LAMN PROPERTY FOR THE DRIFT AND VOLATILITY PARAMETERS OF A SDE DRIVEN BY A STABLE LEVY PROCESS

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February 20, 2017

Abstract

This work focuses on the Local Asymptotic Mixed Normality (LAMN) property from high frequency observations, of a continuous time process solution of a stochastic differential equation driven by a pure jump Lévy process with index $\alpha \in (0, 2)$. The process is observed on the fixed time interval [0,1] and the parameters appear in both the drift coefficient and scale coefficient. This extends the results of [5] where the index $\alpha \in (1, 2)$ and the parameter appears only in the drift coefficient. We compute the asymptotic Fisher information and find that the rate in the LAMN property depends on the behavior of the Lévy measure near zero. The proof relies on the small time asymptotic behavior of the transition density of the process obtained in [6].

MSC2010: primary 60G51; 62F12; 60H07; secondary 60F05; 60G52; 60J75

Keywords. Lévy process, Stable process, Malliavin calculus for jump processes, LAMN property, Parametric estimation.

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1 Introduction

Modeling with pure jump Lévy processes plays an important role in many fields, especially in mathematical finance (see for example [3], [15], [19]) and parametric inference for such processes is a currently active subject.

In this paper, we are concerned with parametric estimation of a stochastic differential equation driven by a pure jump Lévy process, from high-frequency observations on a fixed observation time. More precisely we consider the statistical experiment $(\mathbb{R}^n, \mathcal{B}_n, (\mathbb{P}_n^\beta)_{\beta\in\Theta\subset\mathbb{R}^2})$ corresponding to the observation of a Lévy driven stochastic equation at discrete times $(X^\beta_{\frac{i}{n}})_{1\leq i\leq n}$, solution of

$$X_t^\beta = x_0 + \int_0^t b(X_s^\beta, \theta)ds + \sigma L_t, \quad t \in [0,1], \tag{1.1}$$

where $(L_t)_{t\in[0,1]}$ is a Lévy process whose Lévy measure is similar near zero to the one of an α-stable process with exponent $\alpha \in (0,2)$ and $\beta = (\theta,\sigma)^T \in \mathbb{R}\times(0,\infty)$ is an unknown parameter to be estimated. We prove in this work that the Local Asymptotic Mixed Normality property (LAMN) holds for the parameter β. The LAMN property has been introduced by Jeganathan [13] [14] to extend to the markovian case the LAN property introduced in the pioneer works by Lecam and Hájek (see [20], [9]) in the i.i.d. case. This property permits in particular to identify the optimal estimation rate for the parameters θ and σ and the asymptotic Fisher information.

Parametric inference and LAN property for pure jump Lévy processes based on high frequency observations have been investigated in several papers, see for example Aït-Sahalia and Jacod [1] [2], Kawai and Masuda [16] [17], Masuda [21]. In particular, in [21], the LAN property is established and estimators are proposed for the parameters (θ,σ,α) in the model $X_t = \theta t + \sigma L_t^\alpha$, where (L_t^α) is an α-stable process. Aït-Sahalia and Jacod [1] [2] considered the model $X_t = \sigma L_t^\alpha + \theta Y_t$ where (Y_t) is a Lévy process, independent of (L_t^α) and dominated by (L_t^α). More recently, Ivanenko, Kulik and Masuda [11] proved the LAN property for the parameter (θ,σ) in the model $X_t = \theta t + \sigma Z_t + U_t$, where Z is a locally α-stable process and U is an independent and less active process. In all these works, the increments $(X_{\frac{i}{n}} - X_{\frac{i-1}{n}})_{1\leq i\leq n}$ are independent and the transition density of the discrete time process $(X_{\frac{i}{n}})_{1\leq i\leq n}$ is almost explicit. Extensions to stochastic equations driven by pure jump Lévy processes are not immediate and require a different approach since the transition density of the Markov chain $(X_{\frac{i}{n}})_{1\leq i\leq n}$ is unknown. Moreover they involve a random asymptotic Fisher information and lead to the LAMN property. Concerning the parametric estimation of a stochastic differential equation driven by a pure jump Lévy process from high frequency observations on a fixed observation time, we can

mention the recent paper by Masuda [22] where some estimators of the parameters (θ, σ) are proposed for the general equation

$$X_t = x_0 + \int_0^t b(X_s, \theta)ds + \int_0^t c(X_{s-}, \sigma)dL_s,$$

where L is a locally α-stable process, with $\alpha \in [1, 2)$. However in that case the asymptotic efficiency of the estimators is not yet establish and to our knowledge, the only result in that direction is given in Clément and Gloter [5], where the LAMN property is proved for the estimation of the drift parameter θ for the process solution of (1.1) (with $\sigma = 1$), in the case $\alpha \in (1, 2)$. They show that the LAMN property is satisfied with rate $u_n = n^{\frac{1}{2}-\frac{1}{\alpha}}$ and information $\mathcal{I}_\theta = \int_0^1 \partial_\theta b(X_s^\beta, \theta)^2 ds \int_{\mathbb{R}} \frac{\varphi'_\alpha(u)^2}{\varphi_\alpha(u)} du$, where φ_α is the density of the standard α-stable distribution with characteristic function $u \mapsto e^{-C(\alpha)|u|^\alpha}$.

Based on the main ideas of [5] and using the results of [6], we extend in the present paper these results to $\alpha \in (0, 2)$ and prove that the LAMN property holds for the parameters (θ, σ) with rate $u_n = \begin{pmatrix} n^{\frac{1}{2}-\frac{1}{\alpha}} & 0 \\ 0 & n^{-\frac{1}{2}} \end{pmatrix}$ and information matrix $\mathcal{I} = \begin{pmatrix} \mathcal{I}_{11} & 0 \\ 0 & \mathcal{I}_{22} \end{pmatrix}$ where $\mathcal{I}_{11} = \frac{1}{\sigma^2}\mathcal{I}_\theta$ and $\mathcal{I}_{22} = \frac{1}{\sigma^2} \int_{\mathbb{R}} \frac{(\varphi_\alpha(u) + u\varphi'_\alpha(u))^2}{\varphi_\alpha(u)} du$. The proof is mainly based on the L^2-regularity property of the transition density (see Jeganathan [13]) and on Malliavin calculus (see for example Gobet [8] for the use of Malliavin calculus in the case of a diffusion process). The L^2-regularity property is established here by using the asymptotic behavior of the density of the process solution of (1.1) in small time as well as its derivative with respect to the parameter, given in [6] and based on the Malliavin calculus for jump processes developed by Bichteler, Gravereaux and Jacod [4]. It also requires a careful study of the asymptotic behavior of the information matrix based on one observation of the process, this is the subject of Section 3. This paper contains also an independent and interesting result stating a continuity property with respect to the conditioning variable in a conditional expectation (see Proposition 6.1).

This paper is organized as follows. The main results (asymptotic expansion of the log-likelihood function and LAMN property) are stated in Section 2. Section 3 studies the asymptotic behavior of the Fisher information matrix based on the observation of $X^\beta_{\frac{1}{n}}$ (as n goes to infinity). The proof of the main results are given in Section 4 and Section 5. Finally, Section 6 and Section 7 contain some additional technical proof required to establish the results of Section 3.

2 Main results

We consider the process $(X_t^\beta)_{t\in[0,1]}$ described by

$$X_t^\beta = x_0 + \int_0^t b(X_s^\beta, \theta)ds + \sigma L_t \quad t \in [0,1],$$

where $(L_t)_{t\in[0,1]}$ is a pure jump Lévy process defined on some probability space $(\Omega, \mathcal{A}, \mathcal{P})$ and we assume that the following assumptions are fulfilled.

$\mathbf{H}_1$: (a) The function b has bounded derivatives up to order five with respect to both variables.

