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Abstract

We propose the kl-UCB^{++} algorithm for regret minimization in stochastic bandit models with exponential families of distributions. We prove that it is simultaneously asymptotically optimal (in the sense of Lai and Robbins' lower bound) and minimax optimal. This is the first algorithm proved to enjoy these two properties at the same time. This work thus merges two different lines of research, with simple proofs involving no complexity overhead.

Keywords: Stochastic multi-armed bandits, regret analysis, upper confidence bound (UCB), minimax optimality, asymptotic optimality.

1. Introduction

For regret minimization in stochastic bandit problems, two notions of time-optimality coexist. On the one hand, one may consider a fixed model: the famous lower bound by Lai and Robbins (1985) showed that the regret of any consistent strategy should grow at least as $C(\mu)\log(T)\big(1-o(1)\big)$ when the horizon T goes to infinity. Here, $C(\mu)$ is a constant depending solely on the model. A strategy with a regret upper-bounded by $C(\mu)\log(T)\big(1-o(1)\big)$ will be called in this paper *asymptotically-optimal*. Lai and Robbins provided a first example of such a strategy in their seminal work. Later, Garivier and Cappé (2011) and Maillard et al. (2011) provided finite-time analysis for variants of the UCB algorithm (see Agrawal (1995); Katehakis and Robbins (1995); Auer et al. (2002a)) which imply asymptotic optimality. Since then, other algorithms like Bayes-UCB (Kaufmann et al., 2012) and Thompson Sampling (Korda et al., 2013) have also joined the family.

On the other hand, for a fixed horizon T one may assess the quality of a strategy by the greatest regret suffered in all possible bandit models. If the regret of a bandit strategy is upper-bounded by $C'\sqrt{KT}$ (the optimal rate: see Auer et al. (2002b) and Cesa-Bianchi and Lugosi (2006)) for some numeric constant C', this strategy is called *minimax-optimal*. The MOSS strategy by Audibert and Bubeck (2009) was the first proved to be minimax-optimal.

Hitherto, as far as we know, no algorithm was proved to be *at the same time* asymptotically- and minimax-optimal. Two limited exceptions may be mentioned: the case of two Gaussian arms is treated in Garivier et al. (2016a); and the Optimally Confident UCB algorithm of Lattimore (2015) is proved to be minimax-optimal and almost problem-dependent optimal for Gaussian multi-armed bandit problems.

Contributions. In this work, we put forward the kl-UCB^{++} algorithm, a slightly modified version of kl-UCB^{+} algorithm discussed in Garivier et al. (2016a) as an empirical improvement of UCB, and analyzed in Kaufmann (2016). This bandit strategy is designed for some exponential distribution families, including for example Bernoulli and Gaussian laws. We prove that it is at the same time asymptotically- and minimax-optimal. This work thus merges the progress which has been made in different directions towards the understanding of the optimism principle, finally reconciling the two notions of time-optimality. We present a general yet simple proof, using and improving the best elements of the above-cited sources without any additional complexity. To this end, we develop new deviation inequalities, improving the analysis of the different terms contributing to the regret.

The paper is organized as follows. In Section 2, we introduce the setting and assumptions required for the main results, Theorems 1 and 2, which are presented in Section 3. We give the entire proofs of these results in Sections 4 and 5, with only a few technical lemmas proved in Appendix A. We conclude in Section 6 with some brief references to possible future prospects.

2. Notation and Setting

Exponential families. We consider a simple stochastic bandit problem with K arms indexed by $a \in \{1, \dots, K\}$, with $K \geqslant 2$. Each arm is assumed to be a probability distribution of some canonical one-dimensional exponential family ν_θ indexed by $\theta \in \Theta$. The probability law ν_θ is assumed to be absolutely continuous with respect to a dominating measure ρ on $\mathbb{R}$, with a density given by

$$\frac{\mathrm{d}\nu_\theta}{\mathrm{d}\rho}(x) = \exp(x\theta - b(\theta)), \quad \text{where } b(\theta) = \log \int_{\mathbb{R}} e^{x\theta} \mathrm{d}\rho(x) \text{ and } \Theta = \left\{\theta \in \mathbb{R} : \ b(\theta) < +\infty\right\}.$$

It is well-known that b is convex, two times differentiable on Θ and that $b'(\theta) = E(\nu_\theta)$ and $b''(\theta) = V(\nu_\theta)$, respectively the mean and the variance of the distribution ν_θ. The family can thus be parametrized by the mean $\mu = b'(\theta)$, for $\mu \in I = b'(\Theta) := (\bar{\mu}^-, \bar{\mu}^+)$. The Kullback-Leibler divergence between two distributions is $\mathrm{KL}(\nu_\theta, \nu_{\theta'}) = b(\theta') - b(\theta) - b'(\theta)(\theta' - \theta)$. This permits to define the following divergence on the set of arm expectations: if $\mu = E(\nu_\theta)$ and $\mu' = E(\nu_{\theta'})$ then

$$\mathrm{kl}(\mu, \mu') := \mathrm{KL}(\nu_\theta, \nu_{\theta'}) = b^*(\mu) - b^*(\mu') - b^{*\prime}(\mu')(\mu - \mu'),$$

where b^* is the Fenchel conjugate of b. For a minimax analysis, we need to restrict the set of means to bounded interval: we suppose that each arm ν_θ satisfies $\mu = b'(\theta) \in [\mu-, \mu+] \subset I$ for two fixed real numbers μ^+, μ^-. Our analysis requires some kind of Pinsker inequality; we therefore assume that the variance is bounded in the exponential family: there exists $V > 0$ such that

$$\sup_{\mu \in I} b''\big(b'^{-1}(\mu)\big) = \sup_{\mu \in I} V\big(\nu_{b'^{-1}(\mu)}\big) \leqslant V < +\infty \,.$$

