Another Look at the Implementation of Read/write Registers in Crash-prone Asynchronous Message-Passing Systems (Extended Version)

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Abstract

"Yet another paper on" the implementation of read/write registers in crash-prone asynchronous message-passing systems! Yes..., but, differently from its predecessors, this paper looks for a communication abstraction which captures the essence of such an implementation in the same sense that total order broadcast can be associated with consensus, or message causal delivery can be associated with causal read/write registers. To this end, the paper introduces a new communication abstraction, named SCD-broadcast (SCD standing for "Set Constrained Delivery"), which, instead of a single message, delivers to processes sets of messages (whose size can be arbitrary), such that the sequences of message sets delivered to any two processes satisfies some constraints. The paper then shows that: (a) SCD-broadcast allows for a very simple implementation of a snapshot object (and consequently also of atomic read/write registers) in crash-prone asynchronous message-passing systems; (b) SCD-broadcast can be built from snapshot objects (hence SCD-broadcast and snapshot objects –or read/write registers– are "computationally equivalent"); (c) SCD-broadcast can be built in message-passing systems where any minority of processes may crash (which is the weakest assumption on the number of possible process crashes needed to implement a read/write register).

Keywords: Asynchronous system, Atomicity, Communication abstraction, Linearizability, Message-passing system, Process crash, Read/write atomic register, Snapshot object.

1 Introduction

The "one-shot" terracotta tablets introduced and used at Sumer about 3030 BC [23], and the "multi-shot" palimpsests used in the middle-age, can be considered as ancestors of the *read/write register* abstraction. Such an object provides its users with a write operation which defines a new value of the register, and a read operation which returns its value. When considering sequential computing, read/write registers are universal in the sense that they are assumed to allow solving any problem that can be solved [37].

On the variety of read/write registers and their distributed implementation In a shared read/write memory system, the registers are given for free. The situation is different in a message-passing system, where the computing entities (processes) communicate by sending and receiving messages transmitted through a communication network. Hence, in such a distributed context, a register is not given for free, but constitutes a communication abstraction which must be built by a distributed algorithm with the help of the local memories of the processes and the communication network.

Several types of registers have been proposed. They differ according to (a) their size (from binary registers which contain a single bit, to bounded and unbounded registers); (b) their behavior in the presence of concurrency (safe, regular, atomic [25]); (c) the number of processes which are allowed to read them (Single-Reader -SR- vs Multi-Reader -MR- register); and (d) the number of processes which are allowed to write them (Single-Writer -SR- vs Multi-Writer -MR- register), which gives four possible combinations from SWSR to MWMR. There are algorithms building MWMR atomic (bounded and unbounded) registers from SWSR binary safe registers [25] (see [8, 26, 34] for surveys of such algorithms).

As far as a read/write register is concerned, *atomicity* means that (a) each read or write operation appears as if it had been executed instantaneously at a single point of the time line, (b) this point appears between its start event and its end event, (c) no two operations appear at the same point of the time line, and (d) a read returns the value written by the closest preceding write operation (or the initial value of the register if there is no preceding write) [25, 27]. *Linearizability* is atomicity extended to any object defined from a sequential specification on total operations [18]. In the following, we consider the terms atomicity and linearizability as synonyms. Hence, a sequence of read and write operations satisfying atomicity is said to be linearizable, and is called a linearization. The point of the time line at which an operation appears to have been executed is called its linearization point.

Many distributed algorithms have been proposed, which build a read/write register on top of a message-passing system, be it failure-free or failure-prone. In the failure-prone case, the addressed failure models are the process crash failure model, and the Byzantine process failure model (see textbooks, e.g., [8, 26, 32, 33]). When considering process crash failures (the one considered in this paper[1]), the most famous of these algorithms was proposed by H. Attiya, A. Bar-Noy, and D. Dolev in [5]. This algorithm, usually called ABD according to the names of its authors, considers an n-process asynchronous system in which up to $t < n/2$ processes may crash. As $t < n/2$ is an upper bound of the number of process crashes which can be tolerated (see [5]), this algorithm is t-resilient optimal. Its instances implementing SWMR or MWMR atomic read/write registers rely on (a) quorums [38], and (b) a classical broadcast/reply communication pattern. This communication pattern is used twice in a read operation, and once (twice) in a write operation for an SWMR (MWMR) atomic read/write register.

Other algorithms –each with its own properties– implementing atomic read/write registers on top of crash-prone asynchronous message-passing systems can be found in the literature ([4, 12, 17, 29] to cite a few; see also the analytic presentation given in [36]).

From registers to snapshot objects The snapshot object was introduced in [1, 3]. A snapshot object is an array $REG[1..m]$ of atomic read/write registers which provides the processes with two operations, denoted write() and snapshot(). If the base registers are SWMR the snapshot is called SWMR snapshot (and we have then $m = n$). In this case, the invocation of write(v) by a process p_i assigns v to $REG[i]$, and the invocation of snapshot() by a process p_i returns the value of the full array as if the operation had been executed instantaneously. If the base registers are MWMR, the snapshot is called MWMR snapshot. The invocation of write(r, v), where $1 \leq r \leq m$, by a process p_i assigns v to $REG[r]$, and snapshot() is defined as before. Said another way, the operations write() and snapshot() are atomic, i.e., in any execution of an SWMR (or MWMR) snapshot object, its operations write() and snapshot() are linearizable.

Implementations of both SWMR and MWMR snapshot objects on top of read/write atomic registers have

[1] For Byzantine failures, see for example [28].

been proposed (e.g., [1, 3, 19, 20]). The "hardness" to build snapshot objects in read/write systems and associated lower bounds are presented in the survey [13]. The best algorithm known to implement an SWMR snapshot requires $O(n \log n)$ read/write on the base SWMR registers for both the write() and snapshot() operations [6]. As far as MWMR snapshot objects are concerned, there are implementations where each operation has an $O(n)$ cost[2].

As far as the construction of an SWMR (or MWMR) snapshot object in crash-prone asynchronous message-passing systems where $t < n/2$ is concerned, it is possible to stack two constructions: first an algorithm implementing SWMR (or MWMR) atomic read/write registers (such as ABD), and, on top of it, an algorithm implementing an SWMR (or MWMR) snapshot object. This stacking approach provides objects whose operation cost is $O(n^2 \log n)$ messages for SWMR snapshot, and $O(n^2)$ messages for MWMR snapshot. An algorithm based on the same communication pattern as ABD, which builds an atomic SWMR snapshot object "directly" (i.e., without stacking algorithms) was recently presented in [11] (the aim of this algorithm is to perform better that the stacking approach in concurrency-free executions).

Another look at the implementation of read/write registers and snapshot objects In sequential computing, there are "natural" pairings linking data structures and control structures. The most simple examples are the pair "array and **for** loop", and the pair "tree and recursion".

When we look at the implementation of a causal read/write register [2] on top of a (crash-free or crash-prone) message-passing system, the causal message delivery broadcast abstraction [9, 35] is the appropriate communication abstraction. Namely, given this abstraction for free, the algorithms implementing the read and write operations build on top of it, become very simple, need only a few lines, and are easy to understand and to prove correct. Of course, this is due to the fact that the causal broadcast abstraction captures and abstracts the causality relation needed to implement a causal read/write register. Similarly, total order broadcast is the communication abstraction associated with the consensus object [10]. This is summarized in Table 1.

Concurrent object	Communication abstraction
Causal read/write registers	Causal message delivery [9, 35]
Consensus	Total order broadcast [10]
Snapshot object (and R/W register)	SCD-broadcast (This paper)

Table 1: Associating objects and communication abstractions in a wait-free model

As already said, all the algorithms we know which implement atomic read/write registers, and (by stacking transitivity or directly) SWMR or MWMR snapshots objects, on top of crash-prone asynchronous message-passing systems, are based on a broadcast/reply pattern plus the use of intersecting quorums. Hence, the following question naturally arises: Is this approach the "only" way to implement a snapshot object (or an atomic register), or is there a *specific communication abstraction which captures the essence and simplifies the implementation of snapshot objects (and atomic read/write registers)*?

Content of the paper Informatics in general (and distributed computing in particular) is a science of abstractions, and this paper is *distributed programming abstraction*-oriented. It strives to address a "desired level of abstraction and generality – one that is broad enough to encompass interesting new situations yet specific enough to address the crucial issues" as expressed in [16]. More precisely, it answers the previous question in a positive way. To this end, it presents a simple broadcast abstraction which matches –and therefore captures the essence of– snapshot objects (and atomic read/write registers). We call it *Set-Constrained Delivery Broadcast* (in short SCD-broadcast). Given this communication abstraction, it is possible to quorum-free build snapshot objects, and vice versa. Hence, similarly to consensus and total order broadcast, SCD-broadcast and snapshot objects have the same computational power (Table 1).

The SCD-broadcast communication abstraction allows a process to broadcast messages, and to deliver sets of messages (instead of single messages) in such a way that, if a process p_i delivers a message set[3] ms containing a message m, and later delivers a message set ms' containing a message m', then no process p_j can deliver first a set containing m' and later another set containing m. Let us notice that p_j is not prevented from delivering m and m' in the same set.

[2]Snapshot objects built in read/write models enriched with operations such as Compare&Swap, or LL/SC, have also been considered, e.g.,[21, 19]. Here we are interested in pure read/write models.

[3]In the rest of the paper, the identifiers starting with"*ms*" denote message sets.

The implementation of an instance of SCD-broadcast costs $O(n^2)$ messages. It follows that the cost of a snapshot operation (or a read/write register operation) on top of a message-passing asynchronous system, where any minority of processes may crash, is also $O(n^2)$ for both SWMR and MWMR snapshot objects (i.e., better than the stacking approach for SWMR snapshot objects). Additionally, be the snapshot objects that are built SWMR or MWMR, their implementation differ only in the fact that their underling read/write registers are SWMR or MWMR. This provides us with a noteworthy genericity-related design simplicity.

Of course, there is rarely something for free. The algorithms implementing the snapshot and write operations are simple because the SCD-broadcast abstraction hides enough "implementation details" and provides consequently a high level abstraction (much higher than the simple broadcast used in ABD-like algorithms). Its main interest lies in its capture of the *high level message communication abstraction* that, despite asynchrony and process failures, allows simple message-passing implementations of shared memory objects such as snapshot objects and atomic read/write registers.

