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# AN EXPLICIT VIEW OF THE HITCHIN FIBRATION ON THE BETTI SIDE FOR $\mathbb{P}^1$ MINUS 5 POINTS

CARLOS T. SIMPSON

*Dedicated to Nigel Hitchin*

ABSTRACT. The dual complex of the divisor at infinity of the character variety of local systems on  $\mathbb{P}^1 - \{t_1, \dots, t_5\}$  with monodromies in prescribed conjugacy classes  $C_i \subset SL_2(\mathbb{C})$ , was shown by Komyo to be the sphere  $S^3$ . We compare in some detail the projection from a tubular neighborhood to this dual complex, with the corresponding Hitchin fibration at infinity.

## 1. INTRODUCTION

The Hitchin fibration, source of a profoundly rich amount of structure, has been studied intensively over the past 30 years. This fibration is defined on Hitchin's moduli space of Higgs bundles  $M_H$  [25, 24, 39]. The Hitchin equations [25] give the nonabelian Hodge correspondence between this moduli space and the character variety  $M_B$ , which we call the “Betti side” to use a motivic terminology, and we obtain a topological map defined on  $M_B$ .

There have recently been signs of a deep relationship between the Hitchin fibration and the structure of the compactification of  $M_B$  as an algebraic variety, notably the  $P = W$  conjecture [9, 21, 22, 23], wallcrossing [31, 32] and Gaiotto-Moore-Neitzke's theory [13]. Given the highly transcendental nature of the solutions of Hitchin's nonlinear partial differential equations, the existence of such a relationship is very surprising.

In this paper, we would like to consider a specific example: the case of  $\mathbb{P}^1$  with five singular points. The Hitchin fibration will be defined on a moduli space of parabolic Higgs bundles [6, 16, 30, 38, 41, 44]. The character variety or Betti moduli space is the space of local systems on  $\mathbb{P}^1 - \{t_1, \dots, t_5\}$  with given conjugacy classes at the five punctures. Our goal is to describe an explicit compactification of  $M_B$  within which we can see some of the fine topological structure of the Hitchin fibration.

A global result has already been established by Komyo [29] who shows that the dual complex of the compactification of  $M_B$  is homotopy equivalent to  $S^3$ . Presumably this should be the same as the  $S^3$  at infinity in the

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base of the Hitchin fibration, although that remains conjectural for now. Without addressing that conjecture, we would like to identify more precisely a fibration structure over this sphere, which (again conjecturally) would correspond to the structure of the fibration on  $M_H$ . The case of 5 points is the first one where the discriminant locus in the Hitchin base comes into play.

In order to motivate this investigation, it is important to understand what happens in the case of  $\mathbb{P}^1$  minus 4 points. Let us fix the conjugacy classes  $C_i = C \subset SL(2, \mathbb{C})$  of matrices of trace zero. The character variety  $M_B$  for  $\mathbb{P}^1$  minus 4 points is the *Fricke-Klein cubic* given by the equation

$$(1) \quad xyz + x^2 + y^2 + z^2 - 4 = 0$$

in  $\mathbb{A}^3$ . It has four singular points corresponding to reducible representations—our choice of conjugacy classes is not generic here. These don't affect the behavior in a neighborhood of the divisor at infinity, and they go away when generic  $C_i$  are chosen; in that case the equation acquires a general linear term as has been discussed recently in the papers of Boalch [4] and Goldman and Toledo [17] among others.

The divisor at infinity, given by the highest degree term  $xyz$ , is a triangle of  $\mathbb{P}^1$ 's. It is already a divisor with normal crossings, whose dual complex is a real triangle homotopic to the circle  $S^1$ . There are six strata  $S_1, S_2, S_3, S_{12}, S_{23}, S_{31}$ . The punctured neighborhoods at infinity denoted  $T_1^*, T_2^*, T_3^*, T_{12}^*, T_{23}^*, T_{31}^*$  are all homotopic to  $(S^1)^2$ , indeed  $T_{ij}^* \cong \Delta^* \times \Delta^*$  (a product of two punctured disks) whereas  $T_i^* \cong \mathbb{G}_m \times \Delta^*$ .

Let  $T^* \subset M_B$  be the union of all these, the punctured neighborhood at infinity of the character variety. It maps to the dual complex  $S^1$  and the homotopy fiber is  $(S^1)^2$ . A calculation about how the pieces fit together as we go around the triangle shows that the monodromy operation is multiplication by  $-1$  on the torus  $(S^1)^2$ .

Let's compare now with the Hitchin fibration: the moduli space of parabolic Higgs bundles [6, 16, 30, 38, 44] (in this case, with rational parabolic weights  $1/4$  and  $3/4$  so the orbifold picture [41] applies)<sup>1</sup> is two-dimensional so the Hitchin base is just  $\mathbb{A}^1$  and the Hitchin fibration

$$M_H \rightarrow \mathbb{A}^1$$

has general fiber an elliptic curve  $E$ , namely the one branched over the given four points in  $\mathbb{P}^1$ . The only non-smooth fiber is the nilpotent cone over  $0 \in \mathbb{A}^1$  and the other fibers are all isomorphic by the  $\mathbb{C}^*$ -action. One can see that the monodromy operation of going once around the punctured disk at infinity, corresponds to the hyperelliptic involution of  $E$  namely it is multiplication by  $-1$ . This picture is readily identified with the Betti picture: the dual complex of the triangle of  $\mathbb{P}^1$ 's corresponds to the circle at

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<sup>1</sup>The Hitchin integrable systems in the case of  $\mathbb{P}^1$  date back, in a birational sense, to Garnier [14, 43] but that doesn't take into account semistability and the nonabelian harmonic theory.

infinity in  $\mathbb{A}^1$ , and the punctured neighborhoods  $T_i^*$  and  $T_{ij}^*$  correspond to the fibers of the Hitchin fibration.

Something special was happening here: the Hitchin base being 1-dimensional, the discriminant locus doesn't meet the sphere at infinity, so the Hitchin map is topologically a fiber bundle and its identification with the picture on the Betti side becomes a homotopy-theoretical aspect.

As soon as the moduli space has dimension  $\geq 4$ , there will be a discriminant locus in the base, over which the Hitchin map will have degenerate fibers. It is natural to ask, to what extent can we still see the topology of this map by looking closely at the Betti side?

The purpose of the present paper is to attempt to shine some light on that question, by considering the next case—the first in which the Hitchin fibration has degenerate fibers, the case of  $\mathbb{P}^1$  minus 5 points.

Our discussion will take place mostly on the Betti side. To motivate it, let us however look in general terms at what to expect by considering the actual Hitchin fibration. We'll maintain the choice of  $C_i = C$  conjugacy classes of matrices in  $SL(2, \mathbb{C})$  with trace zero; for 5 points that is generic so the moduli space is a smooth 4-dimensional variety. The Lagrangian Hitchin fibration goes to affine space of half the dimension:

$$\varphi : M_H \rightarrow \mathbb{A}^2.$$

The general fibers are 2-dimensional abelian varieties, thus topologically  $(S^1)^4$ . The discriminant locus  $\text{Disc} \subset \mathbb{A}^2$  is conical, being invariant under the action of  $\mathbb{C}^*$ . The general spectral curve is a genus 2 curve branched over the given five points, plus a variable sixth point, and this new point provides the coordinate in  $\mathbb{P}^1 = \mathbb{A}^2 - \{0\}/\mathbb{C}^*$ . Degenerate fibers therefore occur when the sixth point coincides with one of the five others. In other words, the discriminant locus consists of five lines in  $\mathbb{A}^2$  whose slopes are given by the five original points.

