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INVARIANCE PRINCIPLE VIA ORTHOMARTINGALE APPROXIMATION

DAVIDE GIRAUDO

ABSTRACT. We obtain a necessary and sufficient condition for the orthomartingale-coboundary decomposition. We establish a sufficient condition for the approximation of the partial sums of a strictly stationary random fields by those of stationary orthomartingale differences. This condition can be checked under multidimensional analogues of the Hannan condition and the Maxwell-Woodroffe condition.

1. INTRODUCTION AND NOTATIONS

In all the paper, we shall use the following notations. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space.

- For a function $f: \Omega \rightarrow \mathbb{R}$, $\|f\|$ will denote the \mathbb{L}^2 -norm of f . The subspace of centered square integrable functions is denoted as \mathbb{L}_0^2 .
- If d is a positive integer, we denote by $[d]$ the set $\{1, \dots, d\}$.
- If $\mathbf{n} = (n_1, \dots, n_d)$ is an element of \mathbb{N}^d , we denote by $\min \mathbf{n}$ the quantity $\min_{1 \leq q \leq d} n_q$ and $|\mathbf{n}| := \prod_{q=1}^d n_q$. Moreover, we shall write $\mathbf{2}^{\mathbf{n}} = \sum_{q=1}^d 2^{n_q} \mathbf{e}_q$.
- If $q \in [d]$, then \mathbf{e}_q is the element of \mathbb{N}^d such that the q th coordinate is equal to 1, and all the others to 0.
- We denote for an element \mathbf{i} of \mathbb{Z}^d and a non-empty subset J of $[d]$ the multiindex $\mathbf{i}_J \in \mathbb{Z}^d$ defined as $\sum_{q \in J} i_q \mathbf{e}_q$.
- Let $(a_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d}$ be a family of real numbers. We define

$$\limsup_{\mathbf{n} \rightarrow +\infty} a_{\mathbf{n}} := \lim_{i \rightarrow +\infty} \sup_{\mathbf{n}: \min \mathbf{n} \geq i} a_{\mathbf{n}}. \quad (1.0.1)$$

- Let $T_q, q \in [d]$ be bijective, bi-measurable and measure preserving maps from Ω to itself which are pairwise commuting. For $\mathbf{i} \in \mathbb{Z}^d$, we denote by $T^{\mathbf{i}}$ the map $T_1^{i_1} \circ \dots \circ T_d^{i_d}$, $U^{\mathbf{i}}: \mathbb{L}^1 \rightarrow \mathbb{L}^1$ the operator defined by $(U^{\mathbf{i}}f)(\omega) = f(T^{\mathbf{i}}\omega)$ and

$$S_{\mathbf{n}}(f) = \sum_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{n}-1} U^{\mathbf{i}}(f) = \sum_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{n}-1} (f) \circ T^{\mathbf{i}}. \quad (1.0.2)$$

We also use the notation $U_q := U^{\mathbf{e}_q}$.

- We shall write as a product the composition of operators U_q and we use the convention $\prod_{q \in \emptyset} U_q = \text{I}$.
- If I is a subset of $[d]$, then $\varepsilon(I)$ is the element of \mathbb{Z}^d whose q th coordinate is -1 if q belong to I and 0 otherwise.
- The product, sum and minimum of two elements of \mathbb{Z}^d is understood to be coordinatewise.
- Let $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ denote a filtration. For $J \subset [d]$, we denote by $\mathcal{F}_{\infty \mathbf{1}_J}$ the σ -algebra generated by $\bigcup_{\mathbf{j} \in \mathbb{Z}^d, \mathbf{j}_J \leq \mathbf{0}_J} \mathcal{F}_{\mathbf{j}}$.

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1.1. The invariance principle. For $\mathbf{i} \succcurlyeq \mathbf{1}$, we denote the unit cube with upper corner at $\mathbf{i} = (i_1, \dots, i_d)$ that is,

$$R_{\mathbf{i}} := \prod_{q=1}^d (i_q - 1, i_q]. \quad (1.1.1)$$

For a measurable function $f: \Omega \rightarrow \mathbb{R}$, we consider the partial sum process defined by

$$S_{\mathbf{n}}(f, \mathbf{t}) := \sum_{\mathbf{i} \in [\mathbf{1}, \mathbf{n}]} \lambda([\mathbf{0}, \mathbf{n} \cdot \mathbf{t}] \cap R_{\mathbf{i}}) U^{\mathbf{i}} f, \quad \mathbf{t} \in [0, 1]^d, \mathbf{n} \in (\mathbb{N}^*)^d, \quad (1.1.2)$$

where λ denotes the Lebesgue measure on \mathbb{R}^d , $[\mathbf{0}, \mathbf{n} \cdot \mathbf{t}] = \prod_{q=1}^d [0, n_q t_q]$ and

$$[\mathbf{1}, \mathbf{n}] = \left\{ \mathbf{i} \in \mathbb{Z}^d, 1 \leq i_q \leq n_q \text{ for each } q \in \{1, \dots, d\} \right\}. \quad (1.1.3)$$

We are interested in the functional central limit theorem in $C([0, 1]^d)$ for the net $(S_{\mathbf{n}}(f, \cdot))_{\mathbf{n} \in (\mathbb{N}^*)^d}$ in order to understand the asymptotic behavior of the partial sums of $(f \circ T^{\mathbf{i}})$ over rectangles. By "functional central limit theorem in $C([0, 1]^d)$ ", we mean that for each continuous bounded functional $F: (C([0, 1]^d), \|\cdot\|_{\infty}) \rightarrow \mathbb{R}$, the convergence $F(S_{\mathbf{n}}(f, \cdot)/a_{\mathbf{n}}) \rightarrow F(W)$ holds as $\min \mathbf{n}$ goes to infinity, where W is a Gaussian process (or a mixture of a Gaussian process). Usually, the normalizing term $a_{\mathbf{n}}$ will be chosen as $|\mathbf{n}| := \prod_{q=1}^d n_q$.

The question of the functional central limit theorem in the space of continuous functions (endowed with the uniform norm) for strictly stationary random fields has been studied. Wichura [WIC69] established such a result for an i.i.d. centered random field with finite variance, which generalized Donsker's one dimensional result [DON51]. Wichura's result was extended to a class of stationary ergodic martingale differences random fields [BD79, PR98], and Dedecker found a projective condition [DED01]. Wang and Woodroffe [WW13] attempted to extend the Maxwell and Woodroffe condition [MW00] but found a weaker condition, which was improved by Volný and Wang [VW14]. The latter is a multidimensional extension of Hannan's condition [HAN73]. In the context of the mentioned works, the limiting process is a standard Brownian sheet when the considered random field is ergodic, that is, a Gaussian process $(W_{\mathbf{t}})_{\mathbf{t} \in [0, 1]^d}$ such that $\text{Cov}(W_{\mathbf{t}}; W_{\mathbf{s}}) = \prod_{i=1}^d \min\{t_i, s_i\}$.

1.2. Orthomartingales. Let $(T_q)_{q=1}^d$ be bijective, bi-measurable and measure preserving transformations on $(\Omega, \mathcal{F}, \mu)$. Assume that $T_q \circ T_{q'} = T_{q'} \circ T_q$ for each $q, q' \in \{1, \dots, d\}$. Let \mathcal{F}_0 be a sub- σ -algebra of \mathcal{F} such that for each $q \in \{1, \dots, d\}$, $\mathcal{F}_0 \subset T_q^{-1} \mathcal{F}_0$. In this way, $\mathcal{F}_{\mathbf{i}} := T^{-\mathbf{i}} \mathcal{F}_0$, $\mathbf{i} \in \mathbb{Z}^d$, yields a filtration. If for each $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^d$ and each integrable and $\mathcal{F}_{\mathbf{l}}$ -measurable random variable Y ,

$$\mathbb{E}[Y | \mathcal{F}_{\mathbf{k}}] = \mathbb{E}[Y | \mathcal{F}_{\mathbf{k} \wedge \mathbf{l}}] \text{ almost surely,} \quad (1.2.1)$$

the transformations $(T_q)_{q=1}^d$ are said to be *completely commuting*.

Recall that $\mathbf{i} \preccurlyeq \mathbf{j}$ means that $i_q \leq j_q$ for each $q \in \{1, \dots, d\}$. The collection of random variables $\{M_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is said to be an orthomartingale random field with respect to the completely commuting filtration $(T^{-\mathbf{i}} \mathcal{F}_0)_{\mathbf{i} \in \mathbb{Z}^d}$ if for each $\mathbf{n} \in \mathbb{N}^d$, $M_{\mathbf{n}}$ is $\mathcal{F}_{\mathbf{n}}$ -measurable, integrable and for each $\mathbf{i}, \mathbf{j} \in \mathbb{Z}_+^d$ such that $\mathbf{i} \preccurlyeq \mathbf{j}$,

$$\mathbb{E}[M_{\mathbf{j}} | \mathcal{F}_{\mathbf{i}}] = M_{\mathbf{i}}. \quad (1.2.2)$$

Definition 1.1. Let $m: \Omega \rightarrow \mathbb{R}$ be a measurable function. The random field $(m \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ is an orthomartingale difference random field with respect to the completely commuting filtration $(T^{-\mathbf{i}} \mathcal{F}_0)_{\mathbf{i} \in \mathbb{Z}^d}$ if the random field $(M_{\mathbf{n}})_{\mathbf{n} \in \mathbb{N}^d}$ defined by $M_{\mathbf{n}} := \sum_{\mathbf{i} \in [0, \mathbf{n}-1]} m \circ T^{\mathbf{i}}$ is an orthomartingale random field.

Proposition 1.2. Let $(m \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ be an orthomartingale differences random field with respect to the completely commuting filtration $(T^{-\mathbf{i}} \mathcal{F}_0)_{\mathbf{i} \in \mathbb{Z}^d}$. Then for each $\mathbf{n} \in \mathbb{N}^d$ such that $\mathbf{n} \succcurlyeq \mathbf{1}$, the following

inequality holds:

$$\left\| \frac{1}{|\mathbf{n}|^{1/2}} \max_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} |S_{\mathbf{i}}(m)| \right\| \leq 2^d \|m\|. \quad (1.2.3)$$

This shows that the the family of normalized maxima of partial sums is bounded in \mathbb{L}^2 . Lemma 3.1 in [VW14] shows more.

Proposition 1.3. *Let $(m \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ be an orthomartingale differences random field with respect to the completely commuting filtration $(T^{-\mathbf{i}}\mathcal{F}_0)_{\mathbf{i} \in \mathbb{Z}^d}$. Then the family*

$$\left\{ \frac{1}{|\mathbf{n}|} \max_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} |S_{\mathbf{i}}(m)|^2, \mathbf{n} \in (\mathbb{N}^*)^d \right\} \quad (1.2.4)$$

is uniformly integrable.

1.3. Orthomartingale approximation. There are essentially two methods for establishing the invariance principle for a stationary sequence. The first one is approximation by an i.i.d. sequence, which leads to good results but there are processes which cannot be treated in this way. An other method for establishing limit theorems for strictly stationary sequences is a martingale approximation. Since it is known that a stationary martingale difference sequence satisfies the invariance principle, one can try to prove an invariance principle by martingale approximation. More formally, given a square integrable centered function $f: \Omega \rightarrow \mathbb{R}$, one can wonder whether there exists a square integrable martingale differences sequence $(m \circ T^{\mathbf{i}})_{\mathbf{i} \geq 0}$ such that $\lim_{n \rightarrow +\infty} n^{-1/2} \|\max_{\mathbf{1} \leq i \leq n} |S_{\mathbf{i}}(f - m)|\| = 0$. The existence of such an approximation without the max has been investigated in [GOR69]. A necessary and sufficient condition has been given in [WW04, ZW08B] in the adapted case, then extended to the adapted case in [VOL06]. The question of the choice of filtration has also been considered in [QV12].

This approach was also used for other limit theorems, like the quenched weak invariance principle [CV13, CM14] or the law of the iterated logarithms [ZW08A].

A multidimensional analogue of the martingale approximation has not been so intensively studied. There are various way to define martingales random fields in dimension greater than one (cf. [NP92, CAI69]).

In this paper, we shall work on orthomartingale approximation, since it is known [VOL15, CDV15] that when T_1 is ergodic, the invariance principle takes place.

Definition 1.4. *We say that the function f admits an orthomartingale approximation if there exists a square integrable function m such that $(m \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ is an orthomartingale differences random field*

$$\limsup_{\mathbf{n} \rightarrow +\infty} \frac{1}{\sqrt{|\mathbf{n}|}} \left\| \max_{\mathbf{1} \preceq \mathbf{i} \preceq \mathbf{n}} |S_{\mathbf{i}}(f - m)| \right\| = 0. \quad (1.3.1)$$

The uniform norm of the function $t \mapsto S_{\mathbf{n}}(f, t)$ can be controlled by the maxima of partial sums. Moreover, a stationary orthomartingale differences random field with respect to a completely commuting filtration such that one of the maps T_1, \dots, T_d is ergodic satisfies the invariance principle. Therefore, when ergodicity in one direction holds, an orthomartingale approximation entails the invariance principle. In the other cases, an invariance principle may still hold, but the limiting process may not be a Brownian sheet (see Remark 5.5 in [WW13]).

The paper is organizes as follows. Section 2 contains the main results of the paper, namely, a necessary and sufficient condition for the orthomartingale-coboundary decomposition, a sufficient condition for the existence of an approximating orthomartingale and the verification of the latter under two projective condition: Hannan and Maxwell-Woodrooffe. Section 3 is devoted the proofs.

