Embeddability into relational lattices is undecidable ⋆

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Abstract. The natural join and the inner union operations combine relations of a database. Tropashko and Spight realized that these two operations are the meet and join operations in a class of lattices, known by now as the relational lattices. They proposed then lattice theory as an algebraic approach to the theory of databases alternative to the relational algebra. Litak et al. proposed an axiomatization of relational lattices over the signature that extends the pure lattice signature with a constant and argued that the quasiequational theory of relational lattices over this extended signature is undecidable.
We prove in this paper that embeddability is undecidable for relational lattices. More precisely, it is undecidable whether a finite subdirectly-irreducible lattice can be embedded into a relational lattice. Our proof is a reduction from the coverability problem of a multimodal frame by a universal product frame and, indirectly, from the representability problem for relation algebras.
As corollaries we obtain the following results: the quasiequational theory of relational lattices over the pure lattice signature is undecidable and has no finite base; there is a quasiequation over the pure lattice signature which holds in all the finite relational lattices but fails in an infinite relational lattice.

1 Introduction

The natural join and the inner union operations combine relations (i.e. tables) of a database. Most of today's web programs query their databases making repeated use of the natural join and of the union, of which the inner union is a mathematically well behaved variant. Tropashko and Spight realized [23,22] that these two operations are the meet and join operations in a class of lattices, known by now as the class of relational lattices. They proposed then lattice theory as an algebraic approach, alternative to Codd's relational algebra [3], to the theory of databases.

An important first attempt to axiomatize these lattices is due to Litak, Mikulás, and Hidders [13]. These authors propose an axiomatization, comprising equations and quasiequations, in a signature that extends the pure lattice

* Extended abstract, see [21] for a full version of this paper.

signature with a constant, the header constant. A main result of that paper is that the quasiequational theory of relational lattices is undecidable in this extended signature. Their proof mimics Maddux's proof that the equational theory of cylindric algebras of dimension $n \geq 3$ is undecidable [14].

We have investigated in [20] equational axiomatizations for relational lattices using as tool the duality theory for finite lattices developed in [19]. A conceptual contribution from [20] is to make explicit the similarity between the developing theory of relational lattices and the well established theory of combination of modal logics, see e.g. [11]. This was achieved on the syntactic side, but also on the semantic side, by identifying some key properties of the structures dual to the finite atomistic lattices in the variety generated by the relational lattices, see [20, Theorem 7]. These properties make the dual structures into frames for commutator multimodal logics in a natural way.

In this paper we exploit this similarity to transfer results from the theory of multidimensional modal logics to lattice theory. Our main result is that *it is undecidable whether a finite subdirectly irreducible lattice can be embedded into a relational lattice*. We prove this statement by reducing to it the coverability problem of a frame by a universal $S5^3$-product frame, a problem shown to be undecidable in [10]. As stated there, the coverability problem is—in light of standard duality theory—a direct reformulation of the representability problem of finite simple relation algebras, problem shown to be undecidable by Hirsch and Hodkinson [9].

Our main result and its proof allow us to derive further consequences. Firstly, we refine the undecidability theorem of [13] and prove that *the quasiequational theory of relational lattices in the pure lattice signature is undecidable* as well and *has no finite base*. Then we argue that *there is a quasiequation that holds in all the finite relational lattices, but fails in an infinite one*. For the latter result, we rely on the work by Hirsch, Hodkinson, and Kurucz [10] who constructed a finite 3-multimodal frame which has no finite p-morphism from a finite universal $S5^3$-product frame, but has a p-morphism from an infinite one. On the methodological side, we wish to point out our use of generalized ultrametric spaces to tackle these problems. A key idea in the proof of the main result is the characterization of universal $S5^A$-product frames as pairwise complete generalized ultrametric spaces with distance valued in the Boolean algebra $P(A)$, a characterization that holds when A is finite.

The paper is structured as follows. We recall in Section 2 some definitions and facts on frames and lattices. Relational lattices are introduced in Section 3. In Section 4 we outline the proof of our main result—embeddability of a finite subdirectly-irreducible lattice into a relational lattice is undecidable—and derive then the other results. In Section 5 we show how to construct a lattice from a frame and use functoriality of this construction to argue that such lattice embeds into a relational lattice whenever the frame is a p-morphic image of a universal product frame. The proof of the converse statement is carried out in Section 7. Among the technical tools needed to prove the converse, the theory of generalized ultrametric spaces over a powerset Boolean algebra and the aforementioned characterization of universal $S5^A$-product frames as pairwise complete spaces over $P(A)$ are developed in Section 6.

Due to the lack of space, we omit most of the technical proofs on lattices and ultrametric spaces; these proofs are accessible via the preprint [21].

2 Frames and lattices

Frames. Let A be a set of actions. An *A-multimodal frame* (briefly, an *A-frame* or a *frame*) is a structure $\mathfrak{F} = \langle X_{\mathfrak{F}}, \{R_a \mid a \in A\}\rangle$ where, for each $a \in A$, R_a is a binary relation on $X_{\mathfrak{F}}$. We say that an A-frame is S4 if each R_a is reflexive and transitive. If $\mathfrak{F}_0$ and $\mathfrak{F}_1$ are two A-frames, then a p-morphism from $\mathfrak{F}_0$ to $\mathfrak{F}_1$ is a function $\psi : X_{\mathfrak{F}_0} \longrightarrow X_{\mathfrak{F}_1}$ such that, for each $a \in A$,

- if xR_ay, then $\psi(x)R_a\psi(y)$,
- if $\psi(x)R_az$, then xR_ay for some y with $\psi(y) = z$.

Let us mention that A-multimodal frames and p-morphisms form a category.

A frame $\mathfrak{F}$ is said to be *rooted* (or *initial*, see [18]) if there is $f_0 \in X_{\mathfrak{F}}$ such that every other $f \in X_{\mathfrak{F}}$ is reachable from f_0. We say that an A-frame $\mathfrak{F}$ is *full* if, for each $a \in A$, there exists $f, g \in X_{\mathfrak{F}}$ such that $f \neq g$ and fR_ag. If $G = (V, D)$ is a directed graph, then we shall say that G is rooted if it is rooted as a unimodal frame.

A particular class of frames we shall deal with are the *universal* $S5^A$*-product frames*. These are the frames $\mathfrak{U}$ with $X_{\mathfrak{U}} = \prod_{a\in A} X_a$ and xR_ay if and only if $x_i = y_i$ for each $i \neq a$, where $x := \langle x_i \mid i \in A\rangle$ and $y := \langle y_i \mid i \in A\rangle$.

Orders and lattices. We assume some basic knowledge of order and lattice theory as presented in standard monographs [4,7]. Most of the tools we use in this paper originate from the monograph [6] and have been further developed in [19].