The sphere at infinity in the Hitchin base is  $S^3$ . It intersects the discriminant locus in five circles. They are fibers of the Hopf fibration  $S^3 \rightarrow S^2$ , so they are pairwise linked.

As we have noted above, Komyo has already shown that the dual complex of any compactification of  $M_B$  is homotopy equivalent to  $S^3$  [29]. It didn't seem immediately apparent how to get an explicit description of the components of his compactification.

We pursue an approach that is probably rather special<sup>2</sup> to the case of 5 points: by choosing a good collection of trace coordinates, we get to a hypersurface defined by a generalization of the Fricke-Klein equation (1), and then using some computer-algebra we can investigate explicitly the resolution of singularities at infinity. This is complicated by the fact that our hypersurface expression will not be for  $M_B$  but rather for its quotient by an

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<sup>2</sup>It isn't clear whether or not to expect a series of nice polynomials corresponding to any number of points. Boalch points out in [5] a whole series of polynomial equations dating back to Euler, somewhat similar to Fricke-Klein, defining wild character varieties.

involution, so the information of a  $2 : 1$  covering needs to be brought along and the hypersurface itself will have a singular locus.

We are able to identify five circles in the dual complex, over which the Betti Hitchin fibers (i.e. the punctured tubular neighborhoods of the strata at infinity) degenerate. One problem with this picture is that the circles intersect a little bit—they don't seem to split apart entirely in any Betti compactification. But there exist non-intersecting pairs and those circles are indeed linked. The monodromy of the smooth part of the fibration around the circles is as expected.

At the end of the paper we discuss briefly some further directions of study.

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## 2. A FRICKE-KLEIN EQUATION

Let  $X := \mathbb{P}^1 - \{t_1, t_2, t_3, t_4, t_5\}$ . Choose a basepoint  $x_0 \in X$ , and let  $\gamma_i$  be paths going from the basepoint around  $t_i$  and back, so we have the relation  $\gamma_1 \cdots \gamma_5 = 1$ . A local system on  $X$ , with framing over  $x_0$ , is given by a quintuple of matrices  $(A_1, \dots, A_5)$  such that

$$(2) \quad A_1 A_2 A_3 A_4 A_5 = 1.$$

We consider  $SL(2, \mathbb{C})$ -local systems, and impose the conditions that  $A_i \in C$  where  $C$  is the conjugacy class of matrices whose eigenvalues are  $\pm\sqrt{-1}$ . Equivalently  $C$  is the set of matrices  $A \in SL(2, \mathbb{C})$  with  $\text{Tr}(A) = 0$ . The variety  $R_B$  of such representations is therefore the variety of quintuples of matrices with  $\det(A_i) = 1$ ,  $\text{Tr}(A_i) = 0$  and satisfying (2). The moduli space  $M_B := R_B/PSL(2, \mathbb{C})$  is the GIT quotient by the conjugation action of the group, an action that factors through  $PSL(2, \mathbb{C})$ .

One should of course envision a choice of conjugacy classes  $C_1, \dots, C_5$  and ask  $A_i \in C_i$ . For the purposes of the present paper, we are making a very specific choice of this collection, with all  $C_i$  equal to the conjugacy class  $C$

of matrices of trace 0. We are doing this in order to simplify as much as possible the equations.

Notice that our collection of conjugacy classes is Kostov-generic: for any choice of one eigenvalue at each point, the product of all five of these is  $\pm\sqrt{-1} \neq 1$  so a local system as above cannot contain a rank 1 subsystem. All the points of  $R_B$  are automatically irreducible. The quotient  $M_B := R_B/PSL(2, \mathbb{C})$  is therefore a geometric quotient, and  $M_B$  is a smooth variety of dimension 4.

The character variety  $M_B$  is affine, and we know in general that an embedding can be obtained by using a finite collection of functions of the form  $\rho \mapsto \text{Tr}(\rho(\xi))$  for group elements  $\xi \in \pi_1(X, x_0)$ . However, getting a practically useable expression requires some luck.

After a certain amount of experimentation, it would seem that one good way to proceed is as follows. Use the group elements  $\xi_i = \gamma_i\gamma_{i+1}$  (in cyclic ordering) to define the coordinate functions

$$\begin{aligned} x &:= \text{Tr}(A_1A_2) & y &:= \text{Tr}(A_2A_3) & z &:= \text{Tr}(A_3A_4) \\ u &:= \text{Tr}(A_4A_5) & v &:= \text{Tr}(A_5A_1). \end{aligned}$$

A generalization of the Fricke-Klein equation to this setup goes as follows.

**Proposition 2.1.** *The map  $(A_1, \dots, A_5) \mapsto (x, y, z, u, v)$  defines a finite 2 : 1 ramified covering.*

$$\phi : M_B \rightarrow H \subset \mathbb{A}^5.$$

*Its image is the hypersurface  $H$  defined by the equation  $f = 0$  where*

$$f(x, y, z, u, v) :=$$

$$xyzuv + (x^2y^2 + y^2z^2 + z^2u^2 + u^2v^2 + v^2x^2) - 4(x^2 + y^2 + z^2 + u^2 + v^2) + 16.$$

*The singular locus of  $H$  is a smooth two-dimensional subvariety  $B \subset H$ , equal to the ramification locus of  $\phi$ .*

*Proof.* The equation was found with some guesswork, using the parametrization discussed below, and a computer algebra program. I used SINGULAR [10]. The equation may be checked directly by the computer program.

To see that  $\phi$  is a 2 : 1 covering, it is useful to have some kind of a parametrization. Komyo's description [29] using GIT didn't seem immediately to yield explicit equations, although it would certainly be interesting to look more closely there. Cluster coordinates would be another option. We'll use some kind of "algebraic Fenchel-Nielsen coordinates" [26, 27].

View  $X$  as being glued from three pieces: one of them contains the punctures  $t_1$  and  $t_2$  with boundary curve  $\alpha = \gamma_1\gamma_2$ ; the other one contains the punctures  $t_4$  and  $t_5$  with boundary curve  $\beta = \gamma_4\gamma_5$ ; and the third piece contains the puncture  $t_3$  with two boundary curves  $\alpha, \beta$  (having opposite orientation). Each piece is a 3-punctured sphere, on which we consider a rank 2 local system: it must be the hypergeometric system, determined by the monodromy traces at the punctures except in degenerate cases. The

traces along the boundaries  $\alpha$  and  $\beta$  are the coordinates  $x$  and  $u$  respectively. If we fix these, then the full local system is determined by glueing parameters along the boundary curves. We should first choose a standard basis in each piece, then express the glueing.