Roadmap The paper is composed of 7 sections. Section 2 presents the two base computation models concerned in this paper, (read/write and message-passing). Section 3 presents the SCD-broadcast communication abstraction. Then, Section 4 presents a simple algorithm which implements a snapshot object on top of an asynchronous system enriched with SCD-broadcast, in which any number of processes may crash. Section 6 addresses the other direction, namely, it presents an algorithm building the SCD-broadcast abstraction on top of an asynchronous system enriched with snapshot objects and where any number of processes may crash. Section 7 concludes the paper. A noteworthy feature of the algorithms that are presented lies in their simplicity, which is a first class property.

Appendix A describes an implementation of SCD-broadcast suited to asynchronous message-passing systems where any minority of processes may crash. Hence, being implementable in the weakest[4] message-passing system model in which a read/write register can be built, SCD-broadcast is not "yet another oracle" which makes things simpler to understand but cannot be implemented. Appendix B presents simplified SCD-based algorithms which build atomic and sequentially consistent read/write registers.

2 Basic Computation Models

This section presents two basic computation models. In both cases, the process model is the same.

2.1 Processes

The computing model is composed of a set of n asynchronous sequential processes, denoted p_1, ..., p_n. "Asynchronous" means that each process proceeds at its own speed, which can be arbitrary and always remains unknown to the other processes.

A process may halt prematurely (crash failure), but it executes its local algorithm correctly until its possible crash. The model parameter t denotes the maximal number of processes that may crash in a run. A process that crashes in a run is said to be *faulty*. Otherwise, it is *non-faulty*. Hence a faulty process behaves as a non-faulty process until it crashes.

2.2 Basic crash-prone asynchronous shared memory model

Atomic read/write register The notion of an atomic read/write register has been formalized in [25, 27]. An MWMR *atomic* register (say REG) is a concurrent object which provides each process with an operation denoted $REG.\mathsf{write}()$, and an operation denoted $REG.\mathsf{read}()$. When a process invokes $REG.\mathsf{write}(v)$ it defines v as being the new value of REG. An MWMR atomic register is defined by the following set of properties.

- Liveness. An invocation of an operation by a non-faulty process terminates.
- Consistency (safety). All the operations invoked by the processes, except possibly –for each faulty process– the last operation it invoked, appear as if they have been executed sequentially and this sequence of operations is such that:
  - each read returns the value written by the closest write that precedes it (or the initial value of REG if there is no preceding write),

[4] From the point of view of the maximal number of process crashes that can be tolerated, assuming failures are independent.

– if an operation $op1$ terminates before an operation $op2$ starts, then $op1$ appears before $op2$ in the sequence.

This set of properties states that, from an external observer point of view, the read/write register appears as if it is accessed sequentially by the processes, and this sequence (a) respects the real-time access order, and (b) belongs to the sequential specification of a register.

Notation The previous computation model is denoted $\mathcal{CARW}_{n,t}[\emptyset]$ ($\mathcal{C}$rash $\mathcal{A}$synchronous $\mathcal{R}$ead-$\mathcal{W}$rite). This basic read/write model is also called *wait-free* read/write model. The symbol $\emptyset$ means there is no specific constraint on t, which is equivalent to $t < n$, as it is always assumed that not all processes crash.

Snapshot object This object was defined in the introduction. As we have seen, snapshot objects can be built in $\mathcal{CARW}_{n,t}[\emptyset]$. As we have seen there are two types of snapshot objects. SWMR snapshot objects (whose base registers are SWMR), and MWMR snapshot objects (whose base registers are MWMR). In the following we consider MWMR snapshot objects, but the algorithms can be trivially adapted to work with SWMR snapshot objects.

$\mathcal{CARW}_{n,t}[\emptyset]$ enriched with snapshot objects is denoted $\mathcal{CARW}_{n,t}[\text{snapshot}]$. As a snapshot object can be built in $\mathcal{CARW}_{n,t}[\emptyset]$ this model has the same computational power as $\mathcal{CARW}_{n,t}[\emptyset]$. It only offers a higher abstraction level.

2.3 Basic crash-prone asynchronous message-passing model

Communication Each pair of processes communicate by sending and receiving messages through two unidirectional channels, one in each direction. Hence, the communication network is a complete network: any process p_i can directly send a message to any process p_j (including itself). A process p_i invokes the operation "send TYPE(m) to p_j" to send to p_j the message m, whose type is TYPE. The operation "receive TYPE() from p_j" allows p_i to receive from p_j a message whose type is TYPE.

Each channel is reliable (no loss, corruption, nor creation of messages), not necessarily first-in/first-out, and asynchronous (while the transit time of each message is finite, there is no upper bound on message transit times).

Let us notice that, due to process and message asynchrony, no process can know if another process crashed or is only very slow.

Notation and necessary and sufficient condition This computation model is denoted $\mathcal{CAMP}_{n,t}[\emptyset]$ ($\mathcal{C}$rash $\mathcal{A}$synchronous $\mathcal{M}$essage-$\mathcal{P}$assing).

The constraint $(t < n/2)$ is a necessary and sufficient condition to implement an atomic read/write register in $\mathcal{CAMP}_{n,t}[\emptyset]$ [5]. Hence, the model $\mathcal{CAMP}_{n,t}[\emptyset]$ whose runs are constrained by $t < n/2$ is denoted $\mathcal{CAMP}_{n,t}[t < n/2]$.

3 A Broadcast Abstraction: Set-Constrained Message Delivery

Definition The set-constrained broadcast abstraction (SCD-broadcast) provides the processes with two operations, denoted scd_broadcast() and scd_deliver(). The first operation takes a message to broadcast as input parameter. The second one returns a non-empty set of messages to the process that invoked it. Using a classical terminology, when a process invokes scd_broadcast(m), we say that it "scd-broadcasts a message m". Similarly, when it invokes scd_deliver() and obtains a set of messages ms, we say that it "scd-delivers a set of messages ms". By a slight abuse of language, we also say that a process "scd-delivers a message m" when it delivers a message $m \in ms$.

SCD-broadcast is defined by the following set of properties, where we assume –without loss of generality– that all the messages that are scd-broadcast are different.

- Validity. If a process scd-delivers a set containing a message m, then m was scd-broadcast by some process.
- Integrity. A message is scd-delivered at most once by each process.
- MS-Ordering. If a process p_i scd-delivers first a message m belonging to a set ms_i and later a message m' belonging to a set $ms'_i \neq ms_i$, then no process scd-delivers first the message m' in some scd-delivered set ms'_j and later the message m in some scd-delivered set $ms_j \neq ms'_j$.

- Termination-1. If a non-faulty process scd-broadcasts a message m, it terminates its scd-broadcast invocation and scd-delivers a message set containing m.
- Termination-2. If a non-faulty process scd-delivers a message m, every non-faulty process scd-delivers a message set containing m.

Termination-1 and Termination-2 are classical liveness properties (found for example in Uniform Reliable Broadcast). The other ones are safety properties. Validity and Integrity are classical communication-related properties. The first states that there is neither message creation nor message corruption, while the second states that there is no message duplication.

The MS-Ordering property is new, and characterizes SCD-broadcast. It states that the contents of the sets of messages scd-delivered at any two processes are not totally independent: the sequence of sets scd-delivered at a process p_i and the sequence of sets scd-delivered at a process p_j must be mutually consistent in the sense that a process p_i cannot scd-deliver first $m \in ms_i$ and later $m' \in ms'_i \neq ms_i$, while another process p_j scd-delivers first $m' \in ms'_j$ and later $m \in ms_j \neq ms'_j$. Let us nevertheless observe that if p_i scd-delivers first $m \in ms_i$ and later $m' \in ms'_i$, p_j may scd-deliver m and m' in the same set of messages.

An example Let m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, ... be messages that have been scd-broadcast by different processes. The following scd-deliveries of message sets by p_1, p_2 and p_3 respect the definition of SCD-broadcast:

- at p_1: $\{m_1, m_2\}$, $\{m_3, m_4, m_5\}$, $\{m_6\}$, $\{m_7, m_8\}$.
- at p_2: $\{m_1\}$, $\{m_3, m_2\}$, $\{m_6, m_4, m_5\}$, $\{m_7\}$, $\{m_8\}$.
- at p_3: $\{m_3, m_1, m_2\}$, $\{m_6, m_4, m_5\}$, $\{m_7\}$, $\{m_8\}$.

Differently, due to the scd-deliveries of the sets including m_2 and m_3, the following scd-deliveries by p_1 and p_2 do not satisfy the MS-broadcast property:

- at p_1: $\{m_1, m_2\}$, $\{m_3, m_4, m_5\}$, ...
- at p_2: $\{m_1, m_3\}$, $\{m_2\}$, ...

A containment property Let ms_i^ℓ be the ℓ-th message set scd-delivered by p_i. Hence, at some time, p_i scd-delivered the sequence of message sets $ms_i^1, \cdots, ms_i^x$. Let $MS_i^x = ms_i^1 \cup \cdots \cup ms_i^x$. The following property follows directly from the MS-Ordering and Termination-2 properties:

- Containment. $\forall\, i, j, x, y$: $(MS_i^x \subseteq MS_j^y) \vee (MS_j^y \subseteq MS_i^x)$.

Remark 1: Weakening SCD-broadcast If the messages in a message set are delivered one at a time, and the MS-Ordering property is suppressed, SCD-broadcast boils down to *Reliable Broadcast*.

Remark 2: On the partial order created by the message sets The MS-Ordering and Integrity properties establish a partial order on the set of all the messages, defined as follows. Let $\mapsto_i$ be the local message delivery order at a process p_i defined as follows: $m \mapsto_i m'$ if p_i scd-delivers the set containing m before the set containing m'. As no message is scd-delivered twice, it is easy to see that $\mapsto_i$ is a partial order (locally know by p_i). The reader can check that there is a total order (which remains unknown to the processes) on the whole set of messages, that complies with the partial order $\cup_{1 \leq i \leq n} \mapsto_i$. This is where SCD-broadcast can be seen as a weakening of total order broadcast.

