# Gödel and the Paradox in *Max Phil* X

Gabriella Crocco[1] and Julien Bernard[2*]
[1 2]Aix Marseille University, CEPERC, UMR 7304, CNRS, France
[2]University of Konstanz, Germany

There is a main thesis in the *Russell paper* which postulates that concepts —conceived as properties and relations in intension— exist independently of our definitions and constructions (Gödel 1990 p. 128). This thesis has three remarkable consequences that are developed throughout the entire article:

a1. concepts exist separately from their arguments. In other words, concepts don't presuppose their domain of application, neither for their definition nor for their being (Gödel 1990 pp. 125-6);

a2. concepts are independent of the propositions in which they occur. They don't presuppose the semantic unity of the proposition from which they would be extracted as pieces, neither for their definition nor for their being (Gödel 1990 p. 137); and

a3. as intensional structures, differing from classes, concepts can be applied to themselves (Gödel 1990 p. 130).

This last consequence seems to give a foundation to the autonomy of concepts, more than the other two do. It sharply distinguishes concepts from classes and sets for which self-belonging (self-membership) is excluded.

There are many reasons why Gödel insists on this self-belonging or self-application. Indeed, concepts are not only structures of the real and the possible, which generate space-temporal objects, but they are also what make the space-temporal world intelligible.[1] Therefore, they are involved in every process of perception and understanding. The fact that Gödel stressed that a kind of self-application is involved in the incompleteness theorem (Gödel 1990 p. 130) is a symptom of the importance of self-application that Gödel considers as a real principle of intelligibility. Human knowledge, i.e. knowledge of being capable of self-consciousness, must be based on concepts, which cannot be reduced to material entities. Sets of objects and mathematical objects are quasi-material entities. The self-applicability of a concept is indeed the only property that seems to justify, from a logical point of view, dealing differently with concepts in intension than with sets. The persistence of this thesis, until the discussions with Wang in the 1970's (Wang 1996 chap. 8) explains in

* Julien Bernard proposed the proof sketched out in section 2, during his post-doc stage at CEPERC financed by the ANR directed by G. Crocco. The interpretation of this proof is due to the intensive collaboration between the two authors, in the last few years.

1 Cf. Gabriella Crocco's "Sinn/Bedeutung and intensio/extensio" in this collective book.

particular Gödel's lasting interest in the paradox of concepts. This paradox remains, according to Gödel, the biggest open problem in logic. According to what he said to Wang, sets are quasi-physical objects (Wang 1996, 8.2.4). That's why there is no self-applicability for them. Or, again:

> 8.6.3 […] while no set can belong to itself, some concepts can apply to themselves: the concept of *concept*, the concept of being applicable to only one thing (or one object), the concept of being distinct from the set of all finite mathematical sets, the concept of being a concept with an infinite range, and so on. It is erroneous to think that to each concept there corresponds a set.

*Max Phil* IX confirms the importance of this question for Gödel. It repeats the diagnosis that concludes the *Russell paper*: paradoxes are the sign of our defective and incomplete understanding of the notions of concept and of class that are at the foundation of logic. This notebook also confirms something which had been speculated in a previous paper: [2] the independence of concepts from their arguments and values implies the existence of a relation of applying (the *esti* about which Wang and Gödel discussed (Wang 8.6.17- 8.6.18)), whose properties have to be understood, in order to solve the paradoxes. This is explicitly expressed by the following *Bemerkung* in *Max Phil* IX:

> [48] *Bem<erkung>* (*Gr<ammatik>*): Letzter Grund für *Antin<omien>* ist, dass wir nicht sehen, was die ∊-Relation eigentlich ist [im Reich der Begriffe], sondern wir sehen einen Ersatz in dem, was wir konstruiert haben. Ebensowenig sehen wir, was der Begriff „Begriff" ist.

Gödel says that to understand the notion of concept means to understand the concept of application (die ∊-Relation) between a concept and its argument(s). This is possible only if we distinguish the subjective concept of application, as we use it in our finite understanding, from the objective concept, as it realizes itself in the realm of concepts. If we clearly grasp this, we release ourselves from the paradoxes.

Actually, Gödel had already explained in the *Russell Paper* that his own diagnosis of the paradoxes was opposed to two of Russell's solutions to the (intensional and extensional) paradoxes, which are discussed in the same paper. Indeed, the reality of concepts, with its three consequences, is not compatible with the strategy of Russell's *no-class theory*, which is presented by Gödel as being essentially based on the *Vicious Circle Principle*, and which is central for the construction of the ramified theory of types presented in the first edition of the *Principia Mathematica*. It is also not compatible with the extensional solution given by Russell in the second edition of the *Principia*, that is an extensional simple theory of types, where the propositional functions occur in propositions only through their values, and where the leading principle is that of unsaturatedness (or *typical ambiguity*), according to which propositional functions are nothing but pieces of propositions.

But what about the other possible solutions to the (intensional and extensional) paradoxes that are indicated in the *Russell paper*, i.e. the Zig-Zag and the limited range of significance strategies? Russell's Zig-Zag theory is, according to Gödel, a theory based on the idea of denying that every propositional function expresses a

2 (Crocco 2006).

concept. The association of a concept to a propositional function is possible only if some conditions are fulfilled.[3]

Gödel also mentions the strategy of the *limited ranges of significance*, based on the idea of denying that a concept can always be meaningfully applied to any argument. Rather, for some concepts, there can be singular points of inapplicability, which if not avoided lead to paradoxes. In so far as our understanding is finite, these singular points can only be apprehended by us *a posteriori*.

What is Gödel's position regarding these two solutions, given that they can both be arranged with the self-application of concepts? In [Crocco 2006], the author conjectured that Gödel's position should be to prefer the second solution. Is this conjecture confirmed by *Max Phil* IX and X? Yes, without any doubt as we will see later on. At the same time, we find—and it was not expected—that Gödel explored the first strategy, more precisely a version of the Zig-Zag strategy linked to the ramified theory of types and to the axiom of reducibility. This article is devoted to the analysis of the remarks that present such a solution, and to the analysis of the context that permits us to understand its interest. It will be divided in two parts. The first part will contain the two remarks from *Max* IX, (p. 48b, 51) on the question of the strategies for the solution of paradoxes. They clearly show Gödel's preference for the strategy of *limited ranges of significance.* We will also present the passages that could explain why Gödel was also interested in the other strategy suggested by Russell (*Max Phil* IX 69 and 72b). The second part presents a remark, from *Max Phil* X (pp. 28-29), and explains it from the perspective of our discussion.