A *lattice* is a poset L such that every finite non-empty subset $X \subseteq L$ admits a smallest upper bound $\bigvee X$ and a greatest lower bound $\bigwedge X$. A lattice can also be understood as a structure $\mathfrak{A}$ for the functional signature $(\vee, \wedge)$, such that the interpretations of these two binary function symbols both give $\mathfrak{A}$ the structure of an idempotent commutative semigroup, the two semigroup structures being connected by the absorption laws $x \wedge (y \vee x) = x$ and $x \vee (y \wedge x) = x$. Once a lattice is presented as such structure, the order is recovered by stating that $x \leq y$ holds if and only if $x \wedge y = x$.

A lattice L is *complete* if any subset $X \subseteq L$ admits a smallest upper bound $\bigvee X$. It can be shown that this condition implies that any subset $X \subseteq L$ admits a greatest lower bound $\bigwedge X$. A lattice is *bounded* if it has a least element $\perp$ and a greatest element $\top$. A complete lattice (in particular, a finite lattice) is bounded, since $\bigvee \emptyset$ and $\bigwedge \emptyset$ are, respectively, the least and greatest elements of the lattice.

If P and Q are partially ordered sets, then a function $f : P \longrightarrow Q$ is *order-preserving* (or *monotone*) if $p \leq p'$ implies $f(p) \leq f(p')$. If L and M are lattices, then a function $f : L \longrightarrow M$ is a *lattice morphism* if it preserves the lattice operations $\vee$ and $\wedge$. A lattice morphism is always order-preserving. A lattice morphism $f : L \longrightarrow M$ between bounded lattices L and M is *bound-preserving* if $f(\bot) = \bot$ and $f(\top) = \top$. A function $g : Q \longrightarrow P$ is said to be *left adjoint* to an order-preserving $f : P \longrightarrow Q$ if $g(q) \leq p$ holds if and only if $q \leq f(p)$ holds; such a left adjoint, when it exists, is unique. If L is finite, M is bounded, and $f : L \longrightarrow M$ is a bound-preserving lattice morphism, then a left adjoint to f always exists and preserves the constant $\bot$ and the operation $\vee$.

A *Moore family on a set* U is a collection $\mathcal{F}$ of subsets of U which is closed under arbitrary intersections. Given a Moore family $\mathcal{F}$ on U, the correspondence sending $Z \subseteq U$ to $\overline{Z} := \bigcap\{Y \in \mathcal{F} \mid Z \subseteq Y\}$ is a closure operator on U, that is, an order-preserving inflationary and idempotent endofunction of $P(U)$. The subsets in $\mathcal{F}$, called the *closed sets*, are exactly the fixpoints of this closure operator. We can give to a Moore family $\mathcal{F}$ a lattice structure by defining

$$\bigwedge X := \bigcap X\,, \qquad\qquad \bigvee X := \overline{\bigcup X}\,. \tag{1}$$

Let L be a complete lattice. An element $j \in L$ is *completely join-irreducible* if $j = \bigvee X$ implies $j \in X$, for each $X \subseteq L$; the set of completely join-irreducible elements of L is denoted here $\mathcal{J}(L)$. A complete lattice is *spatial* if every element is the join of the completely join-irreducible elements below it. An element $j \in \mathcal{J}(L)$ is said to be *join-prime* if $j \leq \bigvee X$ implies $j \leq x$ for some $x \in X$, for each finite subset X of L. If x is not join-prime, then we say that x is *non-join-prime*. An *atom* of a lattice L is an element of L such that $\bot$ is the only element strictly below it. A spatial lattice is *atomistic* if every element of $\mathcal{J}(L)$ is an atom.

For $j \in \mathcal{J}(L)$, a *join-cover* of j is a subset $X \subseteq L$ such that $j \leq \bigvee X$. For $X, Y \subseteq L$, we say that X *refines* Y, and write $X \ll Y$, if for all $x \in X$ there exists $y \in Y$ such that $x \leq y$. A join-cover X of j is said to be *minimal* if $j \leq \bigvee Y$ and $Y \ll X$ implies $X \subseteq Y$; we write $j \lhd_{\mathtt{m}} X$ if X is a minimal join-cover of j. In a spatial lattice, if $j \lhd_{\mathtt{m}} X$, then $X \subseteq \mathcal{J}(L)$. If $j \lhd_{\mathtt{m}} X$, then we say that X is a *non-trivial* minimal join-cover of j if $X \neq \{j\}$. Some authors use the word *perfect* for a lattice which is both spatial and dually spatial. We need here something different:

Definition 1. *We say that a complete lattice is* pluperfect *if it is spatial and for each $j \in \mathcal{J}(L)$ and $X \subseteq L$, if $j \leq \bigvee X$, then $Y \ll X$ for some Y such that $j \lhd_{\mathtt{m}} Y$. The* OD-graph *of a pluperfect lattice L is the structure $\langle \mathcal{J}(L), \leq, \lhd_{\mathtt{m}} \rangle$.*

That is, in a pluperfect lattice every cover refines to a minimal one. Notice that every finite lattice is pluperfect. If L is a pluperfect lattice, then we say that $X \subseteq \mathcal{J}(L)$ is *closed* if it is a downset and $j \lhd_{\mathtt{m}} C \subseteq X$ implies $j \in X$. Closed subsets of $\mathcal{J}(L)$ form a Moore family. The interest of considering pluperfect lattices stems from the following representation theorem stated in [16] for finite lattices; its generalization to pluperfect lattices is straightforward.

Theorem 2. Cf. [21, Theorem 2]. *Let L be a pluperfect lattice and let $\mathsf{L}(\mathcal{J}(L), \leq, \lhd_{\mathtt{m}})$ be the lattice of closed subsets of $\mathcal{J}(L)$. The mapping $l \mapsto \{j \in \mathcal{J}(L) \mid j \leq l\}$ is a lattice isomorphism from L to $\mathsf{L}(\mathcal{J}(L), \leq, \lhd_{\mathtt{m}})$.*

3 The relational lattices $\mathsf{R}(D, A)$

Throughout this paper we shall use the notation Y^X for the set of functions of domain Y and codomain X, for X and Y any two sets.

Let A be a collection of attributes (or column names) and let D be a set of cell values. A *relation* on A and D is a pair (α, T) where $\alpha \subseteq A$ and $T \subseteq D^\alpha$. Elements of the relational lattice $\mathsf{R}(D, A)$ [1] are relations on A and D. Informally, a relation (α, T) represents a table of a relational database, with α being the header, i.e. the collection of names of columns, while T is the collection of rows.

