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# Periodic Motions With Overshooting Phases of a Two-Mass Stick–Slip Oscillator

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We investigate the dynamics of a two degrees-of-freedom oscillator excited by dry friction. The system consists of two masses connected by linear springs and in contact with a belt moving at a constant velocity. The contact forces between the masses and the belt are given by Coulomb's laws. Several periodic orbits including slip and stick phases are obtained. In particular, the existence of periodic orbits involving a part where one of the masses moves at a higher speed than the belt is proved.

Keywords: coupled oscillators, dry friction, periodic motions, stick–slip motions

## 1 Introduction

In this paper, a two degrees-of-freedom oscillator excited by dry friction is considered. The system consists of two masses connected by linear springs and in contact with a belt moving at a constant velocity. The contact forces between the masses and the belt are given by Coulomb's laws.

This model of stick–slip system has been the subject of several recent publications [1–4]. Several friction characteristics have been used, among them, the Coulomb's friction laws are the most frequently assumed. In this case, the system is a piecewise linear system and it is possible to investigate the behavior of the system by using analytical methods instead of numerical ones. Periodic orbits including stick–slip phases are obtained. An interesting phenomenon, which was in the past obtained only for more complex models of dry friction forces [3], is also observed for this model. Periodic orbits including stick phases followed by slip phases during which one of the mass in contact with the belt goes faster than the belt are found.

## 2 Description of the Model

The system (Fig. 1) consists of two masses  $m_1$  and  $m_2$  connected by linear springs of stiffnesses  $k_1$  and  $k_2$ . The masses are in contact with a driving belt moving at a constant velocity  $W$ . Friction forces act between the masses and the belt.

This dry friction oscillator is governed by the following differential system:

$$M\ddot{X} + KX = R, \quad X = (x_1, x_2)^t, \quad R = (F_1, F_2)^t \quad (1)$$

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad K = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}$$

$x_1$  and  $x_2$  are the displacements of the masses, and  $F_1$  and  $F_2$  are the contact friction forces obtained from Coulomb's laws

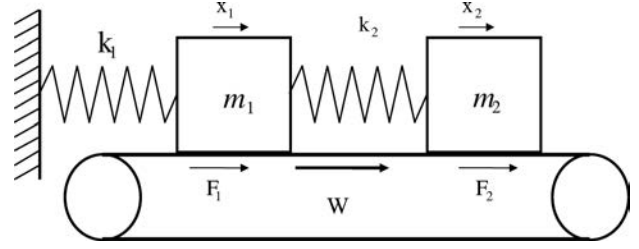


Fig. 1 Description of the model

- $W - \dot{x}_i \neq 0$  and  $F_i = F_{si} \text{sign}(W - \dot{x}_i)$  ( $i = 1, 2$ ) (slipping motion of the masses)  
 $W - \dot{x}_1 = 0$
- $F_1 = \begin{cases} (k_1 + k_2)x_1 - k_2x_2 & \text{if } |(k_1 + k_2)x_1 - k_2x_2| < F_{r1} \\ \varepsilon F_{s1} & \text{if } \varepsilon[(k_1 + k_2)x_1 - k_2x_2] > F_{r1} \ (\varepsilon = \pm 1) \end{cases}$  (sticking motion of  $m_1$ )  
 $W - \dot{x}_2 = 0$
- $F_2 = \begin{cases} k_2(x_2 - x_1) & \text{if } |k_2(x_2 - x_1)| < F_{r2} \\ \varepsilon F_{s2} & \text{if } \varepsilon k_2(x_2 - x_1) > F_{r2} \ (\varepsilon = \pm 1) \end{cases}$  (sticking motion of  $m_2$ )

$F_{s1}$  and  $F_{s2}$  are the friction forces when slip motion occurs, while  $F_{r1}$  and  $F_{r2}$  are the static friction forces ( $F_{si} < F_{ri}$  ( $i = 1, 2$ )).

System (1) is normalized using

$$t = \tilde{\omega}t', \quad \tilde{\omega} = \sqrt{\frac{k_1 + k_2}{m_1}}, \quad V = W/\tilde{\omega} \quad (2)$$

From Eq. (1), it follows:

$$x_1'' + x_1 - \chi x_2 = u_1, \quad \chi = \frac{k_2}{k_1 + k_2}, \quad \eta = \frac{m_1}{m_2}$$

$$x_2'' + \chi\eta(x_2 - x_1) = \eta u_2, \quad u_i = \frac{F_i}{k_1 + k_2} \ (i = 1, 2), \quad (o') = \frac{d(o)}{dt} \quad (3)$$

In the following, the unit length is chosen in order that  $V = 1$ .