## Section 1. The diagnosis of paradoxes

*Max Phil* IX contains two very explicit remarks about Gödel's diagnosis regarding the intensional paradoxes. They are presented on two different pages of this notebook but both of them make reference to the analysis that is contained in the course on logic that Gödel gave at the University of Notre Dame in 1939, thanks to Karl Menger's invitation.[4]

> **[48]** *Bem<erkung>* (*Gr*<*ammatik*>): Auf Grund meiner *Anal*<*yse*> der (*intens*<*ionalen*>) Antinomie (in der *Notre D*<*ame*> Vorlesung) gibt es (wenn der Aussagekalkül beibehalten wird) zwei Möglichkeiten, einen Fehler zu sehen:

3 In (Russell, 1905), the author considered two sub-strategies within the strategy consisting in searching for a condition for the association of a concept to a propositional function. The first substrategy was the *Zigzag theory*, for which the condition is expressed in term of the "*simplicity*" of the propositional function. The second sub-strategy was the theory of the *limitation of size*, for which the criterion was expressed in term of the number of individual objects to which the propositional function applies. In the following we will use (like Gödel) the term "Zigzag strategy" in a wider sense than Russell. It includes all kinds of strategy proposing to search a criterion for the association of a concept to a propositional function. As we will see, even the *no-class theory* (associated with the ramified theory of types) will be included in this category, as a limiting case. Only the strategy of the *limited range of significance* will be considered by Gödel as a radically different strategy.

4 P. Cassou-Nogues, "Gödel's introduction to logic in 1939", *History and Philosophy of logic*, 30-1, pp. 69-90, 2009.

> 1.) Nicht jede definierte Aussagefunktion definiert einen Begriff (das ist anscheinend nicht das Richtige, denn es hätte zur Folge, dass $x \notin x$ tatsächlich für <u>alle</u> Gegenstände definiert ist und dennoch keinen Begriff definiert).
>
> 2.) Der Begriff „definierter Begriff“ ist nicht definiert. Das bedeutet, es gibt „Randwerte“ oder „singuläre Stellen“, [5] worauf er nicht anwendbar <ist>. Es muss aber außerdem der Begriff „sinnvoll anwendbar“ nicht definiert sein [sonst könnte man definieren: falsch überall, wo sinnlos.]
>
> Die Typentheorie macht die Annahme 2., aber bei ihr entsprechen außerdem solchen Begriffen wie „Begriff“ eine unendliche Menge verschiedener Begriffe mit verschiedenen Sinnbereichen.
>
> <u>*Bem<erkung>*</u> (*Gr<ammatik>*): Die Auflösung der Antinomie (nach 2, p. 48) erfordert eine gewissermaßen nicht-mat<hematische> Einstellung, nämlich die Einstellung der „Wahrnehmung“ in einem fest gegebenen Bereich, nicht der Konstruktion. Nur in diesem Sinn kann „sinnvoll“ nicht definiert sein **[51]** und kann eine Unmöglichkeit, den Begriff für alle Argumente zu ergänzen, bestehen.

Gödel reminds us that, when one accepts logical calculus without any restriction on the logical rules (in contrast to what happens in intuitionist logic), there are only two possible solutions left to the paradoxes. Both are explained by Gödel in terms of the notion of a good definition.

The first consists in denying the status of well-defined concepts—and therefore the status of proper entity—to some propositional functions, which are grammatically correct but generate paradoxes (for example: de x∉x). These propositional functions are only pure linguistic expressions, well-formed but without any meaning. It is possible to recognize here a form of the Zig-Zag strategy, mentioned above.

The second consists in denying that the notion of “good definition” (i.e. a definition that is in accordance with grammatical rules, being able to be expressed in a formal language) could be applied to the notion of concept and to the notion of application of a concept to its argument. We can have, indeed, propositional functions that express concepts that are totally meaningful, but whose application to certain arguments takes us outside of the range of meaningfulness. In the case of arithmetical operations, nobody would deny that the process of division is justified, just because it is “meaningless” when applied to zero. Analogously, Russell’s concept x∉x, when applied to itself, takes us outside of the range of meaningfulness, without compelling us to deny that non-self-applicability has the status of a concept. This is clearly a case of the limited ranges of significance strategy.

The gap pointed out by the first strategy is between grammatical rules and rules constructing a meaningful language. Grammatical rules governing the correct formation of sentences seem in this case unable to prevent well-formed but meaningless expressions. The gap that is denounced by the second strategy is between objective concepts, as structures that generate the spatio-temporal objects and subjective concepts, as rules of construction of meaning for a finite mind. Whatever rule we can choose for meaningful subjective concepts, it will be impossible for us to anticipate paradoxical applications. They can only been seen *a posteriori*.

5 Es könnte aber auch bedeuten, dass es stattdessen eine Menge von scharf definierten *Approx*<*imationen*> gibt: Begriffe für Individuen, für Begriffe *etc*. (das ist ganz eigentlich<?> das Wesen der Typentheorie). *Gödel’s footnote*.

These last two solutions are both mentioned in the *Russell paper*, in almost the same terms (Gödel CW 1990, respectively on pages 124 and 137). Nevertheless, a remarkable difference is to be noticed with *Max Phil* IX: the first strategy is clearly viewed negatively in the notebook, whereas such a negative judgment is absent from the *Russell paper*. Gödel blames the Zig-Zag theory because it compels us to deny to a propositional function like $x \notin x$ the status of a concept, even if it is well-defined for every object.

The following remark, at page 51, adds a reason to explain Gödel's preference: the second solution is the only one to express a genuine, realistic conception of concepts, considering them as entities that are independent of their arguments and values. Concepts exceed the range of meaningfulness, of recursive definitions (even transfinite), and of constructions. Concepts come to our perception by means of their realizations in the sensible world, but we cannot reduce them to such realizations. Indeed, the realm of concepts cannot be thought of in terms of operations nor in terms of domains of application of those operations (whatever the degree of idealization is with which we conceive the notion of operation). This is probably the way to understand the expression "nicht-mathematische Einstellung", which is used by Gödel. Gödel seems here to suggest, by the reference to mathematics, a recurrent theme in his conversations with Wang in the 1970s concerning the difference between mathematics and logic. On the contrary to logic, mathematics is always the result of operations that presuppose a domain of objects, which is determined in advance. They develop themselves always from this determinate domain, although they can use transfinite and not-constructive processes. Therefore, the relation between mathematics and logic seems to be deep-rooted in an ontological opposition. According to this, mathematics is devoted to the description of what is invariant in the spatio-temporal realization of concepts. It is the realm of what comes from "iteration"[6] which presupposes the temporal succession, whereas logic is totally independent of it. This is the meaning that Gödel gives to the phenomenon of incompletability in his article of 1951. This is also the deep meaning of the difference between the solutions of the intensional paradoxes and the solutions of the mathematical paradoxes. Finally, this is the meaning of the difference between the relation *esti*, considered from the intensional point of view as an application of the concept to any argument, and the relation *esti*, considered from the point of view of the set-theoretical membership (which excludes every self-belonging). From Gödel's point of view, the idea that concepts do not need a determinate domain of application—and therefore that the notion of meaningful application is in some way undetermined—is the expression of an achieved realism of concepts, which Frege and Russell were unable to reach, with their respective theories of unsaturatedness and of typical ambiguity. Concepts as structures of the real and of the possible are previous to their applications and are independent of them. A concept does not intrinsically have a fixed and foreseeable range of significance. It spreads out in its different realizations but does not presuppose them.