2. MAIN RESULTS

2.1. Orthomartingale-coboundary decomposition. The following operators will be used in the sequel.

Definition 2.1. Let T be a measure preserving \mathbb{Z}^d -action and let \mathcal{F}_0 be a σ -algebra such that $(T^{-i}\mathcal{F}_0)_{i \in \mathbb{Z}^d}$ is a completely commuting filtration. Let $E \subsetneq [d]$ and $\mathbf{i} \in \mathbb{N}^d$. We define the operators $P_{d,E}$ and $P_{d,[d]}$ by

$$P_{d,E}^{\mathbf{i}}(f) := \sum_{J \subset E} (-1)^{|J|+|\mathbf{i}|} \mathbb{E} [U^{\mathbf{i} \cdot \varepsilon(E)} f \mid \mathcal{F}_{\infty \mathbf{1}_J}], \quad f \in \mathbb{L}^1, \quad (2.1.1)$$

$$P_{d,[d]}^{\mathbf{i}}(f) = U^{-\mathbf{i}} f + \sum_{J \subsetneq [d]} (-1)^{|J|+d} \mathbb{E} [U^{-\mathbf{i}} f \mid \mathcal{F}_{\infty \mathbf{1}_J}], \quad f \in \mathbb{L}^1, \quad (2.1.2)$$

and the closed subspaces of \mathbb{L}^2

$$\mathcal{H}_{d,E} := \{h \in \mathbb{L}_0^2 \mid h \text{ is } \mathcal{F}_{\infty \mathbf{1}_E}\text{-measurable and } \mathbb{E}[h \mid \mathcal{F}_{\infty \mathbf{1}_{E'}}] = 0 \text{ if } E' \subsetneq E\}, \quad E \subsetneq [d], \quad (2.1.3)$$

$$\mathcal{H}_{d,[d]} = \{h \in \mathbb{L}^2 \mid \mathbb{E}[h \mid \mathcal{F}_{\infty \mathbf{1}_{E'}}] = 0 \text{ if } E' \subsetneq [d]\}. \quad (2.1.4)$$

When the integer d does not need to be specified, we shall simply denote $P_E^{\mathbf{i}}$ for $E \subset [d]$ and \mathcal{H}_E . In dimension one, we have

$$P_{\emptyset}^{\mathbf{i}}(f) := \mathbb{E}[U^{\mathbf{i}} f \mid \mathcal{F}_0] \text{ and } P_{\{1\}}^{\mathbf{i}}(f) := U^{-i} f - \mathbb{E}[U^{-i} f \mid \mathcal{F}_0], \quad (2.1.5)$$

and these operators have been used in [Vol07, CCD⁺14, Gir17]. In dimension two, the operators $P_{2,E}$ are given by

$$P_{\emptyset}^{i,j}(f) = \mathbb{E}[U^{(i,j)} f \mid \mathcal{F}_{0,0}], \quad (2.1.6)$$

$$P_{\{1\}}^{i,j}(f) = \mathbb{E}[U^{(-i,j)} f \mid \mathcal{F}_{\infty,0}] - \mathbb{E}[U^{(-i,j)} f \mid \mathcal{F}_{0,0}], \quad (2.1.7)$$

$$P_{\{2\}}^{i,j}(f) = \mathbb{E}[U^{(i,-j)} f \mid \mathcal{F}_{0,\infty}] - \mathbb{E}[U^{(i,-j)} f \mid \mathcal{F}_{0,0}] \text{ and} \quad (2.1.8)$$

$$P_{\{1,2\}}^{i,j}(f) = U^{(-i,-j)} f - \mathbb{E}[U^{(-i,-j)} f \mid \mathcal{F}_{\infty,0}] \\ - \mathbb{E}[U^{(-i,-j)} f \mid \mathcal{F}_{0,\infty}] + \mathbb{E}[U^{(-i,-j)} f \mid \mathcal{F}_{0,0}]. \quad (2.1.9)$$

We are now in position to state a necessary and sufficient condition for the orthomartingale-coboundary decomposition.

Theorem 2.2. Let f be a square integrable centered function and $d \geq 1$. Let T be a measure preserving \mathbb{Z}^d -action and let \mathcal{F}_0 be a σ -algebra such that $(T^{-i}\mathcal{F}_0)_{i \in \mathbb{Z}^d}$ is a completely commuting filtration. The following conditions are equivalent:

(1) for each $E \subset [d]$,

$$\sup_{\mathbf{n} \in \mathbb{N}^d} \left\| \sum_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{n}} P_E^{\mathbf{i}} f \right\| < +\infty; \quad (2.1.10)$$

(2) there exists square integrable functions m_J , $J \subset [d]$ such that

$$f = \sum_{J \subset [d]} \prod_{q \in J} (1 - U_q) m_J \quad (2.1.11)$$

and for each $J \neq [d]$, m_J is $\mathcal{F}_{\infty \mathbf{1}_J}$ -measurable and if $I \subsetneq J$, then $\mathbb{E}[m_J \mid \mathcal{F}_{\infty \mathbf{1}_I}] = 0$.

Remark 2.3. In dimension one, Theorem 2.7 reads as follows: a function f can be written as $f = m + g - g \circ T$, where m is \mathcal{F}_0 -measurable and $\mathbb{E}[m \mid T\mathcal{F}_0] = 0$ if and only if

$$\sup_{n \geq 0} \|\mathbb{E}[S_n(f) \mid \mathcal{F}_0]\| < +\infty \text{ and } \sup_{n \geq 0} \|S_n(f) - \mathbb{E}[S_n(f) \mid T^{-n+1}\mathcal{F}_0]\| < +\infty \quad (2.1.12)$$

This can be viewed as a nonadapted version of Proposition 4.1 in [CCD⁺14].

Remark 2.4. This improves the main result in [EMG16] since Theorem 2.2 does not require the function f to be \mathcal{F}_0 -measurable. Moreover, even for such functions, the condition is less restrictive. Indeed, in this case, condition (2.1.10) is equivalent to boundedness of the quantity $\|\mathbb{E}[S_n(f) \mid \mathcal{F}_0]\|$ independently of $\mathbf{n} \succcurlyeq \mathbf{1}$, while that of [EMG16] read $\sum_{\mathbf{k} \in \mathbb{N}^d} \mathbb{E}[U^{\mathbf{k}} f \mid \mathcal{F}_0] < +\infty$.

A related result has been obtained in [GOR09], where reversed martingales are obtained in the decomposition. However, a kind of regularity assumption is made. This is also the case in [VOL17].

2.2. A sufficient condition for orthomartingale approximation. In order to express a sufficient condition for the orthomartingale approximation (1.3.1), we define a blocking operator.

Definition 2.5. Let $f: \Omega \rightarrow \mathbb{R}$ be a measurable function and let k be an integer greater or equal to 1. The blocking operator is defined by

$$B_k(f) := \frac{1}{k^d} \sum_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq k\mathbf{1}} \sum_{E \subset [d]} \prod_{q=1}^d \left(I - P_E^{i_q e_q} \right) P_E^0(f). \quad (2.2.1)$$

Definition 2.6. Let $f: \Omega \rightarrow \mathbb{R}$ be a measurable function. The plus semi-norm, denoted by $\|\cdot\|_+$, is defined by

$$\|f\|_+ := \limsup_{\mathbf{n} \rightarrow +\infty} \frac{1}{|\mathbf{n}|^{1/2}} \left\| \max_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}} |S_{\mathbf{i}}(f)| \right\|. \quad (2.2.2)$$

Theorem 2.7. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $T: \Omega \rightarrow \Omega$ be a measure preserving \mathbb{Z}^d -action. Assume that \mathcal{F}_0 is a sub- σ -algebra of \mathcal{F}_0 such that $(T^{-i}\mathcal{F}_0)_{i \in \mathbb{Z}^d}$ is a completely commuting filtration. Let $f: \Omega \rightarrow \mathbb{R}$ be a measurable function. If

$$\lim_{k \rightarrow +\infty} \|B_k(f) - f\|_+ = 0, \quad (2.2.3)$$

then there exists a function m such that $(m \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ is an orthomartingale differences random field with respect to the filtration $(T^{-i}\mathcal{F}_0)_{i \in \mathbb{Z}^d}$ and

$$\limsup_{\mathbf{n} \rightarrow +\infty} \frac{1}{\sqrt{|\mathbf{n}|}} \left\| \max_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}} |S_{\mathbf{i}}(f - m)| \right\| = 0. \quad (2.2.4)$$

In particular, the conclusion holds if (2.2.3) is replaced by the following one:

$$\forall \emptyset \subsetneq J \subset [d], \forall E \subset [d], \quad \lim_{k \rightarrow +\infty} \frac{1}{k^{|J|}} \left\| \sum_{\mathbf{1}_{\mathbf{J}} \preccurlyeq \mathbf{j} \preccurlyeq k\mathbf{1}_{\mathbf{J}}} P_E^{\mathbf{j}} f \right\|_+ = 0. \quad (2.2.5)$$

Remark 2.8. In dimension one, Theorem 2.7 reads as follows: the condition

$$\lim_{k \rightarrow +\infty} \frac{1}{k} \limsup_{n \rightarrow +\infty} \frac{1}{\sqrt{n}} \left\| \max_{1 \leq i \leq n} \left| S_i \left(\mathbb{E}[S_k(f) \mid \mathcal{F}_0] + \sum_{i=1}^k U^{-i} f - \mathbb{E}[U^{-i} f \mid \mathcal{F}_0] \right) \right| \right\| = 0. \quad (2.2.6)$$

is sufficient for the existence of a function m such that $(m \circ T^i)_{i \geq 0}$ is a martingale differences sequence and $\limsup_{n \rightarrow +\infty} n^{-1/2} \|\max_{1 \leq i \leq n} |S_i(f - m)|\| = 0$. This can be viewed as a nonadapted version of Theorem 1 in [GP11].

Remark 2.9. Theorem 2.7 can be used even if none of the maps T_q , $q \in [d]$ is ergodic. In this case, the function f may not satisfy the central limit theorem because the approximating martingale itself may not satisfy it (see Remark 5.5 in [WW13]).

Remark 2.10. In all the paper, we assume the filtration to be completely commuting. In [PZ17], partially commuting filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}^d}$ are considered, in the sense that if $i \geq i'$ and $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^{d-1}$, then for any integrable random variable Y ,

$$\mathbb{E}[\mathbb{E}[X | \mathcal{F}_{i', \mathbf{v}}] | \mathcal{F}_{i, \mathbf{u}}] = \mathbb{E}[X | \mathcal{F}_{i', \min\{\mathbf{u}, \mathbf{v}\}}]. \quad (2.2.7)$$

It does not seem that our results apply in this context because complete commutativity of the filtration is used in the proof of Theorem 2.7.

2.3. Applications: projective conditions.

2.3.1. *Hannan's condition.* Assume that $d = 1$, $T: \Omega \rightarrow \Omega$ is a bijective bimeasurable measure preserving map and \mathcal{F}_0 is a sub- σ -algebra such that $T\mathcal{F}_0 \subset \mathcal{F}_0$. Assume that $f: \Omega \rightarrow \mathbb{R}$ is measurable with respect to the σ -algebra generated by $\bigcup_{k \in \mathbb{Z}} T^k \mathcal{F}_0$ and such that $\mathbb{E}[f | \bigcap_{k \in \mathbb{Z}} T^k \mathcal{F}_0] = 0$ and let us consider the condition

$$\sum_{i \in \mathbb{Z}^d} \left\| \mathbb{E}[f \circ T^i | \mathcal{F}_0] - \mathbb{E}[f \circ T^i | T\mathcal{F}_0] \right\| < +\infty. \quad (2.3.1)$$

That the central limit theorem is implied by (2.3.1) is contained in [HEY74] (see also Theorem 6 in [VOL93]). When f is \mathcal{F}_0 -measurable, the central limit theorem and the weak invariance principle were proved by Hannan [HAN73, HAN79] under the assumption that T is weakly mixing. Dedecker and Merlevède [DM03] showed that (2.3.1) itself implies the weak invariance principle. Finally, the invariance principle when f satisfies (2.3.1) but is not necessarily \mathcal{F}_0 -measurable was established in [DMV07].

The generalization of condition (2.3.1) to random field has been obtained by Volný and Wang. Let us recall the notations and results of [VW14]. The projection operators with respect to a commuting filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}^d}$ are defined by

$$\pi_{\mathbf{j}} := \prod_{q=1}^d \pi_{j_q}^{(q)}, \quad \mathbf{j} \in \mathbb{Z}^d, \quad (2.3.2)$$

where for $l \in \mathbb{Z}$, $\pi_l^{(q)}: \mathbb{L}^1(\mathcal{F}) \rightarrow \mathbb{L}^1(\mathcal{F})$ is defined for $f \in \mathbb{L}^1$ by

$$\pi_l^{(q)}(f) = \mathbb{E}_l^{(q)}[f] - \mathbb{E}_{l-1}^{(q)}[f] \quad (2.3.3)$$

and

$$\mathbb{E}_l^{(q)}[f] = \mathbb{E} \left[f \mid \bigvee_{\substack{i \in \mathbb{Z}^d \\ i_q \leq l}} \mathcal{F}_i \right], \quad q \in [d], l \in \mathbb{Z}. \quad (2.3.4)$$

Theorem 2.11 ([VW14]). *Let $(\mathcal{F}_i)_{i \in \mathbb{Z}^d} := (T^{-i}\mathcal{F}_0)_{i \in \mathbb{Z}^d}$ be a completely commuting filtration. Let f be a function such that for each $q \in [d]$, $\mathbb{E}[f | T_q^l \mathcal{F}_0] \rightarrow 0$ as $l \rightarrow +\infty$, measurable with respect to the σ -algebra generated by $\bigcup_{i \in \mathbb{Z}^d} T^i \mathcal{F}_0$ and such that $\sum_{i \in \mathbb{Z}^d} \|\pi_i(f)\| < +\infty$. Then there exists a function m such that $(m \circ T^i)_{i \in \mathbb{Z}^d}$ is an orthomartingale differences random field with respect to the completely commuting filtration $(T^{-i}\mathcal{F}_0)_{i \in \mathbb{Z}^d}$ and such that (1.3.1) holds.*

We can recover this result via Theorem 2.7.