4 From SCD-broadcast to an MWMR Snapshot Object

Let $\mathcal{CAMP}_{n,t}$[SCD-broadcast] denote $\mathcal{CAMP}_{n,t}[\emptyset]$ enriched with the SCD-broadcast abstraction. Hence, this abstraction is given for free. This section presents and proves correct a simple algorithm building an MWMR snapshot object on top of $\mathcal{CAMP}_{n,t}$[SCD-broadcast]. The same algorithm with very few simple modifications can be used to build SWMR or MWMR atomic registers in $\mathcal{CAMP}_{n,t}$[SCD-broadcast] (see Appendix B).

4.1 Building an MWMR snapshot object on top of $\mathcal{CAMP}_{n,t}$[**SCD-broadcast**]

Let $REG[1..m]$ denote the MWMR snapshot object that is built.

Local representation of REG at a process p_i At each register p_i, $REG[1..m]$ is represented by three local variables $reg_i[1..m]$ (data part), plus $tsa_i[1..m]$ and $done_i$ (control part).

- $done_i$ is a Boolean variable.
- $reg_i[1..m]$ contains the current value of $REG[1..m]$, as known by p_i.
- $tsa_i[1..m]$ is an array of timestamps associated with the values stored in $reg_i[1..m]$. A timestamp is a pair made of a local clock value and a process identity. Its initial value is $\langle 0, -\rangle$. The fields associated with $tsa_i[r]$ are denoted $\langle tsa_i[r].date, tsa_i[r].proc\rangle$.

Timestamp-based order relation We consider the classical lexicographical total order relation on timestamps, denoted $<_{ts}$. Let $ts1 = \langle h1, i1\rangle$ and $ts2 = \langle h2, i2\rangle$. We have $ts1 <_{ts} ts2 \stackrel{def}{=} (h1 < h2) \vee ((h1 = h2) \wedge (i1 < i2))$.

Algorithm 1: snapshot operation (Lines 1-4) When p_i invokes REG.snapshot(), it first sets $done_i$ to `false`, and invokes scd_broadcast SYNC(i). SYNC() is a synchronization message, whose aim is to entail the refreshment of the value of $reg_i[1..m]$ (lines 11-17) which occurs before the setting of $done_i$ to `true` (line 18). When this happens, p_i returns the value of its local variable $reg_i[1..m]$ and terminates its snapshot invocation.

```
operation snapshot() is
(1)  done_i ← false;
(2)  scd_broadcast SYNC(i);
(3)  wait(done_i);
(4)  return(reg_i[1..m]).

operation write(r, v) is
(5)  done_i ← false;
(6)  scd_broadcast SYNC(i);
(7)  wait(done_i);
(8)  done_i ← false;
(9)  scd_broadcast WRITE(r, v, ⟨tsa_i[r].date + 1, i⟩);
(10) wait(done_i).

when the message set { WRITE(r_{j_1}, v_{j_1}, ⟨date_{j_1}, j_1⟩), ··· , WRITE(r_{j_x}, v_{j_x}, ⟨date_{j_x}, j_x⟩),
                       SYNC(j_{x+1}), ··· , SYNC(j_y) } is scd-delivered do
(11) for each r such that WRITE(r, −, −) ∈ scd-delivered message set do
(12)     let ⟨date, writer⟩ be the greatest timestamp in the messages WRITE(r, −, −);
(13)     if (tsa_i[r] <_ts ⟨date, writer⟩)
(14)       then let v the value in WRITE(r, −, ⟨date, writer⟩);
(15)            reg_i[r] ← v; tsa_i[r] ← ⟨date, writer⟩
(16)     end if
(17) end for;
(18) if ∃ℓ : j_ℓ = i then done_i ← true end if.
```

Algorithm 1: Construction of an MWMR snapshot object $\mathcal{CAMP}_{n,t}$[SCD-broadcast] (code for p_i)

Algorithm 1: write operation (Lines 5-10) When a process p_i wants to assign a value v to $REG[r]$, it invokes REG.write(r, v). This operation is made up of two parts. First p_i executes a re-synchronization (lines 5-7, exactly as in the snapshot operation) whose side effect is here to provide p_i with an up-to-date value of $tsa_i[r].date$. In the second part, p_i associates the timestamp $\langle tsa_i[r].date + 1, i\rangle$ with v, and invokes scd_broadcast WRITE$(r, v, \langle tsa_i[r].date + 1, i\rangle)$ (line 9). In addition to informing the other processes on its write of $REG[r]$, this message WRITE() acts as a re-synchronization message, exactly as a message SYNC(i). When this synchronization terminates (i.e., when the Boolean $done_i$ is set to `true`), p_i returns from the write operation (line 10).

Algorithm 1: scd-delivery of a set of messages When p_i scd-delivers a message set, namely,

$\{$ WRITE$(r_{j_1}, v_{j_1}, \langle date_{j_1}, j_1\rangle)$, $\cdots$, WRITE$(r_{j_x}, v_{j_x}, \langle date_{j_x}, j_x\rangle)$, SYNC$(j_{x+1})$, $\cdots$, SYNC(j_y) $\}$

it first looks if there are messages WRITE(). If it is the case, for each register $REG[r]$ for which there are

messages WRITE$(r, -, -)$ (line 11), p_i computes the maximal timestamp carried by these messages (line 12), and updates accordingly its local representation of $REG[r]$ (lines 13-15). Finally, if p_i is the sender of one of these messages (WRITE() or SYNC()), $done_i$ is set to `true`, which terminates p_i's re-synchronization (line 18).

Message cost An invocation of snapshot() involves one invocation of scd_broadcast(), and an invocation of write() involves two such invocations. It is shown in Appendix A that, in a message-passing system, scd_broadcast() costs $O(n^2)$ protocol messages. It follows that, in such systems, the message cost of both operations of a snapshot object is $O(n^2)$. (This remains true for SWMR snapshot objects, see Appendix B.)

4.2 Proof of Algorithm 1

As they are implicitly used in the proofs that follow, let us recall the properties of the SCD-broadcast abstraction. The non-faulty processes scd-deliver the same messages (exactly one each), and each of them was scd-broadcast. As a faulty process behaves correctly until it crashes, it scd-delivers a subset of the messages scd-delivered by the non-faulty processes.

Without loss of generality, we assume that there is an initial write operation issued by a non-faulty process. Moreover, if a process crashes in a snapshot operation, its snapshot is not considered; if a process crashes in a write operation, its write is considered only if the message WRITE() it sent at line 9 is scd-delivered to at least one non-faulty process (and by the Termination-2 property, at least to all non-faulty processes). Let us notice that a message SYNC() scd-broadcast by a process p_i does not modify the local variables of the other processes.

5 Proof of Lemmas for Theorem 1

Lemma 1 *If a non-faulty process invokes an operation, it returns from its invocation.*

Proof Let p_i be a non-faulty process that invokes a read or write operation. By the Termination-1 property of SCD-broadcast, it eventually receives a message set containing the message SYNC() or WRITE() it sends at line 2, 6 or 9. As all the statements associated with the scd-delivery of a message set (lines 11-18) terminate, it follows that the synchronization Boolean $done_i$ is eventually set to `true`. Consequently, p_i returns from the invocation of its operation. $\square_{Lemma\ 1}$

Extension of the relation $<_{ts}$ The relation $<_{ts}$ is extended to a partial order on arrays of timestamps, denoted $\leq_{tsa}$, defined as follows: $tsa1[1..m] \leq_{tsa} tsa2[1..m] \stackrel{def}{=} \forall r : (tsa1[r] = tsa2[r] \vee tsa1[r] <_{ts} tsa2[r])$. Moreover, $tsa1[1..m] <_{tsa} tsa2[1..m] \stackrel{def}{=} (tsa1[1..m] \leq_{tsa} tsa2[1..m]) \wedge (tsa1[1..m] \neq tsa2[1..m])$. **Definition** Let TSA_i be the set of the array values taken by $ts_i[1..m]$ at line 18 (end of the processing of a message set by process p_i). Let $TSA = \cup_{1 \leq i \leq n} TSA_i$.

Lemma 2 *The order $\leq_{tsa}$ is total on TSA.*

Proof Let us first observe that, for any i, all values in TSA_i are totally ordered (this comes from $ts_i[1..m]$ whose entries can only increase, lines 13 and 15). Hence, let $tsa1[1..m]$ be an array value of TSA_i, and $tsa2[1..m]$ an array value of TSA_j, where $i \neq j$.

Let us assume, by contradiction, that $\neg(tsa1 \leq_{tsa} tsa2)$ and $\neg(tsa2 \leq_{tsa} tsa1)$. As $\neg(tsa1 \leq_{tsa} tsa2)$, there is a registers r such that $tsa2[r] < tsa1[r]$. According to lines 13 and 15, there is a message WRITE$(r, -, tsa1[r])$ received by p_i when $tsa_i = tsa1$ and not received by p_j when $tsa_j = tsa2$ (because $tsa2[r] < tsa1[r]$). Similarly, there is a message WRITE$(r', -, tsa2[r'])$ received by p_j when $tsa_j = tsa2$ and not received by p_i when $tsa_i = tsa1$. This situation contradicts the MS-Ordering property, from which we conclude that either $tsa1 \leq_{tsa} tsa2$ or $tsa2 \leq_{tsa} tsa1$. $\square_{Lemma\ 2}$

Definitions Let us associate a timestamp $ts(\mathsf{write}(r, v))$ with each write operation as follows. Let p_i be the invoking process; $ts(\mathsf{write}(r, v))$ is the timestamp of v as defined by p_i at line 9, i.e., $\langle tsa_i[r].date + 1, i \rangle$.

Let op1 and op2 be any two operations. The relation $\prec$ on the whole set of operations is defined as follows: op1 $\prec$ op2 if op1 terminated before op2 started. It is easy to see that $\prec$ is a real-time-compliant partial order on all the operations.

Lemma 3 *No two distinct write operations on the same register* $\mathsf{write1}(r, v)$ *and* $\mathsf{write2}(r, w)$ *have the same timestamp, and* $(\mathsf{write1}(r, v) \prec \mathsf{write2}(r, w)) \Rightarrow (ts(\mathsf{write1}) <_{ts} ts(\mathsf{write2}))$.