In spite of this severe criticism of Russell's position, Gödel uses and takes Russell's analysis seriously, as he does in the case of his work on the axioms of Set Theory. He makes use of it freely as a useful instrument of thoughts, in order to analyze the

6 One has to understand the notion of iteration in a highly idealized way, as explained by Gödel through the iterative notion of set in (Gödel 1951).

fundamental logical concepts. This is the reason for Gödel's interest in the ramified theory and in the axiom of reducibility, which considered together can be seen as a useful borderline case of the Zig-Zag strategy.

The ramified theory of types is, according to Gödel, the most extreme case of the application of the Vicious Circle Principle, and of the *no-class theory*, which he considers as radical versions of the first strategy of solution to the paradoxes that are indicated in the remark on page 48 in *Max Phil* IX. Instead of denying the status of being a concept or a set (that is, the status of being an entity with a true unity and with identity conditions) only to some propositional functions and to some classes, the theory of order—as Gödel likes to call it—claims directly from the beginning that concepts and classes are generally only mere "*façons de parler*", which are dependent on our definitions and constructions. Although for Gödel such an assertion was inacceptable, already in the *Russell paper* he seems to be interested in one of the aspects of this Russellian strategy: the axiom of reducibility. Russell needs it to define in a general way the notion of natural number and the mathematical principle of induction. In the *Russell paper*, Gödel shows a peculiar interest in this axiom of reducibility, since he seems to be above all interested in a metaphysical version of it, able to represent the gap between subjective and objective concepts. Indeed, Russell's axiom of reducibility claims that, for every propositional function of any order, there exists a first order propositional function having the same extension. Therefore, it stipulates the possibility to reduce every "construction", in the sense of Russell, to properties of objects. Besides, Gödel mentions Russell's interpretation of those primitive properties as Universals, while criticizing the idea that such Universals could be a "concept of sense perception" (Gödel 1990 p. 128). Gödel's interpretation of the axiom of reducibility is metaphysical in the sense that these primitive concepts should correspond to the primitive logical and philosophical concepts that Gödel is searching for. The following remark on page 69 of *Max Phil* X seems to suggest this interpretation:

**[69]**

> *Bem<erkung>* (*Gr<ammatik>*): Wenn man ernst macht mit der These, dass alle „Vielheiten" *non entities* sind (und nur <eine> *faç<on> de parler*) so heißt das: Ein Paar ist zwei Dinge als eines gedacht, dadurch wird es aber nicht eines, sondern bleibt zwei. Allerdings kann man in vieler Hinsicht über zwei Dinge ebenso sprechen wie über eines (z. B. Quantor<en> und entsprechende Relationen und Funktionen definieren wie das *Peano* tut). Daraus entnimmt man dann die unberechtigte Verallgemeinerung, dass das immer möglich ist, d. h. dass ein Paar „kein Gegenstand" ist. Insbesondere vielleicht sind auch manche Begriffe in Wahrheit zwei Begriffe (wenn nur künstlich zusammengefasst) und sollte man das Reduzibilitäts*ax<iom>* so formulieren, dass es für jedes $\phi\ (x)\ a_1\ a_2 ...\ a_n$ gibt, so dass $x \in a_1 ...\ x \in a_n \equiv_x \phi\ (x)$ oder eventuell sogar eine unendliche Menge solcher „Merkmale".

The explicit occurrence of the axiom of reducibility takes place here in a remark that seems to explore the interesting aspect of the "non constructivist" aspect of the theory of order. As Gödel said in the remark on page 48 of *Max Phil* IX, quoted at the beginning, in order to solve the paradoxes, we have to "see" more clearly the difference between the relation *esti* in the objective sense, and its *Ersatz*, i.e. its substitute constructed by us in order to adequately represent reality. Now, says Gödel, it is true that in many cases what at first seems to us to be a concept proves to be only an artificial assembling of two different concepts. It is only our incorrect perception of the realm of concepts that leads us to suppose a unity where there are in fact two distinct entities. The possibility to free ourselves from our *Ersatz* in order to grasp the true concepts is described by Gödel in Wang's book[7] in terms of a process of distinct perception, in which it seems evident that there is an echo of Leibniz's *Meditations on Knowledge, Truth, and Ideas*. Now the axiom of reducibility illustrates one way to represent this replacement of an *Ersatz* (a human linguistic construction) with a true concept, because it postulates the existence of a "clear and distinct" concept (because of level one) for every propositional function of any complexity. It is probably one reason for Gödel's interest in Russell's construction. If we do not consider these concepts of level 1 as properties of individuals, but consider them intensionally as *Grundbegriffe*, the axiom of reducibility asserts that all human concepts are ultimately composed by *Grundbegriffe*, that is primitive ideas *in mente Dei.*

We saw, in the remark discussed in this volume in the other paper by Gabriella Crocco, that the comparative analysis of concepts understood as objective Ideas, and concepts understood as "subjective" processes, finished with three points about the axiom of reducibility. In particular, point n° 9 seems to give a second reason, which we have to examine before turning to our second section.

