(h(sz)^q)]$$
$$A_6 := 8h(sz)^q \Re[\partial((1-|z|^2)^{p+1}) \bar{\partial}(|R(sz)|^2)]$$
$$A_7 := 8 |R(sz)|^2 \Re[\partial((1-|z|^2)^{p+1}) \bar{\partial}(h(sz)^q)] \ ;$$

and we get

$$\Delta g_s(z) = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7.$$

It remains to see that grouping these terms in the right way, this was already treated by the finite case or by the infinite one.

As usual now we group the terms containing $\log^+ |f(sz)|$ and we set

$$A_+(s,z) := \sum_{j=1}^{7} A_j \log^+ |f(sz)|.$$

Set

$$P_+(s,z) := (1-|z|)^{p-1} |R(sz)|^2 h(sz)^q \log^+ |f(sz)| ,$$

and

$$P_{\mathbb{D},+}(s) := \int_{\mathbb{D}} P_+(s,z) dm(z).$$

Proposition 7.1 *We have:*

$$\int_{\mathbb{D}} A_+(s,z) \leq c_+ P_{\mathbb{D},+}(s).$$

Proof.

We first group the terms

$$B_1 := A_1 \log^+ |f(sz)| + A_4 \log^+ |f(sz)| + A_7 \log^+ |f(sz)| ,$$

these terms contain no derivatives of $|R(sz)|^2$ and so verify proposition 5.3 with $P_+(s,z)$, i.e. that

$$\int_{\mathbb{D}} B_1(s,z) \leq c_1(s) P_{\mathbb{D},+}(s).$$

Now we group the terms

$$B_2 := A_2 \log^+ |f(sz)| + A_3 \log^+ |f(sz)| + A_6 \log^+ |f(sz)| ,$$

these terms contain no derivatives of $h(sz)$ and so verify proposition 3.2

$$\int_{\mathbb{D}} B_2(s,z) \leq c_2 P_{\mathbb{D},+}(s),$$

with the same $P_{\mathbb{D},+}(s)$.

So it remains $A_5 \log^+ |f(sz)|$ but again the homogeneity is the right one and we get

$$\int_{\mathbb{D}} A_5(s,z) \leq c_3 P_{\mathbb{D},+}(s). \ \blacksquare$$

Now we group the terms containing $\log^- |f(sz)|$ and we set

$$A_-(s,z) := \sum_{j=1}^{7} A_j \log^- |f(sz)|.$$

Proposition 7.2 *We have:*

$$\int_{\mathbb{D}} A_-(s,z) \leq c_6 P_{\mathbb{D},+}(s).$$

Proof.
We first group the terms

$$B_1 := A_1 \log^- |f(sz)| + A_4 \log^- |f(sz)| + A_7 \log^- |f(sz)|,$$

these terms contain no derivatives of $|R(sz)|^2$ and so verify proposition 5.11:

$$\int_{\mathbb{D}} B_1(s,z) \leq c_1 P_{\mathbb{D},+}(s),$$

still with the same $P_{\mathbb{D},+}(s)$.

Now we group the terms

$$B_2 := A_2 \log^- |f(sz)| + A_3 \log^- |f(sz)| + A_6 \log^- |f(sz)|,$$

these terms contain no derivatives of $h(sz)$ and so verify proposition 3.3:

$$\int_{\mathbb{D}} B_2(s,z) \leq c_2 P_{\mathbb{D},+}(s).$$

So it remains $A_5 \log^- |f(sz)|$, and, in order to separate the points, we consider:

$$\forall j = 1, ..., n,\ G_j := \{z \in \bar{\mathbb{D}} :: \left| \frac{z}{|z|} - \eta_j \right| < \delta\}\ ;\ G := \bigcup_{j=1}^{n} G_j.$$

Then we have

Lemma 7.3 *There are two constants $a(\mu)$, $b(\mu)$, just depending on μ, such that:*

$$\forall z \in G,\ \partial h(sz) \simeq a(\mu).$$

And

$$\forall z \notin G, \bar{\partial}\ |R(sz)|^2 \simeq b(\mu).$$

Proof.
Recall that

$$h(\rho e^{i\theta}) := 1 - \rho + \chi(\frac{1-\rho}{\delta})\delta^{1+\epsilon} H(\theta/\delta),$$

then an easy computation gives $\forall z \in G,\ \partial h(sz) \simeq a(\mu)$ because z is far from E.
And with $R(z) = \prod_{j=1}^{n} (z - \eta_j)^{q_j}$, again an easy computation gives $\forall z \notin G, \bar{\partial}\ |R(sz)|^2 \simeq b(\mu)$ because z is far from $\bigcup_{j=1}^{n} \{\eta_j\}$. ■

We can treat the $A_5 \log^- |f(sz)|$ term easily now ; recall

$$A_5 \log^- |f(sz)| := 8s^2(1-|z|^2)^{p+1} \Re[\bar{\partial} |R(sz)|^2 \times \partial(h(sz)^q)] \log^- |f(sz)|\ ;$$

cut the disc $\mathbb{D} = G \cup (\mathbb{D} \backslash G)$, so

$$\int_{\mathbb{D}} A_5 \log^- |f(sz)| = \int_G A_5 \log^- |f(sz)| + \int_{\mathbb{D}\backslash G} A_5 \log^- |f(sz)|.$$

On G we have, by lemma 7.3, $\partial h(sz) \simeq a(\mu)$ and we win a $(1-|z|^2)$ so we can apply the substitution lemma 10.4 to get

$$\int_G A_5 \log^- |f(sz)| \leq c_5 P_{\mathbb{D},+}(s).$$

On $\mathbb{D}\backslash G$ we have, by lemma 7.3, $\bar{\partial}\ |R(sz)|^2 \simeq b(\mu)$ and we win again a $(1-|z|^2)$ so we can apply the substitution lemma 10.1 to get

$$\int_{\mathbb{D}\backslash G} A_5 \log^- |f(sz)| \leq c_5' P_{\mathbb{D},+}(s),$$

so finally we get

$$\int_{\mathbb{D}} A_-(s,z) \leq c_6 P_{\mathbb{D},+}(s),$$

which ends the proof of the proposition. ■

So we are lead to

Definition 7.4 *Let* $E = \bar{E} \subset \mathbb{T}$ *and* $R(z) = \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ q_j \in \mathbb{R}$ *with* $\forall j = 1,...,n,\ \eta_j \notin E$. *Set* $\varphi(z) = |R(z)|^2 h(z)^q$. *We say that an holomorphic function* f *such that* $|f(0)| = 1$ *is in the generalised Nevanlinna class* $\mathcal{N}_{\varphi,p}(\mathbb{D})$ *if* $\exists \delta > 0,\ \delta < 1$ *such that*

$$\|f\|_{\mathcal{N}_{\varphi,p}} := \sup_{1-\delta<s<1} \int_{\mathbb{D}} (1-|z|)^{p-1} \varphi(sz) \log^+ |f(sz)|.$$