More precisely, let us choose bases for the two end pieces. With  $a := x/2$ , write

$$\rho(\alpha) = \begin{pmatrix} a & a-1 \\ a+1 & a \end{pmatrix} \text{ and } \rho(\gamma_1) = \frac{1}{\sqrt{a^2-1}} \begin{pmatrix} 0 & 1-a \\ 1+a & 0 \end{pmatrix}.$$

Similarly with  $b := u/2$  put

$$\rho'(\beta) = \begin{pmatrix} b & b-1 \\ b+1 & b \end{pmatrix} \text{ and } \rho'(\gamma_5) = \frac{1}{\sqrt{b^2-1}} \begin{pmatrix} 0 & 1-b \\ 1+b & 0 \end{pmatrix}.$$

Here  $\rho'$  means the representation in a different basis. One may check with these expressions that  $\text{Tr}(\rho(\gamma_2)) = 0$  and  $\text{Tr}(\rho(\gamma_4)) = 0$ .

We don't need a basis for the middle piece, but rather write directly the glueing matrix between the two bases above as

$$g = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Set  $\rho(\gamma_i) := g\rho'(\gamma_i)g^{-1}$  for  $i = 4, 5$  with  $\rho'$  as above. The condition for the middle piece is to require  $g$  to satisfy the equation  $\text{Tr}(\rho(\gamma_3)) = 0$ .

Define

$$\begin{aligned} P &:= (a+1)(b-1)p^2, & Q &:= (a+1)(b+1)q^2, \\ R &:= (a-1)(b-1)r^2, & S &:= (a-1)(b+1)s^2. \end{aligned}$$

With these notations we have

$$\begin{aligned} y &= \frac{1}{\sqrt{a^2-1}}(P - Q + R - S), \\ z &= \frac{1}{\sqrt{b^2-1}}(-P - Q + R + S), \end{aligned}$$

and

$$v = -\frac{1}{\sqrt{(a^2-1)(b^2-1)}}(P + Q + R + S)$$

with finally

$$\text{Tr}(\rho(\gamma_3)) = 2ab + P - Q - R + S.$$

If  $x, y, z, u, v$  are given, with  $x, u \neq \pm 2$ , then we get  $a, b$  and can choose determinations of the square-roots to make the above expressions well-defined. From there, the values of  $P, Q, R, S$  are determined, and these give  $p^2, q^2, r^2, s^2$ . The matrix  $g$  is therefore determined up to changing the sign of its coefficients. Also  $g$  is subject to the condition  $\det(g) = 1$ , and the only sign changes that preserve this condition are: multiply the whole matrix by  $-1$ ; or  $p, s \mapsto -p, -s$  with  $q, r$  fixed; or the composition of these which is  $q, r \mapsto -q, -r$  with  $p, s$  fixed. However,  $g$  is also to be considered as an element in  $PSL(2, \mathbb{C})$  since we only care about conjugation by  $g$ . So  $-g$  represents the same element as  $g$ . We conclude that there are at most

2 points in  $M_B$  with the given  $x, y, z, u, v$ , and generically the number is 2. This proves that the map  $\phi$  is quasi-finite and generically 2 : 1 over the subset where  $x, u \neq \pm 2$ .

One notices from the equation  $f = 0$  that the coordinates cannot all be  $\pm 2$ , so there is at least one different one, say (using the cyclic symmetry)  $x \neq \pm 2$ . In the case where the matrices  $\rho(\gamma_3\gamma_4)$  and  $\rho(\gamma_4\gamma_5)$  are both the identity we can conclude that the local system is uniquely determined. If both traces are  $\pm 2$  then we may therefore assume that one of them, say  $\rho(\gamma_4\gamma_5)$  is nontrivially unipotent. Then we proceed using much the same analysis as above, but choosing a frame where

$$\rho'(\beta) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho'(\gamma_5) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The other cases, for example where  $\text{Tr}(\rho(\beta)) = -2$  or where the upper eigenvalue of  $\rho'(\gamma_5)$  is  $-i$  are very similar. Proceeding as before under the assumption that  $x, y, z, u, v$  are fixed, with  $x \neq \pm 2$ , we first get that  $p^2$  and  $r^2$  are determined, hence  $p, r$  determined up to a choice of sign; and then  $q, s$  are determined from them. We again get quasi-finiteness of the map  $\phi$  in this case. By cyclic and other symmetries, this covers all the cases so it shows that the map  $\phi$  is quasi-finite.

Now, since  $M_B$  is affine, if the map  $\phi$  were not proper then it would be non-proper along a codimension 1 subset of the hypersurface  $H$ . However, proceeding as in the above discussions, we can rule out that possibility, essentially by noting that the number of elements in the fiber of  $\phi$  is always 2 outside of codimension 2. We conclude that  $\phi$  is a finite 2 : 1 covering.

Again following the above procedure we can also identify the ramification locus and see that it is the same as the subvariety  $B \subset H$  where  $H$  is not smooth. One can then compute that  $B$  satisfies 10 equations which are cyclic permutations of

$$(x^2 - 4)(y^2 - 4) - 4u^2 = 0 \text{ and } xy^2z + 2uvy - 4xz = 0.$$

The ideal of  $3 \times 3$  minors of the jacobian matrix for this set of equations, together with  $f$ , yields the unit ideal. Hence,  $B$  is smooth.  $\square$

### 3. STRUCTURE OF THE COMPACTIFICATION

Our hypersurface  $H \subset \mathbb{A}^5$  compactifies to a projective hypersurface  $\overline{H} \subset \mathbb{P}^5$  given by the homogenization of the polynomial  $f$ . Let  $\overline{B} \subset \mathbb{P}^5$  be the closure of the branch locus  $B \subset H$ .

Let  $X_0, X_1, X_2, X_3, X_4, X_5$  denote the homogeneous coordinates, with  $x = X_1/X_0, y = X_2/X_0, z = X_3/X_0, u = X_4/X_0, v = X_5/X_0$ .

Since the highest order term of  $f$  is the monomial  $xyzuv$ , it follows that the intersection of  $\overline{H}$  with the  $\mathbb{P}^4 := \mathbb{P}^5 - \mathbb{A}^5$  at infinity, decomposes as a union of divisors

$$\overline{H} \cap \mathbb{P}^4 = \bigcup_{j=1}^5 D_j$$



where  $D_j$  is given by  $X_0 = X_j = 0$ . We have  $D_j \cong \mathbb{P}^3$ . Where appropriate, indices  $j = 1, \dots, 5$  will be considered in cyclic ordering, so for example if  $j = 4$  then  $j + 2 := 1$ .

**Lemma 3.1.** *The singular locus of  $\overline{H}$  decomposes as*

$$\text{Sing}(\overline{H}) = \overline{B} \cup \bigcup_{j=1}^5 (M_j \cup N_j)$$

where  $M_j, N_j \subset D_{j+2} \cap D_{j-2}$  are lines given by the following equations: both satisfy  $X_0 = X_{j+2} = X_{j-2} = 0$ , then for  $M_j$  we have  $X_{j+1} = \sqrt{-1}X_{j-1}$  and for  $N_j$  we have  $X_{j+1} = -\sqrt{-1}X_{j-1}$ .

*Proof.* In the computer algebra program we take the jacobian ideal of the equation for  $\overline{H}$ , then calculate its primary decomposition.  $\square$

A resolution of singularities will therefore require blowing up the lines  $M_j$  and  $N_j$ . However, we also notice the following:

**Lemma 3.2.** *The singular locus of  $\overline{B}$  is the union of five lines*

$$\text{Sing}(\overline{B}) = \bigcup_{j=1}^5 L_j$$

where  $L_j = D_j \cap D_{j+2} \cap D_{j-2}$  is the line given by equations  $X_0 = X_j = X_{j+2} = X_{j-2} = 0$ . This union of five lines is also equal to the intersection of  $\overline{B}$  with the divisor at infinity.