> Das Reduzibilitäts*ax<iom>* besagt, dass es zu jedem Begriff eine umfanggleiche Idee (*Universale*) gibt; was man damit begründen kann, dass die Möglichkeit, die betreffende Klasse von Dingen auszusondern,[8] ihren Grund in einer objektiv gemeinsamen Beschaffenheit der Dinge dieser Klasse haben muss. Man könnte sagen, diese gemeinsame Beschaffenheit bestehe eben in dem, was die *Def<inition>* [besser das *Definiens*] sagt. Aber eine objektive Beschaffenheit kann nur im Vorhandensein gewisser Merkmale in den betrachteten Dingen bestehen, während durch *Def<inition>* diese Dinge gewissermaßen „von außen" beschrieben sind. Eine Universale ist etwas Einfaches und die Beschaffenheit könnte höchstens im Vorhandensein mehrerer Universalien bestehen, aber nicht in einer Struktur von „Sinnen", wie es das Definierende ist. Das *Def<iniens>* sagt, dass gewisse Operationen an dem Ding ausgeführt, ein gewisses Resultat haben. Das ist nicht eine Beschaffenheit (insofern diese Teil des Dinges ist), sondern höchstens Kriterium für eine solche. Die Aussage, dass es zu jedem Begriff eine umfanggleiche Idee gibt, ist allerdings sicher falsch, wenn in voller Allgemeinheit formuliert.

In this same remark, at point n° 7, Gödel distinguished Aristotle's conception, which considers universals as aspects of things, from Plato's conception, which considers them as entities independent of things. Moreover, at the same time, he distinguished between two types of relations of application. The first one is relative to Plato's ideas and is a relation of participation (Teil-haben). It means that "the concept *b* has chosen the thing *a*". The consequence is that there is a mark of *b* in *a* that characterizes this choice. The properties exhibited by sensible things are the sensible manifestation of these conceptual choices. From this perspective, the axiom of reducibility expresses the fact that the possibility of collecting the objects of the world in classes, isolated in the mind with a characteristic property, is ultimately founded on an objective property.

Suppose that we do not identify these (first order) objective properties with their corresponding classes, but rather attribute to these properties a metaphysical status of foundation of what is thinkable and intelligible. Then, the entire structure of the

7 [Wang 1974] p. 84.

8 Mit Hilfe des Begriffes „alle" und den logischen Operationen. *Gödel's footnote.*

theory of orders seems to be a logical illustration of Plato's point of view. In front of the fake unities of the world of appearances, the realm of ideas rises up, composed of true entities and generative principles in which each individual thing participates. Interpreted as such, the axiom of reducibility guarantees the reduction of the unstable and incomplete multiplicity of what exists, to the stable and complete multiplicity of Ideas. Clearly, Plato's solution is unacceptable for Gödel, because it compels us to an excessive devaluation of both existence and individuals. Gödel reminds us, in his proof of the existence of God, that the existence is a positive property. He also remarks that the primitive concepts are to be simple, which implies that the individual concepts, the complete concepts to which monads refer, cannot correspond to a unique Universal but only, at best, to a plurality of them. [9] It is also clear that such a position can only lead us to the first strategy of solution to the paradoxes, the one which Gödel considers as very likely incorrect. But, as is usual, these philosophical defects don't prevent Gödel from seeking to represent this form of realism, and all its consequences, in a logical framework. It is plausible to interpret the *Bemerkung* of *Max Phil* X, presented in the following section, as trying to realize such a task.

## Section 2. A pseudo-Russellian solution

The following remark seems to occupy a very special place in Gödel's *Max Phil* X. On the front page of the notebook we can see a reference to this remark. Gödel, probably during a new reading of his notebook, notices:

> p. 28 Auflösung der Antinomie durch Ersetzung der Existenz einer Menge m, so dass $x \in m \equiv \phi(x)$ durch ein k-Tupel von Mengen.
>
> *p. 28 Dissolution of the antinomy by replacing the existence of a set m such as* $[(x \in m) \equiv \phi(x)]$, *by a k-tuple of sets.*

The remark from page 28 is the following:

> ***Bemerkung* (*Grammatik*)**: Ausgehend von $\in$ als „Teilrelation“ könnte man vermuten, dass die Bedingung für die Existenz einer Menge einfach ist, dass $(\exists x)\, \phi(x)$. Das ist auch widerspruchsfrei, aber es folgt, dass es nur ein Ding geben kann. Denn es folgt $\neg(\exists x)\, x \notin x$ (sonst Widerspruch), also immer $x \in x$. Außerdem aber gibt es für jedes b ein a, so dass:
>
> $x \in a \equiv x = b$, [10] daher $a \in a \equiv a = b$, also $a = b$, daher für jedes $b$, $x \in b$ <ist> nur wahr für $x = b$, gäbe es aber zwei verschiedene Dinge, so gäbe es auch ihr Paar p und dies hätte nicht die eben bewiesene Eigenschaft. Was wäre aber, wenn man verlangte:
>
> $(\exists x, y)\ x \neq y \,.\, \phi(x) \,.\, \phi(y)$, damit die Menge der $\phi$ existiert? Für alle außer einem Einzigen gilt dann $x \in x$. Wie viele Dinge kann es dann höchstens geben? Es ergibt sich daraus, die Antinomie dadurch aufzulösen, dass man nicht die Existenz <u>einer</u> Menge, sondern bloß einer endlichen Anzahl von Merkmalen $a_1 \ldots a_k$ verlangt, derart, dass $x \in a_1 \ldots x \in a_k \equiv \phi(x)$. Mengen von Mengen wären dann $\infty$-stellige Relationen *etc.* {Am ober;en Rand eingefügt: Zerlegung der Eigenschaften in Merkmale}

9 This interpretation gives a possible hint for the understanding of point no.11 of the same *Bemerkung*, in which Gödel refers to Russell's version of the axiom of reducibility. He says that the axiom of reducibility is *sinnlos* for the *a*-concepts (i.e. the defining concepts of an individual *a*).

10 $x = y \,.\equiv.\, (z)\,(z \in x \equiv z \in y)$. Gödel's footnote.

Let's begin with a first reading. If we identify the antinomy, mentioned in the text and on the front page of the notebook, with Russell's antinomy, then this *Bemerkung* seems to present a new solution, not mentioned in the *Russell paper*. In a first reading, we can read the *esti* in an extensional way, and try to understand in which sense the formula can contain a solution to the paradox.