(b_i) The Lévy process $(L_t)_{t\in[0,1]}$ is given by $L_t = \int_0^t \int_{[-1,1]} z\{\overline{\mu}(ds,dz) - \overline{\upsilon}(ds,dz)\} + \int_0^t \int_{[-1,1]^c} z\overline{\mu}(ds,dz)$ where $\overline{\mu}$ is a Poisson random measure, with compensator $\overline{\upsilon}(dt,dz) = dt \times F(z)dz$ where $F(z)$ is given on $\mathbb{R}$ by $F(z) = \frac{1}{|z|^{\alpha+1}} 1_{|z|\neq 0}\tau(z), \quad \alpha \in (0,2)$. We assume that τ is a non negative smooth function equal to 1 on [-1,1], vanishing on $[-2,2]^c$ such that $0 \leq \tau \leq 1$.

(b_{ii}) We assume that $\forall p \geq 1, \int_{\mathbb{R}} \left|\frac{\tau'(u)}{\tau(u)}\right|^p \tau(u)du < \infty$, $\int_{\mathbb{R}} \left|\frac{\tau''(u)}{\tau(u)}\right|^p \tau(u)du < \infty$.

Remark 2.1. *The introduction of the truncation function τ in the density of the Lévy measure is a technical tool to ensure the integrability of $|L_t|^p, \forall p \geq 1$. The above assumption will guarantee that* (1.1) *has an unique solution belonging to $\boldsymbol{L}^p, \forall p \geq 1$, moreover, X_t^β admits a smooth density, for $t > 0$.*

We are interested in the statistical properties of the process (X_t^β), based on the discrete time observations $(X_{i/n}^\beta)_{i=0,...n}$. Before stating our main results, we introduce some notations which are used throughout this paper. For a vector $h \in \mathbb{R}^2$, h^T denotes the transpose of h, and $|h|$ denotes the euclidean norm. For a function f defined on $\mathbb{R} \times \mathbb{R}^2$ depending on both variables (x, β), here $\beta = (\theta, \sigma)^T \in \mathbb{R} \times (0, +\infty)$, we denote by f' the derivative of f with respect to the variable x, by $\partial_\theta f$ the derivative of f with respect to the parameter θ, by $\partial_\sigma f$ the derivative of f with respect to the parameter σ, and $\nabla_\beta f = \begin{pmatrix} \partial_\theta f \\ \partial_\sigma f \end{pmatrix}$.

We denote by $p_{i/n}^\beta(x,y)$ the transition density of the homogeneous Markov chain $(X_{i/n}^\beta)_{i=0,...n}$, by $(\mathcal{F}_{i/n})_i$ the σ-field such that $\mathcal{F}_{i/n} = \sigma(X_s^\beta, s \leq i/n) = \sigma(L_s, s \leq i/n)$ and by $\mathbb{P}_n^\beta$ the law of the vector $(X_{\frac{1}{n}}^\beta, ..., X_1^\beta)$ on $\mathbb{R}^n$.

Our first result is an asymptotic expansion of the log-likelihood ratio.

Theorem 2.1. *We assume that $\boldsymbol{H}_1$ holds. Let $u_n = \begin{pmatrix} n^{\frac{1}{2}-\frac{1}{\alpha}} & 0 \\ 0 & n^{-\frac{1}{2}} \end{pmatrix}$, then for every $h \in \mathbb{R}^2$*

$$\log \frac{d\mathbb{P}_n^{\beta+u_n h}}{d\mathbb{P}_n^\beta}(X_{\frac{1}{n}}^\beta, ..., X_1^\beta) = h^T J_n(\beta)^{\frac{1}{2}} N_n(\beta) - \frac{1}{2} h^T J_n(\beta) h + o_{\mathcal{P}}(1), \tag{2.1}$$

with

$$J_n(\beta) = u_n \sum_{i=0}^{n-1} \mathbb{E}\left[\xi_{i,n}(\beta)\xi_{i,n}^T(\beta)|\mathcal{F}_{i/n}\right] u_n,$$

$$N_n(\beta) = J_n(\beta)^{\frac{-1}{2}} u_n \sum_{i=0}^{n-1} \xi_{i,n}(\beta),$$

$$\xi_{i,n}(\beta) = \begin{pmatrix} \frac{\partial_\theta p^\beta_{\frac{1}{n}}}{p^\beta_{\frac{1}{n}}}\left(X^\beta_{\frac{i}{n}}, X^\beta_{\frac{i+1}{n}}\right) \\ \frac{\partial_\sigma p^\beta_{\frac{1}{n}}}{p^\beta_{\frac{1}{n}}}\left(X^\beta_{\frac{i}{n}}, X^\beta_{\frac{i+1}{n}}\right) \end{pmatrix}.$$

We can precise the asymptotic behavior of $J_n(\beta)$ and $N_n(\beta)$. Let φ_α be the density of L_1^α, where (L_1^α) is a centered α-stable Lévy process whose Lévy measure is $\frac{dz}{|z|^{1+\alpha}}1_{|z|\neq 0}$. We define the following quantity which will be the random asymptotic information of the statistical model:

$$\mathcal{I}(\beta) = \begin{pmatrix} \mathcal{I}_{11} & 0 \\ 0 & \mathcal{I}_{22} \end{pmatrix} \tag{2.2}$$

where

$$\mathcal{I}_{11} = \frac{1}{\sigma^2}\int_0^1 \partial_\theta b(X_s^\beta, \theta)^2 ds \times \int_{\mathbb{R}} \frac{\varphi'_\alpha(u)^2}{\varphi_\alpha(u)} du, \quad \mathcal{I}_{22} = \frac{1}{\sigma^2} \times \int_{\mathbb{R}} \frac{\left(\varphi_\alpha(u) + u\varphi'_\alpha(u)\right)^2}{\varphi_\alpha(u)} du.$$

Remark 2.2. *i) Aït-Sahalia and Jacod [2] proved that the parameter θ of the process $X_t^\theta = \theta t + L_t$ is estimated with rate $n^{\frac{1}{2}-\frac{1}{\alpha}}$ and Fisher information $\int_{\mathbb{R}} \frac{\varphi'_\alpha(u)^2}{\varphi_\alpha(u)} du$. The parameter σ of the process $X_t^\sigma = \sigma L_t$ is estimated with the usual rate $n^{-1/2}$ and Fisher information $\int_{\mathbb{R}} \frac{(\varphi_\alpha(u)+u\varphi'_\alpha(u))^2}{\varphi_\alpha(u)} du$.*
ii) It is worth to notice that the information does not depend on the truncation function τ, but depends on α through the Fisher information of the translated α-stable process and multiplicative α-stable process.

Theorem 2.2. *With the notations and assumptions of Theorem 2.1, the following convergences hold:*

$$J_n(\beta) \xrightarrow{n\to\infty} \mathcal{I}(\beta) \quad \textit{in probability,} \tag{2.3}$$

$$\forall \epsilon > 0, \quad \sum_{i=0}^{n-1} \mathbb{E}\left[|u_n \xi_{i,n}(\beta)|^2 1_{\{|u_n\xi_{i,n}(\beta)|\geq\epsilon\}}\right] \xrightarrow{n\to\infty} 0. \tag{2.4}$$

Theorem 2.3. *We have the convergence in law*

$$J_n(\beta)^{\frac{1}{2}} N_n(\beta) = u_n \sum_{i=0}^{n-1} \xi_{i,n}(\beta) \Rightarrow \mathcal{N}(0, \mathcal{I}(\beta)) \tag{2.5}$$

where the limit variable is conditionally Gaussian and the convergence is stable with respect to the σ-field $\sigma(L_s, s \leq 1)$.

The stable convergence in law (2.5) and the convergence in probability (2.3) yield the convergence in law of the couple $(J_n(\beta), N_n(\beta))$:

$$(J_n(\beta), N_n(\beta)) \Rightarrow (\mathcal{I}(\beta), N),$$

where N is a standard Gaussian variable independent of $\mathcal{I}(\beta)$. As a consequence of the asymptotic expansion given in Theorem 2.1 and the preceding limit theorems, we deduce the LAMN property.