*Proof.* Again in the computer program we calculate the primary decomposition of the ideal given by  $3 \times 3$  minors of the  $10 \times 6$  jacobian matrix of the set of generators for the ideal of  $\overline{B}$ , homogenizations of the polynomials written at the end of the proof of Proposition 2.1. For the intersection of  $\overline{B}$  with  $\mathbb{P}^4$  at infinity, look at the highest order terms of the equations for  $B$ : they are cyclic permutations of  $x^2y^2$  and  $xy^2v$ , so their radical is generated by cyclic permutations of  $xy$ . It says that no two consecutive coordinates should be nonzero, and that constrains onto the union of  $L_j$ .  $\square$

Our resolution strategy will consist of first blowing up the lines  $L_j$  in some order. The strict transforms of  $M_j$  and  $N_j$  become disjoint, and we can resolve them separately.

#### 4. DOUBLE COVERING OF THE SIMPLEX AT INFINITY

Recall that  $M_B$  is a  $2 : 1$  covering of  $H$  branched along  $B$ . To get a compactification of  $M_B$ , we extend this to a normal double covering

$$\overline{\phi} : \overline{M}_B \rightarrow \overline{H}$$

and then to a double covering of the resolution

$$\tilde{\phi} : \tilde{M}_B \rightarrow \tilde{H}.$$

Before getting to the resolution process, let us first consider what the double cover looks like on the simplex made out of the divisors  $D_j$ .

**Lemma 4.1.** *The inverse image of  $D_j$  in  $\overline{M}_B$  is a union of two distinct isomorphic components  $D_j^+$  and  $D_j^-$ . The double intersections  $D_j \cap D_k$  similarly decompose into two pieces. The triple intersection lines decompose into two pieces except for the lines  $L_j$ , and the union of these five lines constitutes the ramification locus at infinity.*

*Proof.* Extend to infinity the description that was used in the proof of Proposition 2.1. Letting  $a, b \rightarrow \infty$  there is a canonical choice of determination of  $\sqrt{a^2 - 1}$  (resp.  $\sqrt{b^2 - 1}$ ), namely the one that is asymptotically equivalent to  $a$  (resp.  $b$ ).

Consider a curve in  $M_B$  approaching a general point of  $D_5$ , with parameter  $w \rightarrow 0$ . Assume given the homogeneous coordinates  $X_i(w)$  with  $X_0 = w$ . Write  $a = a'/w$ ,  $b = b'/w$ , and set  $1/\sqrt{a^2 - 1} = \sigma w$  (resp.  $1/\sqrt{b^2 - 1} = \tau w$ ). We have

$$P = (a' + w)(b' - w)p^2/w^2 =: P'/w^2, \dots$$

and our affine coordinates become

$$y = X_2/X_0 = \sigma(P' - Q' + R' - S')/w, \dots$$

so that

$$X_2(w) = \sigma(P' - Q' + R' - S'), \quad X_3(w) = \tau(-P' - Q' + R' + S'),$$

whereas, because of the extra term in front of  $v$ ,

$$X_5(w) = -\sigma\tau(P' + Q' + R' + S')w.$$

The equation  $\text{Tr}(\rho(\gamma_3)) = 0$  becomes

$$2a'b' + P' - Q' - R' + S' = 0.$$

It is easy now to consider  $X_5 \rightarrow 0$  so as to approach a general point of  $D_5$ : it just means that  $a', b', \sigma, \tau, P', Q', R', S'$  should have generic bounded limiting values. The limiting values of  $X_1, X_2, X_3, X_4$  and  $X_5/w$  together with the equation  $\text{Tr}(\rho(\gamma_3)) = 0$  yield limiting values for  $P', Q', R', S'$ . These are subject to the equation (homogenization of  $f = 0$ ), corresponding to existence of a solution with  $\det(g) = 1$ . Now, the limiting values determine  $p^2, q^2, r^2, s^2$  and as before, there are two distinct choices of matrix  $g$  with  $\det(g) = 1$ , up to multiplication by  $-1$ .

This discussion proves that the covering  $\overline{\phi} : \overline{M}_B \rightarrow \overline{H}$  doesn't ramify at generic points of the divisors  $D_j$ . Now, purity of the branch locus says that over the smooth part of  $\overline{H}$ , the ramification locus has pure codimension 1. But as it is empty in the interior, and doesn't contain general points of the divisors at infinity, it implies that  $\overline{\phi}$  is unramified outside of the singular locus of  $\overline{H}$ . As we have seen in Lemmas 3.1 and 3.2, this singular locus intersected with the divisor at infinity, consists of a collection of lines. In particular, within any of the divisors  $D_j$  it has codimension 2, but as these

are smooth, it follows that the covering must decompose over each  $D_j$ . We make a choice of components to label them by  $D_j^+$  and  $D_j^-$ .

From Lemma 3.1, the singular locus of  $\overline{H}$  contains lines  $M_j, N_j$ . We would like to see that each of these splits into two irreducible components in  $\overline{M}_B$ . For this, we refer to the resolution  $\widetilde{M}_B \rightarrow \widetilde{H}$  to be calculated below. Near a general point of  $M_j$  it consists of blowing up the line  $M_j$  once. The strict transform of  $D_{j-2}$  intersects the exceptional divisor transversally. As we know that the covering has no monodromy over  $D_{j-2}$ , it follows that the covering  $\tilde{\phi}$  doesn't ramify along this exceptional divisor. We shall furthermore see that the exceptional divisor, over the open subset  $\mathbb{G}_m \subset M_j$ , is just  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{G}_m$  and the intersection with  $D_{j-2}$  is of the form  $\mathbb{P}^1 \times \mathbb{G}_m$ . Hence, the covering  $\tilde{\phi}$  has trivial monodromy over the exceptional divisor, i.e. its inverse image in the covering splits into two irreducible components; choosing  $\overline{M}_B$  as a maximal covering extending  $M_B$ , the inverse image of the line  $M_j$  splits into two irreducible components. The same argument holds for  $N_j$ .

This shows that  $M_j, N_j$  are not part of the ramification locus, hence the ramification locus at infinity is made up of the five lines  $L_j$ .  $\square$

We may now form a picture of the dual complex of the divisor at infinity in  $\overline{M}_B$ , even if it isn't normal crossings. There are 10 irreducible components  $D_j^\pm$ , forming a double cover of the simplex  $\bigcup D_j$ . This double cover is ramified along the reducible curve formed out of the five lines  $L_j$ . Notice that their order is changed:  $L_1$  given by  $X_0 = X_1 = X_3 = X_4 = 0$  intersects  $L_3$  given by  $X_0 = X_3 = X_5 = X_1 = 0$  at the point  $[0 : 0 : 1 : 0 : 0 : 0]$ , and  $L_1$  intersects  $L_4$  given by  $X_0 = X_4 = X_1 = X_2 = 0$  at the point  $[0 : 0 : 0 : 1 : 0 : 0]$ . Hence, the five lines form a pentagon in the order

$$(3) \quad L_1 \rightarrow L_3 \rightarrow L_5 \rightarrow L_2 \rightarrow L_4 \rightarrow L_1.$$

The dual complex of the divisor  $\bigcup D_j^\pm$  is still the sphere  $S^3$ : it is a double cover of the original simplex with ramification along the circle of lines. This will be the basic shape of the dual complex of our resolution: the further steps of blowing up lines don't modify its homotopy type.