The strategy seems clear. We are not searching for a criterion for the permission of attributing or not a class to a propositional function $\phi$, which could in turn be associated or not to a concept. Indeed, from the beginning, the classes and concepts that are obtained from the propositional functions are removed in favor of their simplest elements, the "characteristic marks" to which Gödel refers on the top of page 29 (*Zerlegung der Eigenschaften in Merkmale*, which means: Decomposition of properties in their characteristic marks). With the help of the previous remarks presented in the last section, which refer to the axiom of reducibility, we can stipulate that Gödel is playing the role of a Russell-minded thinker, and he tries to take the assertion seriously, according to which classes and concepts are mere "*façons de parler*", mere constructions of fake unities by the mind, fictions. They can however be reduced to the fundamental concepts in which the propositional function can be analyzed, thanks to an intensional version of the axiom of reducibility. Therefore, instead of associating to each propositional function a class (its extension), we have to associate it to a plurality of "characteristic marks", a plurality of primitive Ideas: $a_1, \ldots, a_n$. The axiom schema of comprehension (separation) will be replaced by the following axiom schema (from the figure at the end of the remark). It looks like the formula, mentioned above, which is evoked at the end of the *Bemerkung* on page 69 in *Max Phil* IX, quoted above:

(Axiom schema of characteristic marks (or axiom schema of reducibility

$$\exists a_1 \ldots a_n \forall x \ [(x \in a_1) \wedge \ldots \wedge (x \in a_n)] \Leftrightarrow \phi(x)$$

It is natural to see a parallelism between the axiomof reducibility and the axiom of separation as soon as we read the *esti*, appearing on the left side of the equivalence sign, as the extensional belonging (membership). [11] The parallelism with the axiom of comprehension could incite us to interpret the $a_1, \ldots, a_n$ as classes. Nevertheless, several clues suggest that the solution to the paradoxes that Gödel is presenting here uses an intensional analysis of concepts. Indeed, the "characteristic marks" that participate in the new axiom schema evoke the simple concepts that are used in the composition of every notion (concept), in the logical tradition. Finally, the fact that Gödel permits the reflexivity of the "$\in$" relation ($x \in x$) during the entire note, shows that we are not in the framework of Set Theory as Gödel conceives it in the *Russell paper* (with the axiom of foundation). Rather, we have to adopt a conception in which each primitive notion, which can be the characteristic mark of an object, "is to itself its own mark". [12]

The complex framework in which this Gödelian thought has to be understood is evoked by Gödel in the first lines of the remark. There, he asserts that his solution (to the paradox) can be expressed from an interpretation of the $\in$ as a "Teilrelation". This assertion is linked to the remark in *Max Phil* IX, on pages 68-69, commented

11 This is precisely what Gödel remarks in his article of 1944, p. 140-1, (Gödel 1990) p. 131.

12 G.W. Leibniz, "*Méditations sur la connaissance la vérité et les idées*", Schrecker Ed., on page 10.

on above. The applying relation of an Idea to a thing ($x \in a$) must be thought of as a Teilrelation, that is a relation that indicates that a property *a* is in an object *x* as a mark, or that *x* has *a* as a constitutive property. In this framework, a set is nothing but a multiplicity of objects, collected together by the mean of a certain number of fundamental properties, but without a true unity. From this premise, the analysis is developed in a Russell-like way of thinking, by ambiguously using the relation, with both an intensional interpretation as well as an extensional one. In Rusell 1905, p. 31), the author remarks:

> It is no way essential to the argument [concerning the possible solutions to the transfinite paradoxes] to suppose that classes and relations are taken *in extension*. [One has just to posit that] two norms [=propositional functions) which ar not equivalent (*i*, *e*., suche that, for any value of the variable, both are true or both faise] do not determine the same class [...]

It is based on the idea that classes are reducible to propositional functions, which are in turn reducible to logical primitive concepts. In any case, even when Gödel uses the extensional interpretation of the *esti*, he never presupposes from the beginning all the properties that we usually attached to the usual belonging relation $\in$ of classical Set Theory.

Then, we understand why the plurality of the characteristic marks involved in the new schema prevents Russell's paradox. Indeed, if we take for $\phi$ Russell's propositional function $\phi(x) =_{def} (x \in x)$, we then have:

$$\exists a_1 ... a_n \ \forall x \ [(x \in a_1) \wedge ... \wedge (x \in a_n)] \Leftrightarrow (x \notin x)$$

If n=1, by substituting *x* by $a_1$, we have the contradictory proposition $(a_1 \in a_1) \Leftrightarrow (a_1 \notin a_1)$.
Nevertheless, if n >1, by substituting *x* by any $a_i$, there is no longer a contradiction. We simply have:

$$[(a_i \in a_1) \wedge ... \wedge (a_i \in a_n)] \Leftrightarrow (a_i \notin a_i)$$

This proposition can be true, namely when $(a_i \in a_i)$ and $(a_i \notin a_j)$ for at least one $j \neq i$.
In particular, the paradox will be prevented if we posit that:

1. The concept "not to apply to itself" must necessarily be composed of *several* characteristic marks.
2. Every characteristic mark applies to itself.
3. One characteristic mark never applies to a different characteristic mark.

The first point has been explicitly mentioned in point n° 9 of the remark on pages 68-69 of *Max Phil* IX, quoted above. The two last points are totally faithful to the traditional—in particular the Leibnizian—conception of primitive concepts as positive and independent of each other. We can also remark that, in this framework, $x \notin x$ is defined for every object even if, like any other complex propositional function, it is not a concept. Indeed, the status of concept would be truly attributed only to the characteristic marks.

Interpreted from this perspective, the axiom schema of characteristic marks shows its affinity with the axiom of reducibility, when interpreted in an intensional and metaphysical sense. If there is a decomposition of each propositional function in primitive concepts, then we can deal with sets as the extensions of those propositional functions without the risk of falling in the paradoxes. Of course, this solution is not satisfactory from Gödel's point of view because it does not rely on the strategy of the *limited ranges of significance*, but rather on the one indicated in point n° 8 of the remark on page 48 of *Max Phil* IX quoted above. Here, the paradoxes are prevented by denying that to every propositional function there corresponds a concept. On the contrary, the only true concepts (in the objective sense), the only true unities are the primitive concepts, whose characteristic marks are in the things and are capable of self-application. In front of them, classes and propositional functions would be on the contrary reduced to mere *façons de parler*. They would be only convenient constructions for thought and discourse, but without a true being. A propositional function is nothing but a conjunction of primitive concepts:

$$\exists k, \exists a_{1 \ldots} a_k[(x \in a_1) \wedge \ldots \wedge (x \in a_k)] \Leftrightarrow \phi(x)$$

The membership relation to a set *m* can be reduced to a $(k+1)$-ary relation where *k* might possibly be transfinite:

$$(x \in m) \equiv \phi\ (x) \equiv R(x, a_1 \ldots a_k)$$

A set S of sets is a relation with a (possibly transfinite) number of arguments because

$$\forall t \in S\ \exists k_t \in \mathbb{N}, \exists \left(a_{t_i}\right)_{i=1 \ldots k_t} \left[\forall i \in \{1, \ldots, k_t\}\ \left(x \in a_{t_i}\right)\right] \equiv (x \in t)$$

And we can consider that the arguments of this relation are:

$$x, a_{t_1}, \ldots, a_{t_{k_t}},\ a_{t'_1}, \ldots, a_{t'_{k_{t'}}}, \ldots$$

Where t, t', … are the successive elements of S. This set S has $1+\sum_{s \in S} k_s$ elements, and, therefore, it can be infinite as soon as S is itself infinite, even if the total number of different characteristic marks would itself be finite.