Proof Let $\langle date1, i\rangle$ and $\langle date2, j\rangle$ be the timestamp of write1(r, v) and write2(r, w), respectively. If $i \neq j$, write1(r, v) and write2(r, w) have been produced by different processes, and their timestamp differ at least in their process identity.

So, let us consider that the operations have been issued by the same process p_i, with write1(r, v) first. As write1(r, v) precedes write2(r, w), p_i first invoked scd_broadcast WRITE$(r, v, \langle date1, i\rangle)$ (line 9) and later WRITE$(r, w, \langle date2, i\rangle)$. It follows that these SCD-broadcast invocations are separated by a local reset of the Boolean $done_i$ at line 16. Moreover, before the reset of $done_i$ due to the scd-delivery of the message $\{\cdots$,WRITE$(r, v, \langle date1, i\rangle), \cdots\}$, we have $tsa_i[r].date_i \geq date1$ (lines 12-16). Hence, we have $tsa_i[r].date \geq date1$ before the reset of $done_i$ (line 18). Then, due to the "+1" at line 9, WRITE$(r, w, \langle date2, i\rangle)$ is such that $date2 > date1$, which concludes the proof of the first part of the lemma.

Let us now consider that write1$(r, v) \prec$ write2(r, w). If write1(r, v) and write2(r, w) have been produced by the same process we have $date1 < date2$ from the previous reasoning. So let us assume that they have been produced by different processes p_i and p_j. Before terminating write1(r, v) (when the Boolean $done_i$ is set `true` at line 18), p_i received a message set $ms1_i$ containing the message WRITE$(r, v, \langle date1, i\rangle)$. When p_j executes write2(r, w), it first invokes scd_broadcast SYNC(j) at line 6. Because write1(r, v) terminated before write2(r, w) started, this message SYNC(j) cannot belong to $ms1_i$.

Due to Integrity and Termination-2 of SCD-broadcast, p_j eventually scd-delivers exactly one message set $ms1_j$ containing WRITE$(r, v, \langle date1, i\rangle)$. Moreover, it also scd-delivers exactly one message set $ms2_j$ containing its own message SYNC(j). On the the other side, p_i scd-delivers exactly one message set $ms2_i$ containing the message SYNC(j). It follows from the MS-Ordering property that, if $ms2_j \neq ms1_j$, p_j cannot scd-deliver $ms2_j$ before $ms1_j$. Then, whatever the case ($ms1_j = ms2_j$ or $ms1_j$ is scd-delivered at p_j before $ms2_j$), it follows from the fact that the messages WRITE() are processed (lines 11-17) before the messages SYNC(j) (line 18), that we have $tsa_j[r] \geq \langle date1, i\rangle$ when $done_j$ is set to `true`. It then follows from line 9 that $date2 > date1$, which concludes the proof of the lemma. $\square_{Lemma\ 3}$

Associating timestamp arrays with operations Let us associate a timestamp array $tsa(\mathsf{op})[1..m]$ with each operation op() as follows.

- Case op() = snapshot(). Let p_i be the invoking process; $tsa(\mathsf{op})$ is the value of $tsa_i[1..m]$ when p_i returns from the snapshot operation (line 4).
- Case op() = write(r, v). Let $\mathsf{min}_{tsa}(\{A\})$, where A is a set of array values, denote the smallest array value of A according to $<_{tsa}$. Let $tsa(\mathsf{op}) \stackrel{def}{=} \mathsf{min}_{tsa}(\{tsa[1..m] \in TSA$ such that $ts(\mathsf{op}) \leq_{ts} tsa[r]\})$. Hence, $tsa(\mathsf{op})$ is the first $tsa[1..m]$ of TSA, that reports the operation op() = write(r, v).

Lemma 4 *Let* op *and* op′ *be two distinct operations such that* op $\prec$ op′. *We have* $tsa(\mathsf{op}) \leq_{tsa} tsa(\mathsf{op}')$. *Moreover, if* op′ *is a write operation, we have* $tsa(\mathsf{op}) <_{tsa} tsa(\mathsf{op}')$.

Proof Let p_i and p_j be the processes that performed op and op′, respectively. Let SYNC$_j$ be the SYNC(j) message sent by p_j (at line 2 or 6) during the execution of op′. Let $term\_tsa_i$ be the value of $tsa_i[1..m]$ when op terminates (line 4 or 10), and $sync\_tsa_j$ the value of $tsa_j[1..m]$ when $done_j$ becomes true for the first time after p_j sent SYNC$_j$ (line 3 or 7). Let us notice that $term\_tsa_i$ and $sync\_tsa_j$ are elements of the set TSA.

According to lines 13 and 15, for all r, $tsa_i[r]$ is the largest timestamp carried by a message WRITE$(r, v, -)$ received by p_i in a message set before op terminates. Let m be a message such that there is a set sm scd-delivered by p_i before it terminated op. As p_j sent SYNC$_j$ after p_i terminated, p_i did not receive any set containing SYNC$_j$ before it terminated op. By the properties Termination-2 and MS-Ordering, p_j received message m in the same set as SYNC$_j$ or in a message set sm' received before the set containing SYNC$_j$. Therefore, we have $term\_tsa_i \leq_{tsa} sync\_tsa_j$.

If op is a snapshot operation, then $tsa(\mathsf{op}) = term\_tsa_i$. Otherwise, op() = write(r, v). As p_i has to wait until it processes a set of messages including its WRITE() message (and executes line 18), we have $ts(\mathsf{op}) <_{ts} term\_tsa_i[r]$. Finally, due to the fact that $term\_tsa_i \in TSA$ and Lemma 2, we have $tsa(\mathsf{op}) \leq_{tsa} term\_tsa_i$.

If op′ is a snapshot operation, then $sync\_tsa_j = tsa(\mathsf{op}')$ (line 4). Otherwise, op() = write(r, v) and thanks to the $+1$ in line 9, $sync\_tsa_j[r]$ is strictly smaller than $tsa(\mathsf{op}')[r]$ which, due to Lemma 2, implies $sync\_tsa_j <_{tsa} tsa(\mathsf{op}')$.

It follows that, in all cases, we have $tsa(\mathsf{op}) \leq_{tsa} term\_tsa_i \leq_{tsa} sync\_tsa_j \leq_{tsa} tsa(\mathsf{op}')$ and if op' is a write operation, we have $tsa(\mathsf{op}) \leq_{tsa} term\_tsa_i \leq_{tsa} sync\_tsa_j <_{tsa} tsa(\mathsf{op}')$, which concludes the proof of the lemma. $\square_{Lemma\ 4}$

The previous lemmas allow the operations to be linearized (i.e., totally ordered in an order compliant with both the sequential specification of a register, and their real-time occurrence order) according to a total order extension of the reflexive and transitive closure of the $\rightarrow_{lin}$ relation defined thereafter.

Definition 1 *Let* $\mathsf{op}, \mathsf{op}'$ *be two operations. We define the* $\rightarrow_{lin}$ *relation by* $\mathsf{op} \rightarrow_{lin} \mathsf{op}'$ *if one of the following properties holds:*

- $\mathsf{op} \prec \mathsf{op}'$,
- $tsa(\mathsf{op}) <_{tsa} tsa(\mathsf{op}')$,
- $tsa(\mathsf{op}) = tsa(\mathsf{op}')$, *op is a write operation and* op' *is a snapshot operation,*
- $tsa(\mathsf{op}) = tsa(\mathsf{op}')$, op *and* op' *are two write operations on the same register and* $ts(\mathsf{op}) <_{ts} ts(\mathsf{op}')$,

Lemma 5 *The snapshot object built by Algorithm* 1 *is linearizable.*

Proof We recall the definition of the $\rightarrow_{lin}$ relation: $\mathsf{op} \rightarrow_{lin} \mathsf{op}'$ if one of the following properties holds:

- $\mathsf{op} \prec \mathsf{op}'$,
- $tsa(\mathsf{op}) <_{tsa} tsa(\mathsf{op}')$,
- $tsa(\mathsf{op}) = tsa(\mathsf{op}')$, *op* is a write operation and op' is a snapshot operation,
- $tsa(\mathsf{op}) = tsa(\mathsf{op}')$, op and op' are two write operations on the same register and $ts(\mathsf{op}) <_{ts} ts(\mathsf{op}')$,

We define the $\rightarrow^{\star}_{lin}$ relation as the reflexive and transitive closure of the $\rightarrow_{lin}$ relation.

Let us prove that the $\rightarrow^{\star}_{lin}$ relation is a partial order on all operations. Transitivity and reflexivity are given by construction. Let us prove antisymmetry. Suppose there are $\mathsf{op}_0, \mathsf{op}_2, ..., \mathsf{op}_m$ such that $\mathsf{op}_0 = \mathsf{op}_m$ and $\mathsf{op}_i \rightarrow_{lin} \mathsf{op}_{i+1}$ for all $i < m$. By Lemma 4, for all $i < m$, we have $tsa(\mathsf{op}_i) \leq_{tsa} tsa(\mathsf{op}_{i+1})$, and $tsa(\mathsf{op}_m) = tsa(\mathsf{op}_0)$, so the timestamp array of all operations are the same. Moreover, if op_i is a snapshot operation, then $\mathsf{op}_i \prec \mathsf{op}_{(i+1)\%m}$ is the only possible case ($\%$ stands for "modulo") , and by Lemma 4 again, $\mathsf{op}_{(i+1)\%m}$ is a snapshot operation. Therefore, only two cases are possible.

- Let us suppose that all the op_i are snapshot operations and for all i, $\mathsf{op}_i \prec \mathsf{op}_{(i+1)\%m}$. As $\prec$ is a partial order relation, it is antisymmetric, so all the op_i are the same operation.
- Otherwise, all the op_i are write operations. By Lemma 4, for all $\mathsf{op}_i \not\prec \mathsf{op}_{(i+1)\%m}$. The operations op_i and $\mathsf{op}_{i+1\%m}$ are ordered by the fourth point, so they are write operations on the same register and $ts(\mathsf{op}_i) <_{ts} ts(\mathsf{op}_{i+1\%m})$. By antisymmetry of the $<_{ts}$ relation, all the op_i have the same timestamp, so by Lemma 3, they are the same operation, which proves antisymmetry.