This implies that for all $\mu, \mu' \in I$,

$$d(\mu, \mu') \geqslant \frac{1}{2V}(\mu - \mu')^2 \,. \tag{1}$$

In the sequel, we denote by $\mathcal{F}$ the set of bandit problems ν satisfying these assumptions. This setting includes in particular Bernoulli bandits, with $V = 1/4$ and $\mathrm{kl}(\mu, \mu') = \mu \log(\mu/\mu') + (1 - \mu) \log\big((1 - \mu)/(1 - \mu')\big)$ (by convention, $0 \log 0 = 0 \log 0/0 = 0$). This also includes (bounded) Gaussian bandits with known variance σ^2, with the choice $V = \sigma^2$ and $\mathrm{kl}(\mu, \mu') = (\mu - \mu')^2/(2\sigma^2)$.

Regret. At each round $1 \leqslant t \leqslant T$, the player pulls an arm A_t and receives an independent draw Y_t of the distribution ν_{A_t}. This reward is the only piece of information available to the player. The best mean is $\mu^\star = \max_{a=1,\dots,K} \mu_a$. We denote by $N_a(T) = \sum_{t=1}^T \mathbb{I}_{\{A_t=a\}}$ the number of draws of arm a up to and including time T. In this work, the goal is to minimize the *expected regret*

$$R_T = T\mu^\star - \mathbb{E}\left[\sum_{t=1}^T Y_t\right] = \mathbb{E}\left[\sum_{t=1}^T \big(\mu^\star - \mu_{A_t}\big)\right] = \sum_{a=1}^K \big(\mu^* - \mu_a\big)\,\mathbb{E}\big[N_a(T)\big]\,.$$

Lai and Robbins (1985) proved if a strategy is uniformly efficient, that is if it is such that under any bandit model of a sufficiently rich family (such as an exponential family described above) $R_T = o(T^\alpha)$ for every $\alpha > 0$, then it needs to draw any suboptimal arm a at least as often as

$$\mathbb{E}\big[N_a(T)\big] \geqslant \frac{\log(T)}{\mathrm{kl}(\mu_a, \mu^*)}\,\big(1 - o(1)\big)\,.$$

In light of the previous equality, this directly implies an asymptotic lower bound on $R_T/\log(T)$.

On the other side, a straightforward adaptation of the the proof of Theorem A.2 of Auer et al. (2002b) shows that there exists a constant C' depending only on the considered family $\mathcal{F}$ of distributions such that

$$\sup_{\nu\in\mathcal{F}} R_T \geqslant C' \min\big(\sqrt{KT}, T\big)\,,$$

where the supremum is taken over all bandit problems ν in $\mathcal{F}$. Note that the notion of minimax-optimality is defined here up to a multiplicative constant, in contrast to the definition of (problem-dependent) asymptotic optimality. For a discussion on the minimax and asymptotic lower bounds, we refer to Garivier et al. (2016b) and references therein.

3. The KL-UCB^{++} Algorithm

We denote by $\widehat{\mu}_{a,n}$ the empirical mean of the first n rewards from arm a. The empirical mean of arm a after t rounds is

$$\widehat{\mu}_a(t) = \widehat{\mu}_{a,N_a(t)} = \frac{1}{N_a(t)} \sum_{s=1}^t Y_s\,\mathbb{I}_{\{A_s=a\}}\,.$$

The KL-UCB^{++} algorithm is a slight modification of algorithm KL-UCB$^+$ of Garivier and Cappé (2011) and of the KL-UCB-H$^+$ analyzed in Kaufmann (2016). It uses the exploration function g given by

$$g(n) = \log_+\!\left(\frac{T}{Kn}\left(\log_+^2\!\left(\frac{T}{Kn}\right) + 1\right)\right), \tag{2}$$

where $\log_+(x) := \max\big(\log(x), 0\big)$.

Initialization: Pull each arm of $\{1,..,K\}$ once.
For $t = K$ to $T-1$, **do**

1. Compute for each arm a the quantity

$$U_a(t) = \sup\left\{\mu \in I \,:\, \mathrm{kl}\big(\widehat{\mu}_a(t), \mu\big) \leqslant \frac{g\big(N_a(t)\big)}{N_a(t)}\right\}. \tag{3}$$

2. Play $A_{t+1} \in \arg\max_{a\in\{1,..,K\}} U_a(t)$.

The following results state that the Kl-UCB^{++} algorithm is simultaneously minimax- and asymptotically-optimal.

Theorem 1 (Minimax optimality) *For any family $\mathcal{F}$ satisfying the assumptions detailed in Section 2, and for any bandit model $\nu \in \mathcal{F}$, the expected regret of the Kl-UCB^{++} algorithm is upper-bounded as*

$$R_T \leqslant 76\sqrt{VKT} + (\mu^+ - \mu^-)K\,. \tag{4}$$

Theorem 2 (Asymptotic optimality) *For any bandit model $\nu \in \mathcal{F}$, for any suboptimal arm a and any δ such that $22VK/T \leqslant \delta^2 \leqslant (\mu^* - \mu_a)^2/9$,*

$$\mathbb{E}\big[N_a(T)\big] \leqslant \frac{\log(T)}{\mathrm{kl}(\mu_a + \delta, \mu^* - \delta)} + O_\delta\big(\,\mathrm{loglog}(T)\big) \tag{5}$$

(see the end of the proof in Section 5 for an explicit bound). Therefore,

$$\limsup_{T\to\infty} \frac{\mathbb{E}_a\big[N_a(T)\big]}{\log(T)} \leqslant \frac{1}{\mathrm{kl}(\mu_a, \mu^*)}\,. \tag{6}$$

Theorem 1 and 2 are proved in Section 4 and 5 respectively. The main differences between the two proofs are discussed at the beginning of Section 5.