How did Gödel arrive at this idea? He explains it in the remark by a partly implicit reasoning, which constitutes most of the content of the *Bermerkung*. In order to clarify Gödel's argument, we have to come back to the key pages of the *Russell paper*, where Gödel compared concepts and classes relatively to the three previously presented forms of the vicious circle principle. Indeed, on page 127, Gödel distinguished the following three forms:

a) *No totality can contain terms definable only in terms of this totality;*
b) *No totality can contain terms involving this totality; and*
c) *No totality can contain terms presupposing this totality.*

If "to presuppose" means "to presuppose for its being", then the third form expresses a minimal rational requirement that every totality must satisfy. On the contrary, the first form is only justified from a nominalist or constructivist point of view. Namely, it is justified only if the concerned totality is supposed to be merely the result of our definitions and constructions. What about the second form? Its justification seems to rest in the Russell paper on the distinction between concepts and classes. Indeed, after having asserted that the second form does not apply to concepts, [13] Gödel turns his attention toward classes. He then asserts that the solution that is based on Zermelo's axiomatic—and therefore on the iterative concept of set as developed in Gödel's 1951 article—is the most efficient and general one for mathematics. According to Gödel, Zermelo's theory is in accordance with b), in so far as $x\varepsilon y$ is possible only if the order of $y$ is superior to the order of $x$. Now he says:

> I even think that there exists an interpretation of the term "class" (namely as a certain kind of structures) where it does not apply in the second form either.

And he adds in note 30:

> Ideas tending in this direction are contained in Mirimanoff 1917, 1917a and 1920. Cf. in particular 1917a, p. 212.

Mirimanoff's solution admits the existence of non founded sets. Therefore, it seems that in 1944 Gödel considered, for non necessarily mathematical uses, some solutions of the extensional paradoxes, where the notion of class was interpreted in the sense of a structure. As such, it did not have to obey the second form of the vicious circle principle. Indeed, this "non mathematical" conception of the notion of class reveals itself to be fecund for a pseudo-Russellian solution of the intensional paradoxes that would be acceptable from the point of view of the first strategy indicated in the remark on page 48 of *Max Phil* IX. Mirimanoff's analysis, particularly in the first two articles published in 1917 in *L'enseignement mathématique*, suggests a useful argument to Gödel. It is not directly given by Mirimanoff himself, but is based on the same presuppositions. Mirimanoff takes a universe that consists of indecomposable elements (we will call them Ur-elements, following a German tradition). From them, sets are formed by a process of reunion or association, which is expressed by parentheses. Besides the usual concept of identity, based on the principle of extensionality, Mirimanoff uses a structural concept of isomorphism. He says this about it:

> This notion can be defined by a recursive operation. I will say that two sets, E and F, are isomorphic if one can establish a perfect correlation between the elements of E and those of F, such that: 1) to every indecomposable element of E, there corresponds an indecomposable element of F and 2) to every set-element E' of E, there corresponds an isomorphic set-element F' of F, and conversely. Two different isomorphic sets cannot differ but by their kernels, [14] but not by the operations of association or reunion that are expressed by the parentheses. (Mirimanoff 1917 a, p. 211).

Then, following a suggestion of Russell's in *Principles*, Mirimanoff distinguished between two types of sets. In the sets of the first kind, no element is isomorphic to the set itself; whereas, in the sets of the second kind, at least one element is isomorphic to the set itself.

The set that he gives in 1917a is particularly interesting in relation to the self-applying mechanism that is permitted by the classes of the second kind:

> I remember to have seen, a few years ago, a book for children whose front page was decorated with a big colored picture. This picture, which I call *J*, represented two

13 (Gödel 1990) p. 130.

14 For Mirimanoff, the kernel of a set is the set of the indecomposable terms to which we arrive when we go down, step after step, into the set.

> children who were looking at the same book we are talking about, or rather its image, that is the image *J'* of *J*. On this image *J'*, we could perceive or rather hardly distinguish the two children, at a smaller scale, and the image of the book, deformed by the perspective. All this process was theoretically supposed to continue ad infinitum. Now the primitive image *J* can be considered as a set, whose elements are the two children $e_1$ and $e_2$, the surrounding *f*, and the picture *J'* of *J*. *J'* is itself composed of $e'_1$, $e'_2$, *f'* and *J''*... etc. Therefore, if we agree to look at the elements $e_1$, $e_2$ , and *f* and their transformed version as indecomposable elements, then the isomorphism between *J* and *J'* is manifest […].

The concept J, when it is extensionally represented, is the basic structure from which the Ur-elements $c_1$, $c_2$, … can be collected together. It organizes them, taking part, at the same time, in the set and its structure. Each of the Ur-elements takes part in the set of which they are elements and which is exemplified by the concept. Mirimanoff showed how Russell's paradox can be translated in this context: the set R of all sets of the first kind cannot exist. If it existed, on the one hand, it should be of the first kind. Indeed, a set composed of sets of the first kind is itself of the first kind (Mirimanoff's easy Lemma). On the other hand, the set R would contain by definition every set of the first kind, including itself. Contradiction!

Gödel's reasoning is developed from this master idea of Mirimanoff, but it changes the conclusion. Gödel's idea is suggested by the iteration of a process that consists in: 1) searching for a criterion according to which we could associate a set to a propositional function $\phi$ (the class that is the extension of the concept); and 2) positing that this criterion is not fulfilled in the case of Russell's propositional function $\phi(x) =_{def} (x \notin x)$. These are precisely the same criteria that are posited by Mirimanoff, at the beginning of his first article. But the solution is totally different. Let's follow Gödel's argument.

He begins by testing what happens if the condition of existence of a set consists in the existence of at least one object that satisfies the function $\phi$. According to this framework, the usual axiom schema of comprehension would be replaced by:

(Hypothesis 1)

$\{x|\phi(x)\}$ exists $\Leftrightarrow \exists x\, \phi(x)$

Therefore, such a theory would not admit any null set. This possibility is considered as acceptable and Gödel himself considered it as a hypothesis in "Russell's mathematical logic". [15] In order to avoid Russell's paradox, this new theory must deny the existence of $\{x|x \notin x\}$. According to hypothesis 1, it implies:

$\neg\exists x\, x \notin x$,

which is in turn logically equivalent to $\forall x\, x \in x$. Therefore, our theory should claim that every class belongs to itself. Gödel now proves that, in this case, the theory must be contradictory, apart from the case where our domain contains only one object.

