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Volume entropy

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Abstract. Building on a technical result by Brunnemann and Rideout on the spectrum of the Volume operator in Loop Quantum Gravity, we show that the dimension of the space of the quadrivalent states –with finite-volume individual nodes– describing a region with total volume smaller than V, has *finite* dimension, bounded by $V \log V$. This allows us to introduce the notion of "volume entropy": the von Neumann entropy associated to the measurement of volume.

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I. Introduction

Thermodynamical aspects of the dynamics of spacetime have first been pointed out by Bekenstein's introduction of an entropy associated to the horizon of a black hole [1]. This led to the formulation of the "laws of black holes thermodynamics" by Bardeen, Carter, and Hawking [2] and to Hawking's discovery of black role radiance, which has reinforced the geometry/thermodynamics analogy [3]. The connection between Area and Entropy suggests that it may be useful to treat aspects of space-time statistically at scales large compared to the Planck length [4], whether or not we expect the relevant microscopic elementary degrees of freedom to be simply the quanta of the gravitational field [5], or else. Black hole entropy, in particular, can be interpreted as cross-horizon entanglement entropy (see [6] for recent results reinforcing this interpretation, and references therein), or –most likely equivalently– as the von Neumann entropy of the statistical state representing a macrostate with given horizon Area. In the context of Loop Quantum Gravity (LQG), this was considered in [7] and later extensively analyzed; for a recent review and full references see [8, 9].

All such developments are based on the assignment of thermodynamic properties to spacetime *surfaces*. This association has motivated the holographic hypothesis: the conjecture that the degrees of freedom of a region of space are somehow encoded in its boundary.

In this paper, instead, we study statistical properties associated to spacetime *regions*. We show that it is possible to define a Von Neumann entropy for the quantum gravitational field, associated to the Volume of a region, and that this entropy is (under suitable conditions) finite. The existence of an entropy associated to bulk degrees of freedom of a spin network was already considered in [10].

To this aim, we prove a finiteness result on the number of quantum states of gravity describing a region of finite volume. More precisely, we work in the context of LQG, and we prove that the dimension of the space of diffeomorphism invariant quadrivalent states without zero-volume nodes, describing a region of total volume smaller than V is finite. We give explicitly the upper bound of the dimension as a function of V. The proof is based on a result on the spectrum of the LQG Volume operator proven by Brunnemann and Rideout [11, 12]. Using this, we define the Von Neumann entropy of a quantum state of the gravitational field associated to Volume measurements.

II. Counting spin networks

Consider the measurement of the volume of a 3d spacelike region Σ. The physical system measured is the gravitational field. In the classical theory, this is given by the metric q on Σ: the volume is $V = \int_\Sigma \sqrt{\det q}\ d^3x$. In the quantum context, using the LQG formalism, the geometry of Σ is described by a state in the kinematical Hilbert space $\mathcal{H}_{diff}$. The volume measurement of Σ are described by a volume operator $\hat{V}$ on this state space. We refer to [13, 14] for details on basic LQG results and notation.

We restrict $\mathcal{H}_{diff}$ to four-valent graphs Γ where the nodes n have non-vanishing (unoriented) volume v_n. The spin network states $|\Gamma, j_l, v_n\rangle \in \mathcal{H}_{diff}$, where j_l is the link quantum number or spin, form a countable, orthonormal basis of $\mathcal{H}_{diff}$. (We disregard here eventual additional quantum numbers such as the orientation, that have no bearing on our result.) The intertwiner basis at each node is chosen so that the local volume operator $\hat{V}_n$, acting on a single node, is diagonal and is labelled by the eigenvalues v_n, of the node volume operator $\hat{V}_n$ associated to the node n.

$$\hat{V}_n\ |\Gamma, j_l, v_n\rangle = v_n |\Gamma, j_l, v_n\rangle \tag{1}$$

The states $|\Gamma, j_l, v_n\rangle$ are also eigenstates of the total volume operator $\hat{V} = \sum_{n=1}^{N} \hat{V}_n$, where N is the number of

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nodes in Γ, with eigenvalue

$$v = \sum_{n=1}^{N} v_n, \tag{2}$$

the sum of the node volume eigenvalues v_n.