## 5. RESOLUTION PROCESS

**Theorem 5.1.** *First blow up the lines  $L_j$  in some order. After that, the strict transforms of the  $M_j$  and  $N_j$  become disjoint and we can blow them up. Let  $\tilde{H} \subset \mathbb{P}^5$  denote the resulting hypersurface, and let  $\tilde{B}$  be the strict transform of  $\overline{B}$ . Let  $\mathbf{D}(\tilde{H})$  denote the reduced inverse image of the divisor at infinity  $\overline{H} \cap \mathbb{P}^4$ . Then:*

- (a)— $\mathbf{D}(\tilde{H}) = \bigcup_j \tilde{D}_j \cup \tilde{L}_j \cup \tilde{M}_j \cup \tilde{N}_j$  where  $\tilde{D}_j$  is the strict transform of  $D_j$ , and  $\tilde{L}_j$  (resp.  $\tilde{M}_j, \tilde{N}_j$ ) is the exceptional divisor over  $L_j$  (resp.  $M_j, N_j$ );
- (b)— $\tilde{B}$  is smooth, and the only components of  $\mathbf{D}$  that it intersects are the  $\tilde{L}_j$  intersected smoothly;

- (c)—the  $\tilde{D}_j$ ,  $\tilde{M}_j$  and  $\tilde{N}_j$  are smooth;
- (d)— $\tilde{L}_j$  have curves of ordinary double points along their intersections with  $\tilde{B}$  but are smooth otherwise;
- (e)— $\tilde{L}_j \cap \tilde{L}_{j+2}$  is isomorphic to the Fricke-Klein cubic of equation (1);
- (f)—and the other multiple intersections of divisor components are irreducible and smooth, or empty.

*Proof.* This is checked in the coordinate charts at infinity obtained by dehomogenizing the homogenized equation of  $f$  at some other variable. By symmetry only one is needed. Luckily, the monoidal transformations we need are done along linear centers, essentially coordinate lines. To treat most easily the  $M_j$  and  $N_j$  it is convenient to multiply two of our coordinates by  $\sqrt{-1}$ , then the equations of some examples of  $M_j$  and  $N_j$  become defined over  $\mathbb{Q}$  (translations by 1 of coordinate lines) and the blow-ups can easily be calculated by computer. I kept track of the charts somewhat manually. One important point to notice is that near points  $L_j \cap L_{j+2}$  one of the lines is blown up first, the other one second. The choices are symmetrical. Point (e) is computed on the second exceptional divisor.  $\square$

Let  $\tilde{\phi} : \tilde{M}_B \rightarrow \tilde{H}$  be the  $2 : 1$  covering extending  $\phi$ . Let  $\mathbf{D}(\tilde{M}_B)$  be the inverse image of  $\mathbf{D}(\tilde{H})$ . Then it decomposes as

$$\mathbf{D}(\tilde{M}_B) = \bigcup_j \tilde{D}_j^\pm \cup \tilde{L}_j^M \cup \tilde{M}_j^\pm \cup \tilde{N}_j^\pm.$$

Here  $\tilde{D}_j^+$  and  $\tilde{D}_j^-$  are the two components of the inverse image of  $\tilde{D}_j$  in  $\tilde{M}_B$ , the same for  $\tilde{M}_j^+$ ,  $\tilde{M}_j^-$ , and  $\tilde{N}_j^+$ ,  $\tilde{N}_j^-$ . On the other hand the inverse image of  $\tilde{L}_j$  is a single divisor denoted  $\tilde{L}_j^M$ . It is a double cover of  $\tilde{L}_j$  branched along  $\tilde{B} \cap \tilde{L}_j$ . The intersection of  $\tilde{L}_j^M$  and  $\tilde{L}_{j+2}^M$  is the double cover of the Fricke-Klein cubic, compactifying the double cover  $\mathbb{G}_m \times \mathbb{G}_m$  over the affine cubic of equation (1) that corresponds to the elliptic curve covering of  $\mathbb{P}^1$  branched over 4 points (see [36, §13.4]). From this discussion we obtain the desingularized compactification of  $M_B$ :

**Theorem 5.2.** *The resulting pair  $(\tilde{M}_B, \mathbf{D}(\tilde{M}_B))$  is a simple normal crossings compactification of  $M_B$ . The dual complex of  $\mathbf{D}(\tilde{M}_B)$  is a triangulation of  $S^3$ .*

Our compactification depends on a choice of the order in which the  $L_j$  are blown up. Establish a choice that is almost canonical (except for an orientation): blow them up in the cyclic order (3) doing  $L_{j+2}$  after  $L_j$ . By cyclicity it isn't well-defined globally, however it is well-defined locally in the Zariski topology, yielding a compactification  $(\tilde{M}_B, \mathbf{D}(\tilde{M}_B))$  in the category of (not necessarily projective) schemes. We'll recall below that a different global ordering could be chosen that would give a projective scheme.

## 6. STRATIFICATION

The next objective is to stratify the divisor at infinity  $\mathbf{D}(\widetilde{M}_B)$  in a nice way corresponding to the Hitchin fibration. The stratification shall be denoted

$$\mathbf{D}(\widetilde{M}_B) = \coprod_{\eta} \mathbf{S}_{\eta}.$$

It is obtained from the standard stratification of the divisor with normal crossings, by grouping together certain groups of strata. In general when we say “open stratum” this means the open stratum in the standard stratification.

1. Define  $\mathbf{S}_{\lambda(j)}$  to be the open stratum of  $\widetilde{L}_j^M$ .
2. Let  $\mathbf{S}_{\lambda(j,j+2)}$  denote the open stratum of  $\widetilde{L}_j^M \cap \widetilde{L}_{j+2}^M$ .
3. Let  $\mathbf{S}_{\alpha(j,+)}$  (resp.  $\mathbf{S}_{\delta(j,-)}$ ) denote the open stratum of  $\widetilde{D}_j^+$  (resp.  $\widetilde{D}_j^-$ ).
4. Let  $\mathbf{S}_{\alpha(j,j+2,+)}$  denote the open stratum of  $\widetilde{D}_j^+ \cap \widetilde{D}_{j+2}^+$  and similarly for  $\mathbf{S}_{\delta(j,j+2,-)}$ .
5. Let  $\mathbf{S}_{\alpha(j,j+1,j+2,+)}$  denote the open stratum of  $\widetilde{D}_j^+ \cap \widetilde{D}_{j+1}^+ \cap \widetilde{D}_{j+2}^+$  and similarly for  $\mathbf{S}_{\alpha(j,j+1,j+2,-)}$ .
6. Group together the pieces  $\widetilde{D}_{j-2}^+ \cap \widetilde{D}_{j+2}^+$  with  $\widetilde{M}_j^+$  and  $\widetilde{N}_j^+$ , then take the open part of this in other words the complement of the intersections with other strata we have already considered. Call this  $\mathbf{S}_{\beta(j,+)}$  and define similarly  $\mathbf{S}_{\beta(j,-)}$ .
7. The boundary of  $\mathbf{S}_{\beta(j,+)}$  intersects  $\widetilde{L}_j^M$  in a piece whose open part is denoted  $\mathbf{S}_{\zeta(j,+)}$  and similarly for  $\mathbf{S}_{\zeta(j,-)}$ .
8. This part corresponds to the point  $X_j = 1$ , all the rest 0. The boundary of  $\mathbf{S}_{\beta(j,+)}$  intersects  $\widetilde{L}_{j-1}^M$  in a piece whose open part is denoted  $\mathbf{S}_{\xi(j,+)}$  and similarly for  $\mathbf{S}_{\xi(j,-)}$ . This uses the cyclic ordering of resolution of the  $L_j$ , in general the intersection will be with whichever of the pieces corresponding to  $L_{j-1}$  or  $L_{j+1}$  was blown up first.
9. There are a few extra isolated points for which no notation is needed.