4. Proof of Theorem 1

This proof merges merges ideas presented in Bubeck and Liu (2013) for the analysis of the MOSS algorithm and from the analysis of kl-UCB in Cappé et al. (2013) (see also Kaufmann (2016)). It is divided into the following steps:

Decomposition of the regret. Let a^* be the index of an optimal arm. Since by definition of the strategy $U_{a^*}(t) \leqslant U_{A_{t+1}}(t)$ for all $t \geqslant K-1$, the regret can be decomposed as follows:

$$R_T \leqslant K(\mu^+ - \mu^-) + \underbrace{\sum_{t=K}^{T-1} \mathbb{E}\big[\mu^* - U_{a^*}(t)\big]}_{A} + \underbrace{\sum_{t=K}^{T-1} \mathbb{E}\big[U_{A_{t+1}}(t) - \mu_{A_{t+1}}\big]}_{B}\,. \tag{7}$$

We define $\delta_0 = \sqrt{22VK/T}$; since the bound (4) is otherwise trivial, we assume in the sequel that $\delta_0 \leqslant 1$.

Step 1: Upper-bounding term A. Term A is concerned with the optimal arm a^* only. Two words of intuition: since $U_{a^*}(t)$ is meant to be an upper confidence bound for μ^*, this term should not be too large, at least as long as the the confidence level controlled by function g is large enough – but when the confidence level is low, the number of draws is large and deviations are unlikely.

Upper-bounding term A boils down to controlling the probability that μ^* is under-estimated at time t. Indeed,

$$\mathbb{E}\big[\mu^* - U_{a^*}(t)\big] \leqslant \delta_0 + \int_{\delta_0}^{+\infty} \mathbb{P}\big(U_{a^*}(t) \leqslant \mu^* - u\big)\mathrm{d}u\,, \tag{8}$$

and we need to upper bound the left-deviations of the mean of arm a^*. On the event $\{U_{a^*}(t) \leqslant \mu^* - u\}$, we have that $\widehat{\mu}_{a^*}(t) \leqslant U_{a^*}(t) \leqslant \mu^* - u < \mu^*$, and by definition of $U_{a^*}(t)$ it holds that

$$\mathrm{kl}\big(\widehat{\mu}_{a^*}(t), \mu^*\big) \geqslant \frac{g\big(N_{a^*}(t)\big)}{N_{a^*}(t)}\,.$$

Consequently,

$$\begin{aligned}\mathbb{P}\big(U_{a^*}(t) \leqslant \mu^* - u\big) &\leqslant \mathbb{P}\Big(\widehat{\mu}_{a^*}(t) \leqslant \mu^* - u \text{ and } \mathrm{kl}\big(\widehat{\mu}_{a^*}(t), \mu^*\big) \geqslant g\big(N_{a^*}(t)\big)/N_{a^*}(t)\Big) \\ &\leqslant \mathbb{P}\big(\exists 1 \leqslant n \leqslant T,\ \widehat{\mu}_{a^*,n} \leqslant \mu^* - u \text{ and } \mathrm{kl}(\widehat{\mu}_{a^*,n}, \mu^*) \geqslant g(n)/n\big)\,. \end{aligned} \tag{9}$$

For small values of n, the dominant term is given by $\mathrm{kl}(\widehat{\mu}_{a^*,n}, \mu^*) \geqslant g(n)/n$, whereas for large n the event $\widehat{\mu}_{a^*,n} \leqslant \mu^* - u$ is quite unlikely. This is why we split the probability in two terms, proceeding as follows. Let f be the function defined, for $u \geqslant \delta_0$, by

$$f(u) = \frac{2V}{u^2}\log\!\left(\frac{Tu^2}{2VK}\right)\,.$$

By definition of δ_0, $f(u)K/T \leqslant \exp(-3/2)$, and thus

$$f(u) < \frac{T}{K} \qquad \text{and} \qquad \log\!\left(\frac{T}{Kf(u)}\right) \geqslant 3/2\,. \tag{10}$$

In particular, for $n \leqslant f(u)$ it holds that

$$g(n) = \log\!\left(\frac{T}{Kn}\left(1 + \log^2\!\left(\frac{T}{Kn}\right)\right)\right)\,.$$

It appears that $f(u)$ is the right place where to split the probability of Equation (9): defining $\mathrm{kl}_+(p,q) := \mathrm{kl}(p,q)\mathbb{I}_{\{p\leqslant q\}}$, we write

$$\begin{aligned}&\mathbb{P}\big(\exists 1 \leqslant n \leqslant T,\ \widehat{\mu}_{a^*,n} \leqslant \mu^* - u \text{ and } \mathrm{kl}(\widehat{\mu}_{a^*,n}, \mu^*) \geqslant g(n)/n\big) \leqslant \\ &\qquad \underbrace{\mathbb{P}\big(\exists 1 \leqslant n \leqslant f(u),\ \mathrm{kl}_+(\widehat{\mu}_{a^*,n}, \mu^*) \geqslant g(n)/n\big)}_{A_1} + \underbrace{\mathbb{P}\big(\exists f(u) \leqslant n \leqslant T,\ \widehat{\mu}_{a^*,n} \leqslant \mu^* - u\big)}_{A_2}\,. \end{aligned} \tag{11}$$

Controlling terms A_1 and A_2 is a matter of deviation inequalities.