2.3.2. Maxwell and Woodroffe condition. In the one dimensional case, conditions on the quantities $\mathbb{E}[S_n(f) | T\mathcal{F}_0]$ and $S_n(f) - \mathbb{E}[S_n(f) | T^{-n}\mathcal{F}_0]$ have been investigated. The first result in this direction was obtained by Maxwell and Woodroffe [MW00]: if f is \mathcal{F}_0 -measurable and

$$\sum_{n=1}^{+\infty} \frac{\|\mathbb{E}[S_n(f) | \mathcal{F}_0]\|}{n^{3/2}} < +\infty, \quad (2.3.5)$$

then $(n^{-1/2}S_n(f))_{n \geq 1}$ converges in distribution to $\eta^2 N$, where N is normally distributed and independent of η . Then Volný [VOL06] proposed a method to treat the nonadapted case. Peligrad and Utev [PU05] proved the weak invariance principle under condition (2.3.5). The nonadapted case was addressed in [VOL07]. Peligrad and Utev also showed that condition (2.3.5) is optimal among conditions on the growth of the sequence $(\|\mathbb{E}[S_n(f) | \mathcal{F}_0]\|)_{n \geq 1}$: if

$$\sum_{n=1}^{+\infty} a_n \frac{\|\mathbb{E}[S_n(f) | \mathcal{F}_0]\|}{n^{3/2}} < \infty \quad (2.3.6)$$

for some sequence $(a_n)_{n \geq 1}$ converging to 0, the sequence $(n^{-1/2}S_n(f))_{n \geq 1}$ is not necessarily stochastically bounded (Theorem 1.2. of [PU05]). Volný constructed [VOL10] an example satisfying (2.3.6) and such that the sequence $(\|S_n(f)\|^{-1}S_n(f))_{n \geq 1}$ admits two subsequences which converge weakly to two different distributions. In dimension one, these results are the consequence of a existence of an approximating martingale (see Proposition 3 in [GP11]). We are able to formulate an analogous result in the multidimensional setting.

Theorem 2.12. *Let T be a measure preserving \mathbb{Z}^d -action and let \mathcal{F}_0 be a sub- σ -algebra such that $(T^{-i}\mathcal{F}_0)_{i \in \mathbb{Z}^d}$ is a completely commuting filtration. Let f be a square integrable function such that for any $E \subset [d]$,*

$$\sum_{\mathbf{n} \neq \mathbf{1}} \frac{1}{|\mathbf{n}|^{3/2}} \left\| \sum_{\mathbf{0} \leq i \leq \mathbf{n}-\mathbf{1}} P_E^i f \right\| < +\infty. \quad (2.3.7)$$

Then there exists a function m such that $(m \circ T^i)_{i \in \mathbb{Z}^d}$ is an orthomartingale differences random field and

$$\limsup_{\mathbf{n} \rightarrow +\infty} \frac{1}{\sqrt{|\mathbf{n}|}} \left\| \max_{\mathbf{1} \leq i \leq \mathbf{n}} |S_i(f - m)| \right\| = 0. \quad (2.3.8)$$

Remark 2.13. In dimension one, we recover the result of [PU05, VOL07]. In dimension two, condition (2.3.7) reads as follows: if the series

$$A_\emptyset := \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^{3/2} n_2^{3/2}} \|\mathbb{E}[S_{n_1, n_2}(f) | \mathcal{F}_{0,0}]\| \quad (2.3.9)$$

$$A_{\{1\}} := \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^{3/2} n_2^{3/2}} \|\mathbb{E}[S_{n_1, n_2}(f) | \mathcal{F}_{\infty,0}] - \mathbb{E}[S_{n_1, n_2}(f) | \mathcal{F}_{\infty, n_2-1}]\| \quad (2.3.10)$$

$$A_{\{2\}} := \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^{3/2} n_2^{3/2}} \|\mathbb{E}[S_{n_1, n_2}(f) | \mathcal{F}_{0,\infty}] - \mathbb{E}[S_{n_1, n_2}(f) | \mathcal{F}_{n_1-1, \infty}]\| \quad (2.3.11)$$

$$A_{\{1,2\}} := \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^{3/2} n_2^{3/2}} \|S_{n_1, n_2}(f) - \mathbb{E}[S_{n_1, n_2}(f) | \mathcal{F}_{n_1-1, \infty}] - \mathbb{E}[S_{n_1, n_2}(f) | \mathcal{F}_{\infty, n_2-1}] + \mathbb{E}[S_{n_1, n_2}(f) | \mathcal{F}_{n_1-1, n_2-1}]\| \quad (2.3.12)$$

are convergent, then there exists an orthomartingale differences random fields satisfying (1.3.1).

Remark 2.14. Using an adaptation of the construction given in [DV08, DUR09], we can construct an example of function f which satisfies the assumption of Proposition 2.11 but not that of Theorem 2.12 and vice-versa. Let $(\Omega_1, \mathcal{A}_1, \mu_1, T_1)$ be the dynamical system considered in [DV08, DUR09] and for $2 \leq i \leq d$, let $(\Omega_i, \mathcal{A}_i, \mu_i, T_i)$ be Bernoulli dynamical systems. Then consider $\Omega := \prod_{q=1}^d \Omega_q$, $\mathcal{A} := \bigotimes_{q=1}^d \mathcal{A}_q$, $T^i((\omega_q)_{q=1}^d) := \left(T_q^{i_q} \omega_q\right)_{q=1}^d$. For $2 \leq q \leq d$, let $e^{(q)}: \Omega_q \rightarrow \mathbb{R}$ be such that $(e^{(q)} \circ T_q^i)_{i \in \mathbb{Z}}$ is i.i.d. If f is the function defined in [DV08, DUR09], then let $F = f \cdot \prod_{q=2}^d e^{(q)}$ and $\mathcal{F}_0 := \mathcal{F}_0 \otimes \bigotimes_{q=2}^d \sigma(e^{(q)} \circ T_q^i, i \leq 0)$. In this way, the F satisfies the multidimensional Hannan and Maxwell and Woodroffe conditions if and only if so does f for the unidimensional ones.

Remark 2.15. Using the same construction as previously, but where $(\Omega_1, \mathcal{A}_1, \mu_1, T_1)$ is the dynamical system involved in the proof of Theorem [PU05], we can see that the weight $|\mathbf{n}|^{-3/2}$ in condition (2.3.7) is in some sense optimal.

Remark 2.16. If f is an \mathcal{F}_0 -measurable function, then condition (2.3.7) holds as soon as

$$\sum_{\mathbf{n} \geq \mathbf{1}} \frac{1}{|\mathbf{n}|^{1/2}} \|\mathbb{E}[U^{\mathbf{n}}(f) \mid \mathcal{F}_0]\| < +\infty. \quad (2.3.13)$$

It was proven in [WW13] that when the filtration $(\mathcal{F}_i)_{i \in \mathbb{Z}^d}$ is generated by an i.i.d. random field, condition (2.3.13) implies the central limit theorem. Moreover, if (2.3.13) holds when the \mathbb{L}^2 -norm is replaced by the \mathbb{L}^p -norm for some $p > 2$, then the invariance principle holds. Our result thus extend these ones, since only a finite moment of order two is required and the condition on the dependence is weaker. If f is a function such that for each $q \in [d]$, $\mathbb{E}[f \mid T_q^l \mathcal{F}_0] \rightarrow 0$ as $l \rightarrow +\infty$, measurable with respect to the σ -algebra generated by $\bigcup_{i \in \mathbb{Z}^d} T^i \mathcal{F}_0$ and satisfies (2.3.13), then by Lemma 6.2 in [VW14], $\sum_{i \in \mathbb{Z}^d} \|\pi_i(f)\| < +\infty$.

Remark 2.17. In [PZ17], a central limit theorem has been obtain for an \mathcal{F}_0 -measurable function f satisfying (2.3.7). Their result applies in the context of partially commuting filtrations (see (2.2.7)), which includes a larger class of filtrations than completely commuting ones. Nevertheless, our results include the nonadapted case and lead to an invariance principle.

Remark 2.18. Condition (2.3.7) is much less restrictive than admitting an orthomartingale coboundary decomposition in \mathbb{L}^2 (see Theorem 2.2).

A key step for proving that a function satisfying the Maxwell and Woodroffe condition also satisfies the conditions of Theorem 2.7 is a maximal inequality, which is of independent interest. Note that a similar inequality has been obtained in [WW13] but without the maxima.

Proposition 2.19. *Let $d \geq 1$ be an integer. There exists a constant $C(d)$ such that for any \mathbb{Z}^d -measure preserving action T , any \mathcal{F}_0 such that $(T^{-i} \mathcal{F}_0)_{i \in \mathbb{Z}^d}$ be a completely commuting filtration, any $\mathbf{n} \geq \mathbf{0}$, any $E \subset [d]$ and any $f \in \mathcal{H}_E$:*

$$\begin{aligned} \left\| \max_{1 \leq i \leq 2^{\mathbf{n}}} |S_i(f)| \right\| &\leq C(d) |2^{\mathbf{n}}|^{1/2} \sum_{\mathbf{0} \leq i \leq \mathbf{n}} 2^{-i/2} \left\| \sum_{\mathbf{0} \leq j \leq 2^i} P_E^j f \right\| \\ &\leq C(d)^2 |2^{\mathbf{n}}|^{1/2} \sum_{\mathbf{n} \geq \mathbf{1}} \frac{1}{|\mathbf{n}|^{3/2}} \left\| \sum_{\mathbf{0} \leq j \leq \mathbf{n}-1} P_E^j f \right\|. \end{aligned} \quad (2.3.14)$$

Examples 5 and 6 in [PZ17] are formulated in the context of completely commuting filtration. Our results can be used to treat non causal linear and Volterra random fields. We derive from Theorem 2.12 a sufficient condition for a linear random field to satisfy the weak invariance principle.

Corollary 2.20. *Let $(\varepsilon_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ be an i.i.d. random field where $\varepsilon_{\mathbf{0}}$ is centered and square integrable. Let $T^{\mathbf{j}}: \mathbb{R}^{\mathbb{Z}^d} \rightarrow \mathbb{R}^{\mathbb{Z}^d}$ be defined as $T^{\mathbf{j}}((x_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}) = ((x_{\mathbf{i}+\mathbf{j}})_{\mathbf{i} \in \mathbb{Z}^d})$, $\mathcal{F}_{\mathbf{0}} = \sigma(\varepsilon_{\mathbf{i}}, \mathbf{i} \preceq \mathbf{0})$ and*

$$f = \sum_{\mathbf{i} \in \mathbb{Z}^d} a_{\mathbf{i}} \varepsilon_{-\mathbf{i}}, \quad (2.3.15)$$

where $a_{\mathbf{i}} \in \mathbb{R}$ and $\sum_{\mathbf{i} \in \mathbb{Z}^d} a_{\mathbf{i}}^2 < +\infty$. Define for $E \subset [d]$ and $\mathbf{n} \succcurlyeq \mathbf{1}$,

$$\Delta_{E, \mathbf{n}} := \sum_{\mathbf{j} \in \mathbb{N}^d, \mathbf{j} \cdot \mathbf{1}_E \succcurlyeq \mathbf{1}_E} \left(\sum_{\mathbf{0} \prec \mathbf{k} \prec \mathbf{n} - \mathbf{1}} a_{(\mathbf{k}+\mathbf{j}) \cdot \varepsilon(E)} \right)^2. \quad (2.3.16)$$

Assume that for any $E \subset [d]$, the convergence

$$\sum_{\mathbf{n} \succcurlyeq \mathbf{1}} |\mathbf{n}|^{-3/2} \Delta_{E, \mathbf{n}}^{1/2} < +\infty \quad (2.3.17)$$

holds. Then f satisfies the invariance principle in $C([0, 1]^d)$.

Remark 2.21. Corollary 2.20 also holds when we define the linear process by (2.3.15) but the innovations $\varepsilon_{\mathbf{i}}$ are only supposed to be orthomartingale differences with respect to a completely commuting filtration $(\mathcal{F}_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ (where $\mathcal{F}_{\mathbf{i}}$ is not supposed to be generated by i.i.d.).

3. PROOFS

3.1. Proof of Theorem 2.2.

3.1.1. *Contractions.* In the next proposition, we collect some properties of the operators $P_{d, E}$ and of the spaces $\mathcal{H}_{d, E}$.