We seek a bound on the dimension of the subspace $\mathcal{H}_V$ spanned by the states $|\Gamma, j_l, v_n\rangle$ such that $v \leq V$. That is, we want to count the spin-networks with volume less than V. We do this by bounding the number N_Γ of four valent graphs in $\mathcal{H}_V$, the number $N_{\{j_l\}}$ of possible spin assignments, and the number of the volume quantum numbers assignments $N_{\{v_n\}}$ on each such graph. Clearly

$$\dim \mathcal{H}_V \leq N_\Gamma \; N_{\{j_l\}} N_{\{v_n\}}. \tag{3}$$

Crucial to this bound is the analytical result on the existence of a volume gap in four-valent spin networks found in [11, 12]. The result is the following. In a node bounded by four links with maximum spin j_{max} all non-vanishing volume eigenvalues are larger than

$$v_{gap} \geq \frac{1}{4\sqrt{2}} \, \ell_{\rm P}^3 \gamma^{\frac{3}{2}} \sqrt{j_{max}} \tag{4}$$

Where $\ell_{\rm P}$ is the Planck constant and γ the Immirzi parameter. Numerical evidence for equation (4) was first given in [15] and a compatible result was estimated in [16]. Since the minimum non-vanishing spin is $j = \frac{1}{2}$, this implies that

$$v_{gap} \geq \frac{1}{8} \, \ell_{\rm P}^3 \gamma^{\frac{3}{2}} \equiv v_o \tag{5}$$

From the existence of the volume gap, it follows that there is a maximum value of N_Γ, because there is a maximum number of nodes for graphs in $\mathcal{H}_V$, as every node carries a minimum volume v_o. Therefore a region of volume equal or smaller than V contains at most

$$n = \frac{V}{v_o} \tag{6}$$

nodes. Equation (4) bounds also the number of allowed area quantum numbers, because too large a j_{max} would force too large a node volume. Therefore $N_{\{j_l\}}$ is also finite. Finally, since the dimension of the space of the intertwiners at each node is finite and bounded by the value of spins, it follows that also the number $N_{\{v_n\}}$ of individual volume quantum numbers is bounded. Then (3) shows immediately that the dimension of $\mathcal{H}_V$ is finite. Let us bound it explicitely.

We start by the number of graphs. The number of nodes must be smaller than n, given in (6). The number N_Γ of 4-valent graphs with n nodes is bounded by

$$N_\Gamma \leq n^{4n} \tag{7}$$

because each node can be connected to each other n^n four times $(n^n)^4$.

Equation (4) bounds the spins. Since we must have $V \geq v_{gap}$, we must also have

$$j \leq j_{max} \leq 32 \frac{V^2}{\ell_{\rm P}^6 \gamma^3} = \frac{1}{2} n^2 \tag{8}$$

In a graph with n nodes there are at most $4n$ links (the worst case being all boundary links), and therefore there are at most $(2j_{max}+1)^{4n}$ spin assignments, or, in the large j limit, $(2j_{max})^{4n}$. That is

$$N_{\{j_l\}} \leq (2j_{max})^{4n} \leq n^{8n} \tag{9}$$

Finally, the dimension of the intertwiner space at each node is bounded by the areas associated to that node:

$$\begin{aligned}
&\dim \mathcal{K}_{j_1,j_2,j_3,j_4} = \\
&= \dim \; \mathtt{Inv}_{SU(2)} \left(\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4}\right) \\
&= \mathtt{min}\left(j_1 + j_2, j_3 + j_4\right) - \mathtt{max}\left((j_1 - j_2), (j_3 - j_4)\right) + 1 \\
&\leq 2 \, \mathtt{max}(j_{l \in n}) + 1 \leq 4 \, \mathtt{max}(j_{l \in n})
\end{aligned}$$

with the last step following from $\mathtt{max}(j_{l\in n}) \geq 1/2$. Thus on a graph with n nodes, the maximum number of combination of eigenvalues is limited by:

$$N_{\{v_n\}} \leq (4j_{max})^n = 2^n n^{2n} \tag{10}$$

Combining equations (3), (7), (9) and (10), we have an explicit bound on the dimension of the space of states with volume less than $V = nv_o$:

$$\dim \mathcal{H}_V \leq (cn)^{14n} \tag{11}$$

where c is a number. For large n we can write

$$S_V \equiv \log \dim \mathcal{H}_V \leq 14 \, n \log n \tag{12}$$

which is the entropy associated to Hilbert space. Explicitly

$$S_V \leq 14 \, \frac{V}{v_o} \log \frac{V}{v_o} \sim V \log V. \tag{13}$$

In the large volume limit, when the eigenvalues become increasingly dense, this corresponds to a density of states $\nu(V) \equiv d(\dim \mathcal{H}_V)/dV$ similarly bounded

$$\nu(V) < 14 \left[\log(n) + C\right] (cn)^{14n}. \tag{14}$$

III. Von Neumann proper volume entropy

In the previous section, we have observed that the dimension of the space of (with four-valent, finite-volume nodes) quantum states with total volume less than V is finite. This results implies that there is a finite von

Neumann volume entropy associated to statistical states describing to volume measurements.