Step 1.1: Upper-bounding A_1. The self-normalized probability A_1 can be upper-bounded thanks

to a 'peeling trick'. We assume that $f(u) \geqslant 1$, for otherwise $A_1 = 0$. We use the grid $f(u)/\beta^{l+1} \leqslant n \leqslant f(u)/\beta^l$, where the real $\beta > 1$ will be chosen later. We write

$$A_1 \leqslant \sum_{l=0}^{+\infty} \underbrace{\mathbb{P}\left(\exists \frac{f(u)}{\beta^{l+1}} \leqslant n \leqslant \frac{f(u)}{\beta^l},\ \ \mathrm{kl}_+(\widehat{\mu}_{a^*,n}, \mu^*) \geqslant \gamma_l \right)}_{A_1^l}, \tag{12}$$

where

$$\gamma_l = \frac{\log\left(\frac{T\beta^l}{Kf(u)} \left(1 + \log^2\left(\frac{T}{Kf(u)} \right) \right) \right)}{f(u)/\beta^l}.$$

Thanks to Doob's maximal inequality (see Lemma 4 in Appendix A),

$$A_1^l \leqslant \exp\left(-\frac{1}{\beta} \log\left(\frac{T\beta^l}{Kf(u)} \left(1 + \log^2\left(\frac{T}{Kf(u)} \right) \right) \right) \right) = e^{-l\log(\beta)/\beta - C/\beta},$$

where

$$C := \log\left(\frac{T}{Kf(u)} \left(1 + \log^2\left(\frac{T}{Kf(u)} \right) \right) \right). \tag{13}$$

Plugging this last inequality into (12), together with the numerical inequality of Lemma 3 (see Appendix A), we get

$$\begin{aligned} A_1 \leqslant \sum_{l=0}^{+\infty} e^{-l\log(\beta)/\beta - C/\beta} &\leqslant \frac{1}{1 - e^{-\log(\beta)/\beta}} e^{-C/\beta} \\ &\leqslant \frac{e}{e^{\log(\beta)/\beta} - 1} e^{-C/\beta} \leqslant 2e \max\big(\beta, \beta/(\beta-1)\big) e^{-C/\beta}. \end{aligned}$$

But thanks to Equation (10),

$$C = \log\left(\frac{T}{Kf(u)} \left(1 + \log^2\left(\frac{T}{Kf(u)} \right) \right) \right) \geqslant \log\left(\frac{T}{Kf(u)} \right) \geqslant \frac{3}{2}.$$

It is now time to choose $\beta := C/(C-1)$, so that $\beta \leqslant 2C$ and $\beta/(\beta-1) = C$. This choice yields

$$A_1 \leqslant 4e^2 C e^{-C} = 4e^2 \frac{\log\left(\frac{T}{Kf(u)} \left(1 + \log^2\left(\frac{T}{Kf(u)} \right) \right) \right)}{1 + \log^2\left(\frac{T}{Kf(u)} \right)} \frac{Kf(u)}{T}, \tag{14}$$

and therefore

$$A_1 \leqslant 4e^2 \frac{Kf(u)}{T} = \frac{16e^2 VK}{Tu^2} \log\left(\sqrt{\frac{T}{2VK}} u \right) \tag{15}$$

as, for all $x \geqslant 1$,

$$\frac{\log\Big(x\big(1+\log^2(x)\big)\Big)}{1+\log^2(x)} \leqslant 1\,.$$

Step 1.2: Upper-bounding A_2. The term A_2 is more simple to handle, as it does not involve self-normalized deviations. Thanks to the maximal inequality recalled in Equation (34) of Appendix A,

$$A_2 \leqslant e^{-u^2 f(u)/2V} = \frac{2VK}{Tu^2}\,. \tag{16}$$

Putting Equations (8) to (16) together, we obtain that

$$\mathbb{E}\big[\mu^* - U_{a^*}(t)\big] \leqslant \delta_0 + \int_{\delta_0}^{+\infty} \frac{16e^2VK}{Tu^2}\log\left(\sqrt{\frac{T}{2VK}}u\right) + \frac{2VK}{Tu^2}\mathrm{d}u\,. \tag{17}$$

It remains only to conclude with some calculus:

$$\begin{aligned}\int_{\delta_0}^{+\infty} \frac{16e^2VK}{Tu^2}\log\left(\sqrt{\frac{T}{2VK}}u\right)\mathrm{d}u &= \left[-\frac{16e^2VK}{Tu}\log\left(e\sqrt{\frac{T}{2VK}}u\right)\right]_{\delta_0}^{+\infty} \\ &= \frac{16e^2\sqrt{V}}{\sqrt{22}}\log\big(e\sqrt{11}\big)\sqrt{\frac{K}{T}}\,.\end{aligned}$$

Similarly,

$$\int_{\delta_0}^{+\infty} \frac{2VK}{Tu^2}\mathrm{d}u = 2\sqrt{\frac{V}{22}}\sqrt{\frac{K}{T}}\,,$$

and replacing δ_0 by its value we obtain from Equation (17) the following relation:

$$\mathbb{E}\big[\mu^* - U_{a^*}(t)\big] \leqslant \sqrt{V}\left(\sqrt{22} + \frac{16e^2}{\sqrt{22}}\log\big(e\sqrt{11}\big) + \frac{2}{\sqrt{22}}\right)\sqrt{\frac{K}{T}}\,.$$