Proposition 3.1. (1) *For any $E \subset [d]$ and any square integrable function f , $\|P_E^{\mathbf{0}} f\| \leq \|f\|$.*

(2) *Let $d \geq 1$, $E \subset [d]$ and $\mathbf{j}, \mathbf{k} \in \mathbb{N}^d$. For any function $f \in \mathcal{H}_E$, the function $P_E^{\mathbf{k}} f$ belongs to \mathcal{H}_E and*

$$P_E^{\mathbf{j}} \circ P_E^{\mathbf{k}} f = P_E^{\mathbf{j}+\mathbf{k}} f. \quad (3.1.1)$$

(3) *For any $d \geq 1$ and $E \subset [d+1]$,*

$$\tilde{P}_{d, E \setminus \{d+1\}}^{\mathbf{j}} P_{d+1, E}^{\mathbf{e}_{d+1}} f = P_{d+1, E}^{\mathbf{j}+\mathbf{e}_{d+1}} f, \quad (3.1.2)$$

where $\tilde{P} = P$ if $d+1 \notin E$ and \tilde{P} is defined similarly as (2.1.1) and (2.1.2) but $\mathcal{F}_{\mathbf{0}}$ is replaced by $\mathcal{F}_{\infty \mathbf{1}_{\{d+1\}}}$.

(4) *For any $d \geq 1$ and any positive integer j ,*

$$P_{d+1, E}^{j\mathbf{e}_{d+1}} U_{d+1}^{s(E)} P_{d+1, E}^{\mathbf{e}_{d+1}} f = P_{d+1, E}^{j\mathbf{e}_{d+1}} f \quad (3.1.3)$$

where $s(E) = 1$ if $d+1 \in E$ and -1 otherwise,

Proof. (1) It follows from the fact that for any function h and any sub- σ -algebra \mathcal{G} of \mathcal{F} , $\|h - \mathbb{E}[h | \mathcal{G}]\| \leq \|h\|$ combined with an induction argument.

(2) That $P_E^{\mathbf{k}} f$ belongs to \mathcal{H}_E follows from the definition of P_E . Let $E \subsetneq [d]$. By (2.1.1) and complete commutativity of $(T^{-\mathbf{i}} \mathcal{F}_{\mathbf{0}})_{\mathbf{i} \in \mathbb{Z}^d}$, we have

$$P_E^{\mathbf{j}} \circ P_E^{\mathbf{k}} f = \sum_{I, J \subset E} (-1)^{|I|+|J|} \mathbb{E} \left[U^{(\mathbf{k}+\mathbf{j}) \cdot \varepsilon(E)} f \mid \mathcal{F}_{\min\{\mathbf{j} \cdot \varepsilon(E) + \infty \mathbf{1}_I, \infty \mathbf{1}_J\}} \right]. \quad (3.1.4)$$

Moreover, since I is contained in E , we have, by definition of $\varepsilon(E)$ that

$$\min\{\mathbf{j} \cdot \varepsilon(E) + \infty \mathbf{1}_I, \infty \mathbf{1}_J\} = \min\{-\mathbf{j}_{E \setminus I} + \infty \mathbf{1}_I, \infty \mathbf{1}_J\}. \quad (3.1.5)$$

If $E \setminus I$ contains some i_0 , then

$$\sum_{J \subset E} (-1)^{|J|} \mathbb{E} \left[U^{(\mathbf{k}+\mathbf{j}) \cdot \varepsilon(E)} f \mid \mathcal{F}_{\min\{\mathbf{j} \cdot \varepsilon(E) + \infty \mathbf{1}_I, \infty \mathbf{1}_J\}} \right] = \sum_{J \subset E \setminus \{i_0\}} (-1)^{|J|} a_J,$$

where

$$a_J := \mathbb{E} \left[U^{(\mathbf{k}+\mathbf{j}) \cdot \varepsilon(E)} f \mid \mathcal{F}_{\min\{\infty \mathbf{1}_I - \mathbf{j} \mathbf{e}_I, \infty \mathbf{1}_J\}} \right] - \mathbb{E} \left[U^{(\mathbf{k}+\mathbf{j}) \cdot \varepsilon(E)} f \mid \mathcal{F}_{\min\{\infty \mathbf{1}_I - \mathbf{j} \mathbf{e}_I, \infty \mathbf{1}_J \cup \{i_0\}\}} \right]$$

and the latter term equals 0. Consequently, in (3.1.4), only the term where $I = E$ appears, which gives (3.1.1). Assume that $E = [d]$. Define for $\mathbf{i} \in \mathbb{N}^d$, $Q_1^{\mathbf{i}} f := U^{-\mathbf{i}} (f - \mathbb{E}[f \mid \mathcal{F}_{\infty \mathbf{1}_{[d]}]})$ and $Q_2^{\mathbf{i}} f := \sum_{J \subset [d]} (-1)^{|J|} \mathbb{E}[U^{-\mathbf{j}} f \mid \mathcal{F}_{\infty \mathbf{1}_J}]$. Then $P_{[d]}^{\mathbf{i}} f = Q_1^{\mathbf{i}} + Q_2^{\mathbf{i}}$, $Q_1^{\mathbf{j}+\mathbf{k}} f = Q_1^{\mathbf{j}} Q_1^{\mathbf{k}}(f)$ and by similar arguments as in the case where $E \subsetneq [d]$, $Q_2^{\mathbf{j}+\mathbf{k}} f = Q_2^{\mathbf{j}} Q_2^{\mathbf{k}}(f)$. Therefore, establishing (3.1.1) reduces $Q_1^{\mathbf{j}} Q_2^{\mathbf{k}} f = Q_2^{\mathbf{j}} Q_1^{\mathbf{k}} f = 0$ which holds because $\mathbb{E}[U^{\mathbf{a}} Q_1^{\mathbf{b}} f \mid \mathcal{F}_{\infty \mathbf{1}_{[d]}}] = 0$ for any $\mathbf{a} \in \mathbb{Z}^d$ and $\mathbf{b} \in \mathbb{N}^d$.

- (3) When $d+1 \notin E$, this follows from item 2. Assume now that $d+1 \in E$. For any function h , the following equality holds:

$$P_{d, E \setminus \{d+1\}}^{\mathbf{j}} \widetilde{h} = P_{d+1, E}^{\mathbf{j}} h + \sum_{J \subset E \setminus \{d+1\}} (-1)^{|J|+|E|} \mathbb{E}[U^{\mathbf{j} \cdot \varepsilon(E)} h \mid \mathcal{F}_{\infty \mathbf{1}_J}], \quad (3.1.6)$$

hence, by item 2, it suffices to establish that

$$\sum_{J \subset E \setminus \{d+1\}} (-1)^{|J|+|E|} \mathbb{E}[U^{\mathbf{j} \cdot \varepsilon(E)} P_{d+1, E}^{\mathbf{e}_{d+1}} f \mid \mathcal{F}_{\infty \mathbf{1}_J}] = 0, \quad (3.1.7)$$

which follows from the fact that $U^{\mathbf{j} \cdot \varepsilon(E)} P_{d+1, E}^{\mathbf{e}_{d+1}} f$ belongs to $\mathcal{H}_{d+1, E}$.

- (4) Noticing that $P_{d+1, E}^{\mathbf{k}} f = P_{d+1, E}^{\mathbf{0}} (U^{\mathbf{k} \cdot \varepsilon(E)} f)$, we derive that

$$P_{d+1, E}^{\mathbf{j} \mathbf{e}_{d+1}} U_{d+1}^{s(E)} f = P_{d+1, E}^{\mathbf{0}} \left(U^{\mathbf{j} \mathbf{e}_{d+1} \cdot \varepsilon(E)} U_{d+1}^{s(E)} f \right) = P_{d+1, E}^{\mathbf{0}} \left(U^{(\mathbf{j}+1) \mathbf{e}_{d+1} \cdot \varepsilon(E)} f \right), \quad (3.1.8)$$

which entails the wanted result by an application of item 2. \square

3.1.2. *Intermediate steps.* The proof of Theorem 2.2 will require the following lemmas.

Lemma 3.2. *Let $E \subsetneq [d]$ and let h be an integrable $\mathcal{F}_{\infty \mathbf{1}_I}$ -measurable function such that $\mathbb{E}[h \mid \mathcal{F}_{\infty \mathbf{J}}] = 0$ for each $J \subsetneq I$. Then the function $g_E := \prod_{q=1}^d (I - P_E^{\mathbf{e}_q}) h$ admits the decomposition*

$$g_E := m_k^E + \sum_{\substack{J \subset [d] \\ J \neq \emptyset}} \prod_{j \in J} (I - U_j) m_{k, J}^E, \quad (3.1.9)$$

where $(m_k^E \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ is an orthomartingale differences random field with respect to the filtration $(T^{-\mathbf{i}} \mathcal{F}_0)_{\mathbf{i} \in \mathbb{Z}^d}$ and for each nonempty subset J of $[d]$ such that $J \neq [d]$, the random field $(m_{k, J}^E \circ T^{\mathbf{i}_J})_{\mathbf{i} \in \mathbb{Z}^d}$ is an orthomartingale differences random field with respect to the filtration $(T^{-\mathbf{i}_J} \mathcal{F}_{\infty \mathbf{1}_{J^c}})_{\mathbf{i} \in \mathbb{Z}^d}$.

Lemma 3.3. *Let h be an integrable function. Then the function $g_{[d]} := \prod_{q=1}^d (I - P_{[d]}^{\mathbf{e}_q}) h$ admits the decomposition*

$$g_{[d]} := m_k^{[d]} + \sum_{\substack{J \subset [d] \\ J \neq \emptyset}} \prod_{j \in J} (I - U_j) m_{k, J}^{[d]}, \quad (3.1.10)$$

where $(m_k^{[d]} \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ is an orthomartingale differences random field with respect to the filtration $(T^{-\mathbf{i}} \mathcal{F}_0)_{\mathbf{i} \in \mathbb{Z}^d}$ and for each nonempty subset J of $[d]$ such that $J \neq [d]$, the random field $(m_{k, J}^{[d]} \circ T^{\mathbf{i}_J})_{\mathbf{i} \in \mathbb{Z}^d}$ is an orthomartingale differences random field with respect to the filtration $(T^{-\mathbf{i}_J} \mathcal{F}_{\infty \mathbf{1}_{J^c}})_{\mathbf{i} \in \mathbb{Z}^d}$.

Lemma 3.4. For any square integrable function H and any $K \subset [d]$, the function

$$F := \mathbb{E} \left[\prod_{q \in [d] \setminus K} (I - U_q) H \mid \mathcal{F}_{\infty \mathbf{1}_K} \right] \quad (3.1.11)$$

admits the decomposition

$$F = \sum_{K' \subset [d] \setminus K} \prod_{q \in K'} (I - U_q) m_{K'}, \quad (3.1.12)$$

where $m_{K'}$ is $\mathcal{F}_{\infty \mathbf{1}_{[d] \setminus K'}}$ -measurable and if $S \subsetneq K'$, then $\mathbb{E}[m_{K'} \mid \mathcal{F}_{\infty \mathbf{1}_S}] = 0$.

Proof. Define for $\mathbf{i} \in \mathbb{N}^d$,

$$Q^{\mathbf{i}} f := \mathbb{E} [U^{\mathbf{i}} f \mid \mathcal{F}_{\infty \mathbf{1}_K}] \quad (3.1.13)$$

and $Q_q^{\mathbf{i}} f := Q^{\mathbf{i} e_q} f$. Then $F = \prod_{q \in [d] \setminus K} (I - Q_q) H$. Moreover, since $I - Q_q = I - U_q^{-1} Q_q + U_q^{-1} Q_q - Q_q$ and $Q_{q'}$ commutes with $U_q^{-1} Q_q$ for $q \neq q'$, we derive that

$$F = \sum_{K' \subset [d] \setminus K} \prod_{q \in K'} (U_q^{-1} Q_q - Q_q) \prod_{q' \in ([d] \setminus K) \setminus K'} (I - U_{q'}^{-1} Q_{q'}) H. \quad (3.1.14)$$

If $q \neq q'$, then U_q^{-1} commutes with $Q_{q'}$. Therefore, the following equalities hold

$$F = \sum_{K' \subset [d] \setminus K} \prod_{q \in K'} (U_q^{-1} - I) \prod_{q \in K'} Q_q \prod_{q' \in ([d] \setminus K) \setminus K'} (I - U_{q'}^{-1} Q_{q'}) H \quad (3.1.15)$$

$$= \sum_{K' \subset [d] \setminus K} \prod_{q \in K'} (I - U_q) \prod_{q \in K'} U_q^{-1} Q_q \prod_{q' \in ([d] \setminus K) \setminus K'} (I - U_{q'}^{-1} Q_{q'}) H, \quad (3.1.16)$$

which gives the wanted decomposition. \square

Proof of Lemma 3.2. Observe that

$$g_E = \sum_{F \subset [d]} (-1)^{|F|} P_E^{\mathbf{1}_F} h \quad (3.1.17)$$

$$= \sum_{F \subset [d]} (-1)^{|F|} \sum_{J \subset E} (-1)^{|J|} \mathbb{E} [U^{\mathbf{1}_F \cdot \varepsilon(E)} h \mid \mathcal{F}_{\infty \mathbf{1}_J}] \quad (3.1.18)$$

$$= \sum_{J \subset E} (-1)^{|J|} \mathbb{E} \left[\prod_{q=1}^d (I - U_q^{\varepsilon(E)_q}) h \mid \mathcal{F}_{\infty \mathbf{1}_J} \right] \quad (3.1.19)$$

$$= \sum_{J \subset E} (-1)^{|J|} (-1)^{|J \cap E|} \prod_{j \in J} (I - U_j) \mathbb{E} \left[\prod_{q \in [d] \setminus J} (I - U_q^{\varepsilon(E)_q}) \prod_{l \in J} U_l h \mid \mathcal{F}_{\infty \mathbf{1}_J} \right]. \quad (3.1.20)$$

We then apply Lemma 3.4 to each $J \subset E$ with H such that $\prod_{q \in [d] \setminus J} (I - U_q^{\varepsilon(E)_q}) \prod_{l \in J} U_l h = \prod_{q \in [d] \setminus J} (I - U_q) H$. \square