For each of these strata  $\mathbf{S}_{\eta}$ , let  $\mathbf{T}_{\eta}^*$  denote the punctured tubular neighborhood. The link at infinity of  $M_B$  may be viewed as obtained by glueing together these pieces  $\mathbf{T}_{\eta}^*$ .

View  $\mathbf{S}_{\eta}$  as corresponding to locations in the 3-sphere at infinity of the Hitchin base, and the pieces  $\mathbf{T}_{\eta}^*$  as corresponding to the Hitchin fibers over these locations. We’ll call the  $\mathbf{T}_{\eta}^*$  “Betti Hitchin fibers”.

## 7. DESCRIPTION OF THE BETTI HITCHIN FIBERS

The first proposition describes the Betti Hitchin fibers that look like smooth tori. These correspond to the smooth fibers of the Hitchin fibration.

**Proposition 7.1.** *In the above stratification, the following pieces have the structure of tori:*

- $\mathbf{S}_{\lambda(j,j+2)} \cong \mathbb{G}_m \times \mathbb{G}_m$ ;
- $\mathbf{S}_{\alpha(j,\pm)} \cong \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m$ ;
- $\mathbf{S}_{\alpha(j,j+2,+)} \cong \mathbb{G}_m \times \mathbb{G}_m$ ;
- $\mathbf{S}_{\alpha(j,j+1,j+2,+)} \cong \mathbb{G}_m$ ;
- along with the isolated points.

If  $\mathbf{S}_\eta$  is one of these toroidal pieces of dimension  $d$ , then  $\mathbf{T}_\eta^*$  looks like a bundle over it with fibers being  $(\Delta^*)^{4-d}$  so that homotopically  $\mathbf{T}_\eta^* \cong (S^1)^4$ .

*Proof.* These are mostly easy to see. For  $\mathbf{S}_{\lambda(j,j+2)}$ , as was discussed above Theorem 5.2, the affine cubic given by equation (1) has a unique smooth double cover ramified at the singular points, and that cover is  $\mathbb{G}_m \times \mathbb{G}_m$ .  $\square$

The next proposition isolates the structure of degenerate fibers:

**Proposition 7.2.** *The punctured neighborhoods  $\mathbf{T}_{\beta(j,\pm)}^*$ ,  $\mathbf{T}_{\zeta(j,\pm)}^*$ , and  $\mathbf{T}_{\xi(j,\pm)}^*$  are all homotopic to*

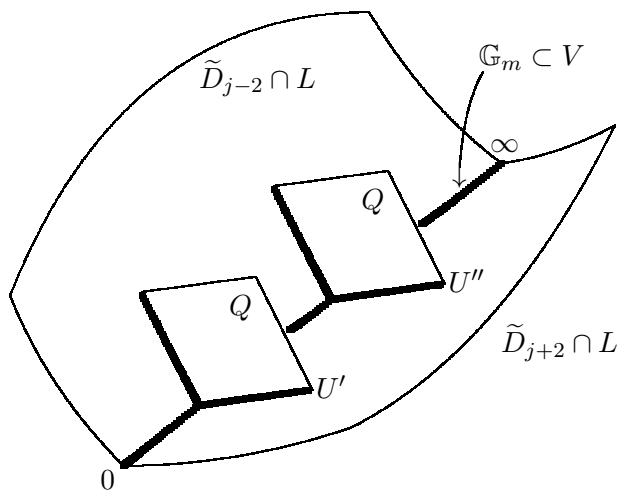
$$S^1 \times S^1 \times (S^2 \vee S^2 \vee S^1).$$

*Proof.* Recall that these strata correspond to starting with  $D_{j-2} \cap D_{j+2} \cong \mathbb{P}^2$ , taking its strict transform under the first transformations along the lines  $L$ , then blowing up  $M_j, N_j$ , and finally going to the covering. The covering just consists of two identical disjoint pieces so everything can be pictured within the blow-up of  $\mathbb{P}^5$ .

After blowing up the line  $M_j$ , the exceptional locus  $\widetilde{M}_j$  has the form  $Q \times M_j$  where  $Q \subset \mathbb{P}^3$  is a quadric surface, isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ . It meets  $\widetilde{L}_j$  at  $Q \times \{p\}$  and  $\widetilde{L}_{j-1}$  at  $Q \times \{p'\}$  for two points  $p, p' \in M_j$ . The same holds for  $\widetilde{N}_j$ . The strata  $\mathbf{S}_{\zeta(j,\pm)}$  and  $\mathbf{S}_{\xi(j,\pm)}$  are isomorphic to  $(\mathbb{G}_m \vee Q \vee Q)$ , a  $\mathbb{G}_m$  with two copies of  $Q$  attached at the points  $\pm\sqrt{-1}$ . The strata  $\mathbf{S}_{\beta(j,\pm)}$  are isomorphic to the product of this diagram, with  $\mathbb{G}_m$ . We may look either at one of the endpoint strata, or at a slice of the bigger product stratum. It meets the other divisors as follows: the line  $\mathbb{G}_m$  corresponds to  $\widetilde{D}_{j-2} \cap \widetilde{D}_{j+2} \cap L$  (here  $L$  denotes either one of the  $\widetilde{L}$ , or a slice for the  $\beta$  stratum) whereas the two copies of  $Q$  meet the divisors in two  $\mathbb{P}^1$ 's. The configuration may be pictured as in Figure 1.

One may then consider the punctured tubular neighborhood of the configuration  $(\mathbb{G}_m \vee Q \vee Q)$ , it turns out<sup>3</sup> to be homotopic to  $S^1 \times (S^2 \vee S^2 \vee S^1)$ . This then should be producted with another copy of  $S^1$ , either for the  $\mathbb{G}_m$  direction in the case of the  $\beta$  strata, or for the juncture with one of the  $\widetilde{L}$  in the case of the  $\zeta$  and  $\xi$  strata.  $\square$

<sup>3</sup>Here is a brief discussion. Factor out a copy of  $S^1$  for the normal direction of the whole configuration. Then, for  $\mathbb{G}_m$  minus the two points we have the complement of four points in  $\mathbb{P}^1$ , that gives a wedge of three circles. That part should be producted with  $S^1$  because it is at the junction of two divisors. Each of the  $Q$  minus the intersection lines is just  $\mathbb{A}^2$ , so contractible. These are attached into the  $\mathbb{G}_m$  part along the neighborhoods of points, that correspond to  $(S^1)^2$ . So our space is  $(S^1 \vee S^1 \vee S^1) \times S^1$ , to which we contract two copies of  $(S^1)^2$ . This in turn gives  $(S^1)^2$  with the same circle contracted two times, and that is  $S^2 \vee S^2 \vee S^1$ .