Summing over t from K to $T-1$, this yields:

$$A \leqslant \sqrt{V}\left(\sqrt{22} + \frac{16e^2}{\sqrt{22}}\log\big(e\sqrt{11}\big) + \frac{2}{\sqrt{22}}\right)\sqrt{KT}\,. \tag{18}$$

Step 2: Upper-bounding term B**.** Term B is of different nature, since typically $U_{A_{t+1}}(t) > \mu_{A_{t+1}}$. However, as for the term A, we first reduce the problem to the upper-bounding of a probability:

$$\begin{aligned}B &\leqslant \sum_{t=K}^{T-1}\delta_0 + \int_{\delta_0}^{+\infty}\mathbb{P}\big(U_{A_{t+1}}(t) - \mu_{A_{t+1}} \geqslant u\big)\mathrm{d}u \\ &\leqslant T\delta_0 + \int_{\delta_0}^{+\infty}\sum_{t=K}^{T-1}\mathbb{P}\big(U_{A_{t+1}}(t) - \mu_{A_{t+1}} \geqslant u\big)\mathrm{d}u\,.\end{aligned} \tag{19}$$

The event $\big\{U_{A_{t+1}}(t) - \mu_{A_{t+1}} \geqslant u\big\}$ is typical if $N_{A_{t+1}}(t)$ is small, and corresponds to a deviation of the sample mean otherwise. In order to handle this correctly, we first get rid of the randomness of $N_{A_{t+1}}(t)$ by the pessimistic trajectorial upper bound

$$\sum_{t=K}^{T-1} \mathbb{I}_{\left\{U_{A_{t+1}} - \mu_{A_{t+1}} \geqslant u\right\}} \leqslant \sum_{n=1}^{T} \sum_{a=1}^{K} \mathbb{I}_{\left\{U_{a,n} - \mu_a \geqslant u\right\}} \,.$$

In addition, we simplify the upper-bound thanks to our assumption (1) that some Pinsker type inequality is available:

$$U_{a,n} \leqslant B_{a,n} := \widehat{\mu}_{a,n} + \sqrt{2V \frac{g(n)}{n}} \,. \tag{20}$$

Hence, B can be upper-bounded as

$$B \leqslant T\delta_0 + \sum_{a=1}^{K} \int_{\delta_0}^{+\infty} \sum_{n=1}^{T} \mathbb{P}(B_{a,n} - \mu_a \geqslant u) \mathrm{d}u \,. \tag{21}$$

Then, we need only to upper bound $\sum_{n=1}^{T} \mathbb{P}(B_{a,n} - \mu_a \geqslant u)$ for each arm $a \in \{1, \dots, K\}$. We cut the sum at the critical sample size $n(u)$ where the event $\big\{B_{a,n} - \mu_a > u\big\}$ becomes atypical: for $u \geqslant \delta_0$, let $n(u)$ be the integer such that

$$n(u) = \left\lceil \frac{8V}{u^2} \log\!\left(\frac{Tu^2}{8VK}\right) \right\rceil \,.$$

For $n \geqslant n(u)$ it holds that

$$\sqrt{2V \frac{g(n)}{n}} \leqslant \frac{u}{\sqrt{2}} \,. \tag{22}$$

Indeed, as $\log(1 + x^2) \leqslant x$ for all $x \geqslant 0$, we have

$$2V \frac{g(n)}{n} \leqslant \frac{4V}{n} \log_+\!\left(\frac{T}{Kn}\right) \,.$$

Observe that $h(x) := \log\big(x/\log(x)\big)/\log(x)$ is such that $h(x) \leqslant 1$ for $x \geqslant 11/4$, and thus for $n \geqslant n(u)$ and $u \geqslant \delta_0$

$$2V \frac{g(n)}{n} \leqslant \frac{4V}{n(u)} \log_+\!\left(\frac{T}{Kn(u)}\right) \leqslant \frac{u^2}{2} h\left(\frac{Tu^2}{8VK}\right) \leqslant \frac{u^2}{2} \,.$$

Therefore, cutting the sum in (21) at $n(u)$, we obtain:

$$\begin{aligned}
\sum_{n=1}^{T} \mathbb{P}(B_{a,n} - \mu_a \geqslant u) &\leqslant n(u) - 1 + \sum_{n=n(u)}^{T} \mathbb{P}\big(\widehat{\mu}_{a,n} - \mu_a \geqslant u - \sqrt{g(n)/n}\big) \\
&\leqslant n(u) - 1 + \sum_{n=n(u)}^{T} \mathbb{P}\Big(\widehat{\mu}_{a,n} - \mu_a \geqslant u\big(1 - 1/\sqrt{2}\big)\Big) \\
&\leqslant \frac{8V}{u^2} \log\!\left(\frac{Tu^2}{8VK}\right) + \sum_{n=n(u)}^{T} \mathbb{P}(\widehat{\mu}_{a,n} - \mu_a \geqslant cu) \,,
\end{aligned} \tag{23}$$

where $c := 1 - 1/\sqrt{2}$. It remains to integrate Inequality (23) from $u = \delta_0$ to $u = 1$. The first summand involves the same integral as we have already met in the upper bound of term A_1:

$$\int_{\delta_0}^{+\infty} \frac{8V}{u^2} \log\left(\frac{Tu^2}{8VK}\right) \mathrm{d}u = 16\sqrt{\frac{V}{22}} \log\left(e\sqrt{\frac{11}{4}}\right) \sqrt{\frac{T}{K}}\,.$$