Proof of Lemma 3.3. We start from the following inequalities:

$$g_{[d]} = \sum_{F \subset [d]} (-1)^{|F|} P_E^{\mathbf{1}_F} h \quad (3.1.21)$$

$$= \sum_{F \subset [d]} (-1)^{|F|} \sum_{J \subsetneq E} (-1)^{|J|} \mathbb{E} [U^{\mathbf{1}_F \cdot \varepsilon(E)} h \mid \mathcal{F}_{\infty \mathbf{1}_J}] + \sum_{F \subset [d]} (-1)^{|F|} U^{-\mathbf{1}_F} h \quad (3.1.22)$$

$$= \sum_{J \subsetneq E} (-1)^{|J|} \mathbb{E} \left[\prod_{q=1}^d (I - U_q^{\varepsilon(E)_q}) h \mid \mathcal{F}_{\infty \mathbf{1}_J} \right] + \prod_{q=1}^d (I - U_q^{-1}) h \quad (3.1.23)$$

and we apply Lemma 3.4 to each $J \subsetneq E$ and a H such that $\prod_{q \in [d] \setminus J} (I - U_q^{\varepsilon(E)_q}) \prod_{l \in J} U_l^h = \prod_{q \in [d] \setminus J} (I - U_q) H$. \square

3.2. Proof of Theorem 2.7.

Lemma 3.5. *Let A_1, \dots, A_d be commuting operators from a closed subspace V of \mathbb{L}^2 to itself, and $A^i := A_1^{i_1} \dots A_d^{i_d}$. Let F be a function such that $\sup_{\mathbf{n} \in \mathbb{N}^d} \left\| \sum_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{n}} A^i F \right\| < +\infty$. Then there exists a function $h \in V$ such that $F = \prod_{q=1}^d (I - A_q) h$.*

Proof. We use the idea of proof of Lemma 5 in [BR058]. We define

$$f_n := \frac{1}{n^d} \sum_{\mathbf{1} \leq \mathbf{k} \leq \mathbf{n}} \sum_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{k} - \mathbf{1}} \prod_{q=1}^d (I - A_q) A^i F \quad (3.2.1)$$

Using the assumption on F , we derive that

$$\lim_{n \rightarrow +\infty} \|F - f_n\| = 0. \quad (3.2.2)$$

Moreover, defining

$$h_n := \frac{1}{n^d} \sum_{\mathbf{1} \leq \mathbf{k} \leq \mathbf{n}} \sum_{\mathbf{0} \leq \mathbf{i} \leq \mathbf{k} - \mathbf{1}} A^i F \quad (3.2.3)$$

we observe that the sequence $(\|h_n\|)_{n \geq 1}$ is bounded. Since \mathbb{L}^2 is reflexive, there exists a subsequence $(h_{n_k})_{k \geq 1}$ which converges weakly in \mathbb{L}^2 to some h . Then the sequence $(f_{n_k})_{k \geq 1} = \left(\prod_{q=1}^d (I - A_q) h_{n_k} \right)_{k \geq 1}$ converges weakly to $\prod_{q=1}^d (I - A_q) h$. By uniqueness of the limit, equality $F = \prod_{q=1}^d (I - A_q) h$ holds. That h belongs to V follows from closedness of V . \square

Proof of Theorem 2.2. We prove sufficiency (necessity can be checked by direct computations). We use the idea of proof of Proposition 4.1 in [CCD⁺14]. Since $f = \sum_{E \subset [d]} P_E f$, it suffices to find an orthomartingale-coboundary decomposition for $P_E f$ for any subset E of $[d]$. To this aim, we apply Lemma 3.5 to the following setting: $V = \mathcal{H}_E$, $F = P_E f$ and $A_q^i := P_E^{i \mathbf{e}_q}$. We then conclude by Lemmas 3.2 and 3.3. \square

3.2.1. *Construction of the approximating martingale.* The combination of Lemma 3.2 and 3.3 shows that $B_k(f)$ admits an orthomartingale-coboundary decomposition.

Lemma 3.6. *For each integer $k \geq 1$ and each integrable and measurable function $f: \Omega \rightarrow \mathbb{R}$, the function $B_k(f)$ can be written in the following way:*

$$B_k(f) = m_k + \sum_{\substack{J \subset [d] \\ J \neq \emptyset}} \prod_{j \in J} (I - U_j) m_{k,J}, \quad (3.2.4)$$

where $(m_k \circ T^{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}^d}$ is an orthomartingale differences random field with respect to the filtration $(T^{-\mathbf{i}} \mathcal{F}_{\mathbf{0}})_{\mathbf{i} \in \mathbb{Z}^d}$ and for each nonempty subset J of $[d]$ such that $J \neq [d]$, the random field $(m_{k,J} \circ T^{\mathbf{i}_J})_{\mathbf{i} \in \mathbb{Z}^d}$ is an orthomartingale differences random field with respect to the filtration $(T^{-\mathbf{i}_J} \mathcal{F}_{\infty \mathbf{1}_J \mathbf{c}})_{\mathbf{i} \in \mathbb{Z}^d}$.

Considering the notations of Lemma 3.6, we introduce the following notation:

$$C_k(f) := \sum_{\substack{J \subset [d] \\ J \neq \emptyset}} \prod_{j \in J} (I - U_j) m_{k,J}, \quad k \geq 1. \quad (3.2.5)$$

Therefore, for any $k \geq 1$, the following equality holds

$$f = f - B_k(f) + m_k + C_k(f). \quad (3.2.6)$$

3.2.2. *Sufficiency.* Lemma 3.6 gave a sequence of functions $(m_k)_{k \geq 1}$ such that $(m_k \circ T^i)_{i \in \mathbb{Z}^d}$ is an orthomartingale differences random field with respect to the filtration $(T^{-i}\mathcal{F}_0)_{i \in \mathbb{Z}^d}$. Now, we have that to show that if (2.2.3) holds, then the sequence $(m_k)_{k \geq 1}$ is convergent, and that the limiting function m satisfies (1.3.1).

Lemma 3.7. *For any function f , and any $k \geq 1$, $\|C_k(f)\|_+ = 0$.*

Proof. It suffices to prove that for any non-empty subset J of $[d]$ and any $k \geq 1$,

$$\left\| \prod_{j \in J} (I - U_j) m_{k,J} \right\|_+ = 0. \quad (3.2.7)$$

Observe that

$$\max_{1 \leq i \leq \mathbf{n}} \left| S_i \left(\prod_{j \in J} (I - U_j) m_{k,J} \right) \right| = \max_{1 \leq i \leq \mathbf{n}} \left| \prod_{j \in J} (I - U_j^{i_j}) S_{i_{\mathbf{J}^c}}(m_{k,J}) \right| \quad (3.2.8)$$

$$\leq 2^{|J|} \max_{0 \leq i \leq \mathbf{n}} |U^{i_{\mathbf{J}}} S_{i_{\mathbf{J}^c} + \mathbf{1}_{\mathbf{J}}}(m_{k,J})|. \quad (3.2.9)$$

Since J is not empty, it contains some q . Using the fact that $\|\max_{i \in I} |Y_i|\| \leq |I|^{1/2} \max_{i \in I} \|Y_i\|$, we derive that for any $\mathbf{n} \geq \mathbf{1}$,

$$\frac{1}{|\mathbf{n}|} \mathbb{E} \left[\max_{1 \leq i \leq \mathbf{n}} \left| S_i \left(\prod_{j \in J} (I - U_j) m_{k,J} \right) \right|^2 \right] \leq \frac{2^{2|J|}}{n_q} \mathbb{E} \left[\max_{0 \leq l \leq n_q} U_q^l Y_{\mathbf{n}} \right], \quad (3.2.10)$$

where

$$Y_{\mathbf{n}} := \frac{1}{|\mathbf{n}_{[d] \setminus J}|} \max_{1_{[d] \setminus J} \leq i \leq \mathbf{n}_{[d] \setminus J}} |S_{i_{\mathbf{J}^c} + \mathbf{1}_{\mathbf{J}}}(m_{k,J})|^2. \quad (3.2.11)$$

Then for any $R > 0$,

$$\frac{2^{2|J|}}{n_q} \mathbb{E} \left[\max_{0 \leq l \leq n_q} U_q^l Y_{\mathbf{n}} \right] \leq \frac{2^{2|J|}}{n_q} R + 2^{2|J|} \mathbb{E} [Y_{\mathbf{n}} \mathbf{1} \{Y_{\mathbf{n}} > R\}]. \quad (3.2.12)$$

By Proposition 1.3, the family $\{Y_{\mathbf{n}}, \mathbf{n} \geq \mathbf{1}\}$ is uniformly integrable. This gives (3.2.7) and ends the proof of Lemma 3.7. \square

Lemma 3.8. *Let $f: \Omega \rightarrow \mathbb{R}$ be a measurable square integrable function such that (2.2.3) holds. Then the sequence $(m_k)_{k \geq 1}$ is convergent in \mathbb{L}^2 to some function m .*

Proof. Let k and l be fixed positive integers. Since $((m_k - m_l) \circ T^i)_{i \in \mathbb{Z}^d}$ is an orthomartingale differences random field with respect to the filtration $(T^{-i}\mathcal{F}_0)_{i \in \mathbb{Z}^d}$, we have for each positive integer n , by orthogonality of increments,

$$\|m_k - m_l\| = \frac{1}{n^{d/2}} \|S_{n\mathbf{1}}(m_k - m_l)\|. \quad (3.2.13)$$

By Lemma 3.6 and (3.2.6), the following equality holds

$$S_{n\mathbf{1}}(m_k - m_l) = S_{n\mathbf{1}}(B_k(f) - f) - S_{n\mathbf{1}}(B_l(f) - f) - S_{n\mathbf{1}}(C_k(f)) + S_{n\mathbf{1}}(C_l(f)) \quad (3.2.14)$$

hence taking the \mathbb{L}^2 norm, we get

$$\|S_{n\mathbf{1}}(m_k - m_l)\| \leq \|S_{n\mathbf{1}}(B_k(f) - f)\| + \|S_{n\mathbf{1}}(B_l(f) - f)\| + \|S_{n\mathbf{1}}(C_k(f))\| + \|S_{n\mathbf{1}}(C_l(f))\|.$$

Dividing on both sides by $n^{d/2}$ and letting n going to infinity, we get by Lemma 3.7 and (3.2.13)

$$\|m_k - m_l\| \leq \|B_k(f) - f\|_+ + \|B_l(f) - f\|_+. \quad (3.2.15)$$

This proves that the sequence $(m_k)_{k \geq 1}$ is Cauchy in \mathbb{L}^2 hence convergent to some function m . This ends the proof of Lemma 3.8. \square

Since for each k , the function m_k is \mathcal{F}_0 -measurable, the function m is \mathcal{F}_0 -measurable. Moreover, we have for each $q \in [d]$, $\mathbb{E}[m_k | T_q \mathcal{F}_0] = 0$ hence $\mathbb{E}[m | T_q \mathcal{F}_0] = 0$, which proves that $(m \circ T^i)_{i \in \mathbb{Z}^d}$ is an orthomartingale differences random field.

The purpose of the following lemma is the verification that m gives the wanted approximation.

Lemma 3.9. *Let $f: \Omega \rightarrow \mathbb{R}$ be a measurable square integrable function such that (2.2.3) holds and let m be the function given by Lemma 3.8. Then (1.3.1) takes place.*

Proof of Lemma 3.9. Let $k \geq 1$ be an arbitrary but fixed integer. For any $\mathbf{n} \in \mathbb{N}^d$ such that $\mathbf{n} \succcurlyeq \mathbf{1}$, we have, using Proposition 1.2 with $M := m - m_k$,

$$\frac{1}{|\mathbf{n}|^{1/2}} \left\| \max_{1 \leq i \leq \mathbf{n}} |S_i(f - m)| \right\| \leq \frac{1}{|\mathbf{n}|^{1/2}} \left\| \max_{1 \leq i \leq \mathbf{n}} |S_i(f - m_k)| \right\| + 2^d \|m - m_k\|. \quad (3.2.16)$$

Now, we use the inequality

$$\frac{1}{|\mathbf{n}|^{1/2}} \left\| \max_{1 \leq i \leq \mathbf{n}} |S_i(f - m_k)| \right\| \leq \frac{1}{|\mathbf{n}|^{1/2}} \left\| \max_{1 \leq i \leq \mathbf{n}} |S_i(f - B_k(f))| \right\| + \frac{1}{|\mathbf{n}|^{1/2}} \left\| \max_{1 \leq i \leq \mathbf{n}} |S_i(m_k - B_k(f))| \right\|$$

and take the lim sup as \mathbf{n} goes to infinity to obtain that for any $k \geq 1$,

$$\limsup_{\mathbf{n} \rightarrow \infty} \frac{1}{|\mathbf{n}|^{1/2}} \left\| \max_{1 \leq i \leq \mathbf{n}} |S_i(f - m)| \right\| \leq \|B_k(f) - f\|_+ + \|C_k(f)\|_+ + 2^d \|m - m_k\|. \quad (3.2.17)$$

By (2.2.3), Lemmas 3.7 and 3.8, we get that the right hand side of (3.2.17) converges to 0 as k goes to infinity. This concludes the proof of Lemma 3.9 and that of Theorem 2.7. \square