## 3 Description of the Different Modes of Oscillations

For each mass, three kinds of motions occur: slipping motion with a velocity less than the belt velocity, overshooting motion with a velocity greater than the belt velocity, and sticking motion with a velocity equal to the belt velocity. The system is not smooth and may include several phases of slipping, overshooting, or sticking motion for each mass. However, the system is piecewise linear and for each kind of configurations, the closed-form solution is available [2]. In the following, the description of the different modes of the system oscillations is detailed.

**3.1 Slip–Slip:**  $x_i' < V$  and  $u_i = u_{si}$  ( $i = 1, 2$ ). This motion is given by

$$Z(t) = H(t)Z_0, \quad Z(t) = \begin{pmatrix} z(t) \\ z'(t) \end{pmatrix}, \quad H(t) = \begin{pmatrix} H_1(t)H_2(t) \\ H_3(t)H_1(t) \end{pmatrix}$$

$$z(t) = x(t) - d_0, \quad d_0 = (d_{01}, d_{02})^t \quad (4)$$

$$x(t) = (x_1(t), x_2(t))^t, \quad Z_0 = Z(0)$$

$$d_{01} = \frac{u_{s1} + u_{s2}}{1 - \chi}, \quad d_{02} = \frac{\chi u_{s1} + u_{s2}}{\chi(1 - \chi)}$$

The  $2 \times 2$  matrices  $H_i(t)$  ( $i = 1, 2, 3$ ) are obtained by a modal analysis of system (1), where  $u_i = u_{si}$  ( $i = 1, 2$ ) [2].

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**3.2 Overshoot–Overshoot:**  $x'_i > V$  and  $u_i = -u_{si}$  ( $i=1, 2$ ). This motion is given by

$$Z(t) = H(t)Z_0 + 2(H(t) - I_4)\Delta_0, \quad \Delta_0 = \begin{pmatrix} d_0 \\ 0 \end{pmatrix} \quad (5)$$

( $I_4$  unitarian matrix of order 4)

**3.3 Slip–Overshoot:**  $x'_1 < V, x'_2 > V, u_1 = u_{s1}$ , and  $u_2 = -u_{s2}$ . The motion is given by

$$Z(t) = H(t)Z_0 + 2u_{s2}(H(t) - I_4)A_0 \quad (6)$$

$$\alpha_0 = \begin{pmatrix} 1/(1-\chi) \\ 1/\chi(1-\chi) \end{pmatrix} \quad A_0 = \begin{pmatrix} \alpha_0 \\ 0 \end{pmatrix}$$

**3.4 Overshoot–Slip:**  $x'_1 > V, x'_2 < V, u_1 = -u_{s1}$ , and  $u_2 = u_{s2}$ . The motion is governed by

$$Z(t) = H(t)Z_0 + 2u_{s1}(H(t) - I_4)B_0 \quad (7)$$

$$\beta_0 = \begin{pmatrix} 1/(1-\chi) \\ 1/(1-\chi) \end{pmatrix} \quad B_0 = \begin{pmatrix} \beta_0 \\ 0 \end{pmatrix}$$

**3.5 Slip–Stick:**  $x'_1 < V, x'_2 = V, u_1 = u_{s1}$ , and  $|\chi(x_2 - x_1)| < u_{r2}$ . This motion is governed by

$$Z(t) = \Gamma(t)Z_0, \quad \Gamma(t) = \begin{pmatrix} \Gamma_1(t)\Gamma_2(t) \\ \Gamma_3(t)\Gamma_1(t) \end{pmatrix} \quad (8)$$

The  $2 \times 2$  matrices  $\Gamma_i(t)$  ( $i = 1, 2, 3$ ) are given in Ref. [2].

**3.6 Stick–Slip:**  $x'_1 = V, x'_2 < V, u_2 = u_{s2}$ , and  $|x_1 - \chi x_2| < u_{r1}$ . This oscillation is described by

$$Z(t) = C(t)Z_0, \quad C(t) = \begin{pmatrix} C_1(t)C_2(t) \\ C_3(t)C_1(t) \end{pmatrix} \quad (9)$$

The  $2 \times 2$  matrices  $C_i(t)$  ( $i = 1, 2, 3$ ) are given also in Ref. [2].