Corollary 2.4. *The family* $(\mathbb{P}_n^\beta)$ *satisfies the LAMN property with rate* $u_n = \begin{pmatrix} n^{\frac{1}{2}-\frac{1}{\alpha}} & 0 \\ 0 & n^{-\frac{1}{2}} \end{pmatrix}$, *and information* $\mathcal{I}(\beta)$ *given by* (2.2).

The rate of estimation of the drift parameter depends on α : when α tends to 2, the rate is extremely low, however, when α goes to zero, it becomes high, especially for $\alpha < 1$ where it is faster than the usual one $\sqrt{n}$. On the other hand, the rate of estimation of the volatility parameter σ is $n^{-1/2}$ and does not depend on α.

The remainder of the paper is devoted to the proofs of the main theorems above. The first step of our approach consists in studying the asymptotic Fisher information matrix by using Malliavin calculus techniques.

3 The asymptotic Fisher information matrix in small time

3.1 The asymptotic properties of the Fisher information matrix

Our main concern in this section is to study the asymptotic properties of the Fisher information carried by the observation of the random variable $X_{1/n}^\beta$. We recall the definition of the Fisher information matrix:

$$\mathcal{I}^{n,\beta,x_0} = \begin{pmatrix} \mathcal{I}_{11}^{n,\beta,x_0} & \mathcal{I}_{12}^{n,\beta,x_0} \\ \mathcal{I}_{12}^{n,\beta,x_0} & \mathcal{I}_{22}^{n,\beta,x_0} \end{pmatrix} \tag{3.1}$$

where

$$\mathcal{I}_{11}^{n,\beta,x_0} = \mathbb{E}\left[\left(\frac{\partial_\theta p_{\frac{1}{n}}^\beta(x_0, X_{1/n}^\beta)}{p_{\frac{1}{n}}^\beta(x_0, X_{1/n}^\beta)}\right)^2\right], \quad \mathcal{I}_{12}^{n,\beta,x_0} = \mathbb{E}\left[\left(\frac{\partial_\theta p_{\frac{1}{n}}^\beta(x_0, X_{1/n}^\beta)}{p_{\frac{1}{n}}^\beta(x_0, X_{1/n}^\beta)} \frac{\partial_\sigma p_{\frac{1}{n}}^\beta(x_0, X_{1/n}^\beta)}{p_{\frac{1}{n}}^\beta(x_0, X_{1/n}^\beta)}\right)\right],$$

and $\mathcal{I}_{22}^{n,\beta,x_0} = \mathbb{E}\left[\left(\frac{\partial_\sigma p_{\frac{1}{n}}^\beta(x_0, X_{1/n}^\beta)}{p_{\frac{1}{n}}^\beta(x_0, X_{1/n}^\beta)}\right)^2\right]$.

The following proposition gives the asymptotic behavior of the Fisher information based on the observation of the random variable $X_{1/n}^\beta$ as $\frac{1}{n}$ tends to zero.

Proposition 3.1. *Let (β_n) be a sequence such that $\beta_n \xrightarrow{n\to\infty} \beta$, $Q \subset \mathbb{R}\times(0,\infty)$ a compact set and* $u_n = \begin{pmatrix} n^{\frac{1}{2}-\frac{1}{\alpha}} & 0 \\ 0 & n^{-\frac{1}{2}} \end{pmatrix}$ *then*

i) $nu_n\mathcal{I}^{n,\beta_n,x_0}u_n \xrightarrow{n\to\infty} \begin{pmatrix} \frac{1}{\sigma^2}\partial_\theta b(x_0,\theta)^2 \int_{\mathbb{R}} \frac{\varphi'_\alpha(u)^2}{\varphi_\alpha(u)}du & 0 \\ 0 & \frac{1}{\sigma^2}\int_{\mathbb{R}} \frac{[\varphi_\alpha(u)+u\varphi'_\alpha(u)]^2}{\varphi_\alpha(u)}du \end{pmatrix}$
and this convergence is uniform with respect to x_0.

ii) $\sup_{n,\beta\in Q,x_0} n^{2-2/\alpha}\mathcal{I}_{11}^{n,\beta,x_0} = \sup_{n,\beta\in Q,x_0} n^{2-2/\alpha}\mathbb{E}\left[\frac{\partial_\theta p^\beta_{\frac{1}{n}}(x_0,X^\beta_{1/n})^2}{p^\beta_{\frac{1}{n}}(x_0,X^\beta_{1/n})^2}\right] < \infty.$

iii) $\sup_{n,\beta\in Q,x_0} \mathcal{I}_{22}^{n,\beta,x_0} = \sup_{n,\beta\in Q,x_0} \mathbb{E}\left[\left(\frac{\partial_\sigma p^\beta_{\frac{1}{n}}(x_0,X^\beta_{1/n})}{p^\beta_{\frac{1}{n}}(x_0,X^\beta_{1/n})}\right)^2\right] < \infty.$

iv) $\sup_{n,\beta\in Q,x_0} n^{1-1/\alpha}\mathcal{I}_{12}^{n,\beta,x_0} = \sup_{n,\beta\in Q,x_0} n^{1-1/\alpha}\mathbb{E}\left[\left(\frac{\partial_\theta p^\beta_{\frac{1}{n}}(x_0,X^\beta_{1/n})}{p^\beta_{\frac{1}{n}}(x_0,X^\beta_{1/n})}\frac{\partial_\sigma p^\beta_{\frac{1}{n}}(x_0,X^\beta_{1/n})}{p^\beta_{\frac{1}{n}}(x_0,X^\beta_{1/n})}\right)\right] < \infty.$

Remark 3.1. *From Proposition 3.1, we see that the Fisher information contained in one observation is of magnitude $n^{-1}u_n^{-2}$ and the Fisher information based on n observations is of magnitude u_n^{-2}. This is consistent with the fact that u_n is the rate in the LAMN property stated in Corollary 2.4.*

The rest of this section is devoted to the proof of Proposition 3.1.

3.2 Proof of Proposition 3.1

The proof of Proposition 3.1 relies on a representation of the score function using Malliavin calculus. We recall in Section 3.2.1 all the necessary definitions and results established in [6].

3.2.1 Representation of the score function using Malliavin calculus

We consider $\mu^e(dt,dz,du)$ a Poisson measure on $[0,\infty)\times\mathbb{R}\times[0,1]$ with compensating measure $\upsilon^e(dt,dz,du) = dt1_{|z|\neq 0}\frac{dz}{|z|^{1+\alpha}}du$ and for $n\geq 1$, we define the Poisson random measure $\mu^{(n)}$ by

$$\forall A\subset[0,\infty)\times\mathbb{R},\quad \mu^{(n)}(A) = \int_{[0,\infty)}\int_{\mathbb{R}}\int_{[0,1]} 1_A(t,z)1_{\{u\leq\tau(\frac{z}{n^{1/\alpha}})\}}\mu^e(dt,dz,du).$$

We note that the compensator of $\mu^{(n)}(dt,dz)$ is $\upsilon^{(n)}(dt,dz) = dt\times\tau(\frac{z}{n^{1/\alpha}})1_{|z|\neq 0}\frac{dz}{|z|^{1+\alpha}} := dt\times F_n(z)dz$ and the compensated Poisson random measure $\tilde{\mu}^{(n)}(dt,dz) = \mu^{(n)}(dt,dz) - \upsilon^{(n)}(dt,dz)$.
We define the process (L_t^n) by:

$$L_t^n = \int_0^t\int_{[-n^{1/\alpha},n^{1/\alpha}]} z\tilde{\mu}^{(n)}(ds,dz) + \int_0^t\int_{[-n^{1/\alpha},n^{1/\alpha}]^c} z\mu^{(n)}(ds,dz). \tag{3.2}$$

We now consider the process $(\overline{Y}_t^{n,\beta,x_0})$ solution of the equation

$$\overline{Y}_t^{n,\beta,x_0} = x_0 + \frac{1}{n}\int_0^t b(\overline{Y}_s^{n,\beta,x_0},\theta)ds + \frac{\sigma}{n^{1/\alpha}}L_t^n \quad t\in[0,1]. \tag{3.3}$$