Before we define the natural join, the inner union operations, and the order on $\mathsf{R}(D, A)$, let us recall some key operations. If $\alpha \subseteq \beta \subseteq A$ and $f \in D^\beta$, then we shall use $f_{\restriction_\alpha} \in D^\alpha$ for the restriction of f to α; if $T \subseteq D^\beta$, then $T\upharpoonright_\alpha$ shall denote projection to α, that is, the direct image of T along restriction, $T\upharpoonright_\alpha := \{f_{\restriction_\alpha} \mid f \in T\}$; if $T \subseteq D^\alpha$, then $i_\beta(T)$ shall denote cylindrification to β, that is, the inverse image of restriction, $i_\beta(T) := \{f \in D^\beta \mid f_{\restriction_\alpha} \in T\}$. Recall that i_β is right adjoint to $\upharpoonright_\alpha$. With this in mind, the natural join and the inner union of relations are respectively described by the following formulas:

$$
\begin{aligned}
(\alpha_1, T_1) \wedge (\alpha_2, T_2) &:= (\alpha_1 \cup \alpha_2, T) \\
&\qquad \text{where } T = \{f \mid f_{\restriction_{\alpha_i}} \in T_i, i = 1, 2\} \\
&\qquad\qquad = i_{\alpha_1 \cup \alpha_2}(T_1) \cap i_{\alpha_1 \cup \alpha_2}(T_2)\,, \\
(\alpha_1, T_1) \vee (\alpha_2, T_2) &:= (\alpha_1 \cap \alpha_2, T) \\
&\qquad \text{where } T = \{f \mid \exists i \in \{1, 2\}, \exists g \in T_i \text{ s.t. } g_{\restriction_{\alpha_1 \cap \alpha_2}} = f\} \\
&\qquad\qquad = T_1\upharpoonright_{\alpha_1 \cap \alpha_2} \cup T_2\upharpoonright_{\alpha_1 \cap \alpha_2}\ .
\end{aligned}
$$

The order is then given by $(\alpha_1, T_1) \leq (\alpha_2, T_2)$ iff $\alpha_2 \subseteq \alpha_1$ and $T_1\upharpoonright_{\alpha_2} \subseteq T_2$.

A convenient way of describing these lattices was introduced in [13, Lemma 2.1]. The authors argued that the relational lattices $\mathsf{R}(D, A)$ are isomorphic to the lattices of closed subsets of $A \cup D^A$, where $Z \subseteq A \cup D^A$ is said to be closed if it is a fixed-point of the closure operator $\overline{(\,-\,)}$ defined as

$$\overline{Z} := Z \cup \{f \in D^A \mid A \setminus Z \subseteq Eq(f, g), \text{ for some } g \in Z\}\,,$$

where in the formula above $Eq(f, g)$ is the equalizer of f and g. Letting $\delta(f, g) := \{x \in A \mid f(x) \neq g(x)\}$, the above definition of the closure operator is obviously equivalent to the following one:

$$\overline{Z} := \alpha \cup \{f \in D^A \mid \delta(f, g) \subseteq \alpha, \text{ for some } g \in Z \cap D^A\}, \text{ with } \alpha = Z \cap A.$$

From now on, we rely on this representation of relational lattices. Relational lattices are atomistic pluperfect lattices. The completely join-irreducible elements of $\mathsf{R}(D, A)$ are the singletons $\{a\}$ and $\{f\}$, for $a \in A$ and $f \in D^A$, see [13]. By

[1] In [13] such a lattice is called *full* relational lattice. The wording "class of relational lattices" is used there for the class of lattices that have an embedding into some lattice of the form $\mathsf{R}(D, A)$.

an abuse of notation we shall write x for the singleton $\{x\}$, for $x \in A \cup D^A$. Under this convention, we have therefore $\mathcal{J}(\mathsf{R}(D, A)) = A \cup D^A$. Every $a \in A$ is join-prime, while the minimal join-covers are of the form $f \lhd_{\mathtt{m}} \delta(f, g) \cup \{g\}$, for each $f, g \in D^A$, see [20]. The only non-trivial result from [20] that we use later (for Lemma 24 and Theorem 29) is the folllowing:

Lemma 3. *Let L be a finite atomistic lattice in the variety generated by the class of relational lattices. If $\{j\} \cup X \subseteq \mathcal{J}(L)$, $j \leq \bigvee X$, and all the elements of X are join-prime, then j is join-prime.*

The Lemma—which is an immediate consequence of Theorem 7 in [20]—asserts that a join-cover of an element $j \in \mathcal{J}(L)$ which is not join-prime cannot be made of join-prime elements only.

4 Overview and statement of the results

For an arbitrary frame $\mathfrak{F}$, we construct in Section 5 a lattice $\mathsf{L}(\mathfrak{F})$; if $\mathfrak{F}$ is rooted and full, then $\mathsf{L}(\mathfrak{F})$ is a subdirectly irreducible lattice, see Proposition 16. The key Theorem leading to the undecidability results is the following one.

Theorem 4. *Let A be a finite set and let $\mathfrak{F}$ be an* S4 *finite rooted full A-frame. There is a surjective p-morphism from a universal* $\mathrm{S5}^A$*-product frame $\mathfrak{U}$ to $\mathfrak{F}$ if and only if $\mathsf{L}(\mathfrak{F})$ embeds into some relational lattice $\mathsf{R}(D, B)$.*

Proof (outline). The construction L defined in Section 5 extends to a contravariant functor, so if $\mathfrak{U}$ is a universal $\mathrm{S5}^A$-product frame and $\psi : \mathfrak{U} \longrightarrow \mathfrak{F}$ is a surjective p-morphism, then we have an embedding $\mathsf{L}(\psi)$ of $L(\mathfrak{F})$ into $\mathsf{L}(\mathfrak{U})$. We can assume that all the components of $\mathfrak{U}$ are equal, i.e. that the underlying set of $\mathfrak{U}$ is of the form $\prod_{a \in A} X$; if this is the case, then $\mathsf{L}(\mathfrak{U})$ is isomorphic to the relational lattice $\mathsf{R}(X, A)$.

The converse direction, developed from Section 6 up to Section 7, is subtler. Considering that $\mathsf{L}(\mathfrak{F})$ is subdirectly-irreducible, we argue that if $\psi : \mathsf{L}(\mathfrak{F}) \longrightarrow \mathsf{R}(D, B)$ is a lattice embedding, then we can suppose it preserves bounds; in this case ψ has a surjective left adjoint $\mu : \mathsf{R}(D, B) \longrightarrow \mathsf{L}(\mathfrak{F})$. Let us notice that there is no general reason for ψ to be the image by L of a p-morphism. Said otherwise, the functor L is not full and, in particular, the image of an atom by μ might not be an atom. The following considerations, mostly developed in Section 7, make it possible to extract a p-morphism from the left adjoint μ. Since both $\mathsf{L}(\mathfrak{F})$ and $\mathsf{R}(D, B)$ are generated (under possibly infinite joins) by their atoms, each atom $x \in \mathsf{L}(\mathfrak{F})$ has a preimage $y \in \mathsf{R}(D, B)$ which is an atom. The set F_0 of non-join-prime atoms of $\mathsf{R}(D, B)$ such that $\mu(f)$ is a non-join-prime atom of $\mathsf{L}(\mathfrak{F})$ is endowed with a $P(A)$-valued distance δ. The pair (F_0, δ) is shown to be a pairwise complete ultrametric space over $P(A)$. Section 6 recalls and develops some observations on ultrametric spaces valued on powerset algebras. The key ones are Theorems 18 and 19, stating that—when A is finite—pairwise complete ultrametric spaces over $P(A)$ and universal $\mathrm{S5}^A$-product frames are essentially the same objects. The restriction of μ to F_0 yields then a surjective p-morphism from F_0, considered as a universal $\mathrm{S5}^A$-product frame, to $\mathfrak{F}$. $\square$

The following problem was shown to be undecidable in [10]: given a finite 3-frame $\mathfrak{F}$, does there exists a surjective p-morphism from a universal $\mathrm{S5}^3$-product frame $\mathfrak{U}$ to $\mathfrak{F}$? In the introduction we referred to this problem as the coverability problem of a 3-frame by a universal $\mathrm{S5}^3$-product frame. The problem was shown to be undecidable by means of a reduction from the representability problem of finite simple relation algebras, shown to be undecidable in [9]. We need to strengthen the undecidability result of [10] with some additional observations—rootedness and fullness—as stated in the following Proposition.