Let $\leq_{lin}$ be a total order extension of $\rightarrow^{\star}_{lin}$. Relation $\leq_{lin}$ is real-time compliant because $\rightarrow^{\star}_{lin}$ contains $\prec$.

Let us consider a snapshot operation op and a register r such that $tsa(\mathsf{op})[r] = \langle date1, i\rangle$. According to line 10, it is associated to the value v that is returned by $\mathsf{read1}()$ for r, and comes from a WRITE$(r, v, \langle date1, i\rangle)$ message sent by a write operation $\mathsf{op}_r = \mathsf{write}(r, v)$. By definition of $tsa(\mathsf{op}_r)$, we have $tsa(\mathsf{op}_r) \leq_{tsa} tsa(\mathsf{op})$ (Lemma 4), and therefore $\mathsf{op}_r \leq_{lin} \mathsf{op}$. Moreover, for any different write operation op'_r on r, by Lemma 3, $ts(\mathsf{op}'_r) \neq ts(\mathsf{op}_r)$. If $ts(\mathsf{op}'_r) <_{ts} ts(\mathsf{op}_r)$, then $\mathsf{op}'_r \leq_{lin} \mathsf{op}_r$. Otherwise, $tsa(\mathsf{op}) <_{tsa} tsa(\mathsf{op}'_r)$, and (due to the first item of the definition of $\rightarrow_{lin}$) we have $\mathsf{op} \leq_{lin} \mathsf{op}'_r$. In both cases, the value written by op_r is the last value written on r before op, according to $\leq_{lin}$. $\square_{Lemma\ 5}$

Theorem 1 *Algorithm* 1 *builds an* MWMR *snapshot object in the system model* $\mathcal{CAMP}_{n,t}$[SCD-broadcast].

Proof The proof follows from Lemmas 1-5. $\square_{Theorem\ 1}$

6 From SWMR Snapshot to SCD-broadcast

This section presents an algorithm which builds the SCD-broadcast abstraction in $\mathcal{CARW}_{n,t}$[snapshot]. This algorithm completes the computational equivalence of snapshot and SCD-broadcast. (SWMR snapshot objects can be easily implemented in $\mathcal{CAMP}_{n,t}$[SCD-broadcast] by instantiating Algorithm 1 with $m = n$, and only allowing p_i to invoke REG.write$(r, -)$.)

6.1 Algorithm 2

Shared objects The shared memory is composed of two SWMR snapshot objects (as defined above). Let ϵ denote the empty sequence.

- $SENT[1..n]$: is a snapshot object, initialized to $[\emptyset, \cdots, \emptyset]$, such that $SENT[i]$ contains the messages scd-broadcast by p_i.
- $SETS\_SEQ[1..n]$: is a snapshot object, initialized to $[\epsilon, \cdots, \epsilon]$, such that $SETS\_SEQ[i]$ contains the sequence of the sets of messages scd-delivered by p_i.

The notation $\oplus$ is used for the concatenation of a message set at the end of a sequence of message sets.

Local objects Each process $p_i$manages the following local objects.

- $sent_i$ is a local copy of the snapshot object $SENT$.
- $sets\_seq_i$ is a local copy of the snapshot object $SETS\_SEQ$.
- $to\_deliver_i$ is an auxiliary variable whose aim is to contain the next message set that p_i has to scd-deliver.

The function members(set_seq) returns the set of all the messages contained in set_seq.

Description of Algorithm 2 When a process p_i invokes scd_broadcast(m), it adds m to $sent_i[i]$ and $SENT[i]$ to inform all the processes on the scd-broadcast of m. It then invokes the internal procedure progress() from which it exits once it has a set containing m (line 1).

A background task T ensures that all messages will be scd-delivered (line 2). This task invokes repeatedly the internal procedure progress(). As, locally, both the application process and the underlying task T can invoke progress(), which accesses the local variables of p_i, those variables are protected by a local fair mutual exclusion algorithm providing the operations enter_mutex() and exit_mutex() (lines 3 and 11).

```
operation scd_broadcast(m) is
(1) sent_i[i] ← sent_i[i] ∪ {m}; SENT.write(sent_i[i]); progress().

(2) background task T is repeat forever progress() end repeat.

procedure progress() is
(3)  enter_mutex();
(4)  catch_up();
(5)  sent_i ← SENT.snapshot();
(6)  to_deliver_i ← (∪_{1≤j≤n} sent_i[j]) \ members(sets_seq_i[i]);
(7)  if (to_deliver_i ≠ ∅) then sets_seq_i[i] ← sets_seq_i[i] ⊕ to_deliver_i;
(8)                             SETS_SEQ[i] ← sets_seq_i[i];
(9)                             scd_deliver(to_deliver_i)
(10) end if;
(11) exit_mutex().

procedure catch_up() is
(12) sets_seq_i ← SETS_SEQ.snapshot();
(13) while (∃j, set : set is the first set in sets_seq_i[j] : set ⊈ members(sets_seq_i[i])) do
(14)       to_deliver_i ← set \ members(sets_seq_i[i]);
(15)       sets_seq_i[i] ← sets_seq_i[i] ⊕ to_deliver_i; SETS_SEQ[i] ← sets_seq_i[i];
(16)       scd_deliver(to_deliver_i)
(17) end while.
```

Algorithm 2: An implementation of SCD-broadcast in $\mathcal{CARW}_{n,t}$[snapshot] (code for p_i)

The procedure progress() first invokes the internal procedure catch_up(), whose aim is to allow p_i to scd-deliver sets of messages which have been scd-broadcast and not yet locally scd-delivered.

To this end, catch_up() works as follows (lines 12-17). Process p_i first obtains a snapshot of $SETS\_SEQ$, and saves it in $sets\_seq_i$ (line 12). This allows p_i to know which message sets have been scd-delivered by all the processes; p_i then enters a "while" loop to scd-deliver as many message sets as possible according to what was scd-delivered by the other processes. For each process p_j that has scd-delivered a message set set containing messages not yet scd-delivered by p_i (predicate of line 13), p_i builds a set $to\_deliver_i$ containing the messages in set that it has not yet scd-delivered (line 14), and locally scd-delivers it (line 16). This local scd-delivery needs to update accordingly both $sets\_seq_i[i]$ (local update) and $SETS\_SEQ[i]$ (global update).

When it returns from catch_up(), p_i strives to scd-deliver messages not yet scd-delivered by the other processes. To this end, it first obtains a snapshot of $SENT$, which it stores in $sent_i$ (line 5). If there are messages that can be scd-delivered (computation of $to\_deliver_i$ at line 6, and predicate at line 7), p_i scd-delivers them and updates $sets\_seq_i[i]$ and $SETS\_SEQ[i]$ (lines 7-9) accordingly.

6.2 Proof of Algorithm 2

Lemma 6 *If a process scd-delivers a set containing a message m, some process invoked* scd_broadcast(m).

Proof The proof follows directly from the text of the algorithm, which copies messages from $SENT$ to $SETS\_SEQ$, without creating new messages. $\square_{Lemma\ 6}$

Lemma 7 *No process scd-delivers the same message twice.*

Proof Let us first observe that, due to lines 7 and 15, all messages that are scd-delivered at a process p_i have been added to $sets\_seq_i[i]$. The proof then follows directly from (a) this observation, (b) the fact that (due to the local mutual exclusion at each process) $sets\_seq_i[i]$ is updated consistently, and (c) lines 6 and 14, which state that a message already scd-delivered (i.e., a message belonging to $sets\_seq_i[i]$) cannot be added to $to\_deliver_i$. $\square_{Lemma\ 7}$

Lemma 8 *Any invocation of* scd_broadcast() *by a non-faulty process p_i terminates.*

Proof The proof consists in showing that the internal procedure progress() terminates. As the mutex algorithm is assumed to be fair, process p_i cannot block forever at line 3. Hence, p_i invokes the internal procedure catch_up(). It then issues first a snapshot invocation on $SETS\_SEQ$ and stores the value it obtains the value of $sets\_seq_i$. There is consequently a finite number of message sets in $sets\_seq_i$. Hence, the "while" of lines 13-17 can be executed only a finite number of times, and it follows that any invocation of catch_up() by a non-faulty process terminates. The same reasoning (replacing $SETS\_SEQ$ by $SENT$) shows that process p_i cannot block forever when it executes the lines 5-10 of the procedure progress(). $\square_{Lemma\ 8}$

Lemma 9 *If a non-faulty process scd-broadcasts a message m, it scd-delivers a message set containing m.*

Proof Let p_i be a non-faulty process that scd-broadcasts a message m. As it is non-faulty, p_i adds m to $SENT[i]$ and then invokes progress() (line 1). As $m \in SENT$, it is eventually added to $to\_deliver_i$ if not yet scd-delivered (line 6), and scd-delivered at line 9, which concludes the proof of the lemma. $\square_{Lemma\ 9}$

Lemma 10 *If a non-faulty process scd-delivers a message m, every non-faulty process scd-delivers a message set containing m.*

Proof Let us assume that a process scd-delivers a message set containing a message m. It follows that the process that invoked scd_broadcast(m) added m to $SENT$ (otherwise no process could scd-deliver m). Let p_i be a correct process. It invokes progress() infinitely often (line 2). Hence, there is a first execution of progress() such that $sent_i$ contains m (line 5). If then follows from line 6 that m will be added to $to\_deliver_i$ (if not yet scd-delivered). If follows that p_i will scd-deliver a set of messages containing m at line 9. $\square_{Lemma\ 10}$

Lemma 11 *Let p_i be a process that scd-delivers a set ms_i containing a message m and later scd-delivers a set ms'_i containing a message m'. No process p_j scd-delivers first a set ms'_j containing m' and later a set ms_j containing m.*

Proof Let us consider two messages m and m'. Due to total order property on the operations on the snapshot object $SENT$, it is possible to order the write operations of m and m' into $SENT$. Without loss of generality, let us assume that m is added to $SENT$ before m'. We show that no process scd-delivers m' before m.[5]

Let us consider a process p_i that scd-delivers the message m'. There are two cases.