We observe that the process $(L_{t/n})$ (recall $\mathbf{H}_1(b_i)$) equals in law $(\frac{1}{n^{1/\alpha}}L_t^n)$ since the associated jump measures have the same compensator. As a consequence, $(X^\beta_{\frac{t}{n}})_{t\in[0,1]}$ equals in law $(\overline{Y}_t^{n,\beta,x_0})_{t\in[0,1]}$. Let q^{n,β,x_0} be the density of $\overline{Y}_1^{n,\beta,x_0}$ then the connection between the densities of $X^\beta_{\frac{1}{n}}$ and $\overline{Y}_1^{n,\beta,x_0}$ is given by

$$p^\beta_{1/n}(x_0,x) = q^{n,\beta,x_0}(x). \tag{3.4}$$

Moreover, when n grows, the process (L_t^n) converges almost surely to an α-stable process defined by

$$L_t^\alpha = \int_0^t\int_{[-1,1]} z\tilde{\mu}(ds,dz) + \int_0^t\int_{[-1,1]^c} z\mu(ds,dz), \tag{3.5}$$

where μ is the Poisson random measure defined by,

$$\forall A\subset[0,\infty)\times\mathbb{R},\quad \mu(A) = \int_{[0,\infty)}\int_{\mathbb{R}}\int_{[0,1]} 1_A(t,z)\mu^e(dt,dz,du).$$

The compensator of $\mu(dt,dz)$ is $\upsilon(dt,dz) = dt\times 1_{|z|\neq 0}\frac{dz}{|z|^{1+\alpha}}$ and we note the compensated Poisson random measure $\tilde{\mu}(dt,dz) = \mu(dt,dz) - \upsilon(dt,dz)$.

The connection between L^n and the α-stable process L^α is given more clearly by the following lemma.

Lemma 3.1. *[Clément-Gloter-Nguyen [6]] On the event* $A_n = \big\{\mu(\{(t,z)|0\le t\le 1, |z|\ge n^{1/\alpha}\}) = 0\big\}$, *we have*

$$\mu^{(n)} = \mu,\quad L_t^n = L_t^\alpha,$$

and $\mathbb{P}(A_n) = 1 + O(1/n)$.

Furthermore, let $(f_n)_{n\in\mathbb{N}}$ *and* f *be measurable functions from* $\Omega\times[0,1]\times\mathbb{R}$ *to* $\mathbb{R}$ *such that there exists* C *with* $\mathbb{P}(C)=1$ *and* $\forall\omega\in C$, $\forall s\in[0,1], \forall|z|>1$ $f_n(\omega,s,z)\xrightarrow{n\to\infty} f(\omega,s,z)$. *Then*

$$\int_0^1\int_{|z|>1} f_n(\omega,s,z)\mu^{(n)}(ds,dz)\xrightarrow[a.s.]{n\to\infty}\int_0^1\int_{|z|>1} f(\omega,s,z)\mu(ds,dz).$$

Moreover, we have $L_1^n\xrightarrow[a.s]{n\to\infty} L_1^\alpha$.

We remark that $\overline{Y}_1^{n,\beta,x_0}$ admits derivatives with respect to the parameters θ and σ, denoted by $(\partial_\theta Y_t^\beta)_t$ and $(\partial_\sigma Y_t^\beta)_t$, respectively, solution of

$$\partial_\theta\overline{Y}_t^{n,\beta,x_0} = \frac{1}{n}\int_0^t b'(\overline{Y}_s^{n,\beta,x_0},\theta)\partial_\theta\overline{Y}_s^{n,\beta,x_0}ds + \frac{1}{n}\int_0^t\partial_\theta b(\overline{Y}_s^{n,\beta,x_0},\theta)ds, \tag{3.6}$$

$$\partial_\sigma\overline{Y}_t^{n,\beta,x_0} = \frac{1}{n}\int_0^t b'(\overline{Y}_s^{n,\beta,x_0},\theta)\partial_\sigma\overline{Y}_s^{n,\beta,x_0}ds + \frac{L_t^n}{n^{1/\alpha}}. \tag{3.7}$$

Furthermore, the derivatives of $\overline{Y}_1^{n,\beta,x_0}$ with respect to the parameters satisfy the following properties.

Lemma 3.2. *[Clément-Gloter-Nguyen [6]] For all $p \geq 1$,*

i) $|\partial_\theta \overline{Y}_1^{n,\beta,x_0}| \leq \frac{C}{n}$,

ii) $\sup_{s\in[0,1]} \left|\partial_\sigma \overline{Y}_s^{n,\beta,x_0}\right| \xrightarrow[\boldsymbol{L}^{2p}]{n\to\infty} 0.$

We now recall the definition of the Malliavin operators L and Γ on the Poisson space associated to the measure $\mu^{(n)}$ and their basic properties (see Bichteler, Gravereaux, Jacod [4], Chapter IV, Section 8-9-10). For a test function $f : [0,1]\times E \mapsto \mathbb{R}$ (f is measurable, $\mathcal{C}^2$ with respect to the second variable, with bounded derivative, and $f \in \cap_{p\geq 1}\mathbf{L}^p(\nu)$) we set $\mu^{(n)}(f) = \int_0^1 \int_E f(t,z)\mu^{(n)}(dt,dz)$. We introduce an auxiliary function ρ^n as

$$\rho^n(z) = \begin{cases} z^4 & \text{if} \quad |z| < 1 \\ \zeta(z) & \text{if} \quad 1 \leq |z| \leq 2 \\ z^2\tau(\frac{z}{2n^{1/\alpha}}) & \text{if} \quad |z| > 2 \end{cases} \tag{3.8}$$

where τ is defined in the assumption $\mathbf{H}_1(b_i)$, and ζ is a non negative function belonging to $\mathbf{C}^\infty$ such that the function ρ^n belongs to $\mathbf{C}^\infty$. Note that ζ is defined such that $\rho^n(z)$ admits a derivative and $\rho^n, (\rho^n)',\ \rho^n\frac{F_n'(z)}{F_n(z)}$ belong to $\cap_{p\geq 1}\mathbf{L}^p(F_n(z)dz)$. From the conditions on τ, we can easily deduce that

$$z^2\tau(\tfrac{z}{2n^{1/\alpha}}) = \begin{cases} z^2 & \text{if} \quad 2 \leq |z| \leq 2n^{1/\alpha} \\ 0 & \text{if} \quad |z| > 4n^{1/\alpha}. \end{cases}$$

Moreover, we can see that $\rho^n(z) \xrightarrow{n\to\infty} \rho(z)$ where

$$\rho(z) = \begin{cases} z^4 & \text{if} \quad |z| < 1 \\ \zeta(z) & \text{if} \quad 1 \leq |z| \leq 2 \\ z^2 & \text{if} \quad |z| > 2. \end{cases} \tag{3.9}$$

Note that from the definition of ρ^n and ρ, we can easily see that $\rho^n(z) = \rho(z)$ if $|z| \leq 2n^{1/\alpha}$.

With these notations, we define the Malliavin operator L, on a simple functional $\mu^{(n)}(f)$, in the same way as in [5] by the following equations :

$$L(\mu^{(n)}(f)) = \frac{1}{2}\mu^{(n)}\left((\rho^n)'f' + \rho^n\frac{F_n'}{F_n}f' + \rho^n f''\right),$$

where f' and f'' are the derivatives with respect to the second variable. For $\Phi = F(\mu^{(n)}(f_1), .., \mu^{(n)}(f_k))$, with F of class $\mathcal{C}^2$, we set

$$L\Phi = \sum_{i=1}^k \frac{\partial F}{\partial x_i}(\mu^{(n)}(f_1), ..., \mu^{(n)}(f_k))L(\mu^{(n)}(f_i)) + \frac{1}{2}\sum_{i,j=1}^k \frac{\partial^2 F}{\partial x_i \partial x_j}(\mu^{(n)}(f_1), ..., \mu^{(n)}(f_k))\mu^{(n)}(\rho^n f_i' f_j').$$

These definitions permit to construct a linear operator L on a space $D \subset \cap_{p\geq 1}\mathbf{L}^p$ whose basic properties are the following.

i) L is self-adjoint: $\forall \Phi, \Psi \in D$, we have $\mathbb{E}\Phi L\Psi = \mathbb{E}L\Phi\Psi$.

ii) $L\Phi^2 \geq 2\Phi L\Phi$.

iii) $\mathbb{E}L\Phi = 0$.