Gödel's proof is based on the fact that the "classes" that are considered in this theory must verify the principle of extensionality. It is what he writes in a note at the bottom of the page. We have to remark that there is no contradiction in this frank acceptation of the principle of extensionality. Indeed, it is totally compatible with Russell's point of view. It is worth noting that Gödel was seriously considering

15 Cf. (Gödel 1990) p. 131.

as a plausible hypothesis the idea that concepts could admit a kind of principle of extensionality, just as classes do. [16]

Once we have accepted the principle of extensionality, the development of the proof is based on the fact that, in this theory, we should have a "principle of the singleton", claiming that for every object x the singleton {x} exists, and a "principle of the pair", claiming that for any two different objects x and y, the pair {x, y} exists.

We understand that those principles are accepted by Gödel as consequences of the hypothesis 1, as soon as we accept the existence of an individuating property $P_a$ for every object *a*, or at least a name *a* (constant of object). Then, the existence of the singleton $\{a\}$ follows from the hypothesis 1, when we take the propositional function $P_a(x)$ (respectively the propositional function "x=a") for (ϕ x).

In the same manner, the existence of the pair $\{a, b\}$ follows from hypothesis 1 when we take "$P_a(x) \vee P_b(x)$" (respectively "$(x=a) \vee (x=b)$") for ϕ(x).

Gödel first shows that, in the obtained theory, *every object should be identified with the singleton that contains it.* Indeed, let's take any object *a*. The singleton $\{a\}$ exists and, by definition, it contains anything but "*a*", i.e.: $x \in \{a\} \Leftrightarrow x=a$. Besides, because of the non existence of Russell's class, we have already shown that any object must contain itself. In particular $\{a\} \in \{a\}$, and then we must posit $a=\{a\}$.

The remaining part of the proof is by contradiction. If there were two different objects *a* and *b* in our universe, then the pair $\{a, b\}$ would exist. It then would be equal to its own singleton $\{\{a, b\}\}$ like any other object. But the equality $\{a, b\}=\{\{a, b\}\}$ is absurd, because it would refer to a set that contains two objects and, at the same time, contains only one. In conclusion, we must refuse the strategy consisting in solving Russell's paradox by hypothesis 1, because it is consistent only in the non-interesting case where there is only one object in the universe, which would then be equal to its own singleton: $a=\{a\}=\{\{a\}\}$…

That's why Gödel tries to improve hypothesis 1. He proposes, as a criterion of existence of a class corresponding to a given propositional function ϕ, the fact that ϕ is verified by at least two different elements. Therefore, hypothesis 1 is replaced by:

(Hypothesis 2)

$\{x|\phi(x)\}$ exists $\Leftrightarrow \exists x \exists y\ (x\neq y) \wedge \phi(x) \wedge \phi(y)$

Such a theory would not only deny the existence of the null set, but would also deny the existence of singletons (because the propositional functions that correspond to a singleton don't verify hypothesis 2), except if we give them existence by a specific axiom, similarly to the attitude taken by Gödel in "Russell's mathematical logic". [17] However such a theory will continue to admit a principle of the pair, a principle of the triplet, etc. For example, for any two objects *a* and *b*, the property $\phi(x)$, defined by $(x=a) \vee (x=b)$, actually fulfills hypothesis 2).

Now the important question is the following one: can we obtain the same phenomenon of dependence between, on the one hand, the number of objects that must verify a

16 Cf. (Gödel 1990) on page 129.

17 Cf. (Gödel 1990) p. 141. In that case, the equivalence in the expression of hypothesis 2 would be transformed into an implication. It would express a sufficient but not necessary condition.

property in order for us to give an existence to the class that is its extension, and, on the other hand, the number of objects that we can admit without falling into contradiction? (*Was wäre aber, wenn man verlangte*: $\exists x \exists y\ (x \neq y) \wedge \phi(x)\ \wedge \phi(y)$, *damit die Menge der* $\phi$ *existiert?)* In other words, does our hypothesis 2 lead us to posit a maximum of 2 objects in our universe, and can this result be generalized? Gödel doesn't directly answer this question. But we can remark that the answer is positive. We can demonstrate it by the same kind of approach as with hypothesis 1.

Let's adopt hypothesis 2. To prevent Russell's paradox, we must posit that $\{x|x \notin x\}$ doesn't exist. It can be written, according to hypothesis 2:

$$\neg \exists x \exists y\ ((x \neq y) \wedge (x \notin x) \wedge (y \notin y))$$

Gödel says that we must conclude that all objects, except one, belong to themselves. However, the logical meaning of the above proposition shows that all objects, except *perhaps* one, belong to themselves:

$$\forall x \forall y\ ((x=y) \vee (x \in x) \vee (y \in y))$$

How can we conciliate Gödel's assertion with that? It is sufficient to place ourselves in a universe like Mirimanoff's one, with Ur-elements (simple and indecomposable logical elements, which don't belong to themselves), and classes (possibly of the second kind) constructed from those Ur-elements. If we refuse such a framework, then the theory that derives from hypothesis 2 is always contradictory or admits only a null universe.[18] That's why we will admit the existence of Ur-elements. We begin by proving the following lemma:

> <u>Lemma :</u> Let's suppose that, in our domain, we can define at least 3 different objects A, B and C, which are not pairs (i.e. they are Ur-elements or classes with strictly more than two objects). Then, our theory is contradictory.
>
> <u>Proof:</u> Let's consider the pairs {A, B} and {A, C}. They are distinct (by the principle of extensionality). According to what we said above, in relation to the non existence of Russell's class, one of those pairs must belong to itself. But it is impossible, since none of A, B and C is a pair. Contradiction!

The lemma applies in particular if there is in our domain at least 4 different objects *a*, *b*, *c* and *d*. Then we take A={*a*, *b*, *c*}, B={*a*, *b*, *d*} and C={*a*, *c*, *d*} for example. The only non contradictory cases are those with 1, 2 or 3 objects inside the domain:

a) Let's suppose that there is exactly 1 object. Then, it can only be a Ur-element (there are not enough objects to form a set, according to hypothesis 2). This case is non contradictory. The element *a* is an object not belonging to itself.
b) Let's suppose that there are exactly 2 objects: *a*, and *b*. Then, the pair {*a*, *b*} exists (hypothesis 2)). Therefore, this pair is identical to one of those objects, let's say

18 In order to be convinced about this point, it suffices to remark that, in the demonstration below, where Ur-elements are permitted, the only non contradictory and non empty cases contain Ur-elements.