Proposition 5. *It is undecidable whether, given a finite set A with* $\mathrm{card} A \geq 3$ *and an* S4 finite rooted full *A-frame $\mathfrak{F}$, there is a surjective p-morphism from a universal* $\mathrm{S5}^A$*-product $\mathfrak{U}$ to $\mathfrak{F}$.*

Proof. Throughout this proof we assume a minimum knowledge of the theory of relation algebras, see e.g. [15].

The Proposition actually holds if we restrict to the case when $\mathrm{card} A = 3$. Given a finite simple relation algebra $\mathfrak{A}$, the authors of [10] construct a 3-multimodal frame $\mathfrak{F}_{\mathfrak{A},3}$ such that $\mathfrak{A}$ is representable if and only if $\mathfrak{F}_{\mathfrak{A},3}$ is a p-morphic image of some universal $\mathrm{S5}^3$-product frame. The frame $\mathfrak{F}_{\mathfrak{A},3}$ is S4 and rooted [10, Claim 8]. We claim that $\mathfrak{F}_{\mathfrak{A},3}$ is also full, unless $\mathfrak{A}$ is the two elements Boolean algebra. To prove this claim, let us recall first that an element of $\mathfrak{F}_{\mathfrak{A},3}$ is a triple (t_0, t_1, t_2) of atoms of $\mathfrak{A}$ such that $t_2^{\smile} \leq t_0 ; t_1$; moreover, if t, t' are two such triples and $i \in \{0, 1, 2\}$, then tR_it' if and only if t and t' coincide in the i-th coordinate. If a is an atom of $\mathfrak{A}$, then $a \leq e_l ; a$ and $a \leq a ; e_r$ for two atoms e_l, e_r below the multiplicative unit of $\mathfrak{A}$. Therefore, the triples $t := (e_l, a, a^{\smile})$ and $t' = (a, e_r, a^{\smile})$ are elements of $\mathfrak{F}_{\mathfrak{A},3}$ and tR_2t'. If, for each atom a, these triples are equal, then every atom of $\mathfrak{A}$ is below the multiplicative unit, which therefore concides with the top element $\top$; since $\mathfrak{A}$ is simple, then relation $\top = \top ; x ; \top$ holds for each $x \neq \bot$. It follows that $x = \top ; x ; \top = \top$, for each $x \neq \bot$, so $\mathfrak{A}$ is the two elements Boolean algebra. Thus, if $\mathfrak{A}$ has more than two elements, then $t \neq t'$ and tR_2t' for some $t, t' \in \mathfrak{F}_{\mathfrak{A},3}$. Using the cycle law of relation algebras, one also gets pairs of distinct elements of $\mathfrak{F}_{\mathfrak{A},3}$, call them u, u' and w, w', such that uR_0u' and wR_1w'.

Therefore, if we could decide whether there is a p-morphism from some universal $\mathrm{S5}^3$-frame to a given S4 finite rooted full frame $\mathfrak{F}$, then we could also decide whether a finite simple relation algebra $\mathfrak{A}$ is representable, by answering positively if $\mathfrak{A}$ has exactly two elements and, otherwise, by answering the existence problem of a p-morphism to $\mathfrak{F}_{\mathfrak{A},3}$. □

Combining Theorem 4 with Proposition 5, we derive the following undecidability result.

Theorem 6. *It is not decidable whether a finite subdirectly irreducible atomistic lattice embeds into a relational lattice.*

Let us remark that Theorem 6 partly answers Problem 7.1 in [13].

In [13] the authors proved that the quasiequational theory of relational lattices (i.e. the set of all definite Horn sentences valid in relational lattices) in the signature $(\wedge, \vee, H)$ is undecidable. Here H is the header constant, which is interpreted in a relational lattice $\mathsf{R}(D, A)$ as the closed subset A of $A \cup D^A$. Problem 4.10 in [13] asks whether the quasiequational theory of relational lattices in the restricted signature $(\wedge, \vee)$ of pure lattice theory is undecidable as well. We positively answer this question.

Theorem 7. *The quasiequational theory of relational lattices in the pure lattice signature is undecidable.*

It is a general fact that if the embeddability problem of finite subdirectly-irreducible algebras in a class $\mathcal{K}$ is undecidable, then the quasiequational theory of $\mathcal{K}$ is undecidable as well. We thank a colleague for pointing out to us how this can be derived from Evans' work [5]. We add here the proof of this fact, since we shall need it later in the proof of Theorem 10.

Proof. Given a finite subdirectly-irreducible algebra A with least non trivial congruence $\theta(\hat{a}, \bar{a})$, we construct a quasiequation ϕ_A with the following property: for any other algebra (in the same signature) K, $K \not\models \phi_A$ if and only if A has an embedding into K.

The construction is as follows. Let $X_A = \{x_a \mid a \in A\}$ be a set of variables in bijection with the elements of A. For each function symbol f in the signature Ω, let $T_{A,f}$ be its table, that is the formula

$$T_{A,f} = \bigwedge_{(a_1,\ldots,a_{ar(f)}) \in A^{ar(f)}} f(x_{a_1}, \ldots, x_{ar(f)}) = x_{f(a_1,\ldots,a_{ar(f)})}.$$

We let ϕ_A be the universal closure of $\bigwedge_{f \in \Omega} T_{A,f} \Rightarrow x_{\hat{a}} = x_{\bar{a}}$. We prove next that an algebra K sastifies ϕ_A if and only if there is no embedding of A into K.

If $K \models \phi_A$ and $\psi : A \longrightarrow K$, then $v(x_a) = \psi(a)$ is a valuation such that $K, v \models \bigwedge_{f \in \Omega} T_{A,f}$, so $\psi(\hat{a}) = v(x_{\hat{a}}) = v(x_{\bar{a}}) = \psi(\bar{a})$ and ψ is not injective.

Conversely, suppose $K \not\models \phi_A$ and let v be a valuation such that $K, v \models \bigwedge_{f \in \Omega} T_{A,f}$ and $K, v \not\models x_{\hat{a}} = x_{\bar{a}}$. Define $\psi : A \longrightarrow K$ as $\psi(a) = v(x_a)$, then ψ is a morphism, since $K, v \models T_{A,f}$ for each $f \in \Omega$. Let $Ker_\psi = \{(a, a') \mid \psi(a) = \psi(a')\}$ so, supposing that ψ is not injective, Ker_ψ is a non-trivial congruence. Then $(\hat{a}, \bar{a}) \in \theta(\hat{a}, \bar{a}) \subseteq Ker_\psi$, so $v(x_{\hat{a}}) = \psi(\hat{a}) = \psi(\bar{a}) = v(x_{\bar{a}})$, a contradiction. We have therefore $Ker_\psi = \{(a, a) \mid a \in A\}$, which shows that ψ is injective.