We associate to L, the symmetric bilinear operator Γ:

$$\Gamma(\Phi, \Psi) = L(\Phi\Psi) - \Phi L\Psi - \Psi L\Phi. \tag{3.10}$$

This operator satisfies the following properties (see [4, equation (8-3)])

$$\Gamma(F(\Phi), \Psi) = F'(\Phi)\Gamma(\Phi, \Psi), \tag{3.11}$$

$$\Gamma(F(\Phi_1, \Phi_2), \Psi) = \partial_{\Phi_1} F(\Phi_1, \Phi_2)\Gamma(\Phi_1, \Psi) + \partial_{\Phi_2} F(\Phi_1, \Phi_2)\Gamma(\Phi_2, \Psi), \tag{3.12}$$

$$|\Gamma(\Phi, \Psi)| \leq \Gamma(\Phi, \Phi)^{1/2}\Gamma(\Psi, \Psi)^{1/2}. \tag{3.13}$$

Remark 3.2. *The operators L and Γ depend on n through the functions ρ^n and F_n but to simplify the notation we omit the dependence in n.*

The operator L and the operator Γ permit to establish the following integration by parts formula (see [4, Propositions 8-10, p.103]).

Proposition 3.2. *For Φ and Ψ in D, and f bounded with bounded derivative up to order two, if $\Gamma(\Phi, \Phi)$ is invertible and $\Gamma^{-1}(\Phi, \Phi) \in \cap_{p\geq 1}\boldsymbol{L}^p$ then we have*

$$\mathbb{E}f'(\Phi)\Psi = \mathbb{E}f(\Phi)\mathcal{H}_\Phi(\Psi), \tag{3.14}$$

with

$$\mathcal{H}_\Phi(\Psi) = -2\Psi\Gamma^{-1}(\Phi, \Phi)L\Phi - \Gamma(\Phi, \Psi\Gamma^{-1}(\Phi, \Phi)) \tag{3.15}$$

$$= -2\Psi\Gamma^{-1}(\Phi, \Phi)L\Phi - \frac{1}{\Gamma(\Phi, \Phi)}\Gamma(\Phi, \Psi) + \frac{\Psi}{\Gamma(\Phi, \Phi)^2}\Gamma(\Phi, \Psi) \tag{3.16}$$

$$= \Phi L\left(\frac{\Psi}{\Gamma(\Phi, \Phi)}\right) - \frac{\Psi L\Phi}{\Gamma(\Phi, \Phi)} - L\left(\frac{\Psi\Phi}{\Gamma(\Phi, \Phi)}\right). \tag{3.17}$$

We recall now the representation of the logarithmic derivative of the density of $\overline{Y}_1^{n,\beta,x_0}$ as a conditional expectation proved in [6] which will be used throughout the proof of the main theorems of this paper.

Theorem 3.1. *[Theorem 2.2 and Theorem 3.1 in Clément-Gloter-Nguyen [6]]*
Let q^{n,β,x_0} be the density of $\overline{Y}_1^{n,\beta,x_0}$ then we have the representation of the logarithmic derivative of the density as a conditional expectation:

$$\frac{\nabla_\beta p^\beta_{\frac{1}{n}}}{p^\beta_{\frac{1}{n}}}(x_0,u)=\frac{\nabla_\beta q^{n,\beta,x_0}}{q^{n,\beta,x_0}}(u)=\begin{pmatrix}\frac{\partial_\theta q^{n,\beta,x_0}}{q^{n,\beta,x_0}}(u)\\ \frac{\partial_\sigma q^{n,\beta,x_0}}{q^{n,\beta,x_0}}(u)\end{pmatrix}=\mathbb{E}(\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\nabla_\beta \overline{Y}_1^{n,\beta,x_0})|\overline{Y}_1^{n,\beta,x_0}=u) \tag{3.18}$$

with

$$\begin{aligned}\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\nabla_\beta \overline{Y}_1^{n,\beta,x_0})&=\begin{pmatrix}\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta,x_0})\\ \mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta,x_0})\end{pmatrix}\\ &=\begin{pmatrix}\partial_\theta \overline{Y}_1^{n,\beta,x_0}\\ \partial_\sigma \overline{Y}_1^{n,\beta,x_0}\end{pmatrix}\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(1)-\frac{1}{\Gamma(\overline{Y}_1^{n,\beta,x_0},\overline{Y}_1^{n,\beta,x_0})}\begin{pmatrix}\Gamma(\overline{Y}_1^{n,\beta,x_0},\partial_\theta \overline{Y}_1^{n,\beta,x_0})\\ \Gamma(\overline{Y}_1^{n,\beta,x_0},\partial_\sigma \overline{Y}_1^{n,\beta,x_0})\end{pmatrix}\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(1)&=\frac{\Gamma(\overline{Y}_1^{n,\beta,x_0},\Gamma(\overline{Y}_1^{n,\beta,x_0},\overline{Y}_1^{n,\beta,x_0}))}{\left(\Gamma(\overline{Y}_1^{n,\beta,x_0},\overline{Y}_1^{n,\beta,x_0})\right)^2}-2\frac{L\overline{Y}_1^{n,\beta,x_0}}{\Gamma(\overline{Y}_1^{n,\beta,x_0},\overline{Y}_1^{n,\beta,x_0})}\\ &=\frac{1}{\sigma}n^{1/\alpha}\widehat{\mathcal{H}}^n_\beta(1)+\mathcal{R}^n_{1,\beta}(1)+\mathcal{R}^n_{2,\beta}(1)+\mathcal{R}^n_{3,\beta}(1).\end{aligned} \tag{3.20}$$

The main term $\widehat{\mathcal{H}}^n_\beta(1)$ is given by

$$\widehat{\mathcal{H}}^n_\beta(1)=\frac{\int_0^1\int_{\mathbb{R}}(\epsilon^n_s)^{-3}\rho^n(z)(\rho^n)'(z)\mu^n(ds,dz)}{\epsilon^n_1\left[\int_0^1\int_{\mathbb{R}}(\epsilon^n_s)^{-2}\rho^n(z)\mu^n(ds,dz)\right]^2}-\frac{\int_0^1\int_{\mathbb{R}}(\epsilon^n_s)^{-1}\left[(\rho^n)'(z)-\frac{1+\alpha}{z}\rho^n(z)\right]\mu^n(ds,dz)}{\epsilon^n_1\int_0^1\int_{\mathbb{R}}(\epsilon^n_s)^{-2}\rho^n(z)\mu^n(ds,dz)} \tag{3.21}$$

with

$$\epsilon^n_s=exp\left(\frac{1}{n}\int_0^s b'(\overline{Y}^{n,\beta,x_0}_u,\theta)du\right). \tag{3.22}$$

The remainder terms satisfies

$$\forall p\geq 2,\quad \mathbb{E}\left|\mathcal{R}^n_{1,\beta}(1)\right|^p\leq\frac{C}{n},\qquad |\mathcal{R}^n_{2,\beta}(1)|\leq\frac{C_1}{n},\qquad |\mathcal{R}^n_{3,\beta}(1)|\leq\frac{C_2}{n}, \tag{3.23}$$

where C, C_1 and C_2 are some deterministic constants.