And we have the Blaschke type condition, still using lemma 10.6 from the appendix, with $\varphi(z) = |R(z)|^2 h(z)^q$:

Theorem 7.5 *Let* $E = \bar{E} \subset \mathbb{T}$ *and* $R(z) = \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ q_j \in \mathbb{R},\ q_j > p/4$, *with* $\forall j = 1,...,n,\ \eta_j \notin E$. *Suppose* $q > 0$ *and* $f \in \mathcal{N}_{\varphi,p}(\mathbb{D})$ *with* $|f(0)| = 1$, *then*

$$\sum_{a \in Z(f)} (1-|a|^2)^{1+p} \varphi(a) |R(a)|^2 \leq c(\varphi) \|f\|_{\mathcal{N}_{\varphi,p}}.$$

As for the case of the rational function R only, we get the

Corollary 7.6 *Let* $E = \bar{E} \subset \mathbb{T}$ *and* $R(z) = \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ q_j \in \mathbb{R}$, *with* $\forall j = 1,...,n,\ \eta_j \notin E$. *Let* $\forall j = 1,...,n$, *if* $q_j > -p/2,\ \tilde{q}_j = q_j$ *else choose* $\tilde{q}_j > -p/2$ *and set* $\tilde{R}(z) := \prod_{j=1}^{n} (z-\eta_j)^{\tilde{q}_j}$, *and* $\varphi(z) = |R(z)|\, h(z)^q,\ \tilde{\varphi}(z) = \left|\tilde{R}(z)\right| h(z)^q$. *Suppose* $q > 0$ *and* $f \in \mathcal{N}_{\varphi,p}(\mathbb{D})$ *with* $|f(0)| = 1$, *then*

$$\sum_{a \in Z(f)} (1-|a|^2)^{1+p} \tilde{\varphi}(a) \leq c(\varphi) \|f\|_{\mathcal{N}_{\varphi,p}}.$$

Corollary 7.7 *Let* $E = \bar{E} \subset \mathbb{T}$ *and* $R(z) = \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ q_j \in \mathbb{R}$, *with* $\forall j = 1,...,n,\ \eta_j \notin E$. *Let* $\forall j = 1,...,n$, *if* $q_j > -p/2,\ \tilde{q}_j = q_j$ *else choose* $\tilde{q}_j > -p/2$ *and set* $\tilde{R}(z) := \prod_{j=1}^{n} (z-\eta_j)^{\tilde{q}_j}$, *and* $\varphi(z) = |R(z)|\, d(z,E)^q,\ \tilde{\varphi}_\epsilon(z) = \left|\tilde{R}(z)\right| d(z,E)^{(q)_+(1+\epsilon)}$. *Suppose* $f \in \mathcal{N}_{\varphi,p}(\mathbb{D})$ *with* $|f(0)| = 1$, *then*

$$\sum_{a \in Z(f)} (1-|a|^2)^{1+p} \tilde{\varphi}_\epsilon(a) \leq c(\varphi,\epsilon) \|f\|_{\mathcal{N}_{\varphi,p}}.$$

Proof.
Still using that $d(z,E)^{1+\epsilon} \lesssim h(z) \lesssim d(z,E)$ by lemma 5.1, and copying the proof of corollary 6.4 we are done. ■

We proceed exactly the same way for the case $p = 0$ to set, with $\gamma(z) := \sum_{j=1}^{n} |q_j| \, |z - \eta_j|^{-1}$:

Definition 7.8 *Let* $E = \bar{E} \subset \mathbb{T}$ *and* $R(z) = \prod_{j=1}^{n} (z - \eta_j)^{q_j}$, $q_j \in \mathbb{R}$ *with* $\forall j = 1, ..., n$, $\eta_j \notin E$. *Set* $\varphi(z) = |R(z)|^2 \, h(z)^q$. *We say that an holomorphic function* f *such that* $|f(0)| = 1$ *is in the generalised Nevanlinna class* $\mathcal{N}_{\varphi,0}(\mathbb{D})$ *if* $\exists \delta > 0$, $\delta < 1$ *such that*

$$\|f\|_{\mathcal{N}_{\varphi,0}} := \sup_{1-\delta<s<1} \int_{\mathbb{T}} \varphi(se^{i\theta}) \log^+ \left|f(se^{i\theta})\right| + \sup_{1-\delta<s<1} \int_{\mathbb{D}} \varphi(z)\gamma(z)h(z)^{-1} \log^+ |f(z)|.$$

And we have the Blaschke type condition, still using lemma 10.6 from the appendix, with $\varphi(z) = |R(z)|^2 \, h(z)^q \gamma(z) h(z)^{-1}$ and $\psi(z) = |R(z)|^2 \, h(z)^q$:

Theorem 7.9 *Let* $E = \bar{E} \subset \mathbb{T}$ *and* φ *as above. Suppose* $q > 0$ *and* $f \in \mathcal{N}_{\varphi,0}(\mathbb{D})$ *with* $|f(0)| = 1$, *then*

$$\sum_{a \in Z(f)} (1 - |a|^2) \varphi(a) \, |R(a)|^2 \leq c(\varphi) \|f\|_{\mathcal{N}_{\varphi,0}}.$$

Corollary 7.10 *Let* $E = \bar{E} \subset \mathbb{T}$ *and* $R(z) = \prod_{j=1}^{n} (z - \eta_j)^{q_j}$, $q_j \in \mathbb{R}$, *with* $\forall j = 1, ..., n$, $\eta_j \notin E$. *Suppose* $\varphi(z) := |R(z)| \, d(z,E)^q$ *and* $f \in \mathcal{N}_{\varphi,0}(\mathbb{D})$ *with* $|f(0)| = 1$, *and set* $\tilde{R}(z) := \prod_{j=1}^{n} (z - \eta_j)^{(q_j)_+}$, *then*

$$\sum_{a \in Z(f)} (1 - |a|^2) d(a,E)^{(q)_+ (1+\epsilon)} \left|\tilde{R}(a)\right|^2 \leq c(\varphi, \epsilon) \|f\|_{\mathcal{N}_{\varphi,0}}.$$

Proof.
Again using that $d(z,E)^{1+\epsilon} \lesssim h(z) \lesssim d(z,E)$ by lemma 5.1, and copying the proof of corollary 4.5 we are done. ■

8 Application : L^∞ estimates.

8.1 Case E finite.

We shall retrieve some of the results of Boritchev, Golinski and Kupin [3], [4].

Suppose the function f verifies $|f(z)| \leq \exp \frac{D}{|R(z)|}$ with $R(z) := \prod_{j=1}^{n} (z - \eta_j)^{q_j}$.