- p_i scd-delivers the message m' at line 9. Hence, p_i obtained m' from the snapshot object $SENT$ (lines 5-6). As m was written in $SENT$ before m', we conclude that $SENT$ contains m. It then follows from line 6 that, if p_i has not scd-delivered m before (i.e., m is not in $sets\_seq_i[i]$), then p_i scd-delivers it in the same set as m'.

- p_i scd-delivers the message m' at line 16. Due to the predicate used at line 13 to build a set of message to scd-deliver, this means that there is a process p_j that has previously scd-delivered a set of messages containing m'.
  Moreover, let us observe that the first time the message m' is copied from $SENT$ to some $SETS\_SEQ[x]$ occurs at line 8. As m was written in $SENT$ before m', the corresponding process p_x cannot see m' and not m. It follows from the previous item that p_x has scd-delivered m in the same message set (as the one including m'), or in a previous message set. It then follows from the predicate of line 13 that p_i cannot scd-delivers m' before m.

  To summarize, the scd-deliveries of message sets in the procedure catch_up() cannot violate the MS-Ordering property, which is established at lines 6-10.

$\square_{Lemma\ 11}$

Theorem 2 *Algorithm* 2 *implements the* SCD-Broadcast *abstraction in the system model $\mathcal{CARW}_{n,t}[t < n]$.*

Proof The proof follows from Lemma 6 (Validity), Lemma 7 (Integrity), Lemmas 8 and 9 (Termination-1), Lemma 10 (Termination-2), and Lemma 11 (MS-Ordering). $\square_{Theorem\ 2}$

7 Conclusion

This paper has introduced a new communication abstraction (SCD-broadcast) providing processes with an abstraction level between reliable broadcast and total order broadcast (which captures the necessary and sufficient constraint on message deliveries which allows consensus objects to be implemented in asynchronous crash-prone message-passing systems).

More precisely, SCD-broadcast captures the abstraction level which is "necessary and sufficient" to implement read/write registers and snapshot objects on top of asynchronous message-passing systems prone to process failures. "Sufficient" means here that no other notion or object[6] is needed to build a register or a snapshot object at the abstraction level provided by SCD-broadcast, while "necessary" means that the objects that are built (registers and snapshot objects) are the weakest from a shared memory computational point of view.

As announced in the Introduction, an algorithm implementing SCD-broadcast in an asynchronous message-passing system where any minority of processes may crash is described in Appendix A. This algorithm requires $O(n^2)$ protocol messages per invocation of scd_broadcast(). It follows that the SCD-broadcast-based MWMR snapshot algorithm presented in the paper requires $O(n^2)$ protocol messages per invocation of snapshot() or write() operation. This is the best read/write snapshot algorithm we know in the context of asynchronous message-passing systems.

[5] Let us notice that it is possible that a process scd-delivers them in two different message sets, while another process scd-delivers them in the same set (which does not contradicts the lemma).

[6] The notion of intersecting quorums is neither provided by the abstraction level offered by SCD-broadcast, nor required –in addition to SCD-broadcast– to implement registers or snapshot objects. Actually, it is hidden and majority quorums appear only in the implementation of SCD-broadcast.

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A An Implementation of SCD-broadcast in Message-Passing Systems

This section shows that the SCD-broadcast communication abstraction is not an oracle-like object which allows us to extend our understanding of computing, but cannot be implemented. It describes an implementation of SCD-broadcast in $\mathcal{CAMP}_{n,t}[t < n/2]$, which is the weakest assumption on process failures that allows a read/write register to be built on top of an asynchronous message-passing system [5] (see footnote 4).

To simplify the presentation, and without loss of generality, we consider that the communication channels are FIFO. The associated communication operations are denoted fifo_broadcast() and fifo_deliver().

A.1 Algorithm 3

Local variables at a process p_i Each process p_i manages the following local variables.

- $buffer_i$: buffer where are stored the messages not yet scd-delivered in a message set.
- $to\_deliver_i$: next set of messages to be scd-delivered.
- sn_i: local sequence number (initialized to 0), which measures the local progress of p_i.
- $clock_i[1..n]$: array of sequence numbers. $clock_i[j]$ is the greatest sequence number x such that the application message identified by $\langle x, j\rangle$ was in a message set scd-delivered by p_i.

Operation scd_broadcast() When p_i invokes scd_broadcast(m), where m is an application message, it sends the message FORWARD(m, i, sn_i, i, sn_i) to itself (this simplifies the writing of the algorithm), and waits until it has no more message from itself pending in $buffer_i$, which means it has scd-delivered a set containing m.

A protocol message FORWARD() (line 1) is made up of five fields: the associated application message m, and two pairs, each made up of a sequence number and a process identity. The first pair (sd, sn) is the identity of the application message, while the second one (f, sn_f) is the local progress (sn_f) of the forwarder process p_f when it forwards this protocol message.

Reception of FORWARD$(m, sd, sn_{sd}, f, sn_f)$ When a process p_i receives such a protocol message, it first invokes forward$(m, sd, sn_{sd}, f, sn_f)$ to participate in the reliable broadcast of this message (line 3), and then invokes try_deliver() to see if a message set can be scd-delivered (line 4).

Procedure forward() This procedure can be seen as an enrichment (with the fields f and sn_f) of the reliable broadcast implemented by the messages FORWARD$(m, sd, sn_{sd}, -, -)$. Considering such a message FORWARD$(m, sd, sn_{sd}, f, sn_f)$, m was scd-broadcast by p_{sd} at its local time sn_{sd}, and relayed by the forwarding process p_f at its local time sn_f. If $sn_{sd} \leq clock_i[sd]$, p_i has already scd-delivered a message set containing m (see lines 18 and 20). If $sn_{sd} > clock_i[sd]$, there are two cases.

- The message m is not in $buffer_i$. In this case, p_i creates a quadruplet msg, and adds it to $buffer_i$ (lines 8-10). This quadruplet $\langle msg.m, msg.sd, msg.f, msg.cl\rangle$ is such that
  - the field $msg.m$ contains the application message m,
  - the field $msg.sd$ contains the id of the sender of this application message,
  - the field $msg.sn$ contains the local date associated with m by its sender,
  - the field $msg.cl$ is an array of size n, such that $msg.cl[x]$ = sequence number (initially $+\infty$) associated with m by p_x when it broadcast FORWARD$(msg.m, -, -, -, -)$. This last field is crucial in the scd-delivery of a message set containing m.

  After the quadruplet msg has been built, p_i first adds it to $buffer_i$ (line 10), and invokes (line 11) fifo_broadcast FORWARD$(m, sd, sn_{sd}, i, sn_i)$ to implement the reliable broadcast of m identified by $\langle sd, sn_{sd}\rangle$. Finally, p_i records its progress by increasing sn_i (line 12).
- There is a quadruplet msg in $buffer_i$ associated with m, i.e., $msg = \langle m, sd, -, -\rangle \in buffer_i$ (predicate of line 6). In this case, p_i assigns sn_f to $msg.cl[f]$ (line 7), thereby indicating that m was known and forwarded by p_f at its local time sn_f.

```
operation scd_broadcast(m) is
(1)  forward(m, i, sn_i, i, sn_i);
(2)  wait(∄ msg ∈ buffer_i : msg.sd = i).

 when the message FORWARD(m, sd, sn_sd, f, sn_f) is fifo-delivered do   % from p_f
(3)  forward(m, sd, sn_sd, f, sn_f);
(4)  try_deliver().

procedure forward(m, sd, sn_sd, f, sn_f) is
(5)  if (sn_sd > clock_i[sd])
(6)    then if (∃ msg ∈ buffer_i : msg.sd = sd ∧ msg.sn = sn_sd)
(7)           then msg.cl[f] ← sn_f
(8)           else threshold[1..n] ← [∞, ..., ∞]; threshold[f] ← sn_f;
(9)                let msg ← ⟨m, sd, sn_sd, threshold[1..n]⟩;
(10)               buffer_i ← buffer_i ∪ {msg};
(11)               fifo_broadcast FORWARD(m, sd, sn_sd, i, sn_i);
(12)               sn_i ← sn_i + 1
(13)       end if
(14) end if;

procedure try_deliver() is
(15) let to_deliver_i ← {msg ∈ buffer_i : |{f : msg.cl[f] < ∞}| > n/2};
(16) while (∃msg ∈ to_deliver_i, msg' ∈ buffer_i \ to_deliver_i : |{f : msg.cl[f] < msg'.cl[f]}| ≤ n/2) do
       to_deliver_i ← to_deliver_i \ {msg}
     end while;
(17) if (to_deliver_i ≠ ∅)
(18)   then for each (msg ∈ to_deliver_i such that clock_i[msg.sd] < msg.sn)
                 do clock_i[msg.sd] ← msg.sn end for;
(19)       buffer_i ← buffer_i \ to_deliver_i;
(20)       ms ← {m : ∃ msg ∈ to_deliver_i : msg.m = m}; scd_deliver(ms)
(21) end if.
```

Algorithm 3: An implementation of SCD-broadcast in $\mathcal{CAMP}_{n,t}[t < n/2]$ (code for p_i)

Procedure try_deliver() When it executes try_deliver(), p_i first computes the set $to\_deliver_i$ of the quadruplets msg containing application messages m which have been seen by a majority of processes (line 15). From p_i's point of view, a message has been seen by a process p_f if $msg.cl[f]$ has been set to a finite value (line 7).

If a majority of processes received first a message FORWARD$(m', -, -, -, -)$ and later another message FORWARD$(m, -, -, -, -)$, it might be that some process p_j scd-delivered a set containing m' before scd-delivering a set containing m. Therefore, p_i must avoid scd-delivering a set containing m before scd-delivering a set containing m'. This is done at line 16, where p_i withdraws the quadruplet msg corresponding to m if it has not enough information to deliver m' (i.e. the corresponding msg' is not in $to\_deliver_i$) or it does not have the proof that the situation cannot happen, i.e. no majority of processes saw the message corresponding to msg before the message corresponding to msg'.