For the remaining summand, Inequality (34) yields

$$\sum_{n=n(u)}^{T} \mathbb{P}(\widehat{\mu}_{a,n} - \mu_a \geqslant cu) \leqslant \sum_{n=n(u)}^{T} e^{-\frac{u^2c^2n}{2V}} \leqslant \frac{1}{e^{\frac{u^2c^2}{2V}} - 1}\,.$$

Thus, as $e^x - 1 \geqslant x$ for all $x \geqslant 0$,

$$\int_{\delta_0}^{+\infty} \frac{1}{e^{\frac{u^2c^2}{2V}} - 1} \mathrm{d}u \leqslant \int_{\delta_0}^{+\infty} \frac{2V}{u^2c^2} \mathrm{d}u = \frac{2}{c^2}\sqrt{\frac{V}{22}}\sqrt{\frac{T}{K}}\,,$$

Putting everything together starting from Inequality (23), we have proved that

$$\int_{\delta_0}^{+\infty} \sum_{n=1}^{T} \mathbb{P}(B_{a,n} - \mu_a \geqslant u) \mathrm{d}u \leqslant \sqrt{\frac{V}{22}} \left(16 \log\left(e\sqrt{\frac{11}{4}}\right) + \frac{2}{c^2}\right) \sqrt{\frac{T}{K}}\,.$$

By Equation (21), replacing δ_0 by its value finally yields

$$B \leqslant \sqrt{V}\left(\sqrt{22} + \frac{16}{\sqrt{22}} \log\left(e\sqrt{\frac{11}{4}}\right) + \frac{2}{\sqrt{22}c^2}\right)\sqrt{KT}\,. \tag{24}$$

Conclusion of the proof. It just remains to plug Inequalities (18) and (24) into Equation (7):

$$\begin{aligned} A + B &\leqslant \sqrt{V}\left(2\sqrt{22} + \frac{16e^2}{\sqrt{22}} \log\left(e\sqrt{11}\right) + \frac{2}{\sqrt{22}} + \frac{16}{\sqrt{22}} \log\left(e\sqrt{\frac{11}{4}}\right) + \frac{2}{\sqrt{22}c^2}\right)\sqrt{KT} \\ &\leqslant 76\sqrt{VKT}\,, \end{aligned}$$

which concludes the proof.

5. Proof of Theorem 2

The analysis of asymptotic optimality shares many elements with the minimax analysis, with some differences however. The decomposition of the regret into two terms A and B is similar, but localized on a fixed sub-optimal arm $a \in \{1, \dots, K\}$: we analyze the number of draws of a and not directly the regret (and we do not need to integrate the deviations at the end). We proceed roughly as in the proof of Theorem 1 for term A, which involves the deviations of an optimal arm. For term B, which stands for the behavior of the sub-optimal arm a, a different (but classical) argument is used, as one cannot simply use the Pinsker-like Inequality (1) if one wants to obtain the correct constant (and thus asymptotic optimality).

Decomposition of $\mathbb{E}\big[N_a(T)\big]$. If arm a is pulled at time $t+1$, then by definition of the strategy $U_{a^*}(t) \leqslant U_a(t)$ for any index a^* of an optimal arm. Thus,

$$\begin{aligned}\big\{A_{t+1}=a\big\} &\subseteq \big\{\mu^*-\delta \geqslant U_a(t)\big\} \cup \big\{\mu^*-\delta < U_a(t) \text{ and } A_{t+1}=a\big\} \\ &\subseteq \big\{\mu^*-\delta \geqslant U_{a^*}(t)\big\} \cup \big\{\mu^*-\delta < U_a(t) \text{ and } A_{t+1}=a\big\}\,.\end{aligned}$$

As a consequence,

$$\mathbb{E}\big[N_a(T)\big] \leqslant 1 + \underbrace{\sum_{t=K}^{T-1} \mathbb{P}\big(U_{a^*}(t) \leqslant \mu^*-\delta\big)}_{\text{A}} + \underbrace{\sum_{t=K}^{T-1} \mathbb{P}\big(\mu^*-\delta < U_a(t) \text{ and } A_{t+1}=a\big)}_{\text{B}}\,, \tag{25}$$

and it remains to bound each of these terms.

Step 1: Upper-bounding term A. As in the proof of Theorem 1, we write

$$\begin{aligned}&\mathbb{P}\big(U_{a^*}(t) \leqslant \mu^*-\delta\big) \leqslant \\ &\qquad \underbrace{\mathbb{P}\big(\exists 1 \leqslant n \leqslant f(\delta),\ \ \mathrm{kl}_+(\widehat{\mu}_{a^*,n}, \mu^*) \geqslant g(n)/n\big)}_{A_1} + \underbrace{\mathbb{P}\big(\exists f(\delta) \leqslant n \leqslant T,\ \ \widehat{\mu}_{a^*,n} \leqslant \mu^*-\delta\big)}_{A_2}\,,\end{aligned} \tag{26}$$

where we use the same function

$$f(\delta) = \frac{2V}{\delta^2}\log\!\left(\frac{T\delta^2}{2KV}\right)\,.$$

Thanks to the Inequality (14) that we saw in the proof of Theorem 1, we obtain that

$$A_1 \leqslant 4e^2 \frac{\log\!\left(\frac{T}{Kf(\delta)}\left(1+\log^2\!\left(\frac{T}{Kf(\delta)}\right)\right)\right)}{\log\!\left(\frac{T}{Kf(\delta)}\right)} \frac{f(\delta)}{\log\!\left(\frac{T}{Kf(\delta)}\right)} \frac{K}{T} \leqslant \frac{16e^2}{\delta^2}\frac{2VK}{T}\,.$$