**3.7 Overshoot–Stick:**  $x'_1 > V, x'_2 = V, u_1 = -u_{s1}$ , and  $|\chi(x_2 - x_1)| < u_{r2}$ . The motion is obtained from the formula

$$Z(t) = \Gamma(t)Z_0 + 2u_{s1}(\Gamma(t) - I_4)B_0 \quad (10)$$

**3.8 Stick–Overshoot:**  $x'_1 = V, x'_2 > V, u_2 = -u_{s2}$ , and  $|x_1 - \chi x_2| < u_{r1}$ . The motion is defined by

$$Z(t) = C(t)Z_0 + 2u_{s2}(C(t) - I_4)A_0 \quad (11)$$

**3.9 Stick–Stick:**  $x'_1 = V, x'_2 = V, |x_1 - \chi x_2| < u_{r1}$ , and  $|\chi(x_2 - x_1)| < u_{r2}$ .

The eventual transition between one configuration to another one leads to solve a set of transcendental equations. For example, a transition from a slip motion of  $m_2$  ( $x'_2 < V$ ) to an overshooting motion of this mass ( $x'_2 > V$ ) occurs if at some time  $t = t_1$ , we have

$$x'_2(t_1) = V, \quad \chi[x_2(t_1) - x_1(t_1)] < -u_{r2} \quad (12)$$

In the following, several sets of periodic motions are obtained. The initial conditions and the duration times of each parts of the motion are computed. The MATLAB software is used to solve the

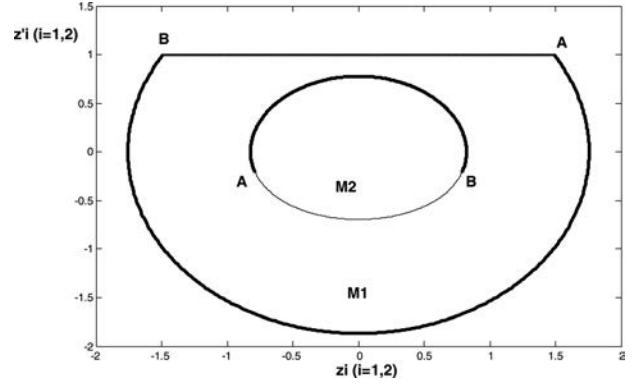


Fig. 2 Symmetrical orbits of the system

transcendental equations related to the eventual transitions between two successive configurations occurring in these orbits.

#### 4 Investigation of Periodic Orbits With Stick–Slip Phases (Symmetrical Solutions)

For each period (Fig. 2), the motion is composed of a global slipping motion (for  $0 < t < t_{AB}$ ) followed by a sticking motion of the first mass and a slipping motion of the second one (for  $0 < t - t_{AB} < t_{BA}$ ). We prove analytically that the phase portraits of the system are symmetrical with respect to the line corresponding to the constant part of the solution.

*Example.* For  $\chi = 0.3, \eta = 0.9, u_{s1} = u_{s2} = 0.8, u_{r1} = 2.5255$ , and  $u_{r2} = 1$ , we obtain  $t_{AB} = 4, t_{BA} = 2.978, z_{10} = 1.4888, z_{20} = -0.789$ , and  $z'_{20} = -0.2114$ .

#### 5 Periodic Orbits With Overshooting Phase of the Second Mass

A periodic solution (Figs. 3 and 4) composed for each period of four configurations is obtained: the first part is a slip–overshoot motion ( $x'_1 < V, x'_2 > V$ , and  $0 < t < t_{AB}$ ), the second one is a slip–stick motion ( $x'_1 < V, x'_2 = V$ , and  $0 < t - t_{AB} < t_{BC}$ ), the third one is a global slip motion ( $x'_1 < V, x'_2 < V$ , and  $0 < t - t_{AB} - t_{BC} < t_{CD}$ ), and the last part is a stick–slip motion ( $x'_1 = V, x'_2 > V$ , and  $0 < t - t_{AB} - t_{BC} - t_{CD} < t_{DA}$ ).

*Example.* For  $\chi = 0.2, \eta = 3.8, u_{s1} = 0.1, u_{s2} = 0.1749, u_{r1} = 1.9044$ , and  $u_{r2} = 0.5943$ , we obtain  $t_{AB} = 2.118, t_{BC} = 0.3, t_{CD} = 1.7, t_{DA} = 2.16, z_{10} = 1.2936$ , and  $z_{20} = -2.5537$ .

#### 6 Periodic Orbits With Overshooting Phase of the First Mass

A periodic solution (Figs. 5 and 6) composed for each period of four configurations is obtained: the first part is an overshoot–slip motion ( $x'_1 > V, x'_2 < V$ , and  $0 < t < t_{AB}$ ), the second one is a stick–slip motion ( $x'_1 = V, x'_2 < V$ , and  $0 < t - t_{AB} < t_{BC}$ ), the third one is a global slip motion ( $x'_1 < V, x'_2 < V$ , and  $0 < t - t_{AB} - t_{BC} < t_{CD}$ ), and the last part is a slip–stick motion ( $x'_1 < V, x'_2 = V$ , and  $0 < t - t_{AB} - t_{BC} - t_{CD} < t_{DA}$ ).