3.3. Proof under projective conditions.

3.3.1. *Hannan's condition.* Lemma 5.2 in [VW14] states the following inequality: for any function f satisfying the conditions of Proposition 2.11 and any $\mathbf{n} \in (\mathbb{N}^*)^d$,

$$\left\| \max_{1 \leq i \leq \mathbf{n}} |S_i(f)| \right\| \leq 2^d |\mathbf{n}|^{1/2} \sum_{i \in \mathbb{Z}^d} \|\pi_i(f)\|. \quad (3.3.1)$$

We shall check (2.2.5). To this aim, we fix a nonempty subset J of $[d]$ and $E \subset [d]$ and we apply (3.3.1) to the function $k^{-|J|} \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq k \mathbf{1}_J} P_E^{\mathbf{j}} f$ (which satisfies the assumptions of Proposition 2.11 because so does f) in order to obtain

$$\left\| k^{-|J|} \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq k \mathbf{1}_J} P_E^{\mathbf{j}} f \right\|_+ \leq 2^d \sum_{i \in \mathbb{Z}^d} \left\| \pi_i \left(k^{-|J|} \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq k \mathbf{1}_J} P_E^{\mathbf{j}} f \right) \right\|. \quad (3.3.2)$$

Define $I_E := \{\mathbf{i} \in \mathbb{Z}^d \mid i_q > 0 \text{ if and only if } q \in E\}$. Then for $\mathbf{i} \in \mathbb{Z}^d \setminus I_E$, $\pi_{\mathbf{i}} \left(k^{-|J|} \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq k \mathbf{1}_J} P_E^{\mathbf{j}} f \right) = 0$ and if \mathbf{i} belongs to I_E , then

$$\left\| \pi_{\mathbf{i}} \left(k^{-|J|} \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq k \mathbf{1}_J} P_E^{\mathbf{j}} f \right) \right\| \leq k^{-|J|} \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq k \mathbf{1}_J} \|\pi_{\mathbf{i} - \mathbf{j} \cdot \varepsilon(E)}\| \quad (3.3.3)$$

hence

$$\sum_{\mathbf{i} \in I_E} \left\| \pi_{\mathbf{i}} \left(k^{-|J|} \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq k \mathbf{1}_J} P_E^{\mathbf{j}} f \right) \right\| \leq k^{-|J|} \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq k \mathbf{1}_J} \sum_{\mathbf{i} \in I_E} \|\pi_{\mathbf{i} - \mathbf{j} \cdot \varepsilon(E)}\|. \quad (3.3.4)$$

Since J is nonempty, we can choose $q \in J$. Observe that

$$\sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq k \mathbf{1}_J} \sum_{\mathbf{i} \in I_E} \|\pi_{\mathbf{i} - \mathbf{j} \cdot \varepsilon(E)}\| \leq k^{|\mathbb{Z}^d| - 1} \sum_{j=1}^k \sum_{\mathbf{i} \in \mathbb{Z}^d: |i_q| > j} \|\pi_{\mathbf{i}}\|, \quad (3.3.5)$$

hence

$$\left\| k^{-|J|} \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq k \mathbf{1}_J} P_E^{\mathbf{j}} f \right\|_+ \leq 2^d \frac{1}{k} \sum_{j=1}^k \sum_{\mathbf{i} \in \mathbb{Z}^d: |i_q| > j} \|\pi_{\mathbf{i}}\|. \quad (3.3.6)$$

That (2.2.5) is satisfied follows from finiteness of $\sum_{\mathbf{i} \in \mathbb{Z}^d} \|\pi_{\mathbf{i}}\|$. This concludes the proof of Theorem 2.11.

3.3.2. Maxwell and Woodroffe condition.

Proof of Proposition 2.19. As in [PU05,PUW07,CUN14], the proof will be done by dyadic induction.

We shall prove by induction on d the following assertion: there exists constants $K(d)$, $C(d, J)$, $\emptyset \subsetneq J \subset [d]$ such that for any commuting invertible measure preserving maps T_1, \dots, T_d , any sub- σ -algebra \mathcal{F}_0 of \mathcal{F} such that $(T^{-i}\mathcal{F}_0)_{\mathbf{i} \in \mathbb{Z}^d}$ is a commuting filtration, any subset E of $[d]$ any function $f \in \mathcal{H}_{d,E}$ and any $\mathbf{n} \in \mathbb{N}^d$,

$$\begin{aligned} \left\| \max_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{2}^{\mathbf{n}}} |S_{\mathbf{i}}(f)| \right\| &\leq |\mathbf{2}^{\mathbf{n}}|^{1/2} K(d) \|f\| \\ &+ |\mathbf{2}^{\mathbf{n}}|^{1/2} \sum_{\emptyset \subsetneq J \subset [d]} C(d, J) \sum_{\mathbf{0}_J \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}_J} |\mathbf{2}^{\mathbf{i}_J}|^{-1/2} \left\| \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} P_E^{\mathbf{j}} f \right\| \end{aligned} \quad (3.3.7)$$

where P_E is defined by (2.1.1).

The constants are defined recursively in the following way:

$$K(d+1) = 6K(d), \quad (3.3.8)$$

if J is a nonempty subset of $[d]$, then

$$C(d+1, J) = 4C(d, J) \quad (3.3.9)$$

$$C(d+1, J \cup \{d+1\}) = 2\sqrt{2}C(d+1, J) + C(d, J). \quad (3.3.10)$$

and

$$C(d+1, \{d+1\}) = \sqrt{2}K(d+1). \quad (3.3.11)$$

When $d = 1$, the result was established in Proposition 2.3. of [PU05] when the function f is \mathcal{F}_0 -measurable and was extended to the nonadapted case in Proposition 1 of [VOL07].

Now, assume that the result holds for some $d \geq 1$ and let us prove it for $d+1$. This will be done by induction on n_{d+1} . More precisely, we consider the following assertion $\mathcal{P}(m)$ defined as "there exists constants $K(d+1)$, $C(d+1, J)$, $\emptyset \subsetneq J \subset [d+1]$ such that for any commuting invertible measure preserving maps T_1, \dots, T_{d+1} , any sub- σ -algebra \mathcal{F}_0 of \mathcal{F} such that $(T^{-i}\mathcal{F}_0)_{\mathbf{i} \in \mathbb{Z}^{d+1}}$ is a completely commuting filtration, any subset E of $[d+1]$, any function $f \in \mathcal{H}_{d+1,E}$, any $\mathbf{n} \in \mathbb{N}^{d+1}$ such that $n_{d+1} \leq m$,

$$\begin{aligned} \left\| \max_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{2}^{\mathbf{n}}} |S_{\mathbf{i}}(f)| \right\| &\leq |\mathbf{2}^{\mathbf{n}}|^{1/2} K(d+1) \|f\| \\ &+ |\mathbf{2}^{\mathbf{n}}|^{1/2} \sum_{\emptyset \subsetneq J \subset [d+1]} C(d+1, J) \sum_{\mathbf{0}_J \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}_J} |\mathbf{2}^{\mathbf{i}}|^{-1/2} \left\| \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} P_E^{\mathbf{j}} f \right\|, \end{aligned} \quad (3.3.12)$$

where P_E is defined by (2.1.1).

The assertion $\mathcal{P}(0)$ holds by the case of the dimension d . Now assume that $\mathcal{P}(m)$ is true for some m and let us prove $\mathcal{P}(m+1)$. We thus know that

- a) inequality (3.3.7) holds for any commuting invertible measure preserving maps T_1, \dots, T_d , any sub- σ -algebra \mathcal{F}_0 of \mathcal{F} such that $(T^{-i}\mathcal{F}_0)_{i \in \mathbb{Z}^d}$ is a commuting filtration, any $E \subset [d]$, any function $f \in \mathcal{H}_{d,E}$ and any $\mathbf{n} \in \mathbb{N}^d$ and
- b) for any commuting invertible measure preserving maps $\widetilde{T}_1, \dots, \widetilde{T}_{d+1}$, any sub- σ -algebra $\widetilde{\mathcal{F}}_0$ of \mathcal{F} such that $(\widetilde{T}^{-i}\widetilde{\mathcal{F}}_0)_{i \in \mathbb{Z}^{d+1}}$ is a commuting filtration, any $E \subset [d+1]$, any function $f \in \mathcal{H}_{d+1,E}$ and any $\widetilde{\mathbf{n}} \in \mathbb{N}^{d+1}$ such that $\widetilde{n}_{d+1} \leq m$,

$$\begin{aligned} \left\| \max_{1 \leq i \leq 2^{\mathbf{n}}} |S_i(\widetilde{f})| \right\| &\leq |\mathbf{2}^{\mathbf{n}}|^{1/2} K(d+1) \left\| \widetilde{f} \right\| \\ &+ |\mathbf{2}^{\mathbf{n}}|^{1/2} \sum_{\emptyset \subseteq J \subset [d+1]} C(d+1, J) \sum_{\mathbf{0}_J \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}_J} |\mathbf{2}^{\mathbf{i}}|^{-1/2} \left\| \sum_{1_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} \widetilde{P}_E^{\mathbf{j}} \widetilde{f} \right\|, \end{aligned} \quad (3.3.13)$$

where the operators \widetilde{P}_E is defined by (2.1.1) with $\mathcal{F}_{\infty \mathbf{1}_J}$ replaced by $\widetilde{\mathcal{F}}_{\infty \mathbf{1}_J}$.

Let T_1, \dots, T_{d+1} be commuting invertible measure preserving maps, \mathcal{F}_0 be a sub- σ -algebra of \mathcal{F} such that $(T^{-i}\mathcal{F}_0)_{i \in \mathbb{Z}^{d+1}}$ is a commuting filtration, $E \subset [d+1]$, $f \in \mathcal{H}_{d+1,E}$, and $\mathbf{n} \in \mathbb{N}^{d+1}$ such that $n_{d+1} \leq m+1$. It suffices to prove (3.3.12) in the case $\mathbf{n} \in \mathbb{N}^{d+1}$ with $n_{d+1} = m+1$. We define

$$g := f - U_{d+1}^{s(E)} P_{d+1,E}^{\mathbf{e}_{d+1}}(f) \quad (3.3.14)$$

where $s(E) = 1$ if $d+1 \in E$ and -1 otherwise. We derive the inequality

$$\begin{aligned} \max_{1 \leq i \leq 2^{\mathbf{n}}} |S_i(f)| &\leq \max_{1 \leq i \leq 2^{\mathbf{n}}} |S_i(g)| \\ &+ \max_{1 \leq i \leq 2^{\mathbf{n} - (\mathbf{m}+1)\mathbf{e}_{d+1}}} \max_{1 \leq j \leq 2^{\mathbf{m}}} U_{d+1}^j |S_i(U_{d+1}^{s(E)} P_{d+1,E}^{\mathbf{e}_{d+1}}(f))| \\ &+ \max_{1 \leq i \leq 2^{\mathbf{n} - \mathbf{e}_{d+1}}} |S_i(T_1, \dots, T_d, T_{d+1}^2, (I+U_{d+1})U_{d+1}^{s(E)} P_E(P_{d+1,E}^{\mathbf{e}_{d+1}}(f)))|, \end{aligned} \quad (3.3.15)$$

where $S_i(T_1, \dots, T_d, T_{d+1}^2, \cdot)$ is defined like in (1.0.2) but T_{d+1} is replaced by T_{d+1}^2 , hence

$$\left\| \max_{1 \leq i \leq 2^{\mathbf{n}}} |S_i(f)| \right\| \leq (I) + (II) + (III), \quad (3.3.16)$$

where

$$(I) := \left\| \max_{1 \leq i \leq 2^{\mathbf{n}}} |S_i(g)| \right\| \quad (3.3.17)$$

$$(II) := \left\| \max_{1 \leq i \leq 2^{\mathbf{n} - (\mathbf{m}+1)\mathbf{e}_{d+1}}} \max_{1 \leq j \leq 2^{\mathbf{m}}} U_{d+1}^j |S_i(U_{d+1}^{s(E)} P_{d+1,E}^{\mathbf{e}_{d+1}}(f))| \right\| \text{ and} \quad (3.3.18)$$

$$(III) := \left\| \max_{1 \leq i \leq 2^{\mathbf{n} - \mathbf{e}_{d+1}}} |S_i(T_1, \dots, T_d, T_{d+1}^2, (I+U_{d+1})U_{d+1}^{s(E)} P_{d+1,E}^{\mathbf{e}_{d+1}}(f))| \right\|. \quad (3.3.19)$$

If $s(E) = 1$, we define the σ -algebra \mathcal{G}_N by

$$\mathcal{G}_N = \sigma \left(\bigcup_{i \in \mathbb{Z}^d} T^i T_{d+1}^{-N} \mathcal{F}_0 \right) \quad (3.3.20)$$

and if $s(E) = -1$,

$$\mathcal{G}_N = \sigma \left(\bigcup_{i \in \mathbb{Z}^d} T^i T_{d+1}^{-N-1} \mathcal{F}_0 \right) \quad (3.3.21)$$

The control of (I) requires the following lemmas.