If $to\_deliver_i$ is not empty after it has been purged (lines 16-17), p_i computes a message set to scd-deliver. This set ms contains all the application messages in the quadruplets of $to\_deliver_i$ (line 20). These quadruplets are withdrawn from $buffer_i$ (line 18). Moreover, before this scd-delivery, p_i needs to updates $clock_i[x]$ for all the entries such that $x = msg.sd$ where $msg \in to\_deliver_i$ (line 18). This update is needed to ensure that the future uses of the predicate of line 17 are correct.

A.2 Proof of Algorithm 3

Lemma 12 *If a process scd-delivers a set containing m, some process invoked* scd_broadcast(m).

Proof If process p_i scd-delivers a set containing a message m, it has previously added into $buffer_i$ a quadruplet msg such that $msg.m = m$ (line 10), for which it has fifo-received at least $\frac{n}{2}$ FORWARD$(m, -, -, -, -)$ messages. The first of these messages ever sent was sent after a process invoked scd_broadcast(m). $\square_{Lemma\ 12}$

Lemma 13 *No process scd-delivers the same message twice.*

Proof After a message m scd-broadcast by p_{sd} with a sequence number sn_{sd} is scd-delivered by p_i, $clock_i[sd] \geq sn_{sd}$ thanks to line 18 and there is no $msg \in buffer_i$ with $msg.sd = sd$ and $msg.sn = sn_{sd}$, as it was re-

moved on line 19. Thanks to line 5, no such msg' will be added again in $buffer_i$. As $to\_deliver_i$ is defined as a subset of $buffer_i$ on line 15, m will never be scd-delivered by p_i again. $\square_{Lemma\ 13}$

Lemma 14 *If a message* FORWARD$(m, sd, sn_{sd}, i, sn_i)$ *is broadcast by a non-faulty process* p_i*, then each non-faulty process* p_j *broadcasts a single message* FORWARD$(m, sd, sn_{sd}, j, sn_j)$.

Proof First, we prove that p_j broadcasts a message FORWARD$(m, sd, sn_{sd}, j, sn_j)$. As p_i is non-faulty, p_j will eventually receive the message sent by p_i. At that time, if $sn_{sd} > clock_j[sd]$, after the condition on line 6 and whatever its result, $buffer_i$ contains a value msg with $msg.sd = sd$ and $msg.sn_{sd} = sn_{sd}$. That msg was inserted at line 10 (possibly after the reception of a different message), just before p_j sent a message FORWARD$(m, sd, sn_{sd}, j, sn_j)$ at line 11. Otherwise, $clock_j[sd]$ was incremented on line 18, when validating some msg' added to $buffer_j$ after p_j received a (first) message FORWARD$(msg'.m, sd, sn_{sd}, f, clock_f[sd])$ from p_f. Because the messages FORWARD() are fifo-broadcast (hence they are delivered in their sending order), p_{sd} sent message FORWARD$(msg.m, sd, sn_{sd}, sd, sn_{sd})$ before FORWARD$(msg'.m, sd, clock_j[sd], sd, clock_j[sd])$, and all other processes only forward messages, p_j received a message FORWARD$(msg.m, sd, sn_{sd}, -, -)$ from p_f before the message FORWARD$(msg'.m, sd, clock_j[sd], -, -)$. At that time, $sn_{sd} > clock_j[sd]$, so the previous case applies.

After p_j broadcasts its message FORWARD$(m, sd, sn_{sd}, j, sn_j)$ on line 11, there is a $msg \in buffer_j$ with $ts(msg) = \langle sd, sn_{sd}\rangle$, until it is removed on line 16 and $clock_j[sd] \geq sn_{sd}$. Therefore, one of the conditions at lines 5 and 6 will stay false for the stamp $ts(msg)$ and p_j will never execute line 11 with the same stamp $\langle sd, sn_{sd}\rangle$ later. $\square_{Lemma\ 14}$

Lemma 15 *Let* p_i *be a process that scd-delivers a set* ms_i *containing a message* m *and later scd-delivers a set* ms'_i *containing a message* m'*. No process* p_j *scd-delivers first a set* ms'_j *containing* m' *and later a set* ms_j *containing* m*.*

Proof Let us suppose there are two messages m and m' and two processes p_i and p_j such that p_i scd-delivers a set ms_i containing m and later scd-delivers a set ms'_i containing m' and p_j scd-delivers a set ms'_j containing m' and later scd-delivers a set ms_j containing m.

When m is delivered by p_i, there is an element $msg \in buffer_i$ such that $msg.m = m$ and because of line 15, p_i has received a message FORWARD$(m, -, -, -, -)$ from more than $\frac{n}{2}$ processes.

- If there is no element $msg' \in buffer_i$ such that $msg'.m = m'$, since m' has not been delivered by p_i yet, p_i has not received a message FORWARD$(m', -, -, -, -)$ from any process (lines 10 and 19). Therefore, because the communication channels are FIFO, more than $\frac{n}{2}$ processes have sent a message FORWARD$(m, -, -, -, -)$ before sending a message FORWARD$(m', -, -, -, -)$.
- Otherwise, $msg' \notin to\_deliver_i$ after line 16. As the communication channels are FIFO, more than $\frac{n}{2}$ processes have sent a message FORWARD$(m, -, -, -, -)$ before a message FORWARD$(m', -, -, -, -)$.

Using the same reasoning, it follows that when m' is delivered by p_j, more than $\frac{n}{2}$ processes have sent a message FORWARD$(m', -, -, -, -)$ before sending a message FORWARD$(m, -, -, -, -)$. There exists a process p_k in the intersection of the two majorities, that has both sent a message FORWARD$(m', -, -, -, -)$ before sending FORWARD$(m, -, -, -, -)$ and sent a message FORWARD$(m', -, -, -, -)$ before sending FORWARD$(m, -, -, -, -)$. However, by Lemma 14, p_k can only send one message FORWARD$(m', -, -, -, -)$ and one message FORWARD$(m, -, -, -, -)$, which leads to a contradiction. $\square_{Lemma\ 15}$

Lemma 16 *If a message* FORWARD$(m, sd, sn_{sd}, i, sn_i)$ *is fifo-broadcast by a non-faulty process* p_i*, this process scd-delivers a set containing* m*.*

Proof Let p_i be a non-faulty process. For any pair of messages msg and msg' ever inserted in $buffer_i$, let $ts = ts(msg)$ and $ts' = ts(msg')$. Let $\rightarrow_i$ be the dependency relation defined as follows: $ts \rightarrow_i ts' \stackrel{def}{=} |\{j : msg'.cl[j] < msg.cl[j]\}| \leq \frac{n}{2}$ (i.e. the dependency does not exist if p_i knows that a majority of processes have seen the first update –due to msg'– before the second –due to msg–). Let $\rightarrow_i^\star$ denote the transitive closure of $\rightarrow_i$.

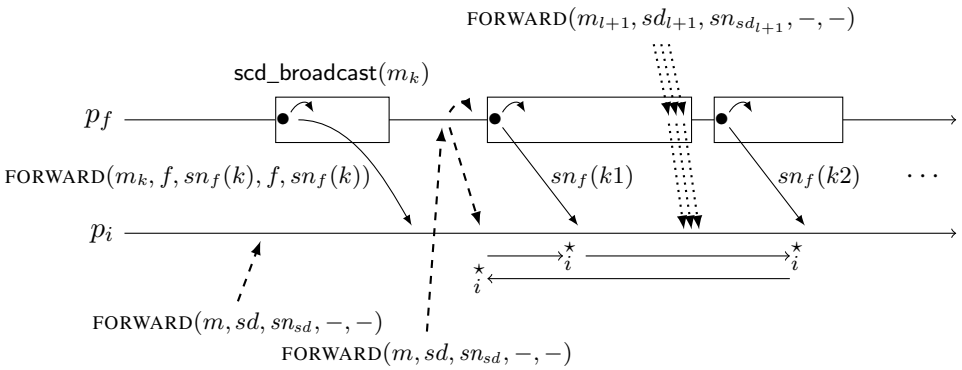

Figure 1: Message pattern introduced in Lemma 16

Let us suppose (by contradiction) that the timestamp $\langle sd, sn_{sd}\rangle$ associated with the message m (carried by the protocol message FORWARD$(m, sd, sn_{sd}, i, sn_i)$ fifo-broadcast by p_i), has an infinity of predecessors according to $\rightarrow_i^\star$. As the number of processes is finite, an infinity of these predecessors have been generated by the same process, let us say p_f. Let $\langle f, sn_f(k)\rangle_{k\in\mathbb{N}}$ be the infinite sequence of the timestamps associated with the invocations of the scd_broadcast() issued by p_f. The situation is depicted by Figure 1.

As p_i is non-faulty, p_f eventually receives a message FORWARD$(m, sd, sn_{sd}, i, sn_i)$, which means p_f broadcast an infinity of messages FORWARD$(m(k), f, sn_f(k), f, sn_f(k))$ after FORWARD$(m, sd, sn_{sd}, f, sn_f)$. Let $\langle f, sn_f(k1)\rangle$ and $\langle f, sn_f(k2)\rangle$ be the timestamps associated with the next two messages sent by p_f, with $sn_f(k1) < sn_f(k2)$. By hypothesis, we have $\langle f, sn_f(k2)\rangle \rightarrow_i^\star \langle sd, sn_{sd}\rangle$. Moreover, all processes received their first message FORWARD$(m, sd, sn_{sd}, -, -)$ before their first message FORWARD$(m(k), f, sn_f(k), -, -)$, so $\langle sd, sn_{sd}\rangle \rightarrow_i^\star \langle f, sn_f(k1)\rangle$. Let us express the path $\langle f, sn_f(k2)\rangle \rightarrow_i^\star \langle f, sn_f(k1)\rangle$:
$\langle f, sn_f(k2)\rangle = \langle sd'(1), sn'(1)\rangle \rightarrow_i \langle sd'(2), sn'(2)\rangle \rightarrow_i \cdots \rightarrow_i \langle sd(m), sn'(m)\rangle = \langle f, sn_f(k1)\rangle$.