"*b*". The object *a* is then a Ur-element, and b is the only object not belonging to itself. This case is not contradictory.

*b*={*a*, *b*}

*a*=Ur-element

c) Let's suppose that there are exactly 3 objects: *a*, *b*, and *c*. The axioms of the pair and of the triplet give the existence of {*a*, *b*}, {*a*, *c*}, {*b*, *c*} and {*a*, *b*, *c*}. They must be 2-by-2 distinct (principle of extensionality). But each of those 4 objects should be identified with one of the objects *a, b* and *c*. Contradiction!

Therefore, the theory that derives from hypothesis 2) is contradictory, unless we accept only 2 objects (or less) in our universe, one of which must be a Ur-element. The other object is then the universal class, which belongs to itself.

Generalization :

Gödel stops at hypothesis 2. Nevertheless, his approach suggests to us the following generalization:

(Hypothesis *n*)

$\{x|\phi(x)\}$ exists $\Leftrightarrow \exists x_{1\ldots} \exists x_n \wedge_{i\neq j}\{(x_i \neq x_j)\} \wedge \phi(x_1) \wedge \phi(x_2) \wedge \ldots \wedge \phi(x_n)$

We can check that we then obtain a limitation of the same kind as in the cases n=1 and n=2. Namely: the theory is contradictory, unless we take only *n* objects, or less, in our domain. In the case where there are exactly *n* objects, then *n*-1 among them are Ur-elements, and the last one is the universal class. In the other cases, all objects are Ur-elements.

Let's return to Gödel's text. From what precedes, Gödel says that we can derive the solution to the paradox presented at the beginning of this section. Nevertheless, the transition from the condition of existence of a set to the number of objects that can be admitted under this condition only indirectly suggests the solution. Indeed, we must reverse, as in a mirror, the conditions of existence, in order to find the solution. The hypothesis *n* tells us that the class of the exists, as long as it collects at least *n* objects. Now we showed above that, if we want at the same time to prevent Russell's paradox and to permit the existence of classes of the second kind, then this hypothesis must be contradictory unless there is only one class of the second type (the universal class) containing itself and the (n-1) other objects that are Ur-elements. In this context, no class of the first kind exists. Neither does the class of the classes of the first kind exist. How can we prevent such a drastic restriction? Following what Gödel says in the remark in *Max Phil* IX, on page 69, let's take the idea seriously that classes and propositional functions are "non entities", mere *façons de parler,* and let's assert the principle of reducibility. To each (well-formed according to syntax), there must correspond a decomposition of it in *n* characteristic marks (kind of Ur-principles). They are also indecomposable but, in opposition to the Ur-elements, they are not individuals but Universals, being able to be applied to themselves.

Then, from the extensional point of view, we would be in a universe that is exclusively composed of classes of objects that participate (in a non exclusive way) in these Universals. These Universals, these Ur-principles would then be fundamental ideas, simple logical elements, each one being its own characteristic mark, and in which every object can participate. They make the discourse on propositional functions and classes possible. And they would be the only true entities that could be applied to themselves.

As we remarked, this solution is clearly unsatisfactory for Gödel. Moreover, as he says in the remark on pages 50-1, in such a framework a propositional function could be meaningfully applied to every object of the domain, without being considered as a concept. Indeed, such a theory would deny (at least intensionally) any being to everything that is not contained in the realm of Ur-principles or Ideas. Then, it seems that such a theory is an interpretation of Plato's universe and its relation of participation.

## Conclusion

The *Bemerkung* of *Max Phil* X discussed in the previous section shows how deep the correlation is between the text of the *Russell paper* and the content of *Max Phil* X. In the latter Gödel explores some of the trails suggested in the Russell paper, expressing himself more freely and explicitly about them. Russell's axiom of reducibility gives to him the opportunity to follow the track of the Zig Zag principle in the context of Mirimanoff's proposal. The result is a solution of the paradox where only the Ur-principle, the fundamental characteristic marks, are considered as real entities, whereas the other propositional functions are mere "*façon de parler*". As far as we know so far there is no other place in the rest of the *Max Phil* where Gödel mentions this solution. As far as we have seen, this solution à la Mirimanoff is not followed anywhere else in the *Max Phil*.

## References


CASSOU-NOGUES, Pierre (2009) "Gödel's introduction to logic in 1939", *History and Philosophy of Logic*, 30-1, pp. 69-90.

CROCCO, Gabriella (2006) "Gödel on concepts", *History and Philosophy of Logic*, 27-2, pp. 171-192.

GÖDEL, Kurt (1944) "Russell's mathematical logic", in (Gödel 1990) pp. 119-141.

GÖDEL, Kurt (1951) "Some basic theorems on the foundations of mathematics and their implications", in (Gödel 1996), pp. 304-323.

GÖDEL, Kurt (1990) *Collected Works*, *vol. II* : *Publications* 1938-1974, S. Feferman *et al.*, eds., Oxford: Oxford University Press.

GÖDEL, Kurt (1996) *Collected Works*, *vol. III : Unpublished Essays and Lectures,* S. Feferman *et al.*, eds., Oxford: Oxford University Press.

LEIBNIZ, Gottfried Wilhelm (1978) "Méditations sur la connaissance la vérité et les idées", in *Opuscules philosophiques choisis*, Paul Schrecker, ed., Paris: Vrin.

MIRIMANOFF, Dmitry (1917) "Les antinomies de Russell et de Burali-Forti et le problème fondamental de la théorie des ensembles", *L'enseignement mathématique*, 1917, pp. 37-52.

MIRIMANOFF, Dmitry (1917a) "Remarques sur la théorie des ensembles et les antinomies cantoriennes -1", *L'enseignement mathématique*, 1917, pp. 209-217.

RUSSELL, Bertrand (1905) “On some difficulties in the theory of transfinite numbers and order types”, *Proceedings of the London Mathematical Society* 4: 29-53. The paper was read to the Society on 14 December 1905.

RUSSELL, Bertrand and WHITEHEAD, Alfred North [1910-1913] *Principia Mathematica*, Cambridge: Cambridge University Press, 1925-1927, 2nd edition.

WANG, Hao (1996) *A Logical Journey: From Gödel to Philosophy,* Cambridge, Mass.: MIT Press.