We deduce that $|R(z)| \log |f(z)|$ is in $L^1(\mathbb{T})$ with a better exponent of almost 1 over the rational function R. Precisely set

$$\forall \epsilon \geq 0,\ R_\epsilon(z) := \prod_{j=1}^{n} (z-\eta_j)^{q_j-1+\epsilon},$$

we have:

Lemma 8.1 *If the function f verifies $|f(z)| \leq \exp \frac{D}{|R(z)|}$ with $R(z) := \prod_{j=1}^{n} (z-\eta_j)^{q_j}$, we have*

$$\forall \epsilon > 0,\ \int_{\mathbb{T}} \left|R_\epsilon(e^{i\theta})\right| \log^+ \left|f(e^{i\theta})\right| \leq DC(\delta,\epsilon).$$

Proof.
The hypothesis gives $|R(z)| \log^+ |f(z)| \leq D$ and

$$\frac{R_\epsilon(z)}{R(z)} = \prod_{j=1}^{n} \frac{(z-\eta_j)^{q_j-1+\epsilon}}{(z-\eta_j)^{q_j}} = \prod_{j=1}^{n} (z-\eta_j)^{-1+\epsilon},$$

so

$$|R_\epsilon(z)| \log^+ |f(z)| \leq \frac{|R_\epsilon(z)|}{|R(z)|} D \leq D \prod_{j=1}^{n} (z-\eta_j)^{-1+\epsilon}.$$

Because the points $\{\eta_k\}$ are separated on the torus $\mathbb{T}$ by $\delta > 0$ say and $|z-\eta_j|^{-1+\epsilon}$ is integrable for the Lebesgue measure on the torus $\mathbb{T}$ because $\epsilon > 0$, we get:

$$\int_{\mathbb{T}} \frac{\left|R_\epsilon(e^{i\theta})\right|}{|R(e^{i\theta})|} \left|R(e^{i\theta})\right| \log^+ \left|f(e^{i\theta})\right| \leq D \int_{\mathbb{T}} \prod_{j=1}^{n} \left|e^{i\theta}-\eta_j\right|^{-1+\epsilon} \leq DC(\delta,\epsilon). \blacksquare$$

Then we have:

Theorem 8.2 *Suppose the holomorphic function f verifies $|f(z)| \leq \exp \frac{D}{(1-|z|^2)^p |R(z)|}$ with $R(z) := \prod_{j=1}^{n} (z-\eta_j)^{q_j}$, $q_j \in \mathbb{R}$, then we have:*

for $p=0$, we set $\tilde{R}_\epsilon(z) := \prod_{j=1}^{n} (z-\eta_j)^{(q_j-1+\epsilon)_+}$ and we get:

$$\sum_{a \in Z(f)} (1-|a|) \left|\tilde{R}_\epsilon(a)\right| \leq Dc(\epsilon,p,R).$$

For $p > 0$, $\forall j = 1,...,n$, if $q_j - 1 > -p/2$ set $\tilde{q}_j = q_j$ else choose $\tilde{q}_j > 1 - p/2$, and set $\tilde{R}_0(z) := \prod_{j=1}^{n} (z-\eta_j)^{\tilde{q}_j - 1}$, then:

$$\forall \epsilon > 0,\ \sum_{a \in Z(f)} (1-|a|)^{1+p+\epsilon} \left|\tilde{R}_0(a)\right| \leq Dc(\epsilon,R).$$

We shall cut the proof in two cases.

- **Case** $p = 0$.

We shall apply the corollary 4.5 with R_ϵ instead of R.
To apply corollary 4.5 we have to show that

$$\sup_{s<1}\int_{\mathbb{D}}|R_\epsilon(sz)|\,s\left|\sum_{j=1}^{n}q_j(z-\eta_j)^{-1}\right|\log^+|f(sz)|<\infty$$

and

$$\sup_{s<1}\int_{\mathbb{T}}\left|R_\epsilon(se^{i\theta})\right|\log^+\left|f(se^{i\theta})\right|<\infty.$$

The hypothesis gives $|R(z)|\log^+|f(z)|\le D$ so we get

$$|R_\epsilon(sz)|\log^+|f(sz)|\le D\prod_{j=1}^{n}|1-s\bar{\eta}_jz|^{-1+\epsilon},$$

because, as already seen, $\dfrac{R_\epsilon(sz)}{R(sz)}=\prod_{j=1}^{n}(1-s\bar{\eta}_jz)^{-1+\epsilon}$,

so we get:

$$|R_\epsilon(sz)|\sum_{k=1}^{n}|1-s\bar{\eta}_kz|^{-1+\epsilon}\log^+|f(z)|\le 2D\,|q|\sum_{k=1}^{n}\prod_{j\neq k}\left(|1-s\bar{\eta}_jz|^{-1+\epsilon}\right)|1-s\bar{\eta}_kz|^{-2+\epsilon}.$$

Because the points $\{\eta_k\}$ are separated by δ and $|1-\bar{\eta}_jz|^{-2+\epsilon}$ is integrable for the Lebesgue measure on the disc $\mathbb{D}$ because $\epsilon>0$, we get:

$$\sup_{s<1}\int_{\mathbb{D}}|R_\epsilon(sz)|\,s\sum_{j=1}^{n}|q_j|\,|1-s\bar{\eta}_jz|^{-1}\log^+|f_s|\,dm(z)\le 2D\,|q|\,c(\delta,\epsilon).$$

Now to apply corollary 4.5 we need also to compute

$$\int_{\mathbb{T}}\left|R_\epsilon(se^{i\theta})\right|\log^+\left|f(se^{i\theta})\right|\le\int_{\mathbb{T}}\frac{\left|R_\epsilon(se^{i\theta})\right|}{|R(e^{i\theta})|}\left|R(se^{i\theta})\right|\log^+\left|f(se^{i\theta})\right|\le$$

$$\le D\int_{\mathbb{T}}\left|\prod_{j=1}^{n}(1-s\bar{\eta}_je^{i\theta})^{-1+\epsilon}\right|.$$

Again the points $\{\eta_k\}$ are separated by δ and $\left|1-\bar{\eta}_je^{i\theta}\right|^{-1+\epsilon}$ is integrable for the Lebesgue measure on the torus $\mathbb{T}$ because $\epsilon>0$. So we get:

$$\sup_{s<1}\int_{\mathbb{T}}\left|R_\epsilon(se^{i\theta})\right|\log^+\left|f(se^{i\theta})\right|\le c(\delta,\epsilon),$$

which ends the proof of the case $p=0$.

- **Case** $p>0$.