Let now $\mathcal{K}$ be a class of algebras in the same signature. We have then

$$\begin{aligned} \mathcal{K} \not\models \phi_A \text{ iff } & K \not\models \phi_A \text{ for some } K \in \mathcal{K} \\ \text{iff } & \text{there is an embedding of } A \text{ into } K, \text{ for some } K \in \mathcal{K} \ . \end{aligned}$$

Thus, if the embeddability problem of finite subdirectly-irreducible algebras into some algebra in $\mathcal{K}$ is undecidable, then the quasiequational theory of $\mathcal{K}$ is undecidable as well. □

Following [10], let us add some further observations on the quasiequational theory of relational lattices.

Lemma 8. *The class of lattices that have an embedding into a relational lattice is closed under ultraproducts.*

Proof. Let us say that a sublattice L of a lattice $\mathsf{R}(D, A)$ is H-closed if the subset A belongs to L. Let $\mathcal{R}$ denote the closure under isomorphisms of the class of H-closed sublattices of some $\mathsf{R}(D, A)$. It is proved in [13, Corollary 4.2] that $\mathcal{R}$ is closed under ultraproducts. It immediately follows from this result that the class of lattices that have an embedding into some relational lattice is closed under ultraproducts, as follows. Let $\{L_i \longrightarrow \mathsf{R}(D_i, A_i) \mid i \in I\}$ be a family of lattice embeddings and let $\mathcal{F}$ be an ultrafilter over I. The ultraproduct constructions on $\{L_i \mid i \in I\}$ and $\{\mathsf{R}(D_i, A_i) \mid i \in I\}$ yield a lattice embedding $\prod_{\mathcal{F}} L_i \longrightarrow \prod_{\mathcal{F}} \mathsf{R}(D_i, A_i)$. Clearly, each $\mathsf{R}(D_i, A_i)$ belongs to $\mathcal{R}$, whence the ultraproduct $\prod_{\mathcal{F}} \mathsf{R}(D_i, A_i)$ belongs to $\mathcal{R}$ as well: thus $\prod_{\mathcal{F}} \mathsf{R}(D_i, A_i)$ embeds into some $\mathsf{R}(D, A)$, and so does $\prod_{\mathcal{F}} L_i$. □

Theorem 9. *The quasiequational theory of relational lattices is not finitely axiomatizable.*

Proof. A known result in universal algebra—see e.g. [2, Theorem 2.25]—states that a subdirectly-irreducible algebra satisfies all the quasiequations satisfied by a class of algebras if and only if it embeds in an ultraproduct of algebras in this class. Lemma 8 implies that the class of lattices that have an embedding into an ultraproduct of relational lattices and the class of lattices that have an embedding into some relational lattices are the same. Therefore a subdirectly-irreducible lattice L embeds in a relational lattice if and only if it satisfies all the quasiequations satisfied by the relational lattices. If this collection of quasiequations was a logical consequence of a finite set of quasiequations, then we could decide whether a finite subdirectly-irreducible L satisfies all these quasiequations, by verifying whether L satisfies the finite set of quasiequations. In this way, we could also decide whether such an L embeds into some relational lattice. □

Finally, the following Theorem, showing that the quasiequational theory of the finite relational lattices is stronger than the quasiequational theory of all the relational lattices, partly answers Problem 3.6 in [13].

Theorem 10. *There is a quasiequation which holds in all the finite relational lattices which, however, fails in an infinite relational lattice.*

Proof. In the first appendix of [10] an S4 finite rooted full 3-frame $\mathfrak{F}$ is constructed that has no surjective p-morphism from a finite universal $\mathrm{S5}^3$-product frame, but has such a p-morphism from an infinite one.

Since $\mathsf{L}(\mathfrak{F})$ is finite whenever $\mathfrak{F}$ is finite, we obtain by using Theorem 4 a subdirectly-irreducible finite lattice L which embeds into an infinite relational lattice, but has no embedding into a finite one.

Let ϕ_L be the quasiequation as in the proof of Theorem 7. We have therefore that, for any lattice K, $K \models \phi_L$ if and only if L does not embed into K.

Correspondingly, any finite relational lattice satisfies ϕ_L and, on the other hand, $K \not\models \phi_L$ if K is the infinite lattice into which L embeds. □

5 The lattice of a multimodal frame

We assume throughout this Section that A is a finite set of actions.

Let $\alpha \subseteq A$, $\mathfrak{F}$ be an A-frame, $x, y \in X_{\mathfrak{F}}$. We define an *$\alpha$-path* from x to y as a sequence $x = x_0 R_{a_0} x_1 \ldots x_{k-1} R_{a_{k-1}} x_k = y$ with $\{a_0, \ldots, a_{k-1}\} \subseteq \alpha$. We use the notation $x \xrightarrow{\alpha} y$ to mean that there is an α-path from x to y. Notice that if $\mathfrak{F}$ is an S4 A-frame, then $x \xrightarrow{\{a\}} y$ if and only if $xR_a y$. Given an A-frame $\mathfrak{F} = \langle X_{\mathfrak{F}}, \{R_a \mid a \in A\}\rangle$, we construct a lattice as follows. For $\alpha \subseteq A$, we say that $Y \subseteq X_{\mathfrak{F}}$ is *α-closed* if $x \in Y$, whenever there is a α-path from x to some $y \in Y$. We say that a subset $Z \subseteq A \cup X_{\mathfrak{F}}$ is *closed* if $Z \cap X_{\mathfrak{F}}$ is $Z \cap A$-closed. It is straightforward to verify that the collection of closed subsets of $A \cup X_{\mathfrak{F}}$ is a Moore family.

Definition 11. *The lattice $\mathsf{L}(\mathfrak{F})$ is the lattice of closed subsets of $A \cup X_{\mathfrak{F}}$.*

The lattice operations on $\mathsf{L}(\mathfrak{F})$ are defined as in the display (1). Actually, $\mathsf{L}(-)$ is a contravariant functor from the category of frames to the category of lattices. Namely, for a p-morphism $\psi : \mathfrak{F}_0 \longrightarrow \mathfrak{F}_1$ and any $Z \subseteq A \cup X_{\mathfrak{F}_1}$, define $\mathsf{L}(\psi)(Z) := (Z \cap A) \cup \psi^{-1}(Z \cap X_{\mathfrak{F}_1})$.