The simplest possibility is to consider the micro-canonical ensemble describing the volume measurement of a region of space. That is, we take Volume to be a macroscopic (or thermodynamic, or "coarse grained") variable, and we write the corresponding statistical microstate that maximizes entropy. If the measured volume is in the interval $I_V = [V - \delta V, V]$, with small δV, then the corresponding micro-canonical state is simply

$$\rho = \frac{\mathcal{P}_{V,\delta V}}{\dim \mathcal{H}_V}. \tag{15}$$

where $\mathcal{P}_{V,\delta V}$ is the projector on

$$\mathcal{H}_{V,\delta V} \equiv Span\{|\Gamma, j_l, v_n> \ : \ v \in I_V\}. \tag{16}$$

namely the span of the eigenspaces of eigenvalues of the volume that are in I_V. Explicitly, the projector can be written in the form

$$\mathcal{P}_{V,\delta V} \equiv \sum_{v \in I_V} |\Gamma, j_l, v_n >< \Gamma, j_l, v_n| \tag{17}$$

The von Neumann entropy of (15) is

$$S = -Tr[\rho \log \rho] = \log \dim \mathcal{H}_V < S_V \sim V \log V. \tag{18}$$

It is interesting to consider also a more generic state where $\rho \sim p(V)$, for an arbitrary distribution $p(V)$ of probabilities of measuring a given volume eigenstate with volume V. For this state, the probability distribution of finding the value V in a volume measurement is

$$P(V) = \nu(V) p(V) \tag{19}$$

and the entropy can be written as the sum of two terms

$$S = \int dV \ \nu(V) p(V) \log(p(V)) = S1 + S2 \tag{20}$$

where the first

$$S_P = -\int dV \ P(V) \log(P(V)) \tag{21}$$

is just the entropy due to the spread in the outcomes of volume measurements, while the second

$$S_{\text{Volume}} \equiv S - S_P = \int dV \ P(V) \log(\nu(V)) \tag{22}$$

can be seen as as a *proper* volume entropy. The bound found in the previous Section on $\nu(V)$, which indicates that $\nu(S)$ grows less that V^2, shows that this proper volume entropy is finite for any distribution $P(V)$ whose variance is finite. S_{Volume} can be viewed as the irreducible entropy associated to any volume measurement.

IV. Lower bound

Let us now bound the dimension of $\mathcal{H}_V$ from below. The crucial step for this is to notice the existence of a maximum δV in the spacing between the eigenvalues of the operator $\hat{V}_n$. For instance, if we take a node between two large spins j and two $\frac{1}{2}$ spins, the volume eigenvalues have decreasing spacing, with maximum spacing for the lowest eigenvalues, of the order v_o. Disregarding irrelevant small numerical factors, let's take v_o as the maximal spacing.

Given a volume V, let, as before, $n = V/v_0$ and consider spin networks with total volume in the interval $I_n = [(n-1)v_o, nv_o]$. Let N_m be the number of spin networks with m nodes that have the total volume v in the interval I_n. For $m = 1$, there is at least one such spin network, because of the minimal spacing. For $m = 2$, the volume v must be split between the two nodes: $v = v_1 + v_2$. This can be done in at least $n - 1$ manners, with $v_1 \in I_p$ and $v_1 \in I_{n-p}$ and p running from 1 to $n - 1$. This possibility is guaranteed again by the existence of the maximal spacing. In general, for m nodes, there are

$$N_{n,m} = \binom{n-1}{m-1} \tag{23}$$

different ways of splitting the total volume among nodes. This is the number of *compositions* of n in m subsets. Finally, the number m of nodes can vary between 1 and the maximum n, giving a total number of possible states larger than

$$N_n = \sum_{m=1}^{n} N_{n,m} = \sum_{m=1}^{n} \binom{n-1}{m-1} = 2^{n-1}. \tag{24}$$