Lemma 3.10. *The sequence $\left(\max_{\mathbf{1}_{[d]} \preccurlyeq \mathbf{i}_{[d]} \preccurlyeq \mathbf{2}_{[d]}^n} \max_{1 \leq i_{d+1} \leq N} |S_i(g)|\right)_{N \geq 1}$ is a submartingale with respect to the filtration $(\mathcal{G}_N)_{N \geq 1}$.*

Proof. For any $E \subset [d+1]$,

$$g = \sum_{J \subset E} (-1)^{|J|+|E|} \left(\mathbb{E}[f \mid \mathcal{F}_{\infty \mathbf{1}_J}] - \mathbb{E}[f \mid \mathcal{F}_{\infty \mathbf{1}_{J-s(E)\mathbf{e}_{d+1}}}] \right) \quad (3.3.22)$$

and since the summand vanishes if $d+1$ belongs to J , we actually have

$$g = \sum_{J \subset E \setminus \{d+1\}} (-1)^{|J|+|E|} \left(\mathbb{E}[f \mid \mathcal{F}_{\infty \mathbf{1}_J}] - \mathbb{E}[f \mid \mathcal{F}_{\infty \mathbf{1}_{J-s(E)\mathbf{e}_{d+1}}}] \right). \quad (3.3.23)$$

Consequently, $\max_{\mathbf{1}_{[d]} \preccurlyeq \mathbf{i}_{[d]} \preccurlyeq \mathbf{2}_{[d]}^n} \max_{1 \leq i_{d+1} \leq N} |S_i(g)|$ is \mathcal{G}_N -measurable and

$$\mathbb{E} \left[\max_{\mathbf{1}_{[d]} \preccurlyeq \mathbf{i}_{[d]} \preccurlyeq \mathbf{2}_{[d]}^n} \max_{1 \leq i_{d+1} \leq N} |S_i(g)| \mid \mathcal{G}_{N-1} \right] \geq \max_{\mathbf{1}_{[d]} \preccurlyeq \mathbf{i}_{[d]} \preccurlyeq \mathbf{2}_{[d]}^n} \max_{1 \leq i_{d+1} \leq N-1} |S_i(g)|, \quad (3.3.24)$$

which ends the proof of Lemma 3.10. \square

Lemma 3.11. *The function $\sum_{k=0}^{2^{m+1}-1} U_{d+1}^k g$ belongs to $\widetilde{\mathcal{H}}_{d,E \setminus \{d+1\}}$, where the latter space is defined like $\mathcal{H}_{d,E \setminus \{d+1\}}$, but the σ -algebra \mathcal{F}_0 is replaced by $\mathcal{F}_{\infty \mathbf{1}_{\{d+1\}}}$.*

Proof. It suffices to prove that for any non-negative integer k , the function $U_{d+1}^k g$ belongs to $\widetilde{\mathcal{H}}_{d,E \setminus \{d+1\}}$. In view of (3.3.23) and complete commutativity of the filtration $(T^{-i}\mathcal{F}_0)_{i \in \mathbb{Z}^d}$, for any $I \subsetneq E \setminus \{d+1\}$, the following equality holds:

$$\mathbb{E} \left[U_{d+1}^k g \mid \mathcal{F}_{\infty \mathbf{1}_{I \cup \{d+1\}}} \right] = U_{d+1}^k \sum_{J \subset E \setminus \{d+1\}} (-1)^{|J|+|E|} \left(\mathbb{E}[f \mid \mathcal{F}_{\infty \mathbf{1}_{J \cap I}}] - \mathbb{E}[f \mid \mathcal{F}_{\infty \mathbf{1}_{J \cap I - s(E)\mathbf{e}_{d+1}}}] \right).$$

Suppose that $d+1 \in E$. In this case, $s(E) = 1$ hence $\mathcal{F}_{\infty \mathbf{1}_{J \cap I - s(E)\mathbf{e}_{d+1}}}$ and $\mathcal{F}_{\infty \mathbf{1}_{J \cap I}}$ are contained in $\mathcal{F}_{\infty \mathbf{1}_E}$ hence $\mathbb{E} \left[U_{d+1}^k g \mid \mathcal{F}_{\infty \mathbf{1}_{I \cup \{d+1\}}} \right] = 0$. If $d+1 \notin E$, then $I \cup \{d+1\} \subsetneq E$ hence we also have $\mathbb{E} \left[U_{d+1}^k g \mid \mathcal{F}_{\infty \mathbf{1}_{I \cup \{d+1\}}} \right] = 0$.

That $U_{d+1}^k g$ is $\mathcal{F}_{\infty \mathbf{1}_{E \cup \{d+1\}}}$ -measurable when $E \neq [d+1]$ follows from (3.3.23). This ends the proof of Lemma 3.11. \square

By Lemma 3.10 and Doob's inequality, we infer that

$$(I) \leq 2 \sum_{E \subset [d+1]} \left\| \max_{\mathbf{1}_{[d]} \preccurlyeq \mathbf{i}_{[d]} \preccurlyeq \mathbf{2}_{[d]}^n} |S_{\mathbf{i}_{[d]}, 2^{m+1}}(g)| \right\| \quad (3.3.25)$$

We use item A) in the following setting: the σ -algebra \mathcal{F}_0 is replaced by $\widetilde{\mathcal{F}}_0 = \sigma \left(\bigcup_{l \in \mathbb{Z}} T_{d+1}^l \mathcal{F}_0 \right)$, the function f is replaced by $\widetilde{f} := \sum_{j=0}^{2^{m+1}-1} U_{d+1}^j g$ (which belongs to $\widetilde{\mathcal{H}}_{d,E \setminus \{d+1\}}$ by Lemma 3.11), and $\widetilde{\mathbf{n}} = (n_1, \dots, n_d)$:

$$(I) \leq 2 \left| \widetilde{\mathbf{2}}^{\mathbf{n}} \right|^{1/2} K(d) \left\| \sum_{j=0}^{2^{m+1}-1} U_{d+1}^j g \right\| + \left| \widetilde{\mathbf{2}}^{\mathbf{n}} \right|^{1/2} \sum_{\emptyset \subsetneq J \subset [d]} C(d, J) \sum_{\mathbf{0}_J \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}_J} \left| \widetilde{\mathbf{2}}^{\mathbf{i}} \right|^{-1/2} \left\| \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\widetilde{\mathbf{2}}^{\mathbf{i}})_J} P_{d,E \setminus \{d+1\}}^{\widetilde{\mathbf{j}}} \sum_{j=0}^{2^{m+1}-1} U_{d+1}^j g \right\|. \quad (3.3.26)$$

The sequence $\left(\sum_{j=0}^{N-1} U_{d+1}^j g\right)_{N \geq 1}$ is a martingale hence, by item 1 of Proposition 3.1,

$$\left\| \sum_{j=0}^{2^{m+1}-1} U_{d+1}^j g \right\| = 2^{\frac{m+1}{2}} \|g\| \leq 2 \cdot 2^{\frac{m+1}{2}} \|f\|. \quad (3.3.27)$$

We now bound the second term of (3.3.26). Let J be a non-empty subset of $[d]$, and let \mathbf{i} and \mathbf{j} be two elements of \mathbb{N}^d such that $\mathbf{0}_J \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}_J$ and $\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J$. Since $\mathcal{F}_{\infty \mathbf{1}_{\mathbf{F} \cup \{d+1\}}}$ is T_{d+1} -invariant,

$$P_{d, E \setminus \{d+1\}} \widetilde{\mathbf{j}} \sum_{k=0}^{2^{m+1}-1} U_{d+1}^k g = \sum_{k=0}^{2^{m+1}-1} U_{d+1}^k P_{d, E \setminus \{d+1\}} \widetilde{\mathbf{j}} g \quad (3.3.28)$$

Since the sequence $\left(\sum_{k=0}^{2^{m+1}-1} U_{d+1}^k P_{d, E \setminus \{d+1\}} \widetilde{\mathbf{j}} g\right)_{N \geq 1}$ is a martingale, we derive that

$$\begin{aligned} & \left\| \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} P_{d, E \setminus \{d+1\}} \widetilde{\mathbf{j}} \sum_{k=0}^{2^{m+1}-1} U_{d+1}^k g \right\| \\ &= 2^{\frac{m+1}{2}} \left\| \sum_{F \subset E \setminus \{d+1\}} (-1)^{|F|} \mathbb{E} \left[\sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} U^{j \cdot \varepsilon(E)} g \mid \mathcal{F}_{\infty \mathbf{1}_{\mathbf{F} \cup \{d+1\}}} \right] \right\|. \end{aligned} \quad (3.3.29)$$

Assume that $d+1 \notin E$. In this case,

$$g = f - U_{d+1} P_E^{\mathbf{e}_{d+1}} f = f - \sum_{I \subset E} (-1)^{|E|+|I|} \mathbb{E} [f \mid \mathcal{F}_{\infty \mathbf{1}_{\mathbf{I} - \mathbf{e}_{d+1}}}], \quad (3.3.30)$$

and since f belongs to $\mathcal{H}_{d+1, E}$, we derive that $\mathbb{E} [f \mid \mathcal{F}_{\infty \mathbf{1}_{\mathbf{I} - \mathbf{e}_{d+1}}}] = 0$ if $I \subsetneq E$ hence

$$g = f - \mathbb{E} [f \mid \mathcal{F}_{\infty \mathbf{1}_{\mathbf{E} - \mathbf{e}_{d+1}}}], \quad (3.3.31)$$

Consequently, using the $\mathcal{F}_{\infty \mathbf{1}_{\mathbf{E}}}$ -measurability of f , we derive that

$$\begin{aligned} & \mathbb{E} \left[\sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} U^{j \cdot \varepsilon(E)} g \mid \mathcal{F}_{\infty \mathbf{1}_{\mathbf{F} \cup \{d+1\}}} \right] \\ &= \mathbb{E} \left[\sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} U^{j \cdot \varepsilon(E)} f \mid \mathcal{F}_{\infty \mathbf{1}_{\mathbf{F}}} \right] - \mathbb{E} \left[\sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} U^{j \cdot \varepsilon(E)} f \mid \mathcal{F}_{\infty \mathbf{1}_{\mathbf{F} - \mathbf{e}_{d+1}}} \right] \end{aligned} \quad (3.3.32)$$

and we get

$$\left\| \sum_{F \subset E \setminus \{d+1\}} (-1)^{|F|} \mathbb{E} \left[\sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} U^{j \cdot \varepsilon(E)} g \mid \mathcal{F}_{\infty \mathbf{1}_{\mathbf{F} \cup \{d+1\}}} \right] \right\| \leq 2 \left\| \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} P_{d+1, E}^{\mathbf{j}} f \right\|.$$

Assume now that $d+1$ belongs to E . In this case,

$$g = \mathbb{E} [f \mid \mathcal{F}_{\infty \mathbf{1}_{\mathbf{E} \setminus \{d+1\}} + \mathbf{e}_{d+1}}}] \quad (3.3.33)$$

and we get

$$\sum_{F \subset E \setminus \{d+1\}} (-1)^{|F|} \mathbb{E} \left[\sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} U^{j \cdot \varepsilon(E)} g \mid \mathcal{F}_{\infty \mathbf{1}_{\mathbf{F} \cup \{d+1\}}} \right] = \mathbb{E} [h \mid \mathcal{F}_{\infty \mathbf{1}_{[d]}}] \quad (3.3.34)$$

where

$$h = \sum_{F \subset E \setminus \{d+1\}} (-1)^{|F|} \mathbb{E} \left[\sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^i)_J} U^{\mathbf{j} \cdot \varepsilon(E)} f \mid \mathcal{F}_{\infty \mathbf{1}_{F \cup \{d+1\}}} \right] - \sum_{F \subset E \setminus \{d+1\}} (-1)^{|F|} \mathbb{E} \left[\sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^i)_J} U^{\mathbf{j} \cdot \varepsilon(E)} f \mid \mathcal{F}_{\infty \mathbf{1}_F} \right] \quad (3.3.35)$$

hence, in both cases,

$$\left\| \sum_{F \subset E \setminus \{d+1\}} (-1)^{|F|} \mathbb{E} \left[\sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^i)_J} U^{\mathbf{j} \cdot \varepsilon(E)} g \mid \mathcal{F}_{\infty \mathbf{1}_{F \cup \{d+1\}}} \right] \right\| \leq 2 \left\| \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^i)_J} P_{d+1, E}^{\mathbf{j}} f \right\|. \quad (3.3.36)$$

The combination of (3.3.26), (3.3.27), (3.3.29) and (3.3.36) yields

$$(I) \leq 4 |\mathbf{2}^n|^{1/2} K(d) \|f\| + 4 |\mathbf{2}^n|^{1/2} \sum_{\emptyset \subsetneq J \subset [d]} C(d, J) \sum_{\mathbf{0}_J \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}_J} |\mathbf{2}^i|^{-1/2} \left\| \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^i)_J} P_{d+1, E}^{\mathbf{j}} f \right\|. \quad (3.3.37)$$

Let us estimate the impact of (II). Using inequality $\|\max_{i \in I} |Y_i|\| \leq \sqrt{|I|} \max_{i \in I} \|Y_i\|$, we infer that

$$(II) \leq 2^{m/2} \left\| \max_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{2}^{n-(m+1)\mathbf{e}_{d+1}}} |S_i(U_{d+1}^{s(E)} P_{d+1, E}^{\mathbf{e}_{d+1}}(f))| \right\| = 2^{m/2} \left\| \max_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{2}^{n-(m+1)\mathbf{e}_{d+1}}} |S_i(P_{d+1, E}^{\mathbf{e}_{d+1}}(f))| \right\|.$$