Proposition 12. Cf. [21, Proposition 17]. *$\mathsf{L}(\psi)$ sends closed subsets of $A \cup X_{\mathfrak{F}_1}$ to closed subsets of $A \cup X_{\mathfrak{F}_0}$. Its restriction to $\mathsf{L}(\mathfrak{F}_1)$ yields a bound-preserving lattice morphism $L(\psi) : \mathsf{L}(\mathfrak{F}_1) \longrightarrow \mathsf{L}(\mathfrak{F}_0)$. Moreover, if $\psi : \mathfrak{F}_0 \longrightarrow \mathfrak{F}_1$ is surjective, then $\mathsf{L}(\psi)$ is injective.*

We state next the main result of this Section.

Theorem 13. *If there exists a surjective p-morphism from a universal* $\mathrm{S5}^A$*-product frame $\mathfrak{U}$ to an A-frame $\mathfrak{F}$, then $\mathsf{L}(\mathfrak{F})$ embeds into a relational lattice.*

Proof. We say that $\mathfrak{U}$ is uniform on X if all the components of $\mathfrak{U}$ are equal to X. Spelled out, this means that $X_{\mathfrak{U}} = \prod_{a \in A} X$. Let $\psi : \mathfrak{U} \longrightarrow \mathfrak{F}$ be a p-morphism as in the statement of the Theorem. W.l.o.g. we can assume that $\mathfrak{U}$ is uniform on some set X. If this is not the case, then we choose $a_0 \in A$ such that X_{a_0} has maximum cardinality and surjective mappings $p_a : X_{a_0} \longrightarrow X_a$, for each $a \in A$. The product frame $\mathfrak{U}'$ on $\prod_{a \in A} X_{a_0}$ is uniform and $\prod_{a \in A} p_a : \mathfrak{U}' \longrightarrow \mathfrak{U}$ is a surjective p-morphism. By pre-composing ψ with this p-morphism, we obtain a surjective p-morphism from the uniform $\mathfrak{U}'$ to $\mathfrak{F}$. Now, if $\mathfrak{U}$ is uniform on X, then $\mathsf{L}(\mathfrak{U})$ is equal to the relational lattice $\mathsf{R}(X, A)$. Then, by functoriality of L, we have a lattice morphism $\mathsf{L}(\psi) : \mathsf{L}(\mathfrak{F}) \longrightarrow \mathsf{L}(\mathfrak{U}) = \mathsf{R}(X, A)$. By Proposition 12 $\mathsf{L}(\psi)$ is an embedding. □

We review next some properties of the lattices $\mathsf{L}(\mathfrak{F})$.

Proposition 14. Cf. [21, Proposition 20]. *The completely join-irreducible elements of $\mathsf{L}(\mathfrak{F})$ are the singletons, so $\mathsf{L}(\mathfrak{F})$ is an atomistic lattice.*

Identifying singletons of with their elements, the previous proposition states that $\mathcal{J}(\mathsf{L}(\mathfrak{F})) = A \cup X_{\mathfrak{F}}$. To state the next Proposition, let us say that an α-path from $x \in X_{\mathfrak{F}}$ to $y \in X_{\mathfrak{F}}$ is *minimal* if there is no β-path from x to y, for each proper subset β of α.

Proposition 15. Cf. [21, Proposition 21]. *$\mathsf{L}(\mathfrak{F})$ is a pluperfect lattice. Each element of A is join-prime, while the minimal join-covers of $x \in X_{\mathfrak{F}}$ are of the form $x \lhd_{\mathtt{m}} \alpha \cup \{y\}$, for a minimal α-path from x to y.*

Before stating the next Proposition, let us recall from [6, Corollary 2.37] that a finite lattice L is subdirectly-irreducible if and only if the directed graph $(\mathcal{J}(L), D)$ is rooted. Here D is the *join-dependency relation* on the join-irreducible elements of L, which, on atomistic finite lattices, can be defined by saying that jDk holds if $j \neq k$ and $j \leq p \vee k$ for some $p \in L$ with $j \not\leq p$.

Proposition 16. *If a finite A-frame $\mathfrak{F}$ is rooted and full, then $\mathsf{L}(\mathfrak{F})$ is a subdirectly-irreducible lattice.*

Proof. We argue that the digraph $(\mathcal{J}(\mathsf{L}(\mathfrak{F})), D)$ is rooted. Observe that $x \in \overline{\{a, y\}} = a \vee y$ whenever $xR_a y$. This implies that xDy and xDa when $x, y \in X_{\mathfrak{F}}$, $a \in A$, $x \neq y$ and $xR_a y$. The fact that of $(\mathcal{J}(\mathsf{L}(\mathfrak{F})), D)$ is rooted follows now from $\mathfrak{F}$ being rooted and full. $\square$

6 Some theory of generalized ultrametric spaces

Generalized ultrametric spaces over a Boolean algebra $P(A)$ turn out to be a useful tool for relational lattices [13,20]—as well as, we claim here, for universal product frames from multidimensional modal logic [11]. The use of metrics is well known in graph theory, where universal product frames are known as Hamming graphs, see e.g. [8]. Generalized ultrametric spaces over a Boolean algebra $P(A)$ were introduced in [17] to study equivalence relations. The main results of this Section are Theorem 18 and Proposition 19 which together substantiate the claim that when A is finite, universal $\mathrm{S5}^A$-product frames are pairwise complete ultrametric spaces valued in the Boolean algebra $P(A)$. It is this abstract point of view that shall allow us to construct a universal product frame given a lattice embedding $\mathsf{L}(\mathfrak{F}) \longrightarrow \mathsf{R}(D, A)$. We shall develop some observations that are not strictly necessary to prove the undecidability result, which is the main result of this paper. Nonetheless we include them since they are part of a coherent set of results and, as far as we know, they are original.

Definition 17. *An* ultrametric space over $P(A)$ *(briefly, a* space*) is a pair (X, δ), with $\delta : X \times X \longrightarrow P(A)$ such that, for every $f, g, h \in X$,*

$$\delta(f, f) \subseteq \emptyset\,, \qquad \delta(f, g) \subseteq \delta(f, h) \cup \delta(h, g)\,.$$

That is, we have defined an ultrametric space over $P(A)$ as a category (with a small set of objects) enriched over $(P(A)^{op}, \emptyset, \cup)$, see [12]. We shall assume in this paper that such a space (X, δ) is also *reduced* and *symmetric*, that is, that the following two properties hold for every $f, g \in X$:

$$\delta(f, g) = \emptyset \text{ implies } f = g\,, \qquad \delta(f, g) = \delta(g, f)\,.$$

A *morphism* of spaces[2] $\psi : (X, \delta_X) \longrightarrow (Y, \delta_Y)$ is a function $\psi : X \longrightarrow Y$ such that $\delta_Y(\psi(f), \psi(g)) \leq \delta_X(f, g)$, for each $f, g \in X$. If $\delta_Y(\psi(f), \psi(g)) = \delta_X(f, g)$, for each $f, g \in X$, then ψ is said to be an *isometry*. For (X, δ) a space over $P(A)$, $f \in X$ and $\alpha \subseteq A$, the ball centered in f of radius α is defined as usual: $B(f, \alpha) := \{g \in X \mid \delta(f, g) \subseteq \alpha\}$. In [1] a space (X, δ) is said to be *pairwise complete* if, for each $f, g \in X$ and $\alpha, \beta \subseteq A$, $B(f, \alpha \cup \beta) = B(g, \alpha \cup \beta)$ implies $B(f, \alpha) \cap B(g, \beta) \neq \emptyset$. This property is easily seen to be equivalent to:

$$\delta(f, g) \subseteq \alpha \cup \beta \text{ implies } \delta(f, h) \subseteq \alpha \text{ and } \delta(h, g) \subseteq \beta\,, \quad \text{for some } h \in X.$$

If (X, δ_X) is a space and $Y \subseteq X$, then the restriction of δ_X to Y induces a space (Y, δ_X); we say then that (Y, δ_X) is a *subspace* of X. Notice that the inclusion of Y into X yields an isometry of spaces.