We shall show that $\forall\epsilon>0,\ f\in\mathcal{N}_{R_0,p+\epsilon}(\mathbb{D})$. For this we have to prove:

$$\|f\|_{R_0,p+\epsilon}:=\sup_{s<1}\Big(\int_{\mathbb{D}}(1-|z|^2)^{p+\epsilon-1}\,|R_0(sz)|\log^+|f(sz)|\Big)<\infty.$$

Because $|f(sz)|\le\exp\dfrac{D}{(1-|sz|^2)^p\,|R(sz)|}$ we get

$$I(s,\epsilon):=\int_{\mathbb{D}}(1-|z|^2)^{p+\epsilon-1}\,|R_0(sz)|\log^+|f(sz)|\Big)\le\int_{\mathbb{D}}(1-|z|^2)^{p+\epsilon-1}\frac{|R_0(sz)|}{|R(sz)|}\,|R(sz)|\log^+|f|\le$$

$$\le\int_{\mathbb{D}}(1-|z|^2)^{p+\epsilon-1}\frac{|R_0(sz)|}{|R(sz)|}\frac{D}{(1-|sz|^2)^p}.$$

Now, as already seen, $\dfrac{R_0(sz)}{R(sz)}=\prod_{j=1}^{n}(1-s\bar{\eta}_jz)^{-1}$, so we get, because $\forall s\le1,\ (1-|z|^2)\le(1-|sz|^2)$,

$$I(s,\epsilon) \le D\int_{\mathbb{D}} (1-|z|^2)^{\epsilon-1}\prod_{j=1}^{n}(1-s\bar{\eta}_j z)^{-1}.$$

Now we apply lemma 10.5 with $p=\epsilon$ to get

$$\sup_{s<1}\int_{\mathbb{D}} (1-|sz|^2)^{-1+\epsilon}\prod_{j=1}^{n}(1-s\bar{\eta}_j z)^{-1} \le c(\epsilon,\delta).$$

Hence

$$\|f\|_{R_0,p+\epsilon} \le Dc(\epsilon,\delta) \Rightarrow f \in \mathcal{N}_{R_0,p+\epsilon}(\mathbb{D}).$$

But then corollary 3.8 gives that

$$\sum_{a\in Z(f)} (1-|a|)^{1+p+\epsilon}\left|\tilde{R}_0(a)\right| \le C\|f\|_{R_0,p+\epsilon} \le CDc(\epsilon,\delta),$$

which ends the proof of the theorem. ■

8.2 Case E infinite.

Again we shall examine two cases.

- **Case** $p>0$.

Let $E=\bar{E}\subset\mathbb{T}$; we define its Ahern-Clark type $\alpha(E)$ the following way:

$$\alpha(E) := \sup\{\alpha\in\mathbb{R} :: |\{t\in\mathbb{T} :: d(t,E)<x\}| = \mathcal{O}(x^\alpha),\ x\to+0\},$$

where $|A|$ denotes the Lebesgue measure of the set A.

Our hypothesis is

$$\log|f(z)| \le \frac{K}{(1-|z|)^p}\frac{1}{d(z,E)^q},\ z\in\mathbb{D},\ p,q\ge 0.$$

We want to apply corollary 5.15 so we have, with $\varphi(z) := d(z,E)^{q-\alpha(E)+\epsilon'}$,

$$\sum_{a\in Z(f)} (1-|a|^2)^{1+p} d(a,E)^{(q-\alpha(E)+\epsilon')_+(1+\epsilon)} \le c(p,q,\epsilon)\|f\|_{\mathcal{N}_{\varphi,p}}$$

and we shall compute $\|f\|_{\mathcal{N}_{\varphi,p}}$, i.e.

$$\|f\|_{\mathcal{N}_{\varphi,p}} := \sup_{1-\delta\le s<1}\int_{\mathbb{D}} (1-|z|^2)^{p-1} d(sz,E)^{q-\alpha(E)+\epsilon'}\log^+|f(sz)|.$$

The hypothesis gives

$$\forall z\in\mathbb{D},\ \log^+|f(z)| \le \frac{K}{(1-|z|^2)^p}\frac{1}{d(z,E)^q(z)},$$

so we have

$$\int_{\mathbb{D}} (1-|z|^2)^{p-1} d(sz,E)^{q-\alpha(E)+\epsilon'}\log^+|f(sz)| \le K\int_{\mathbb{D}} (1-|z|^2)^{\epsilon'-1} d(sz,E)^{-\alpha(E)}.$$

We set $\Gamma_n := E_n\times(1-2^{-n},\ 1)$ and $\gamma_n := \Gamma_n\backslash\Gamma_{n+1}$. Then we get

$$\int_{\mathbb{D}} (1-|z|^2)^{\epsilon'-1} d(sz,E)^{-\alpha(E)} = \sum_{n\in\mathbb{N}}\int_{\gamma_n} (1-|z|^2)^{\epsilon'-1} d(sz,E)^{-\alpha(E)} \le$$

$$\sum_{n\in\mathbb{N}} 2^{-(\epsilon'-1)n} 2^{n\alpha(E)}\int_{\gamma_n} dm(z) \le \sum_{n\in\mathbb{N}} 2^{-(\epsilon'-1)n} 2^{n\alpha(E)}\,|E_n|\,2^{-n} = \sum_{n\in\mathbb{N}} 2^{-\epsilon' n} =: c(\epsilon') < \infty$$

because $\epsilon'>0$.

So corollary 5.15 gives

$$\sum_{a\in Z(f)}(1-|a|^2)^{1+p}d(a,E)^{(q-\alpha(E)+\epsilon')_+(1+\epsilon)}\leq c(p,q,\epsilon)\|f\|_{\mathcal{N}_{\varphi,p}},$$

hence we get

$$\sum_{a\in Z(f)}(1-|a|^2)^{1+p}d(a,E)^{(q-\alpha(E)+\epsilon')_+(1+\epsilon)}\leq c(p,q,\epsilon)\|f\|_{\mathcal{N}_{\varphi,p}}\leq Kc(\epsilon')c(p,q,\epsilon).$$

So we proved:

Theorem 8.3 *Suppose that* $f\in\mathcal{H}(\mathbb{D}),\ |f(0)|=1$ *and*

$$\forall z\in\mathbb{D},\ \log^+|f(z)|\leq\frac{K}{(1-|z|^2)^p}\frac{1}{d(z,E)^q},$$

then we have, with $\epsilon''=\epsilon'+\epsilon(q-\alpha(E)+\epsilon')$ *which can be chosen as small as we wish,*

$$\sum_{a\in Z(f)}(1-|a|^2)^{1+p}d(a,E)^{(q-\alpha(E)+\epsilon'')_+}\leq c(p,q,\epsilon)c(\epsilon')K.$$

- **Case** $p>0$.