Remark 3.3. *i) From the proof of Theorem 3.2 in [6], we get that $\widehat{\mathcal{H}}^n_\beta(1)$ is bounded by a random variable independent of n and belonging to $\cap_{p\geq 1}\boldsymbol{L}^p$ and it converges in $\boldsymbol{L}^p,\forall p\geq 1$ to $\mathcal{H}_{L^\alpha}(1)$ given by*

$$\mathcal{H}_{L^\alpha}(1)=\frac{\int_0^1\int_{\mathbb{R}}\rho(z)\rho'(z)\mu(ds,dz)}{\left[\int_0^1\int_{\mathbb{R}}\rho(z)\mu(ds,dz)\right]^2}-\frac{\int_0^1\int_{\mathbb{R}}\left[\rho'(z)-\frac{1+\alpha}{z}\rho(z)\right]\mu(ds,dz)}{\int_0^1\int_{\mathbb{R}}\rho(z)\mu(ds,dz)}. \tag{3.24}$$

Moreover, $\mathcal{H}_{L^\alpha}(1)$ *and* $L_1^\alpha \mathcal{H}_{L^\alpha}(1)$ *belong to* $\boldsymbol{L}^p, \forall p \geq 1$.

ii) In the case $b \equiv 0$ *and* $\sigma = 1$, *we have* $\epsilon_s^n = 1$ *and from [6] the remainder terms* $\mathcal{R}_{2,\beta}^n(1), \mathcal{R}_{3,\beta}^n(1)$ *are equal to zero. Moreover, we see that* (3.20) *can be rewritten as* $\widehat{\mathcal{H}}_\beta^n(1) + \frac{1}{n^{1/\alpha}}\mathcal{R}_{1,\beta}^n(1) = \mathcal{H}_{L_1^n}(1)$, *then we can deduce that* $\mathcal{H}_{L_1^n}(1) \xrightarrow[p\geq 2]{\boldsymbol{L}^p} \mathcal{H}_{L^\alpha}(1)$.

3.2.2 Preliminary lemmas

The proof of Proposition 3.1 is based on the following three lemmas. The first lemma shows the convergence of the normalized iterated Malliavin weight $\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\nabla_\beta \overline{Y}_1^{n,\beta,x_0})$.

Lemma 3.3. *Let* $(\beta_n)_{n\geq 1}$ *be a sequence such that* $\beta_n \xrightarrow{n\to\infty} \beta$. *Then, the following convergence holds uniformly with respect to* x_0

$$\begin{pmatrix} n^{1-1/\alpha}\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta_n,x_0}) \\ \mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0}) \end{pmatrix} \xrightarrow[\boldsymbol{L}^p,\forall p\geq 2]{n\to\infty} \begin{pmatrix} \frac{1}{\sigma}\partial_\theta b(x_0,\theta)\mathcal{H}_{L^\alpha}(1) \\ \frac{1}{\sigma}\left(L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1\right) \end{pmatrix}, \tag{3.25}$$

where $\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\nabla_\beta \overline{Y}_1^{n,\beta,x_0}) = \begin{pmatrix} \mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta,x_0}) \\ \mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta,x_0}) \end{pmatrix}$ *are given by* (3.19) *and* L_1^α, $\mathcal{H}_{L^\alpha}(1)$ *are defined by* (3.5), (3.24), *respectively.*

The next two lemmas are related to a continuity property with respect to the conditioning variable, in a conditional expectation.

Lemma 3.4. *Let* $(\beta_n)_{n\geq 1}$ *be a sequence such that* $\beta_n \xrightarrow{n\to\infty} \beta$. *Then, the following convergence holds uniformly with respect to* x_0.

i) $n^{2-2/\alpha}\mathbb{E}[\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]^2] \xrightarrow{n\to\infty} \frac{1}{\sigma^2}\left[\partial_\theta b(x_0,\theta)\right]^2 \mathbb{E}\left[\mathbb{E}[\mathcal{H}_{L^\alpha}(1)|L_1^\alpha]^2\right]$,

ii) $\mathbb{E}[\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]^2] \xrightarrow{n\to\infty} \frac{1}{\sigma^2}\mathbb{E}\left[\mathbb{E}[L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1|L_1^\alpha]^2\right]$,

Lemma 3.5. *Let* $(\beta_n)_{n\geq 1}$ *be a sequence such that* $\beta_n \xrightarrow{n\to\infty} \beta$. *Then, the following convergence holds uniformly with respect to* x_0,

$$\begin{aligned} n^{1-1/\alpha}\mathbb{E}[\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]] \\ \xrightarrow{n\to\infty} \frac{1}{\sigma^2}\mathbb{E}\left[\mathbb{E}[\mathcal{H}_{L^\alpha}(1)|L_1^\alpha]\mathbb{E}[L_1^\alpha \mathcal{H}_{L^\alpha}(1) - 1|L_1^\alpha]\right]. \end{aligned} \tag{3.26}$$

The proofs of the above lemmas are postponed to Appendix 1.

3.2.3 Proof of Proposition 3.1

Proof. *i*) We need to prove that for (β_n) a sequence such that $\beta_n \xrightarrow{n\to\infty} \beta$ and $u_n = \begin{pmatrix} n^{\frac{1}{2}-\frac{1}{\alpha}} & 0 \\ 0 & n^{-\frac{1}{2}} \end{pmatrix}$ we have

$$nu_n\mathcal{I}^{n,\beta_n,x_0}u_n \xrightarrow{n\to\infty} \begin{pmatrix} \frac{1}{\sigma^2}\partial_\theta b(x_0,\theta)^2 \int_{\mathbb{R}} \frac{\varphi'_\alpha(u)^2}{\varphi_\alpha(u)}du & 0 \\ 0 & \frac{1}{\sigma^2}\int_{\mathbb{R}} \frac{[\varphi_\alpha(u)+u\varphi'_\alpha(u)]^2}{\varphi_\alpha(u)}du \end{pmatrix}$$

and that this convergence is uniform with respect to x_0.

Since $nu_n\mathcal{I}^{n,\beta_n,x_0}u_n = \begin{pmatrix} n^{2-\frac{2}{\alpha}}\mathcal{I}_{11}^{n,\beta_n,x_0} & n^{1-\frac{1}{\alpha}}\mathcal{I}_{12}^{n,\beta_n,x_0} \\ n^{1-\frac{1}{\alpha}}\mathcal{I}_{12}^{n,\beta_n,x_0} & \mathcal{I}_{22}^{n,\beta_n,x_0} \end{pmatrix}$, the proof of the above convergence reduces to prove the convergence of each entries of the matrix.

Step 1: We prove that $n^{2-2/\alpha}\mathcal{I}_{11}^{n,\beta_n,x_0} \xrightarrow{n\to\infty} \frac{1}{\sigma^2}\partial_\theta b(x_0,\theta)^2\mathbb{E}\left[\left(\frac{\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)}\right)^2\right] = \frac{1}{\sigma^2}\partial_\theta b(x_0,\theta)^2\int_{\mathbb{R}}\frac{\varphi'_\alpha(u)^2}{\varphi_\alpha(u)}du$, and this convergence is uniform with respect to x_0.

First we remark that we have the representation,

$$\frac{\varphi'_\alpha(u)}{\varphi_\alpha(u)} = -\mathbb{E}\left[\mathcal{H}_{L^\alpha}(1)|L_1^\alpha = u\right]. \tag{3.27}$$

Indeed, let us consider the situation where $b(x,\theta) = \theta$ and $x_0 = 0$. In this case, we have $\overline{Y}_1^{n,\beta,x_0} = \frac{\theta}{n} + \frac{\sigma L_1^n}{n^{1/\alpha}}$ and thus the density of $\frac{n^{1/\alpha}}{\sigma}\left(\overline{Y}_1^{n,\beta,x_0} - \frac{\theta}{n}\right)$ is equal to the density of L_1^n. We denote by φ^n the density of L_1^n. Using the integration by parts formula (3.14), we get for any smooth function f,

$$\int (\varphi^n)'(u)f(u)du = -\mathbb{E}[f(L_1^n)\mathcal{H}_{L_1^n}(1)]. \tag{3.28}$$

On the other hand, we have

$$\begin{aligned}\int (\varphi^n)'(u)f(u)du &= -\int \varphi^n(u)f'(u)du \\ &= -\mathbb{E}[f'(L_1^n)] \xrightarrow{n\to\infty} -\mathbb{E}[f'(L_1^\alpha)] \qquad (3.29)\\ &= -\int \varphi_\alpha(u)f'(u)du = \int \varphi'_\alpha(u)f(u)du. \qquad (3.30)\end{aligned}$$

where the convergence (3.29) is obtained by the fact that $\mathbb{E}[f'(L_1^n)] = \mathbb{E}[f'(L_1^\alpha)1_{L_1^n=L_1^\alpha}] + \mathbb{E}[f'(L_1^n)1_{L_1^n\neq L_1^\alpha}]$ and by Lemma 3.1, we have $\mathbb{P}(L_1^n = L_1^\alpha) \xrightarrow{n\to\infty} 1$.