Our main example of space over $P(A)$ is (D^A, δ), with D^A the set of functions from A to D and the distance defined by

$$\delta(f, g) := \{a \in A \mid f(a) \neq g(a)\}\,. \tag{2}$$

A second example is a slight generalization of the previous one. Given a surjective function $\pi : E \longrightarrow A$, let $\mathsf{Sec}(\pi)$ denote the set of all sections of π, that is the functions $f : A \longrightarrow E$ such that $\pi \circ f = id_A$; the formula in (2) also defines a distance on $\mathsf{Sec}(\pi)$. By identifying $f \in \mathsf{Sec}(\pi)$ with a vector $\langle f_a \in \pi^{-1}(a) \mid a \in A\rangle$, we see that

$$\mathsf{Sec}(\pi) = \prod_{a \in A} X_a\,, \quad \text{where } X_a := \pi^{-1}(a). \tag{3}$$

That is, the underlying set of a space $(\mathsf{Sec}(\pi), \delta)$ is that of a universal $\mathrm{S5}^A$-product frame. Our next observations are meant to understand the role of the universal $\mathrm{S5}^A$-product frame among all the spaces.

A space is *spherically complete* if the intersection $\bigcap_{i \in I} B(f_i, \alpha_i)$ of every chain $\{B(f_i, \alpha_i) \mid i \in I\}$ of balls is non-empty, see e.g. [1]. In this work the injective objects in the category of spaces are characterized as the pairwise and spherically complete spaces. The next Theorem shows that such injective objects are, up to isomorphism, the "universal product frames".

Theorem 18. Cf. [21, Proposition 24 and Theorem 25]. *The spaces of the form* $(\mathsf{Sec}(\pi), \delta)$ *are pairwise and spherically complete. Moreover, every space* (X, δ) *over* $P(A)$ *has an isometry into some* $(\mathsf{Sec}(\pi), \delta)$ *and if* (X, δ) *is pairwise and spherically complete, then this isometry is an isomorphism.*

We develop next the minimal theory needed to carry out the proof of undecidability. We shall assume in particular that A is a finite set. It was shown in [17] that, when A is finite, every space over $P(A)$ is spherically complete–so, from now on, this property will not be of concern to us.

[2] As $P(A)$ is not totally ordered, we avoid calling a morphism "*non expanding map*" as it is often done in the literature.

Observe now that in the display (3), the transition relations of the universal product frame $\prod_a X_a$ and the metric of the space $\mathsf{Sec}(\pi)$ are interdefinable. Indeed, for each $a \in A$, we have $f R_a g$ iff $\delta(f,g) \subseteq \{a\}$. On the other hand, since A is finite, the metric is completely determined from the transition relations of the frame, using the notion of α-path introduced in Section 5, as follows: $\delta(f,g) = \bigcap\{\alpha \subseteq A \mid f \xrightarrow{\alpha} g\}$. We cast our observations in a Proposition:

Proposition 19. *If A is finite, then there is a bijective correspondence between spaces over $P(A)$ of the form $(\mathsf{Sec}(\pi),\delta)$ and universal $\mathrm{S5}^A$-product frames. Universal $\mathrm{S5}^A$-product frames are, up to isomorphism, the pairwise complete spaces over $P(A)$.*

We assume in the rest of this section that (X,δ) is a fixed pairwise complete space. We say that a function $v : X \longrightarrow P(A)$ is a *module* if $v(f) \subseteq \delta(f,g) \cup v(g)$. In enriched category theory "module" is a standard naming for an enriched functor (here, a space morphism) from an enriched category to the base category enriched on itself. Here a module can be seen as a space morphism from (X,δ) to the space $(P(A),\Delta)$, where Δ is the symmetric difference. Given a module v, let us define $\mathsf{S}_v := \{x \in X \mid v(x) = \emptyset\}$.

Lemma 20. Cf. [21, Corollary 28]. *For each module v, S_v is a pairwise complete subspace of (X,δ).*

It is possible to directly define a lattice $\mathsf{L}(X,\delta)$ for each space (X,δ). For simplicity, we shall use $\mathsf{L}(X,\delta)$ here to denote the lattice structure corresponding to $\mathsf{L}(\mathfrak{U})$, where $\mathfrak{U}$ is a universal product frame correponding to (X,δ).

7 From lattice embeddings to surjective *p*-morphisms

We prove in this Section the converse of Theorem 13:

Theorem 21. *Let A be a finite set, let $\mathfrak{F}$ be a finite rooted full* S4 *A-frame. If $\mathsf{L}(\mathfrak{F})$ embeds into a relational lattice $\mathsf{R}(D,B)$, then there exists a universal $\mathrm{S5}^A$-product frame $\mathfrak{U}$ and a surjective p-morphism from $\mathfrak{U}$ to $\mathfrak{F}$.*

To prove the Theorem, we study bound-preserving embeddings of finite atomistic lattices into lattices of the form $\mathsf{R}(D,B)$. Let in the following $i : L \longrightarrow \mathsf{R}(D,B)$ be a fixed bound-preserving lattice embedding, with L a finite atomistic lattice. Since L is finite, i has a left adjoint $\mu : \mathsf{R}(D,B) \longrightarrow L$. By abuse of notation, we shall also use the same letter μ to denote the restriction of this left adjoint to the set of completely join-irreducible elements of $\mathsf{R}(D,B)$ which, we recall, is identified with $B \cup D^B$. It is a general fact—and the main ingredient of Birkhoff's duality for finite distributive lattices—that left adjoints to bound-preserving lattice morphism preserve join-prime elements. Thus we have:

Lemma 22. *If $b \in B$, then $\mu(b)$ is join-prime.*

It is not in general true that left adjoints send join-irreducible elements to join-irreducible elements, and this is a main difficulty towards a proof of Theorem 21. Yet, the following statements hold:

Lemma 23. *For each $x \in \mathcal{J}(L)$ there exists $y \in B \cup D^B$ such that $\mu(y) = x$.*

Lemma 24. *Let $g \in D^B$ such that $\mu(g)$ is join-reducible in L. There exists $h \in D^B$ such that $\mu(h) \in \mathcal{J}(L)$ and $\mu(g) = \bigvee \mu(\delta(g,h)) \vee \mu(h)$; moreover, $\mu(h)$ is non-join-prime whenever L is not a Boolean algebra.*

Let A be the set of atoms of L that are join-prime. While (D^B, δ) is a space over $P(B)$, we need to transform D^B into a space over $P(A)$. To this end, we define a $P(A)$-valued distance δ_A on D^B by $\delta_A(f,g) := \mu(\delta(f,g))$. Because of Lemma 22, we have $\delta_A(f,g) \subseteq A$.