For this case we want to apply corollary 6.4 So let

$$\forall z\in\mathbb{D},\ \log^+|f(z)|\leq K\frac{1}{d(z,E)^q}.$$

We have

Theorem 8.4 *Suppose that* $f\in\mathcal{H}(\mathbb{D}),\ |f(0)|=1$ *and*

$$\forall z\in\mathbb{D},\ \log^+|f(z)|\leq K\frac{1}{d(z,E)^q},$$

then

$$\sum_{a\in Z(f)}(1-|a|^2)d(a,E)^{(q-\alpha(E)+\epsilon'')_+}\leq c(q,\epsilon'')K.$$

Proof.

We have to verify

$$\sup_{1-\delta\leq s<1}\int_{\mathbb{T}}d(se^{i\theta},E)^{q-\alpha(E)+\epsilon'}\log^+\left|f(se^{i\theta})\right|d\theta<\infty$$

and

$$\sup_{1-\delta\leq s<1}\int_{\mathbb{D}}d(sz,E)^{q-\alpha(E)-1+\epsilon'}\log^+|f(sz)|<\infty.$$

For the first one, we have

$$\int_{\mathbb{T}}d(se^{i\theta},E)^{q-\alpha(E)+\epsilon'}\log^+\left|f(se^{i\theta})\right|\leq K\int_{\mathbb{T}}d(se^{i\theta},E)^{\epsilon'-\alpha(E)}.$$

Set $E_n:=\{x\in\mathbb{T}::d(x,E)\geq 2^{-n}\}$ and $F_n:=E_n\backslash E_{n+1}$. we have

$$\int_{\mathbb{T}}d(se^{i\theta},E)^{\epsilon'-\alpha(E)}d\theta=\sum_{n\in\mathbb{N}}\int_{F_n}d(se^{i\theta},E)^{\epsilon'-\alpha(E)}d\theta\leq\sum_{n\in\mathbb{N}}\int_{F_n}2^{-n(\epsilon'-\alpha(E))}d\theta\leq$$

$$\sum_{n\in\mathbb{N}}2^{-n(\epsilon'-\alpha(E))}\int_{F_n}d\theta\leq\sum_{n\in\mathbb{N}}2^{-n\epsilon'}<\infty$$

by the very definition of $\alpha(E)$ and because $\epsilon'>0$.

For the second one we set $\Gamma_n:=E_n\times(1-2^{-n},\ 1)$ and $\gamma_n:=\Gamma_n\backslash\Gamma_{n+1}$.

We get

$$\int_{\mathbb{D}} d(sz,E)^{q-\alpha(E)-1+\epsilon'} \log^+ |f(sz)| = \int_{\mathbb{D}} d(sz,E)^{q-\alpha(E)-1+\epsilon'} \log^+ |f(sz)| \leq \int_{\mathbb{D}} d(sz,E)^{\epsilon'-\alpha(E)-1}.$$

But

$$\sum_{n\in\mathbb{N}} \int_{\gamma_n} d(sz,E)^{\epsilon'-\alpha(E)-1} \leq \sum_{n\in\mathbb{N}} \int_{\gamma_n} 2^{-n(\epsilon'-\alpha(E)-1)} \leq \sum_{n\in\mathbb{N}} 2^{-n(\epsilon'-\alpha(E)-1)} \int_{\gamma_n} dm(z) \leq$$
$$\sum_{n\in\mathbb{N}} 2^{-n(\epsilon'-\alpha(E)-1)} |E_n| \times (2^{-n}) \leq \sum_{n\in\mathbb{N}} 2^{-n\epsilon'} < \infty,$$

because $\epsilon' > 0$. We end the proof as in the case $p > 0$. ■

9 Mixed cases.

As in section 7 we can mixed the two previous cases and we get, by a straightforward adaptation of the previous proofs,

Theorem 9.1 *Suppose that* $f \in \mathcal{H}(\mathbb{D}),\ |f(0)| = 1$ *and*

$$\forall z \in \mathbb{D},\ \log^+ |f(z)| \leq \frac{K}{(1-|z|^2)^p} \frac{1}{|R(z)|\, d(z,E)^q},$$

with $p > 0$, *and* $R(z) := \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ q_j \in \mathbb{R}$, *if* $q_j - 1 > -p/2$ *set* $\tilde{q}_j = q_j$ *else choose* $\tilde{q}_j > 1 - p/2$,

and set $\tilde{R}_0(z) := \prod_{j=1}^{n} (z-\eta_j)^{\tilde{q}_j - 1}$, *then we have, with* $\epsilon > 0$,

$$\sum_{a\in Z(f)} (1-|a|^2)^{1+p+\epsilon} \left|\tilde{R}_0(a)\right| d(a,E)^{(q-\alpha(E)+\epsilon)_+} \leq c(p,q,R,E,\epsilon)K.$$

And

Theorem 9.2 *Suppose that* $f \in \mathcal{H}(\mathbb{D}),\ |f(0)| = 1$ *and*

$$\forall z \in \mathbb{D},\ \log^+ |f(z)| \leq K \frac{1}{|R(z)|\, d(z,E)^q},$$

with $p = 0$, *and* $R(z) := \prod_{j=1}^{n} (z-\eta_j)^{q_j},\ q_j \in \mathbb{R}$, *set* $\tilde{R}_\epsilon(z) := \prod_{j=1}^{n} (z-\eta_j)^{(q_j-1+\epsilon)_+}$

then, with $\epsilon > 0$,

$$\sum_{a\in Z(f)} (1-|a|^2) \left|\tilde{R}_\epsilon(a)\right| d(a,E)^{(q-\alpha(E)+\epsilon)_+} \leq c(q,R,E,\epsilon)K.$$

10 Appendix.

Lemma 10.1 *(Substitution) Suppose* $\delta > 0,\ 0 < u < 1$ *and* $|f(0)| = 1$, *then*

$$\int_{\mathbb{D}} (1-|z|^2)^{p-1+\delta} |R(sz)|^2 \log^- |f(sz)| \leq (1-u^2)^\delta \frac{1}{u^2} P_{\mathbb{D},-}(s) + c(\delta,u) P_{\mathbb{D},+}(s),$$

with $c(\delta,u) := 2\times 4^{|q|}(1-u)^{1-\alpha-\beta},\ \alpha := -2 \max_{j=1,\dots,n}(0,-q_j),\ \beta := 2 \max_{j=1,\dots,n}(q_j)$,

and $P_{\mathbb{D},-}(s) := \int_{\mathbb{D}} (1-|z|^2)^{p-1} |z|^2 |R(sz)|^2 \log^- |f(sz)|,\ P_{\mathbb{D},+}(s) := \int_{\mathbb{D}} (1-|z|^2)^{p-1} |R(sz)|^2 \log^+ |f(sz)|.$

We also have:

$$\forall t \in [t_0, 1[,\ \int_{\mathbb{D}} (1-|z|^2)^{\delta-1} \left|R(t\rho e^{i\theta})\right|^2 \log^- |f(tz)| \le c(0,u) P_{\mathbb{T},+}(t_0) + \frac{1}{2\delta}(1-\sqrt{u})^\delta P_{\mathbb{T},-}(t_0),$$

with

$$P_{\mathbb{T},+}(t_0) := \sup_{1-\delta\le s\le t_0} \int_{\mathbb{T}} \left|R(se^{i\theta})\right|^2 \log^+ \left|f(se^{i\theta})\right| d\theta$$

and

$$P_{\mathbb{T},-}(t_0) := \sup_{1-\delta\le s\le t_0} \int_{\mathbb{T}} \left|R(se^{i\theta})\right|^2 \log^- \left|f(se^{i\theta})\right| d\theta.$$

Proof.