By item A) applied to $P_{d+1, E}^{\mathbf{e}_{d+1}}(f)$ instead of f and $\tilde{\mathbf{n}} = (n_1, \dots, n_d)$, \mathcal{F}_0 and $\mathcal{H}_{d, E}$ if $d+1 \notin E$ (and $\mathcal{F}_{\infty \mathbf{1}_{\{d+1\}}}$, $\widetilde{h_{d, E \setminus \{d+1\}}}$ defined in Proposition 3.1), the following inequality holds

$$(II) \leq 2^{m/2} |\mathbf{2}^{\tilde{\mathbf{n}}}|^{1/2} K(d) \|P_{d+1, E}^{\mathbf{e}_{d+1}}(f)\| + 2^{m/2} |\mathbf{2}^{\tilde{\mathbf{n}}}|^{1/2} \sum_{\emptyset \subsetneq J \subset [d]} C(d, J) \sum_{\mathbf{0}_J \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}_J} |\mathbf{2}^i|^{-1/2} \left\| \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^i)_J} \widetilde{P_{d, E \setminus \{d+1\}}^{\mathbf{j}}} P_{d+1, E}^{\mathbf{e}_{d+1}}(f) \right\|,$$

and using item 3 of Proposition 3.1, it follows that

$$(II) \leq 2 \cdot |\mathbf{2}^n|^{1/2} K(d) \|f\| + |\mathbf{2}^n|^{1/2} \sum_{\emptyset \subsetneq J \subset [d]} C(d, J) \sum_{\mathbf{0}_J \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}_J} |\mathbf{2}^i|^{-1/2} \left\| \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^i)_J} P^{\mathbf{j} + \mathbf{e}_{d+1}} f \right\|. \quad (3.3.38)$$

We now bound (III) using item B) in the following setting: we take $\widetilde{T}_i = T_i$ for $i \in [d]$ and $\widetilde{T}_{d+1} = T_{d+1}^2$, $\widetilde{\mathcal{F}}_0 = \mathcal{F}_0$, $\widetilde{f} = (I + U_{d+1})U_{d+1}^{s(E)}P_{d+1,E}^{\mathbf{e}_{d+1}}(f)$ and $\widetilde{\mathbf{n}} = \mathbf{n} - \mathbf{e}_{d+1}$ in order to get

$$(III) \leq \left| \mathbf{2}^{\widetilde{\mathbf{n}}} \right|^{1/2} K(d+1) \left\| (I + U_{d+1})U_{d+1}^{s(E)}P_{d+1,E}^{\mathbf{e}_{d+1}}(f) \right\| \\ + \left| \mathbf{2}^{\widetilde{\mathbf{n}}} \right|^{1/2} \sum_{\emptyset \subsetneq J \subset [d+1]} C(d+1, J) \sum_{\mathbf{0}_J \preccurlyeq \mathbf{i} \preccurlyeq \widetilde{\mathbf{n}}_J} |\mathbf{2}^{\mathbf{i}}|^{-1/2} \left\| \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} \widetilde{P}_{d+1,E}^{\mathbf{j}} U_{d+1}^{s(E)} P_{d+1,E}^{\mathbf{e}_{d+1}}(f) \right\|.$$

We notice that if $d+1$ does not belong to J , then $\sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} \widetilde{P}_{d+1,E}^{\mathbf{j}} = \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} P_{d+1,E}^{\mathbf{j}}$ and if $d+1$ belongs to J , then $\sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} \widetilde{P}_{d+1,E}^{\mathbf{j}} = \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} P_{d+1,E}^{\mathbf{j} \setminus \{d+1\} + 2j_{d+1} \mathbf{e}_{d+1}}$. By item 4 of Proposition 3.1, we derive that

$$(III) \leq 2^{1/2} |\mathbf{2}^{\mathbf{n}}|^{1/2} K(d+1) \left\| P_{d+1,E}^{\mathbf{e}_{d+1}}(f) \right\| \\ + 2^{-1/2} |\mathbf{2}^{\mathbf{n}}|^{1/2} \sum_{\emptyset \subsetneq J \subset [d]} C(d+1, J) \sum_{\mathbf{0}_J \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}_J} |\mathbf{2}^{\mathbf{i}}|^{-1/2} \left\| \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} P_{d+1,E}^{\mathbf{j} + \mathbf{e}_{d+1}} f \right\| + (III'), \quad (3.3.39)$$

where

$$(III') := 2^{-1/2} |\mathbf{2}^{\mathbf{n}}|^{1/2} \sum_{\emptyset \subsetneq J \subset [d]} C(d+1, J \cup \{d+1\}) \sum_{\mathbf{0}_J \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}_J} \sum_{i_{d+1}=0}^m |\mathbf{2}^{\mathbf{i}}|^{-1/2} \left\| \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} \sum_{j_{d+1}=1}^{2^{i_{d+1}+1}} P_{d+1,E}^{\mathbf{j} + j_{d+1} \mathbf{e}_{d+1}} f \right\|,$$

hence, denoting $J' := J \cup \{d+1\}$ for $\emptyset \subsetneq J \subset [d]$, and making the change of index $i'_{d+1} := i_{d+1} + 1$,

$$(III') \leq |\mathbf{2}^{\mathbf{n}}|^{1/2} \sum_{\emptyset \subsetneq J \subset [d]} C(d+1, J \cup \{d+1\}) \sum_{\mathbf{0}_{J'} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}_{J'}} |\mathbf{2}^{\mathbf{i}}|^{-1/2} \left\| \sum_{\mathbf{1}_{J'} \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_{J'}} P_{d+1,E}^{\mathbf{j}} f \right\| \\ - |\mathbf{2}^{\mathbf{n}}|^{1/2} \sum_{\emptyset \subsetneq J \subset [d]} C(d+1, J \cup \{d+1\}) \sum_{\mathbf{0}_J \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}_J} |\mathbf{2}^{\mathbf{i}}|^{-1/2} \left\| \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} P_{d+1,E}^{\mathbf{j}} f \right\|. \quad (3.3.40)$$

We derive the following inequality:

$$(III) \leq 2^{1/2} |\mathbf{2}^{\mathbf{n}}|^{1/2} K(d+1) \left\| P_{d+1,E}^{\mathbf{e}_{d+1}}(f) \right\| \\ + 2^{-1/2} |\mathbf{2}^{\mathbf{n}}|^{1/2} \sum_{\emptyset \subsetneq J \subset [d]} C(d+1, J) \sum_{\mathbf{0}_J \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}_J} |\mathbf{2}^{\mathbf{i}}|^{-1/2} \left\| \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} P_{d+1,E}^{\mathbf{j} + \mathbf{e}_{d+1}} f \right\| \\ + |\mathbf{2}^{\mathbf{n}}|^{1/2} \sum_{\emptyset \subsetneq J \subset [d]} C(d+1, J \cup \{d+1\}) \sum_{\mathbf{0}_{J'} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}_{J'}} |\mathbf{2}^{\mathbf{i}}|^{-1/2} \left\| \sum_{\mathbf{1}_{J'} \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_{J'}} P_{d+1,E}^{\mathbf{j}} f \right\| \\ - |\mathbf{2}^{\mathbf{n}}|^{1/2} \sum_{\emptyset \subsetneq J \subset [d]} C(d+1, J \cup \{d+1\}) \sum_{\mathbf{0}_J \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}_J} |\mathbf{2}^{\mathbf{i}}|^{-1/2} \left\| \sum_{\mathbf{1}_J \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_J} P_{d+1,E}^{\mathbf{j} + \mathbf{e}_{d+1}} f \right\|. \quad (3.3.41)$$

Combining (3.3.16), (3.3.37), (3.3.38) and (3.3.41), we derive that

$$\begin{aligned}
& \left\| \max_{\mathbf{1} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{2}^{\mathbf{n}}} |S_{\mathbf{i}}(f)| \right\| \leq 6 |\mathbf{2}^{\mathbf{n}}|^{1/2} K(d) \|f\| \\
& + |\mathbf{2}^{\mathbf{n}}|^{1/2} \sum_{\emptyset \subsetneq J \subset [d]} 4C(d, J) \sum_{\mathbf{0}_{\mathbf{J}} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}_{\mathbf{J}}} |\mathbf{2}^{\mathbf{i}}|^{-1/2} \left\| \sum_{\mathbf{1}_{\mathbf{J}} \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_{\mathbf{J}}} P_{d+1, E}^{\mathbf{j}} f \right\| \\
& + |\mathbf{2}^{\mathbf{n}}|^{1/2} \sum_{\emptyset \subsetneq J \subset [d]} \widetilde{C}(J) \sum_{\mathbf{0}_{\mathbf{J}} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}_{\mathbf{J}}} |\mathbf{2}^{\mathbf{i}}|^{-1/2} \left\| \sum_{\mathbf{1}_{\mathbf{J}} \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_{\mathbf{J}}} P_{d+1, E}^{\mathbf{j} + \mathbf{e}_{d+1}} f \right\| \\
& + 2^{1/2} |\mathbf{2}^{\mathbf{n}}|^{1/2} K(d+1) \|P_{d+1, E}^{\mathbf{e}_{d+1}}(f)\| \\
& + |\mathbf{2}^{\mathbf{n}}|^{1/2} \sum_{\emptyset \subsetneq J \subset [d]} C(d+1, J \cup \{d+1\}) \sum_{\mathbf{0}_{\mathbf{J}'} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n}_{\mathbf{J}'}} |\mathbf{2}^{\mathbf{i}}|^{-1/2} \left\| \sum_{\mathbf{1}_{\mathbf{J}'} \preccurlyeq \mathbf{j} \preccurlyeq (\mathbf{2}^{\mathbf{i}})_{\mathbf{J}'}} P_{d+1, E}^{\mathbf{j}} f \right\|, \quad (3.3.42)
\end{aligned}$$

where $\widetilde{C}(J) := C(d, J) + 2^{-1/2} C(d+1, J) - C(d+1, J \cup \{d+1\})$ and using (3.3.8), (3.3.9), (3.3.10) and (3.3.11), we obtain (3.3.12) for $n_{d+1} = m + 1$. This proves the first inequality in (2.3.14). The second one follows from a multidimensional extension of Lemma 2.7 in [PU05]. \square

Proof of Theorem 2.12. Using Theorem 2.7, we shall only check that (2.2.5) holds.

Let $\emptyset \subsetneq J \subset [d]$ and $E \subset [d]$. An application of Proposition 2.19 reduces the proof to

$$\lim_{k \rightarrow +\infty} \frac{1}{k^{|J|}} \sum_{\mathbf{n} \succcurlyeq \mathbf{1}} |\mathbf{n}|^{-3/2} \left\| \sum_{\mathbf{0} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n} - \mathbf{1}} P_E^{\mathbf{i}} \left(\sum_{\mathbf{1}_{\mathbf{J}} \preccurlyeq \mathbf{j} \preccurlyeq k \mathbf{1}_{\mathbf{J}}} P_E^{\mathbf{j}} f \right) \right\| = 0 \quad (3.3.43)$$

If q belongs to J , then

$$\frac{1}{k^{|J|}} \left\| \sum_{\mathbf{0} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n} - \mathbf{1}} \sum_{\mathbf{1}_{\mathbf{J}} \preccurlyeq \mathbf{j} \preccurlyeq k \mathbf{1}_{\mathbf{J}}} P_E^{\mathbf{i} + \mathbf{j}} f \right\| \leq \frac{1}{k} \left\| \sum_{\mathbf{0} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n} - \mathbf{1}} \sum_{l=1}^k P_E^{\mathbf{i} + l \mathbf{e}_{\mathbf{q}}} f \right\| \quad (3.3.44)$$

hence it suffices to prove that for any $q \in [d]$,

$$\lim_{k \rightarrow +\infty} \frac{1}{k} \sum_{\mathbf{n} \succcurlyeq \mathbf{0}} a_{\mathbf{n}, k} = 0 \quad (3.3.45)$$

where

$$a_{\mathbf{n}, k} := k^{-1} |\mathbf{n}|^{-3/2} \left\| \sum_{\mathbf{0} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n} - \mathbf{1}} \sum_{l=1}^k P_E^{\mathbf{i} + l \mathbf{e}_{\mathbf{q}}} f \right\|.$$

We first observe that for any fixed $\mathbf{n} \succcurlyeq \mathbf{1}$, $a_{\mathbf{n}, k} \leq k^{-1} \left\| \sum_{l=1}^k P_E^{l \mathbf{e}_{\mathbf{q}}}(f) \right\|$ and applying Lemma 2.8. of [PU05] to the subadditive sequence $\left(\left\| \sum_{l=1}^k P_E^{l \mathbf{e}_{\mathbf{q}}}(f) \right\| \right)_{k \geq 1}$, we derive that $a_{\mathbf{n}, k} \rightarrow 0$ as k goes to infinity. Moreover,

$$\sup_{k \geq 1} a_{\mathbf{n}, k} \leq |\mathbf{n}|^{-3/2} \left\| \sum_{\mathbf{0} \preccurlyeq \mathbf{i} \preccurlyeq \mathbf{n} - \mathbf{1}} P_E^{\mathbf{i}}(f) \right\|, \quad (3.3.46)$$

hence by dominated convergence, (3.3.43) holds. This ends the proof of Theorem 2.12. \square

Proof of Corollary 2.20. The computation of $P_E^{\mathbf{i}} f$ gives

$$P_E^{\mathbf{i}} f = \sum_{\mathbf{j} \in \mathbb{N}^d, \mathbf{j} \cdot \mathbf{1}_{\mathbf{E}} \succcurlyeq \mathbf{1}_{\mathbf{E}}} a_{(\mathbf{i} + \mathbf{j}) \cdot \varepsilon(\mathbf{E})} \cdot \varepsilon_{\mathbf{j}}. \quad (3.3.47)$$

Summing over $i \in [0, \mathbf{n}]$, taking the \mathbb{L}^2 -norm and using orthogonality of ε_j 's, we derive that $\left\| \sum_{0 \leq i \leq \mathbf{n}-1} P_E^i f \right\| \leq \Delta_{E, \mathbf{n}}$ hence (2.3.17) implies (2.3.7). The approximating martingale satisfies the invariance principle since $T^{\mathbf{e}_1}$ is ergodic. \square

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