Proposition 25. *(D^B, δ_A) is a pairwise complete ultrametric space over $P(A)$.*

We define next $v : D^B \longrightarrow P(A)$ by letting $v(f) := \{a \in A \mid a \leq \mu(f)\}$.

Lemma 26. Cf. [21, Proposition 47]. *The map $v : D^B \longrightarrow P(A)$ is a module on (D^B, δ_A). Moreover $v(f) = \emptyset$ if and only if $\mu(f) \in \mathcal{J}(L) \setminus A$.*

Using Lemmas 20 and 26, we derive:

Corollary 27. *The subspace of (D^B, δ_A) induced by $F_0 := \{f \in D^B \mid \mu(f) \in \mathcal{J}(L) \setminus A\}$ is pairwise complete.*

The following Proposition, which ends the study of bound-preserving lattice embeddings into relational lattices, shows that modulo the shift of the codomain to the lattice of a universal product frame, such a lattice embedding can always be normalized, meaning that join-irreducible elements are sent to join-irreducible elements by the left adjoint.

Proposition 28. Cf. [21, Proposition 51]. *Let L be a finite atomistic lattice and let A be the set of its join-prime elements. If L is not a Boolean algebra and $i : L \longrightarrow \mathsf{R}(D,B)$ is a bound-preserving lattice embedding, then there exists a bound-preserving lattice embedding $j : L \longrightarrow \mathsf{L}(F_0, \delta_A)$, where (F_0, δ_A) is the pairwise complete ultrametric space defined in Corollary 27. Moreover, the left adjoint ν to j satisfies the following condition: for each $k \in A \cup F_0$, if $k \in A$ then $\nu(k) = k$ and, otherwise, $\nu(k) \in \mathcal{J}(L) \setminus A$.*

The following Theorem asserts that we can assume that a lattice embedding is bound-preserving, when its domain is a finite subdirectly-irreducible lattice. It is needed in Theorem 7 to exclude the constants $\perp$ and $\top$ from the signature of lattice theory.

Theorem 29. Cf. [21, Section 7]. *If L is a finite subdirectly-irreducible atomistic lattice which has a lattice embedding into some relational lattice $\mathsf{R}(D,A)$, then there exists a bound-preserving embedding of L into some other relational lattice $\mathsf{R}(D,B)$.*

We conclude next the proof of the main result of this Section, Theorem 21.

Proof (of Theorem 21). Since $\mathfrak{F}$ is rooted and full, $\mathsf{L}(\mathfrak{F})$ is a finite atomistic subdirectly-irreducible lattice by Proposition 16. Therefore, if $i : \mathsf{L}(\mathfrak{F}) \longrightarrow \mathsf{R}(D, B)$ is a lattice embedding, then we can assume, using Theorem 29, that i preserves the bounds. Also, if $\mathsf{L}(\mathfrak{F})$ is a Boolean algebra, then it is the two elements Boolean algebra, since we are assuming that $\mathsf{L}(\mathfrak{F})$ is subdirectly-irreducible. But then, $\mathfrak{F}$ is a singleton, and the statement of the Theorem trivially holds in this case.

We can therefore assume that $\mathsf{L}(\mathfrak{F})$ is not a Boolean algebra. Let us recall that A is the set of join-prime elements of $\mathsf{L}(\mathfrak{F})$, see Proposition 15. Let (F_0, δ_A) be the pairwise complete space over $P(A)$ and let $j : \mathsf{L}(\mathfrak{F}) \longrightarrow \mathsf{L}(F_0, \delta_A)$ be the lattice morphism with the properties stated in Proposition 28; let ν be the left adjoint to j. We can also assume that $\mathsf{L}(F_0, \delta_A) = \mathsf{L}(\mathfrak{U})$ for some universal $\mathrm{S5}^A$-product frame $\mathfrak{U}$.

To avoid confusions, we depart from the convention of identifying singletons with their elements. We define $\psi : X_{\mathfrak{U}} \longrightarrow X_{\mathfrak{F}}$ by saying that $\psi(x) = y$ when $\nu(\{x\}) = \{y\}$. This is well defined since in $\mathsf{L}(\mathfrak{U})$ (respectively $\mathsf{L}(\mathfrak{F})$) the non-join-prime join-irreducible-elements are the singletons $\{x\}$ with $x \in X_{\mathfrak{F}_{\mathfrak{U}}}$ (resp. $x \in X_{\mathfrak{F}}$); moreover, we have $X_{\mathfrak{U}} = F_0$ and each singleton $\{x\}$ with $x \in F_0$ is sent by ν to a singleton $\{y\} \in \mathcal{J}(\mathsf{L}(\mathfrak{F})) \setminus \{\{a\} \mid a \in A\} = \{\{x\} \mid x \in X_{\mathfrak{F}}\}$. The function ψ is surjective since every non-join-prime atom $\{x\}$ in $\mathsf{L}(\mathfrak{F})$ has a preimage by ν an atom $\{y\}$ and such a preimage cannot be join-prime, so $y \in X_{\mathfrak{U}}$.

We are left to argue that ψ is a p-morphism. To this end, let us remark that, for each $a \in A$ and $x, y \in X_{\mathfrak{F}}$ (or $x, y \in X_{\mathfrak{U}}$), the relation $xR_a y$ holds exactly when there is an $\{a\}$-path from x to y, i.e. when $\{x\} \subseteq \overline{\{a, y\}} = \{a\} \vee \{y\}$ (we need here that $\mathfrak{F}$ and $\mathfrak{U}$ are S4 frames).

Thus, let $x, y \in X_{\mathfrak{U}}$ be such that $xR_a y$. Then $\{x\} \subseteq \{a\} \vee \{y\}$ and $\nu(\{x\}) \subseteq \nu(\{a\}) \vee \nu(\{y\}) = \{a\} \vee \nu(\{y\})$. We have therefore $\psi(x) R_a \psi(y)$. Conversely, let $x \in X_{\mathfrak{U}}$ and $z \in X_{\mathfrak{F}}$ be such that $\psi(x) R_a z$. We have therefore $\nu(\{x\}) \subseteq \{a\} \vee \{z\}$, whence, by adjointness,

$$
\begin{aligned}
\{x\} \subseteq j(\{a\} \vee \{z\}) &= j(\{a\}) \vee j(\{z\}) \\
&= \{a\} \vee \{y \mid \nu(\{y\}) = \{z\}\} \\
&= \overline{\{a\} \cup \{y \mid \nu(\{y\}) = \{z\}\}}\,.
\end{aligned}
$$

But this means that there is some $y \in X_{\mathfrak{U}}$ with $\psi(y) = z$ and a $\{a\}$-path from x to y. But then, we also have $xR_a y$. $\square$

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