Because this lemma is a key one for us, we shall give a detailed proof of it. We have

$$A := \int_{\mathbb{D}} (1-|z|^2)^{p-1+\delta} |R(sz)|^2 \log^- |f(sz)| = \int_{D(0,u)} (1-|z|^2)^{p-1+\delta} |R(sz)|^2 \log^- |f(z)| +$$

$$+ \int_{\mathbb{D}\setminus D(0,u)} (1-|z|^2)^{p-1+\delta} |R(sz)|^2 \log^- |f(z)| =: B + C.$$

Clearly for the second term we have

$$C := \int_{\mathbb{D}\setminus D(0,u)} (1-|z|^2)^{p-1+\delta} |R(sz)|^2 \log^- |f(sz)| \le$$

$$(1-u^2)^\delta \frac{1}{u^2} \int_{\mathbb{D}\setminus D(0,u)} (1-|z|^2)^{p-1} |z|^2 |R(sz)|^2 \log^- |f(sz)|.$$

For the first one, we have

$$B := \int_{D(0,u)} (1-|z|^2)^{p-1+\delta} |R(sz)|^2 \log^- |f(sz)|$$

and, changing to polar coordinates,

$$B = \int_0^u (1-\rho^2)^{p-1+\delta} \{ \int_{\mathbb{T}} \left|R(s\rho e^{i\theta})\right|^2 \log^- \left|f(s\rho e^{i\theta})\right| d\theta \} \rho d\rho.$$

We set

$$M(\rho) := \sup_{\theta\in\mathbb{T}} \left|R(\rho e^{i\theta})\right|^2 \le 4^{|q|} (1-\rho)^{-2\max_{j=1,\dots,n}(0,-q_j)},$$

because we have $|z-\eta_j| \le 2$ and $\left|\rho e^{i\theta} - \eta_j\right| \ge (1-\rho)$.

So we get

$$C(s\rho) := \int_{\mathbb{T}} |R(sz)|^2 \log^- |f(sz)| \le M(s\rho) \int_{\mathbb{T}} \log^- \left|f(s\rho e^{i\theta})\right| \le M(s\rho) \int_{\mathbb{T}} \log^+ \left|f(s\rho e^{i\theta})\right|$$

because $\log|f(z)|$ is subharmonic, we get

$$0 = \log|f(0)| \le \int_{\mathbb{T}} \log \left|f(s\rho e^{i\theta})\right| = \int_{\mathbb{T}} \log^+ \left|f(s\rho e^{i\theta})\right| - \int_{\mathbb{T}} \log^- \left|f(s\rho e^{i\theta})\right|.$$

So we have

$$C(s\rho) \le M(s\rho) \int_{\mathbb{T}} \log^+ \left|f(s\rho e^{i\theta})\right|. \tag{10.13}$$

Now we set $m(\rho) := \inf_{\theta\in\mathbb{T}} \left|R(\rho e^{i\theta})\right|^2$ and the same way as for $M(\rho)$, we get $m(\rho) \ge (1-\rho)^{2\max_{j=1,\dots,n}(q_j)}$. Putting it in (10.13), we get

$$C(s,\rho) \le M(s\rho) m(\rho)^{-1} \int_{\mathbb{T}} \left|R(s\rho e^{i\theta})\right|^2 \log^+ \left|f(s\rho e^{i\theta})\right|. \tag{10.14}$$

We set

$$c(\delta, s, u) := \sup_{\rho<u} \frac{M(s\rho)}{m(s\rho)}(1-\rho^2)^\delta \leq 2\times 4^{|q|}(1-u)^{1-\alpha-\beta},$$

with

$$\alpha := -2 \max_{j=1,\dots,n}(0, -q_j),\ \beta := 2 \max_{j=1,\dots,n}(q_j).$$

Now we have

$$B \leq \int_0^u (1-\rho^2)^{p-1} C(s,\rho)\rho d\rho, \tag{10.15}$$

hence $B \leq c(\delta, u)P_{\mathbb{D},+}(s)$.
Adding B and C gives the first part of the lemma.

For the second one, from the definition of C with $p = 0$,

$$C := \int_{\mathbb{D}\setminus D(0,u)} (1-|z|^2)^{-1+\delta} \left|R(sz)\right|^2 \log^- |f(sz)|$$

we get passing in polar coordinates and with $s \in [1-\delta, t_0]$,

$$\begin{aligned} C &= \int_u^1 (1-\rho^2)^{\delta-1} \int_{\mathbb{T}} \left|R(s\rho e^{i\theta})\right|^2 \log^- \left|f(s\rho e^{i\theta})\right| d\theta\rho d\rho \\ &\leq P_{\mathbb{T},-}(t_0) \int_u^1 (1-\rho^2)^{\delta-1}\rho d\rho \leq \frac{1}{2\delta}(1-\sqrt{u})^\delta P_{\mathbb{T},-}(t_0). \end{aligned}$$

Now from (10.14) and (10.15) we get

$$B \leq P_{\mathbb{T},+}(t_0)c(0,u)\int_0^u (1-\rho^2)^{\delta-1}\rho d\rho \leq P_{\mathbb{T},+}(t_0)c(0,u).$$

Adding C with B we get the second part of the lemma. ■

Lemma 10.2 *Let $\eta \in \mathbb{T}$, then we have $\Re(\bar{z}(z-\eta)) \leq 0$ iff $z \in \mathbb{D} \cap D(\frac{\eta}{2},\ \frac{1}{2})$.*

Proof.
We set $z = \eta t$, then we have

$$\bar{z}(z-\eta) = \bar{\eta}\bar{t}(\eta t - \eta) = \bar{t}(t-1).$$

Hence

$$\Re(\bar{z}(z-\eta)) = \Re(\bar{t}(t-1)) = \Re(r^2 - re^{i\theta}) = r^2 - r\cos\theta.$$

Hence with $t = x + iy = re^{i\theta,}$ $x = r\cos\theta,\ y = r\sin\theta$, we get

$$\Re(\bar{t}(t-1)) \leq 0 \iff x^2 + y^2 - x \leq 0$$

which means $(x, y) \in D(\frac{1}{2},\ \frac{1}{2})$ hence $z \in \mathbb{D} \cap D(\frac{\eta}{2},\ \frac{1}{2})$. ■

Lemma 10.3 *Let φ be a continuous function in the unit disc $\mathbb{D}$. We have that:*

$$s \leq t \in]0,1[\to \gamma(s) := \int_{\mathbb{T}} \varphi(se^{i\theta}) \log^- \left|f(se^{i\theta})\right| d\theta$$

is a continuous function of $s \in [0, t]$.