Here, we used that for all $x \geqslant 1$

$$\frac{\log\Big(x\big(1+\log^2(x)\big)\Big)}{\log(x)} \leqslant 2 \qquad \text{and} \qquad \frac{\log(x)}{\log\big(x/\log(x)\big)} \leqslant 2\,,$$

and that

$$\frac{f(\delta)}{\log\!\left(\frac{T}{Kf(\delta)}\right)} = \frac{2V}{\delta^2} \frac{\log\!\left(\frac{T\delta^2}{2VK}\right)}{\log\!\left(\frac{T\delta^2}{2VK}\frac{1}{\log\left(T\delta^2/(2VK)\right)}\right)}\,.$$

Thanks to the maximal inequality recalled in Appendix A as Equation (34), it holds that

$$A_2 \leqslant e^{-\delta^2 f(\delta)/(2V)} = \frac{2VK}{T\delta^2}\,. \tag{27}$$

Putting Equations (26) to (27) together yields:

$$A \leqslant (16e^2+1)\frac{2VK}{\delta^2}\,. \tag{28}$$

Step 2: Upper-bounding term B. Thanks to the definition of $U_a(t)$ it holds that

$$\big\{\mu^* - \delta < U_a(t) \text{ and } A_{t+1} = a\big\} \subseteq \Big\{\mathrm{kl}\big(\widehat{\mu}_a(t), \mu^* - \delta\big) \leqslant g\big(N_a(t)\big)/N_a(t) \text{ and } A_{t+1} = a\Big\}$$

Together with the following classical bandit reasoning, this yields:

$$\begin{aligned} B &\leqslant \sum_{t=K}^{T-1} \mathbb{P}\big(\mathrm{kl}\big(\widehat{\mu}_a(t), \mu^* - \delta\big) \leqslant g\big(N_a(t)\big)/N_a(t) \text{ and } A_{t+1} = a\big) \\ &\leqslant \sum_{n=1}^{T} \mathbb{P}\big(\mathrm{kl}(\widehat{\mu}_{a,n}, \mu^* - \delta) \leqslant g(n)/n\big) \\ &\leqslant \sum_{n=1}^{T} \mathbb{P}\Big(\mathrm{kl}(\widehat{\mu}_{a,n}, \mu^* - \delta) \leqslant \log\Big(T/K\big(1+\log^2(T/K)\big)\Big)/n\Big)\,, \end{aligned} \tag{29}$$

as function g is non-increasing. Now, let $n(\delta)$ be the integer defined as

$$n(\delta) = \left\lceil \frac{\log\Big(T/K\big(1+\log^2(T/K)\big)\Big)}{\mathrm{kl}(\mu_a + \delta, \mu^* - \delta)} \right\rceil\,.$$

Recall that by assumption $\delta < (\mu^* - \mu_a)/3$. Then, for $n \geqslant n(\delta)$,

$$\log\Big(T/K\big(1+\log^2(T/K)\big)\Big)/n \leqslant \mathrm{kl}(\mu_a + \delta, \mu^* - \delta)\,.$$

We cut the sum in (29) at $n(\delta)$, so that

$$\begin{aligned} B &\leqslant n(\delta) - 1 + \sum_{n=n(\delta)}^{T} \mathbb{P}\big(\mathrm{kl}(\widehat{\mu}_{a,n}, \mu^* - \delta) \leqslant \mathrm{kl}(\mu_a + \delta, \mu^* - \delta)\big) \\ &\leqslant \frac{\log\Big(T/K\big(1+\log^2(T/K)\big)\Big)}{\mathrm{kl}(\mu_a + \delta, \mu^* - \delta)} + \sum_{n=n(\delta)}^{T} \mathbb{P}\big(\mathrm{kl}(\widehat{\mu}_{a,n}, \mu^* - \delta) \leqslant \mathrm{kl}(\mu_a + \delta, \mu^* - \delta)\big)\,. \end{aligned} \tag{30}$$

Using the inclusion

$$\big\{\mathrm{kl}(\widehat{\mu}_{a,n}, \mu^* - \delta) \leqslant \mathrm{kl}(\mu_a + \delta, \mu^* - \delta)\big\} \subseteq \{\widehat{\mu}_{a,n} \geqslant \mu_a + \delta\}\,,$$

together with Inequality (34), we obtain that

$$\begin{aligned} \sum_{n=n(\delta)}^{T} \mathbb{P}\big(\mathrm{kl}(\widehat{\mu}_{a,n}, \mu^* - \delta) \leqslant \mathrm{kl}(\mu_a + \delta, \mu^* - \delta)\big) \;&\leqslant\; \sum_{n=n(\delta)}^{T} \mathbb{P}\big(\widehat{\mu}_{a,n} \geqslant \mu_a + \delta\big) \\ &\leqslant \sum_{n=1}^{\infty} e^{-n\delta^2/(2V)} = \frac{1}{e^{\delta^2/(2V)} - 1} \leqslant \frac{2V}{\delta^2}\,, \end{aligned}$$

and Equation (30) yields

$$B \leqslant \frac{\log(T)}{\mathrm{kl}(\mu_a + \delta, \mu^* - \delta)} + \frac{\log\Big(1/K\big(1 + \log^2(T/K)\big)\Big)}{\mathrm{kl}(\mu_a + \delta, \mu^* - \delta)} + \frac{2V}{\delta^2}\,. \tag{31}$$

Conclusion of the proof. It just remains to plug Inequalities (28) and (31) into Equation (25):

$$\mathbb{E}\big[N_a(T)\big] \leqslant \frac{\log(T)}{\mathrm{kl}(\mu_a + \delta, \mu^* - \delta)} + \frac{\log\Big(1/K\big(1 + \log^2(T/K)\big)\Big)}{\mathrm{kl}(\mu_a + \delta, \mu^* - \delta)} + (16e^2 + 2)\frac{2VK}{\delta^2} + 1\,,$$

which concludes the proof.