From (3.28), (3.30), the smoothness of f and Remark 3.3 *ii*) we get $\int f(u)\varphi'_\alpha(u)du = -\mathbb{E}[f(L_1^\alpha)\mathcal{H}_{L^\alpha}(1)]$, and we deduce the representation (3.27).

Furthermore, from (3.18) in Theorem 3.1, we have

$$
\begin{aligned}
n^{2-2/\alpha}\mathcal{I}_{11}^{n,\beta_n,x_0} &= n^{2-2/\alpha}\mathbb{E}\left[\mathbb{E}\left[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}\right]^2\right] \\
&\xrightarrow{n\to\infty} \frac{1}{\sigma^2}\partial_\theta b(x_0,\theta)^2\mathbb{E}\left[\mathbb{E}\left[\mathcal{H}_{L^\alpha}(1)|L_1^\alpha\right]^2\right], \quad \text{uniformly with respect to } x_0, \text{ from Lemma 3.4 } ii), \\
&= \frac{1}{\sigma^2}\partial_\theta b(x_0,\theta)^2\mathbb{E}\left[\frac{\varphi'_\alpha(L_1^\alpha)^2}{\varphi_\alpha(L_1^\alpha)^2}\right] \qquad \text{from (3.27)}.
\end{aligned}
$$

Step 2: We prove that $\mathcal{I}_{22}^{n,\beta_n,x_0} \xrightarrow{n\to\infty} \frac{1}{\sigma^2}\mathbb{E}\left[\left(\frac{\varphi_\alpha(L_1^\alpha)+L_1^\alpha\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)}\right)^2\right] = \frac{1}{\sigma^2}\int_{\mathbb{R}}\frac{[\varphi_\alpha(u)+u\varphi'_\alpha(u)]^2}{\varphi_\alpha(u)}du,$
and this convergence is uniform with respect to x_0.

We remark that from the representation (3.27), we have the following representation,

$$
\frac{\varphi_\alpha(u)+u\varphi'_\alpha(u)}{\varphi_\alpha(u)} = -u\mathbb{E}\left[\left(\mathcal{H}_{L^\alpha}(1)\right)|L_1^\alpha = u\right] + 1 = -\mathbb{E}\left[\left(L_1^\alpha\mathcal{H}_{L^\alpha}(1)-1\right)|L_1^\alpha = u\right]. \tag{3.31}
$$

Furthermore, combining (3.18) and Lemma 3.4 $ii)$, we have

$$
\begin{aligned}
\mathcal{I}_{22}^{n,\beta_n,x_0} &= \mathbb{E}\left[\mathbb{E}\left[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}\right]^2\right], \\
&\xrightarrow{n\to\infty} \frac{1}{\sigma^2}\mathbb{E}\left[\mathbb{E}\left[\left(L_1^\alpha\mathcal{H}_{L^\alpha}(1)-1\right)|L_1^\alpha\right]^2\right], \quad \text{uniformly with respect to } x_0, \\
&= \frac{1}{\sigma^2}\mathbb{E}\left[\frac{\left(\varphi_\alpha(L_1^\alpha)+L_1^\alpha\varphi'_\alpha(L_1^\alpha)\right)^2}{\varphi_\alpha(L_1^\alpha)^2}\right] \qquad \text{from (3.31)}.
\end{aligned}
$$

Step 3: We prove that $n^{1-1/\alpha}\mathcal{I}_{12}^{n,\beta_n,x_0} \xrightarrow{n\to\infty} 0$ and that this convergence is uniform with respect to x_0.

We first recall the expression for the Fisher information

$$
\mathcal{I}_{12}^{n,\beta_n,x_0} = \mathbb{E}\left[\left(\frac{\partial_\theta p_{\frac{1}{n}}^{\beta_n}(x_0,X_{1/n}^{\beta_n})}{p_{\frac{1}{n}}^{\beta_n}(x_0,X_{1/n}^{\beta_n})}\frac{\partial_\sigma p_{\frac{1}{n}}^{\beta_n}(x_0,X_{1/n}^{\beta_n})}{p_{\frac{1}{n}}^{\beta_n}(x_0,X_{1/n}^{\beta_n})}\right)\right],
$$

then from (3.18) in Theorem 3.1 and Lemma 3.5 we have

$$
\begin{aligned}
n^{1-1/\alpha}\mathcal{I}_{12}^{n,\beta_n,x_0} &= n^{1-1/\alpha}\mathbb{E}[\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]] \\
&\xrightarrow{n\to\infty} \frac{1}{\sigma^2}\mathbb{E}\left[\mathbb{E}[\mathcal{H}_{L^\alpha}(1)|L_1^\alpha]\mathbb{E}[L_1^\alpha\mathcal{H}_{L^\alpha}(1)-1|L_1^\alpha]\right].
\end{aligned} \tag{3.32}
$$

On the other hand, from (3.27) and (3.31) we get

$$
\frac{\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)}\left(\frac{\varphi_\alpha(L_1^\alpha)+L_1^\alpha\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)}\right) = \mathbb{E}\left[\mathcal{H}_{L^\alpha}(1)|L_1^\alpha\right]\mathbb{E}\left[\left(L_1^\alpha\mathcal{H}_{L^\alpha}(1)-1\right)|L_1^\alpha\right]. \tag{3.33}
$$

Combining (3.32) with (3.33) gives

$$
n^{1-1/\alpha}\mathcal{I}_{12}^{n,\beta_n,x_0} \xrightarrow{n\to\infty} \frac{1}{\sigma^2}\int_{\mathbb{R}}\frac{\varphi'_\alpha(u)\left[\varphi_\alpha(u)+u\varphi'_\alpha(u)\right]}{\varphi_\alpha(u)}du = \frac{1}{\sigma^2}\int_{\mathbb{R}}\varphi'_\alpha(u)du + \frac{1}{\sigma^2}\int_{\mathbb{R}}\frac{u\left(\varphi'_\alpha(u)\right)^2}{\varphi_\alpha(u)}du = 0, \tag{3.34}
$$

where we used the fact that φ_α is a symmetric function, and that the functions under the integral are odd. This completes the proof of the part i).

ii) Using (3.18) in Theorem 3.1 again and Jensen's inequality, we get:

$$n^{2-2/\alpha}\mathcal{I}_{11}^{n,\beta,x_0} = n^{2-2/\alpha}\mathbb{E}\left[\mathbb{E}(\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta,x_0})|\overline{Y}_1^{n,\beta,x_0})^2\right] \leq n^{2-2/\alpha}\mathbb{E}\left[\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta,x_0})^2\right].$$

But it is clear from Lemma 3.3, the fact that $|\partial_\theta \overline{Y}_1^{n,\beta,x_0}| \leq \frac{C}{n}$ (recall Lemma 3.2 i)) and the boundedness of $\widehat{\mathcal{H}}_{\beta_n}^n(1)$ (recall remark 3.3) that for all $p \geq 1$, $\sup_{n,\beta\in Q,x_0}\mathbb{E}|n^{1-1/\alpha}\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta,x_0})|^p \leq C$.