Proof.
Because $s \leq t < 1$, the holomorphic function in the unit disc $f(se^{i\theta})$ has only a finite number of zeroes say $N(t)$. As usual we can factor out the zeros of f to get

$$f(z) = \prod_{j=1}^{N} (z - a_j) g(z)$$

where $g(z)$ has no zeros in the disc $\bar{D}(0,t)$. Hence we get

$$\log|f(z)| = \sum_{j=1}^{N} \log|z - a_j| + \log|g(z)|.$$

Let $a_j = r_j e^{\alpha_j},\ r_j > 0$ because $|f(0)| = 1$, then it suffices to show that

$$\gamma(s) := \int_{\mathbb{T}} \varphi(se^{i\theta}) \log^{-} \left|se^{i\theta} - re^{i\alpha}\right| d\theta$$

is continuous in s near $s = r$, because $\displaystyle\int_{\mathbb{T}} \varphi(se^{i\theta}) \log^{-} \left|g(se^{i\theta})\right| d\theta$ is clearly continuous.

To see that $\gamma(s)$ is continuous at $s = r$, it suffices to show

$\gamma(s_n) \to \gamma(r)$ when $s_n \to r$.

But

$$\forall \theta \neq 0,\ \varphi(se^{i\theta}) \log\left|se^{i\theta} - r\right| \to \varphi(re^{i\theta}) \log\left|re^{i\theta} - r\right|$$

and $\displaystyle\log \frac{1}{|se^{i\theta} - r|} \leq c_\epsilon \left|se^{i\theta} - r\right|^{-\epsilon}$ with $\epsilon > 0$. So choosing $\epsilon < 1$, we get that $\displaystyle\log \frac{1}{|se^{i\theta} - r|} \in L^1(\mathbb{T})$ uniformly in s. Because $\varphi(se^{i\theta})$ is continuous uniformly in $s \in [0,t]$ we get also $\displaystyle\varphi(se^{i\theta}) \log \frac{1}{|se^{i\theta} - r|} \in L^1(\mathbb{T})$ uniformly in s. So we can apply the dominated convergence theorem of Lebesgue to get the result. ■

Lemma 10.4 *(Substitution) We have, provided that* $|f(0)| = 1$,

$$\forall \rho \in (0,1),\ \int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^{-} \left|f(s\rho e^{i\theta})\right| d\theta \leq c(s,\rho) \int_{\mathbb{T}} h(\rho e^{i\theta})^q \log^{+} \left|f(s\rho e^{i\theta})\right| d\theta,$$

with $\displaystyle c(s,\rho) := \frac{2^q}{(1 - s\rho)^q}$.

Proof.

Recall that

$$\forall z = \rho e^{i\theta} \in \Gamma_j,\ h(s\rho e^{i\theta}) = 1 - s\rho + \chi(\frac{1 - s\rho}{\delta_j}) \delta_j^{1+\epsilon} H(\theta/\delta_j),$$

and $\forall z \in \Gamma_E,\ h(s\rho e^{i\theta}) = 1 - s\rho$, hence $\forall z \in \mathbb{D},\ (1 - \rho) \leq h(\rho e^{i\theta}) \leq 2$.

Using $\forall \theta \in \mathbb{T},\ h(s\rho e^{i\theta}) \leq 2$ we get, as in lemma 10.1,

$$\int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^{-} \left|f(s\rho e^{i\theta})\right| d\theta \leq 2^q \int_{\mathbb{T}} \log^{-} \left|f(s\rho e^{i\theta})\right| \leq 2^q \int_{\mathbb{T}} \log^{+} \left|f(s\rho e^{i\theta})\right|.$$

Now using $\forall \theta \in \mathbb{T},\ h(s\rho e^{i\theta}) \geq (1 - s\rho)$, we get

$$\int_{\mathbb{T}} \log^{+} \left|f(s\rho e^{i\theta})\right| \leq \frac{1}{(1 - s\rho)^q} \int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^{+} \left|f(s\rho e^{i\theta})\right|,$$

so

$$\forall \rho \in (0,1),\ \int_{\mathbb{T}} h(s\rho e^{i\theta})^q \log^{-} \left|f(s\rho e^{i\theta})\right| \leq \frac{2^q}{(1 - s\rho)^q} \int_{\mathbb{T}} h(s\rho e^{i\theta}) \log^{+} \left|f(s\rho e^{i\theta})\right|. \blacksquare$$

Lemma 10.5 *The function* $\displaystyle(1 - |z|^2)^{p-1} \prod_{j=1}^{n} |z - \eta_k|^{-1}$, *with* $p > 0$, *is integrable for the Lebesgue measure in the disc* $\mathbb{D}$ *and we have the estimate*

$$\int_{\mathbb{D}} (1-|z|^2)^{p-1} \prod_{j=1}^{n} |z-\eta_k|^{-1} \leq c(p,\delta) < \infty,$$

where the constant δ is twice the length of the minimal arc between the points $\{\eta_j\}_{j=1,...,n} \subset \mathbb{T}$.

Proof.
Because the points η_k are separated on the torus $\mathbb{T}$ we can assume that we have disjoint sectors Γ_j based on the arcs $\{\eta_j - \delta, \eta_j + \delta\}_{j=1,...,n} \subset \mathbb{T}$ for a $\delta > 0$. Let $\Gamma_0 := \mathbb{D} \backslash \bigcup_{j=1}^{n} \Gamma_j$. We have

$$A := \int_{\mathbb{D}} (1-|z|^2)^{p-1} \prod_{j=1}^{n} |z-\eta_k|^{-1} dm(z) = \sum_{j=0}^{n} \int_{\Gamma_j} (1-|z|^2)^{p-1} \prod_{k=1}^{n} |z-\eta_k|^{-1} dm(z).$$

We set

$$A_0 := \int_{\Gamma_0} (1-|z|^2)^{p-1} \prod_{k=1}^{n} |z-\eta_k|^{-1} dm(z),$$

and we get

$$\forall z \in \Gamma_0,\ \forall k = 1, ..., n,\ |z-\eta_k| \geq \delta \Rightarrow \prod_{k=1}^{n} |z-\eta_k|^{-1} \leq \delta^{-n}.$$

So

$$A_0 \leq \delta^{-n} \int_{\Gamma_0} (1-|z|^2)^{p-1} dm(z) \leq \delta^{-n} \int_{\mathbb{D}} (1-|z|^2)^{p-1} dm(z) \leq 2\pi\delta^{-n}.$$