6. Conclusion and Perspectives

We have proved that the kl-UCB^{++} algorithm is both minimax- and asymptotically-optimal for the exponential distribution families described in Section 2. So far, this algorithm requires the horizon T as a parameter: to keep the proofs clear and simple, we have deferred to future work the analysis of an anytime variant. We believe, though, that obtaining such an extension should be possible by using the tools developed in Degenne and Perchet (2016). Similarly, concerning the distribution-dependent bounds, we have focused in this paper on asymptotic optimality without trying to derive explicit finite-time bounds: we believe that this would have impaired the clarity of the reasoning, while adding very little value.

From a more technical point of view, it may be possible to suppress the extra $\log^2$ exploration term in the definition of the confidence bonus g in Equation (2). This is carried out in Garivier et al. (2016a) using some particularities of the Gaussian distributions; using an improved Chernoff bound such as Talagrand (1995) may allow considering more general cases. Finally, we defer the consideration of general bounded probability distributions (with non-parametric upper-confidence bounds) to future work.

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Appendix A. Some Technical Lemmas

Lemma 3 *For all $\beta > 1$ we have*

$$\frac{1}{e^{\log(\beta)/\beta} - 1} \leqslant 2 \max\big(\beta, \beta/(\beta-1)\big)\,. \tag{32}$$

Proof Inequality (32) is equivalent to

$$e^{\log(\beta)/\beta} - 1 \geqslant \frac{1}{2\beta} \min(1, \beta - 1)\,.$$

If $\beta \geqslant 2$, then

$$e^{\log(\beta)/\beta} - 1 \geqslant e^{\log(2)/\beta} - 1 \geqslant \frac{\log(2)}{\beta} \geqslant \frac{1}{2\beta}\,.$$

Otherwise, if $1 < \beta < 2$, as the function $\beta \mapsto \log(\beta)/(\beta-1)$ is non-increasing one gets

$$\frac{\beta}{\beta-1}\Big(e^{\log(\beta)/\beta} - 1\Big) \geqslant \frac{\log(\beta)}{\beta - 1} \geqslant \log(2) \geqslant 1/2\,.$$

■

Lemma 4 *(Maximal Inequality) Let N and M be two real numbers in $\mathbb{R}^+ \times \overline{\mathbb{R}^+}$, let γ be a real number in $\mathbb{R}^{+*}$, and let $\widehat{\mu}_n$ be the empirical mean of n random variables i.i.d. according to the distribution $\nu_{b'^{-1}(\mu)}$. Then*

$$\mathbb{P}\big(\exists N \leqslant n \leqslant M,\ \ \mathrm{kl}_+(\widehat{\mu}_n, \mu) \geqslant \gamma\big) \leqslant e^{-N\gamma}\,. \tag{33}$$

Proof If $\gamma > \mathrm{kl}(\bar{\mu}^-, \mu)$ or $\widehat{\mu}_n \geqslant \mu$ the Inequality (33) is trivial. Else, there exist two real numbers $z < \mu$ and $\lambda < 0$ such that

$$\gamma = \mathrm{kl}(z, \mu) = \lambda z - \varphi_\mu(\lambda)\,,$$

where φ_μ denotes the the log-moment generating function of $\nu_{b'^{-1}(\mu)}$. Since on the event $\big\{\exists N \leqslant n \leqslant M,\ \ \mathrm{kl}_+(\widehat{\mu}_n, \mu) \geqslant \gamma\big\}$ one has at the same time

$$\widehat{\mu}_n \leqslant \mu\,, \quad \lambda\widehat{\mu}_n - \varphi_\mu(\lambda) \geqslant \lambda z - \varphi_\mu(\lambda) = \gamma \quad \text{and} \quad \lambda n\widehat{\mu}_n - n\varphi_\mu(\lambda) \geqslant N\gamma\,,$$

we can write that

$$\begin{aligned}\mathbb{P}\big(\exists N \leqslant n \leqslant M,\ \ \mathrm{kl}_+(\widehat{\mu}_n, \mu) \geqslant \gamma\big) &\leqslant \mathbb{P}\big(\exists N \leqslant n \leqslant M,\ \ \lambda n\widehat{\mu}_n - n\varphi_\mu(\lambda) \geqslant N\gamma\big) \\ &\leqslant \exp(-N\gamma)\,,\end{aligned}$$

by Doobs maximal inequality for the exponential martingale $\exp\big(\lambda n\widehat{\mu}_n - n\varphi_\mu(\lambda)\big)$. ■

As a simple consequence of this Lemma 4 and Inequality (1), it holds that:

$$\text{for every } x \leqslant \mu, \qquad \mathbb{P}(\exists N \leqslant n \leqslant M,\ \ \widehat{\mu}_n \leqslant x) \leqslant e^{-N(x-\mu)^2/(2V)}\,, \tag{34}$$

$$\text{for every } x \geqslant \mu, \qquad \mathbb{P}(\exists N \leqslant n \leqslant M,\ \ \widehat{\mu}_n \geqslant x) \leqslant e^{-N(x-\mu)^2/(2V)}\,. \tag{35}$$