FIGURE 1. A slice of  $\mathbf{S}_{\beta(j,\pm)}$ 

**Discussion:** Recall that a degenerate elliptic curve made up of two copies of  $\mathbb{P}^1$  meeting in two points, is homotopically  $(S^2 \vee S^2 \vee S^1)$ . From the above proposition, one therefore guesses that the degenerate fibers of the Hitchin fibration should look like elliptic curves times degenerate elliptic curves, where the degenerating factor has two irreducible components in this way. Looking on the side of moduli of parabolic Higgs bundles one may see that this is indeed the case: the degenerate Hitchin fibers for our situation have two irreducible components.

We finally note that there is a more complicated piece.

**Proposition 7.3.** *If the resolution of  $L_j$  was done in cyclic ordering, then the stratum  $\mathbf{S}_{\lambda(j)}$  is homotopically  $S^1 \times (S^2 \vee S^2 \vee S^2)$ . The punctured tubular neighborhood  $\mathbf{T}_{\lambda(j)}^*$  is  $(S^1)^2 \times (S^2 \vee S^2 \vee S^2)$ .*

*Proof.* The open stratum is obtained by blowing up  $L_j$  once in an affine chart at infinity (replacing coordinate  $u$  by a coordinate  $w$  at infinity), and using the chart complementary to  $w = 0$ . Here the equation becomes

$$xyzv + x^2y^2 + x^2v^2 - 4x^2 + z^2 + v^2 - 4 = 0.$$

Our stratum is a double cover branched over the singular locus. This may be calculated in the following way: consider the above equation as quadratic in the coordinate  $z$ ; its discriminant decomposes as a product

$$\Delta = (x^2y^2 - 4x^2 - 4)(v + 2)(v - 2).$$

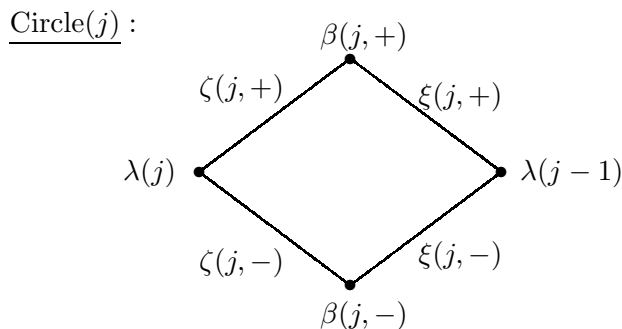
Our hypersurface is a double cover branched over the discriminant but the required stratum is a double cover of that branched over the singular locus. The stratum is thus the product of a surface, double cover of the  $x, y$  plane

branched over  $x^2y^2 - 4x^2 - 4 = 0$ , with the double cover  $\mathbb{G}_m$  over the  $v$ -line branched at  $v = \pm 2$ . It remains just to identify the surface. Consider  $Y := \mathbb{P}^1 \times \mathbb{P}^1$  with two smooth  $(1,1)$ -curves  $C, C' \subset Y$ , and two points  $p', p'' \in C'$  not at the two intersection points. Blow up  $Y$  at  $p', p''$  then remove the strict transforms of  $C, C'$ ; this is our double cover. Without blowing up the points one could see the complement  $Y - C - C'$  is  $S^2$ . Adding in the exceptional  $\mathbb{A}^1$ 's at the blown-up points just adds two more disks over boundary circles, so altogether the surface is homotopy equivalent to  $(S^2 \vee S^2 \vee S^2)$ . Its product with  $\mathbb{G}_m$  gives the stratum  $\mathbf{S}_{\lambda(j)}$ . The stratum is a smooth divisor in  $\tilde{H}$ , defined in its chart by a single equation, so the punctured tubular neighborhood is a trivial  $S^1$  bundle over it.  $\square$

**Discussion:** What is going on here? It turns out that the zone of the Hitchin base covered by  $\mathbf{S}_{\lambda(j)}$  meets two of the discriminant circles, and the Betti Hitchin fiber  $\mathbf{T}_{\lambda(j)}^*$  corresponds to two degenerations attached along their smooth fibers. This phenomenon seems to be unavoidable. If we change the order in which the  $L_j$  are blown up, the pieces of circles are shifted around. For example, if some line is blown up first with both adjoining ones blown up later, then it recovers the homotopy type of a single degeneration as in Proposition 7.2. But in this case there will also be a line that is blown up last, which corresponds to three degenerations glued together.

### 8. FIVE CIRCLES

Let's see how to fit the above pieces together into circles of degeneration in the sphere at infinity. A circle is formed by the following pieces of the stratification, which we indicate by their subscripts using a dual graph notation:



Recall that the  $\beta$ ,  $\zeta$  and  $\xi$  pieces have Betti Hitchin fibers that correspond to the degenerate fibers of the Hitchin fibration, whereas the  $\lambda$  pieces correspond to combinations of two degenerate fibers.

The diagrams we have pictured here, yield 5 circles inside the  $S^3$  dual graph of the resolution; but each one touches two other ones at the  $\lambda$  pieces.

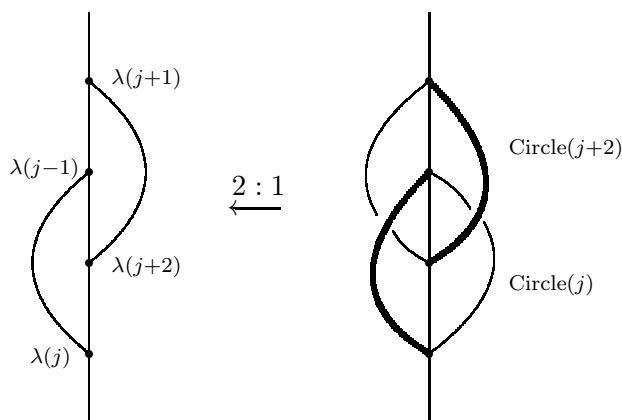


As we discussed above, this touching seems to be an unavoidable phenomenon (it is an interesting theoretical question to make a precise statement and to understand why).

We can, nonetheless, look at pairs of disjoint circles, namely ones as pictured above for indices  $j$  and  $j + 2$ . In the picture of the actual Hitchin fibration, recall that our circles correspond to circles in  $S^3$  that are fibers over the  $\mathbb{P}^1$  at infinity of the Hitchin base, in other words they are fibers of the Hopf fibration  $S^3 \rightarrow \mathbb{P}^1 = S^2$ . Therefore, we expect them to be simply linked.

It is indeed the case that they are linked. To understand this, recall that our  $S^3$  was obtained as a double cover of the original simplex at infinity, ramified along the pentagonal  $S^1$  composed of lines  $L_j$ .

Our  $\text{Circle}(j)$  is given by looking at the preimage of the open piece inside  $D_{j-2} \cap D_{j+2}$  together with the line  $L_j$  and the point  $L_{j-1} \cap L_{j+1}$ . We may therefore view  $\text{Circle}(j)$  as being obtained by joining together two points on the pentagonal  $S^1$  in the original simplex, then taking the preimage by the double cover. And  $\text{Circle}(j + 2)$  is obtained similarly, joining points that alternate, in the order (3), with the points for  $\text{Circle}(j)$ . When going to the double cover we can view the picture as follows. The vertical line represents the pentagonal  $S^1$ , and the right diagram is the double cover of the left one ramified along this vertical line (with thick lines for the  $+$  sheet of the covering pictured in front).