From which it follows that

$$\dim \mathcal{H}_V \geq 2^{n-1}. \tag{25}$$

Can all these states be realised by inequivalent spin networks, with suitable choices of the graph and the spins? To show that this is the case, it is sufficient to display at least one (however peculiar) example of spin network for each sequence of v_n. But given an arbitrary sequence of v_n we can always construct a graph formed by a single one dimensional chain of nodes, each (except the two ends) with two links connecting to the adjacent nodes in the chain and two links in the boundary. All these spin networks exist and are non-equivalent to one another. Therefore we have shown that there are at least 2^{n-1} states with volume between $V - v_o$ and V. In the large volume limit we can write

$$\dim \mathcal{H}_V \geq 2^n = 2^{\frac{V}{v_o}}. \tag{26}$$

so that the entropy satisfies

$$cV \leq S \leq c' V \log V. \tag{27}$$

with c and c' constants.

V. Discussion

Geometrical entropy associated to surfaces of given Area plays a large role in the current discussions of the quantum nature of spacetime. Here we have shown that, under suitable conditions, it is also possible to compute a Von Neumann entropy associated to measurements of the Volume of a region of space. We have not discussed possible physical roles played by this entropy. A number of comments are in order:

(i) Since in the classical low energy limit Volume and area are related by $V \sim A^{\frac{3}{2}}$, the Volume entropy we have considered $S_V \sim V \log V \sim A^{\frac{3}{2}} \log A$ may exceed the Bekenstein bound $S < S_A \sim A$. Volume entropy is accessible only by being in the bulk, and not necessarily from the outside, therefore it does not violate the versions of the Bekenstein bound that only refer to external observables.

(ii) The result presented above depends on the restriction of H_{diff} to four-valent states. We recall that the discussion is currently open in the literature on which of the two theories, with or without this restriction, is physically more interesting, with good arguments on both sides. However, it might be possible to extend the results presented here to the case of higher-valent graphs. Indeed, there is some evidence that there is a volume gap in higher-valent cases too, see for instance [17]. The effect of zero-volume nodes on the Volume entropy will be discussed elsewhere.

(iii) Volume entropy appears to fail to be an extensive quantity. The significance of this conclusion deserves to be explored. This feature is usual for systems with long range interactions, and in particular for systems of particles governed by the gravitational interaction. It is suggestive that gravity could retain this feature even when there are no interacting particle, and the role of long range interactions is taken by "long range" connections between graph nodes[1]. A final word on this behaviour, however, has to wait for a more precise computation of the entropy growth with volume.

(iv) It has been recent pointed out that the interior of an old black old contains surfaces with large volume [18, 19] and that the large volume inside black holes can play an important role in the information paradox [9, 20]. The results presented here may serve to quantify the corresponding interior entropy.

(v) A notion of entropy associated to the volume of space might perhaps provide an alternative to Penrose's Weyl curvature hypothesis [21]. For the second principle of thermodynamics to hold, the initial state of the universe must have had low entropy. On the other hand, from cosmic background radiation observations, the initial state of matter must have been close to having maximal entropy. Penrose addresses this discrepancy by taking into consideration the entropy associated to gravitational degrees of freedom. His hypothesis is that the degrees of freedom which have been activated to bring the increase in entropy from the initial state are the ones associated to the Weyl curvature tensor, which in his hypothesis was null in the initial state of the universe. A definition of the bulk entropy of space, which, as would be expected, grows with the volume, could perhaps perform the same role as the Weyl curvature degrees of freedom do in Penrose's hypothesis: the universe had a much smaller volume close to its initial state, so the total available entropy was low - regardless of the matter entropy content - and has increased since, just because for a space of larger volume we have a greater number of states describing its geometry.

(vi) We close with a very speculative remark. Does the fact that entropy is large for larger volumes imply the existence of an entropic force driving to larger volumes? That is, could there be a statistical bias for transitions to geometries of greater volume? Generically, the growth of the phase space volume is a driving force in the evolution of a system: in a transition process, we sum over *out* states, more available states for a given outcome imply greater probability of that outcome. A full discussion of this point requires the dynamics of the theory to be explicitly taken into account, and we postpone it for future work.

[1] Of course they are not really long range, in the sense that graph connections actually *define* locality.

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