For computing A_j we can assume that $\eta_j = 1$ by rotation and Γ_j based on the arc $(-\delta,\ \delta)$; so we have, because $\prod_{k=1}^{n} |z-\eta_k|^{-1} \leq \delta^{-(n-1)} |1-z|$,

$$A_j := \int_{\Gamma_j} (1-|z|^2)^{p-1} \prod_{k=1}^{n} |z-\eta_k|^{-1} dm(z) \leq \delta^{-(n-1)} \int_{\Gamma_j} (1-|z|^2)^{p-1} |1-z|^{-1} dm(z).$$

Set $\alpha := \dfrac{p}{2} > 0$, then we have $(1-|z|^2)^{\alpha} < 2^{\alpha} |1-z|^{\alpha}$ hence

$$A_j \leq \delta^{-(n-1)} 2^{\alpha} \int_{\Gamma_j} (1-|z|^2)^{\alpha-1} |1-z|^{\alpha-1} dm(z).$$

Changing to polar coordinates, we get

$$A_j \leq \delta^{-(n-1)} 2^{\alpha} \int_0^1 (1-\rho^2)^{\alpha-1} \rho \{ \int_{-\delta}^{\delta} \left|1-\rho e^{i\theta}\right|^{\alpha-1} d\theta \} d\rho.$$

Because $\alpha > 0$, we get

$$\forall \rho \leq 1,\ \int_{-\delta}^{\delta} \left|1-\rho e^{i\theta}\right|^{\alpha-1} d\theta \leq c(\alpha,\delta)$$

and

$$\int_0^1 (1-\rho^2)^{\alpha-1} \rho d\rho \leq c(\alpha).$$

So adding the A_j, we end the proof of the lemma. ■

Lemma 10.6 *Let $\varphi(z)$ be a positive function in $\mathbb{D}$ and $f \in \mathcal{H}(\mathbb{D})$; set $f_s(z) := f(sz)$ and suppose that:*

$$\forall s < 1,\ \sum_{a \in Z(f_s)} (1-|a|^2)^{p+1} \varphi(sa) \leq \int_{\mathbb{D}} (1-|z|^2)^{p-1} \varphi(sz) \log^{+} |f(sz)|,$$

then, for any $1 > \delta > 0$ *we have*

$$\sum_{a\in Z(f)} (1-|a|^2)^{p+1}\varphi(a) \le \sup_{1-\delta<s<1} \int_{\mathbb{D}} (1-|z|^2)^{p-1}\varphi(sz)\log^+|f(sz)|.$$

We have also:

let $\varphi(z),\ \psi(z)$ *be positive continuous functions in* $\mathbb{D}$ *and* $f \in \mathcal{H}(\mathbb{D})$ *such that:*

$$\forall s<1,\ \sum_{a\in Z(f)\cap D(0,s)} (1-|a|^2)\varphi(sa) \le \int_{\mathbb{D}} \varphi(sz)\log^+|f(sz)| + \int_{\mathbb{T}} \psi(se^{i\theta})\log^+\left|f(se^{i\theta})\right|$$

then, for any $1 > \delta > 0$ *we have*

$$\sum_{a\in Z(f)} (1-|a|^2)\varphi(a) \le \sup_{1-\delta<s<1} \int_{\mathbb{D}} \varphi(sz)\log^+|f(sz)| + \sup_{1-\delta<s<1} \int_{\mathbb{T}} \psi(se^{i\theta})\log^+|f(sz)|.$$

Proof.

We have $a \in Z(f_s) \iff f(sa) = 0$, i.e. $b := sa \in Z(f)\cap D(0,s)$. Hence the hypothesis is

$$\forall s<1,\ \sum_{a\in Z(f)\cap D(0,s)} (1-\left|\frac{a}{s}\right|^2)^{p+1}\varphi(a) \le \int_{\mathbb{D}} (1-|z|^2)^{p-1}\varphi(sz)\log^+|f(sz)|.$$

We fix $1-\delta < r < 1,\ r < s < 1$, then, because $Z(f)\cap D(0,r) \subset Z(f)\cap D(0,s)$ and $\varphi \ge 0$, we have

$$\sum_{a\in Z(f)\cap D(0,r)} (1-\left|\frac{a}{s}\right|^2)^{p+1}\varphi(a) \le \sum_{a\in Z(f)\cap D(0,s)} (1-\left|\frac{a}{s}\right|^2)^{p+1}\varphi(a) \le$$

$$\le \sup_{1-\delta<s<1} \int_{\mathbb{D}} (1-|z|^2)^{p-1}\varphi(z)\log^+|f(z)|.$$

In $D(0,r)$ we have a finite fixed number of zeroes of f, and, because $(1-\left|\frac{a}{s}\right|^2)^{p+1}$ is continuous in $s \le 1$ for $a \in \mathbb{D}$, we have

$$\forall a \in Z(f)\cap D(0,r),\ \lim_{s\to 1}(1-\left|\frac{a}{s}\right|^2)^{p+1} = (1-|a|^2)^{p+1}.$$

Hence

$$\sum_{a\in Z(f)\cap D(0,r)} (1-|a|^2)^{p+1}\varphi(a) \le \sup_{1-\delta<s<1} \int_{\mathbb{D}} (1-|z|^2)^{p-1}\varphi(sz)\log^+|f(sz)|.$$

Because the right hand side is independent of $r < 1$ and φ is positive in $\mathbb{D}$ so the sequence

$$S(r) := \sum_{a\in Z(f)\cap D(0,r)} (1-|a|^2)^{p+1}\varphi(a)$$

is increasing with r, we get

$$\sum_{a\in Z(f)} (1-|a|^2)^{p+1}\varphi(a) \le \sup_{1-\delta<s<1} \int_{\mathbb{D}} (1-|z|^2)^{p-1}\varphi(sz)\log^+|f(sz)|.$$

This proves the first part. The proof of the second one is just identical. ■

Remark 10.7 *(i) As can be easily seen by the change of variables* $u = sz$, *if* $p \ge 1$ *we have:*

$$\sup_{1-\delta<s<1} \int_{\mathbb{D}} (1-|z|^2)^{p-1}\varphi(sz)\log^+|f(sz)| \lesssim \int_{\mathbb{D}} (1-|z|^2)^{p-1}\varphi(z)\log^+|f(z)|.$$

(ii) We also have that if $\varphi(z)\log^+|f(z)|$ *is subharmonic, then:*

$$\sup_{1-\delta<s<1} \int_{\mathbb{D}} (1-|z|^2)^{p-1}\varphi(sz)\log^+|f(sz)| \le \int_{\mathbb{D}} (1-|z|^2)^{p-1}\varphi(z)\log^+|f(z)|.$$

But (ii) is not the case in general in our setting.

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