The circles are linked.

**Monodromy:** Let us look at the monodromy around these circles [1]. It corresponds to the monodromy action on the smooth Hitchin fibers as we go around the discriminant locus. Here, it will act as a transformation of the smooth Hitchin fiber  $(S^1)^4$ , obtained when going once around one of the circles. That can be viewed in the slice pictured in Figure 1. Recall that the pieces  $\tilde{D}_{j-2}$  and  $\tilde{D}_{j+2}$  correspond to strata of the form  $\mathbb{G}_m^3$ , multiplied by  $\Delta^*$  for the tubular neighborhood. The points at 0 and  $\infty$  of the  $\mathbb{G}_m$

in Figure 1 also correspond to  $(\Delta^*)^4$ , so we should follow how these glue together as we go from the endpoint 0 through  $\tilde{D}_{j+2}$  to the endpoint  $\infty$  and then back through  $\tilde{D}_{j-2}$  to the endpoint 0 again.

**Proposition 8.1.** *The monodromy transformation acting on  $H_1$  of the fiber, is given by the matrix*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

*Proof.* In Figure 1, there are transverse divisors at the two endpoints  $0, \infty$  (not pictured). Let us calculate in the slice. A punctured neighborhood of 0 (resp.  $\infty$ ) has the form  $(\Delta^*)^3$  with homology generated by loops  $\delta_{j-2}$  and  $\delta_{j+2}$  around  $\tilde{D}_{j-2}$  and  $\tilde{D}_{j+2}$  respectively, and  $\nu_0$  (resp.  $\nu_\infty$ ) around the transverse divisor at 0 (resp.  $\infty$ ). Moving on the side of the divisor  $\tilde{D}_{j+2}$  from 0 to  $\infty$ , the loops  $\delta_{j-2}$  and  $\delta_{j+2}$  stay the same. On the other hand, the endpoint loops undergo

$$(4) \quad \nu_0 \mapsto -\nu_\infty - \delta_{j-2} + \delta_{j+2}.$$

To explain this calculation, consider curves  $V, U', U''$  in  $\tilde{D}_{j+2}$  as pictured in Figure 1:  $V = \tilde{D}_{j+2} \cap \tilde{D}_{j-2}$ , and  $U', U''$  are the two intersections of the pieces  $Q$  with  $\tilde{D}_{j+2}$ . A curve in  $\tilde{D}_{j+2}$  that joins points near 0 to points near  $\infty$ , will be linearly equivalent to  $V + U + U'$ . Such a curve must intersect  $\tilde{D}_{j-2}$  in one point, and that (with a change of orientation also seen in the coefficient of  $\nu_\infty$ ) yields the  $-1$  coefficient of  $\delta_{j-2}$  in (4). On the other hand, the normal bundle to  $\tilde{D}_{j+2}$  in  $\tilde{H}$  has degree  $-1$  on  $V$  (inside  $\tilde{D}_{j-2}$ ,  $V$  is a  $-1$ -curve since it is obtained by blowing up twice a line in projective space). The normal bundle to  $\tilde{D}_{j+2}$  in  $\tilde{H}$  is trivial on  $U'$  and  $U''$  since these are lines inside  $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ . Thus, the normal bundle restricted to  $V + U' + U''$  has degree  $-1$ : this gives the coefficient of  $\delta_{j+2}$  in the expression (4).

Going back from  $\infty$  to 0 but inside the divisor  $\tilde{D}_{j-2}$  has the corresponding effect  $\nu_\infty \mapsto -\nu_0 - \delta_{j+2} + \delta_{j-2}$ . Putting these together gives our monodromy operation:

$$\nu_0 \mapsto -(-\nu_0 - \delta_{j+2} + \delta_{j-2}) - \delta_{j-2} + \delta_{j+2} = \nu_0 - 2\delta_{j-2} + 2\delta_{j+2}.$$

This is the first column of the matrix. The rest of the matrix is the identity, first as we have said on  $\delta_{j-2}, \delta_{j+2}$ ; then also on the extra copy of  $S^1$  gotten by considering that the full picture of our strata before slicing is obtained by either product with a small tube (for the  $\zeta$  and  $\xi$  strata) or product with  $\mathbb{G}_m$  (for the  $\beta$  strata).  $\square$

This matrix corresponds to the monodromy in the actual Hitchin fibration, recalling that the degeneration has two components so it is an elliptic curve product with a two-piece elliptic degeneration. The two components

$Q$  in Figure 1 seem to correspond closely to the vanishing cycles. At the moment we don't have a more precise statement of this relationship.

## 9. FURTHER QUESTIONS

A number of further directions naturally present themselves.

**WKB theories:** It would be good to understand how the geometrical picture presented here corresponds to the WKB approximations near the boundary of the Hitchin moduli space as well as the moduli space of vector bundles with integrable connections. This would provide an example fitting into a number of theories of current interest, such as cluster varieties [12, 18, 19, 34, 45], spectral networks [13], wallcrossing [31, 32] stability conditions [7, 20], buildings [11, 28, 40], abelianization at infinity [35, 37], isomonodromy [4, 15], and others.

It is likely that the WKB geometry will coincide nicely with the picture presented here, only for certain positions of the points  $t_i$ —in other chambers of the moduli space of 5-pointed projective lines, other compactifications of  $M_B$  are probably needed. Optimally this should be taken care of by the theory of [18, 19].

**The  $P = W$  conjecture:** Can one give an explanation of the  $P = W$  phenomenon [9, 21, 22, 23, 33] in terms of the geometrical description? There are several difficulties, for example the fact that we had to combine together several strata for the degenerate Hitchin fibers, and the fact that the circles tend sometimes to meet in the Betti picture.

**Real structures and branes:** In the spirit of [2, 3], it is natural to ask about the position of real subvarieties on both sides of the picture.

Our Fricke-Klein equation is naturally adapted to a real structure on  $\mathbb{P}^1$  whose real circle contains the points  $t_1, \dots, t_5$  in cyclic order. The antiholomorphic involution  $\sigma$  from  $\mathbb{P}^1 - \{t_1, \dots, t_5\}$  to itself provides an algebraic involution of  $M_B$  that preserves our coordinate functions. It is, by the way, the involution associated to our 2 : 1 covering  $\phi$ .

The fixed-point set of the involution is the ramification locus  $B$ . On the Hitchin side, it will correspond to the real locus of  $M_H$  with respect to this real structure. It should be interesting to investigate more precisely the relative positioning of this real locus with respect to the discriminant locus and degenerate Hitchin fibers.

In the other direction,  $M_B$  is defined over  $\mathbb{Q}$  so it also has a real structure given by antiholomorphic involution. In our picture, recall that the lines  $M_j$  and  $N_j$  were defined with a  $\sqrt{-1}$ , so this involution will interchange them. On the Hitchin side it is well-known that this involution corresponds to multiplying the Higgs field by  $-1$ , giving an involution of the spectral curve. From our picture (see the footnote in the proof of Proposition 7.2) we expect that this should exchange the two vanishing cycles in the fiberwise degenerations.

**The Hitchin component:** Continuing on the subject of real structures, it will be interesting to understand the position of the Hitchin section, as well as other components of the locus of real representations, with respect to the Hitchin fibration from the Betti viewpoint.

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