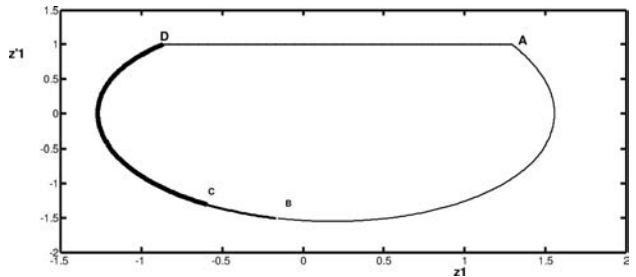


Fig. 3 Phase portrait of  $m_1$  with overshooting of  $m_2$

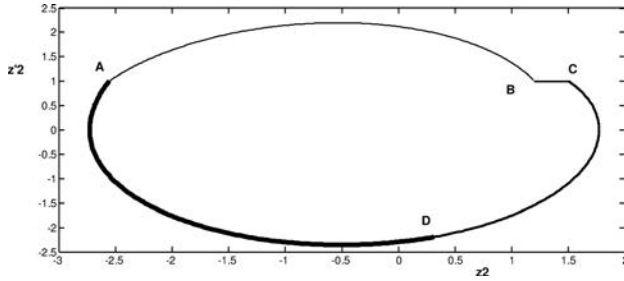


Fig. 4 Phase portrait of  $m_2$  with overshooting part

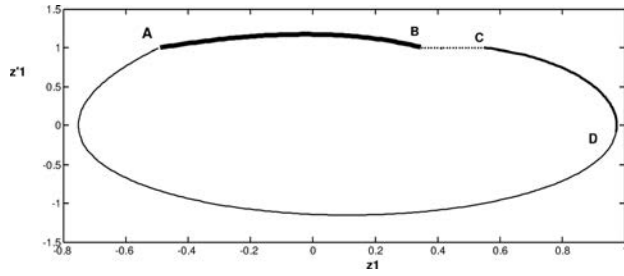


Fig. 5 Phase portrait of  $m_1$  with overshooting part

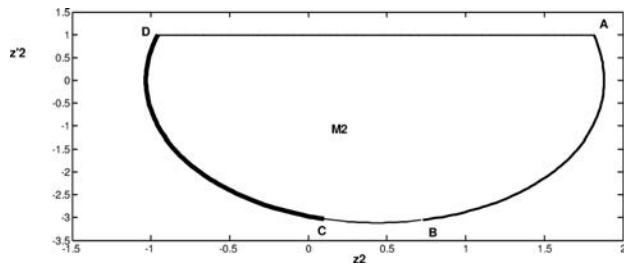


Fig. 6 Phase portrait of  $m_2$  with overshooting of  $m_1$

*Example.* For  $\chi = 0.8, \eta = 4.2, u_{s1} = 0.6482, u_{s2} = 0.1, u_{r1} = 1.1175,$  and  $u_{r2} = 1.928,$  we obtain  $t_{AB} = 0.754, t_{BC} = 0.2, t_{CD} = 0.78, t_{DA} = 2.771, z_{10} = -0.4893,$  and  $z_{20} = 1.8142.$

## 7 Conclusions

In this work, a more complex model of dry friction oscillator is considered. The system consists of two masses connected by linear springs. It is also assumed that each mass is in contact with a belt moving at a constant velocity. This two degrees-of-freedom system includes two friction forces instead of only one as it was assumed in other investigations [4]. Several sets of periodic orbits including stick–slip phases are obtained. Among them, we prove the existence of periodic orbits involving a phase of “overshooting” slip motion for one of the masses. In the past, this kind of orbits was observed only for more complex friction characteristics than Coulomb’s ones, and it is proved that these orbits are not possible for a one degree-of-freedom system excited by Coulomb’s dry friction [3]. Other kind of periodic motions including, for example, a phase of global overshooting motion (overshooting motion for the two masses) can be the subject of future investigations. However, it is possible to prove that for any periodic solutions, a phase of sticking motion for at least one of the masses must occur. This result limits the possible events of such a periodic solution. A more realistic model of dry friction oscillator can also be considered by adding to the system small damping. By using the equivalent damping coefficients [5], a similar investigation of periodic orbits including overshooting phases can be performed.

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