In the time interval starting when p_f sent the message FORWARD$(m(k1), f, sn_f(k1), f, sn_f(k1))$ and finishing when it sent the the message FORWARD$(m(k2), f, sn_f(k2), f, sn_f(k2))$, the waiting condition of line 2 became true, so p_f scd-delivered a set containing the message $m(k1)$, and according to Lemma 12, no set containing the message $m(k2)$. Therefore, there is an index l such that process p_f delivered sets containing messages associated with a timestamp $\langle sd'(l), sn'(l)\rangle$ for all $l' > l$ but not for $l' = l$. Because the channels are FIFO and thanks to lines 15 and 16, it means that a majority of processes have sent a message FORWARD$(-, sd'(l+1), sn'(l+1), -, -)$ before a message FORWARD$(-, sd'(l), sn'(l), -, -)$, which contradicts the fact that $\langle sd'(l), sn'(l)\rangle \rightarrow_i \langle sd'(l+1), sn'(l+1)\rangle$.

Let us suppose a non-faulty process p_i has fifo-broadcast a message FORWARD$(m, sd, sn_{sd}, i, sn_i)$ (line 10). It inserted a quadruplet msg with timestamp $\langle sd, sn_{sd}\rangle$ on line 9 and by what precedes, $\langle sd, sn_{sd}\rangle$ has a finite number of predecessors $\langle sd_1, sn_1\rangle, \ldots, \langle sd_l, sn_l\rangle$ according to $\rightarrow_i^\star$. As p_i is non-faulty, according to Lemma 14, it eventually receives a message FORWARD$(-, sd_k, sn_k, -, -)$ for all $1 \leq k \leq l$ and from all non-faulty processes, which are in majority.

Let $pred$ be the set of all quadruplets msg' such that $\langle msg'.sd, msg'.sn_{sd}\rangle \rightarrow_i^\star \langle sd, sn_{sd}\rangle$. Let us consider the moment when p_i receives the last message FORWARD$(-, sd_k, sn_k, f, sn_f)$ sent by a correct process p_f. For all $msg' \in pred$, either $msg'.m$ has already been delivered or msg' is inserted $to\_deliver_i$ on line 15. Moreover, no $msg' \in pred$ will be removed from $to\_deliver_i$, on line 16, as the removal condition is the same as the definition of $\rightarrow_i$. In particular for $msg' = msg$, either m has already been scd-delivered or m is present in $to\_deliver_i$ on line 17 and will be scd-delivered on line 20. $\square_{Lemma\ 16}$

Lemma 17 *If a non-faulty process scd-broadcasts a message m, it scd-delivers a message set containing m.*

Proof If a non-faulty process scd-broadcasts a message m, it sends a message FORWARD$(m, i, sn_{sd}, i, sn_{sd})$ on line 11, so it scd-delivers a message set containing m by lemma 16. $\square_{Lemma\ 17}$

Lemma 18 *If a non-faulty process scd-delivers a message m, every non-faulty process scd-delivers a message set containing m.*

Proof Suppose a non-faulty process p_i scd-delivers a message m. At line 20, there is $msg \in to\_deliver_i$ such that $msg.m = m$. At line 15, $msg \in buffer_i$, and msg was inserted in $buffer_i$ at line 10, just before p_i sent message FORWARD$(m, sd, sn_{sd}, i, sn_i)$. By Lemma 14, every non-faulty process p_j sends a message FORWARD$(m, sd, sn_{sd}, j, sn_j)$, so by Lemma 16, p_j scd-delivers a message set containing m. $\square_{Lemma\ 18}$

Theorem 3 *Algorithm* 3 *implements the* SCD-broadcast *communication abstraction in* $\mathcal{CAMP}_{n,t}[t < n/2]$. *Moreover, it requires* $O(n^2)$ *messages per invocation of* scd_broadcast().

Proof The proof follows from Lemma 12 (Validity), Lemma 13 (Integrity), Lemma 15 (MS-Ordering), Lemma 17 (Termination-1), and Lemma 18 (Termination-2).

The $O(n^2)$ message complexity comes from the fact that, due to the predicates of line 5 and 6, each application message m is forwarded at most once by each process (line 11). $\square_{Theorem\ 3}$

The next corollary follows from (i) Theorems 1 and 3, and (ii) the fact that the constraint $(t < n/2)$ is an upper bound on the number of faulty processes to build a read/write register (or snapshot object) [5].

Corollary 1 *Algorithm* 3 *is resiliency optimal.*

B Building an MWMR atomic register on top of $\mathcal{CAMP}_{n,t}$[SCD-broadcast]

This appendix shows the genericity dimension of Algorithm 1. It presents trivial simplifications of it, which build MWMR atomic registers and MWMR sequentially consistent registers.

B.1 The algorithm

Let REG denote the MWMR atomic read/write register that is built. The algorithm that builds it is a trivial simplification of the snapshot Algorithm 1, namely its projection on a single MWMR atomic register.

REG is now locally represented by a local variable reg_i and the associated timestamp ts_i initialized to $\langle 0, - \rangle$. The message sent at Line 9 is now WRITE$(v, \langle ts_i.date_i + 1, i \rangle)$, and the predicate of line 11 simplifies to "there are messages WRITE())".

B.2 Proof of the algorithm

The proof is a simplified version of the proof of Theorem 1. For self-completeness, we give here its full proof even if some parts of it are "cut-and-paste" of parts of proofs given in Section 4.2. As in that section, let us associate a timestamp $ts(\mathsf{op})$ with each operation $\mathsf{op}()$ as follows (this is the place where the proof is simplified with respect to a snapshot object).

- Case $\mathsf{op}() = \mathsf{write}(v)$. Let p_i be the invoking process; $ts(\mathsf{op})$ is the timestamp of v as defined by p_i at line 9, i.e., $\langle ts_i.date + 1, i \rangle$.
- Case $\mathsf{op}() = \mathsf{read}()$. Let w be the value returned by the read; $ts(\mathsf{op})$ is then the timestamp associated with w at line 15 by its writer.

Let op1 and op2 be any two operations. The relation $\prec$ on the whole set of operations is defined as follows: op1 $\prec$ op2 if op1 terminated before op2 started. It is easy to see that $\prec$ is a real-time-compliant partial order on all the operations.

The reader can easily check that the statement and the proof of Lemma 1 (applied to the termination of read and write operations), and Lemma 3 (applied to the total order on the write operations, compliant with both the sequential specification of a register, and their real-time occurrence order) remain valid for the algorithm suited to an MWMR atomic read/write register. The next lemma addresses the read operations (which are simpler to manage than snapshot operations).

Lemma 19 *The read/write register* REG *is linearizable.*

Proof Let us now insert each read operation in the previous (real time compliant) total order as follows.

Let read1() be a read operation whose timestamp is $\langle date1, i \rangle$. This operation is inserted just after the write operation write1() that has the same timestamp (this write wrote the value read by read1()). Let us remark that, as read1() obtained the value timestamped $\langle date1, i \rangle$, it did not terminate before write1() started.

It follows that the insertion of read1() into the total order cannot violate the real-time order between write1() and read1().

Let us consider the operation write2() that follows write1() in the write total order. If read1() $\prec$ write2(), the placement of read1() in the total order is real-time-compliant. If $\neg$(read1() $\prec$ write2()), due to the timestamp obtained by read1(), we cannot have write2() $\prec$ read1(). It follows that in this case also, the placement of read1() in the total order is real-time-compliant.

Finally, let us consider two read operations read1() and read2() which have the same timestamp $\langle date, i\rangle$ (hence, they read from the same write operation, say write1()). Both are inserted after write1() in the order of their invocations (if read1() and read2() started simultaneously, they are inserted according to the order on the identities of the processes that invoked them). Hence, the read and write operations are linearizable, which concludes the proof of the lemma. $\square_{Lemma\ 19}$

Theorem 4 *The read/write register REG is an* MWMR *atomic read/write register.*

Proof The proof follows from Lemma 1, Lemma 3, and Lemma 19. $\square_{Theorem\ 4}$

B.3 The case of an SWMR atomic register

When the register REG can be written by a single process (say p_k), the algorithm simplifies. The timestamps disappear at all processes, and as only the writer p_k can invoke REG.write(), it manages a simple date $date_k$ (which is actually a sequence number). The modifications are:

- Line 9 becomes: $date_k \leftarrow date_k + 1$; scd_broadcast WRITE$(v, date_k)$.
- The lines 11-17 become:

```
if (there are messages WRITE())
  then let date be the maximal date in the messages WRITE() received;
       reg_i ← the value associated with date
end if.
```

Let us remark that, due to the Boolean $done_k$, the writer p_k scd-delivers message sets containing at most one message WRITE().

B.4 On sequentially consistency

The case of an MWMR sequentially consistent register As indicated in the Introduction, sequential consistency was introduced in [24]. It is atomicity minus the requirement stating that "if an operation $op1$ terminates before an operation $op2$ starts, then $op1$ must appear before $op2$ in the sequence of the read and write operations". As noticed in [31], sequential consistency can be seen as a weakened form of atomicity, namely lazy linearizability. The composition of sequentially consistent registers is investigated in [30]. The algorithm for sequential consistency presented in [30] and Algorithm 3 are based on similar principles. The constraint $(t < n/2)$ is also a necessary and sufficient condition to implement a sequentially consistent read/write register in $\mathcal{CAMP}_{n,t}[\emptyset]$.

The reader can check that an algorithm building a a sequentially consistent MWMR read/write register can easily be obtained from Algorithm 1 as simplified in Section B.1. One only needs to suppress the synchronization messages SYNC() which ensure the compliance with respect to real-time. The concerned lines are lines 1-3 (read synchronization), and lines 5-7 (write synchronization). In a simple way, this shows the versatility dimension of Algorithm 1.

From sequential consistency to atomicity Given a sequentially consistent snapshot object, Algorithm 2 builds the SCD-broadcast communication abstration. (As the reader can check, this follows from the fact that, when looking at its proof, this algorithm relies only on the fact that the operations on the snapshot object can be totally ordered.) Hence, using on top of it the SCD-broadcast-based Algorithm 1, we obtain an atomic snapshot object. It follows that, thanks to SCD-broadcast, the algorithms presented in the paper allow a sequentially consistent snapshot object to be transformed into an atomic snapshot object (and it is known that –differently from sequential consistent objects– atomic objects are composable for free [18]).