iii) From (3.18) in Theorem 3.1 and Jensen's inequality, we get:

$$\mathcal{I}_{22}^{n,\beta_n,x_0} = \mathbb{E}\left[\mathbb{E}(\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta,x_0})|\overline{Y}_1^{n,\beta,x_0})^2\right] \leq \mathbb{E}\left[\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta,x_0})^2\right].$$

From Lemma 3.3, for all $p \geq 2$, $\sup_{n,\beta\in Q,x_0}\mathbb{E}|\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta,x_0})|^p \leq C$.

iv) Using (3.18) in Theorem 3.1 again with Cauchy-Schwarz inequality and Jensen's inequality we get

$$\begin{aligned}
n^{1-1/\alpha}\left|\mathcal{I}_{12}^{n,\beta,x_0}\right| &= \left|n^{1-1/\alpha}\mathbb{E}[\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta,x_0})|\overline{Y}_1^{n,\beta,x_0}]\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta,x_0})|\overline{Y}_1^{n,\beta,x_0}]]\right| \\
&\leq n^{1-1/\alpha}\left[\mathbb{E}\left(\left(\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta,x_0})|\overline{Y}_1^{n,\beta,x_0}]\right)^2\right)\right]^{1/2}\left[\mathbb{E}\left(\left(\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta,x_0})|\overline{Y}_1^{n,\beta,x_0}]\right)^2\right)\right]^{1/2} \\
&\leq n^{1-1/\alpha}\left[\mathbb{E}\left(\left|\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta,x_0})\right|^2\right)\right]^{1/2}\left[\mathbb{E}\left(\left|\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta,x_0})\right|^2\right)\right]^{1/2}.
\end{aligned}$$

This achieves the proof of Proposition 3.1. □

4 Proof of the asymptotic expansion of the likelihood (Theorems 2.1 - 2.2)

The aim of this section is to prove the asymptotic expansion of the log-likelihood function, stated in Theorem 2.1 as well as the convergence given in Theorem 2.2. The proof of Theorem 2.1 is based essentially on the L^2-regularity property of the transition density $p_{1/n}^\beta(x,y)$. From Jeganathan's article [13], the following four conditions **A1-A4** are sufficient to get the expansion (2.1) of Theorem 2.1.

We recall the notation $\xi_{i,n}(\beta) = \begin{pmatrix} \frac{\partial_\theta p_{\frac{1}{n}}^\beta}{p_{\frac{1}{n}}^\beta}\left(X_{\frac{i}{n}}^\beta, X_{\frac{i+1}{n}}^\beta\right) \\ \frac{\partial_\sigma p_{\frac{1}{n}}^\beta}{p_{\frac{1}{n}}^\beta}\left(X_{\frac{i}{n}}^\beta, X_{\frac{i+1}{n}}^\beta\right) \end{pmatrix}$ and we denote $\chi_n(\beta,x,y) = \begin{pmatrix} \frac{\partial_\theta p_{\frac{1}{n}}^\beta(x,y)}{p_{\frac{1}{n}}^\beta(x,y)^{1/2}} \\ \frac{\partial_\sigma p_{\frac{1}{n}}^\beta(x,y)}{p_{\frac{1}{n}}^\beta(x,y)^{1/2}} \end{pmatrix}$.

A1. L^2**-regularity**

$$\sum_{j=1}^n \mathbb{E}\left[\int_{\mathbb{R}}\left\{p_{\frac{1}{n}}^{\beta+u_nh}\left(X_{\frac{j-1}{n}}^\beta, y\right)^{1/2} - p_{\frac{1}{n}}^\beta\left(X_{\frac{j-1}{n}}^\beta, y\right)^{1/2} - \frac{1}{2}h^T u_n \chi_n(\beta, X_{\frac{j-1}{n}}^\beta, y)\right\}^2 dy\right] \xrightarrow{n\to\infty} 0.$$

A2.

$$J_n(\beta) = u_n \sum_{i=0}^{n-1} \mathbb{E}\left[\xi_{i,n}(\beta)\xi_{i,n}^T(\beta)|\mathcal{F}_{i/n}\right] u_n \xrightarrow{n\to\infty} \mathcal{I}(\beta) \qquad (>0 \quad \text{a.e.}), \quad \text{in probability.}$$

A3.

$$\forall \epsilon > 0, \quad \sum_{i=0}^{n-1} \mathbb{E}\left[|u_n \xi_{i,n}(\beta)|^2 1_{\{|u_n\xi_{i,n}(\beta)|\geq\epsilon\}}\right] \xrightarrow{n\to\infty} 0.$$

A4.

$$\sup_n \sum_{i=0}^{n} \mathbb{E}(\left|u_n \xi_{i,n}(\beta)\xi_{i,n}(\beta)^T u_n\right|) \leq C, \quad \text{for a strictly positive constant } C.$$

The condition **A1** is proved in Section 4.1 and **A2** and **A3** are proved in Section 4.2. The condition **A4** is immediate from Proposition 3.1 $ii-iv$) since

$$n\mathbb{E}(u_n\xi_{i,n}(\beta)\xi_{i,n}(\beta)^T u_n) = \mathbb{E}\begin{pmatrix} n^{2-\frac{2}{\alpha}}\mathcal{I}_{11}^{n,\beta,X_{i/n}^\beta} & n^{1-\frac{1}{\alpha}}\mathcal{I}_{12}^{n,\beta,X_{i/n}^\beta} \\ n^{1-\frac{1}{\alpha}}\mathcal{I}_{12}^{n,\beta,X_{i/n}^\beta} & \mathcal{I}_{22}^{n,\beta,X_{i/n}^\beta} \end{pmatrix}.$$

Note that these conditions do not imply the stable convergence in law (2.5) since in our framework the filtration $(\mathcal{F}_{\frac{i}{n}})_i$ does not satisfy the "nested condition" (see Theorem 3.2 in [10]). The proof of the stable convergence in law will be given in Section 5.

4.1 Proof of the condition A1 (the L^2- regularity condition).

Following [5], the crucial point of the proof is the asymptotic behavior of the transition density of X_t^β established in [6] and recalled below.

Theorem 4.1. *[Clément-Gloter-Nguyen [6]]*

We assume that $\boldsymbol{H}_1$ holds. Let $(\varsigma_t^{n,\theta,x_0})$ be the solution of the ordinary differential equation

$$\varsigma_t^{n,\theta,x_0} = x_0 + \frac{1}{n}\int_0^t b(\varsigma_s^{n,\theta,x_0},\theta)ds \quad t\in[0,1]. \tag{4.1}$$

Let $(\beta_n)_{n\geq 1}$ be a sequence such that $\beta_n \xrightarrow{n\to\infty} \beta$ then for all $(x_0,u)\in\mathbb{R}^2$,

i) $\frac{\sigma_n}{n^{1/\alpha}} p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\theta_n,x_0}) \xrightarrow{n\to\infty} \varphi_\alpha(u)$,

ii) $\sup_{u\in\mathbb{R}} \sup_n \frac{\sigma_n}{n^{1/\alpha}} p_{\frac{1}{n}}^{\beta_n}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\theta_n,x_0}) < \infty$,

where φ_α is the density of L_1^α, a centered α-stable Lévy process.

Theorem 4.2. *[Clément-Gloter-Nguyen [6]]*

Under the assumptions of Theorem 4.1,

i) $\frac{\sigma_n^2}{n^{\frac{2}{\alpha}-1}}\partial_\theta p^{\beta_n}_{\frac{1}{n}}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\theta_n,x_0}) \xrightarrow{n\to\infty} -\partial_\theta b(x_0,\theta) \times \varphi'_\alpha(u)$,
$\frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma p^{\beta_n}_{\frac{1}{n}}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\theta_n,x_0}) \xrightarrow{n\to\infty} -\varphi_\alpha(u) - u\varphi'_\alpha(u)$,

ii) $\sup_{u\in\mathbb{R}}\sup_n \left|\frac{\sigma_n^2}{n^{\frac{2}{\alpha}-1}}\partial_\theta p^{\beta_n}_{\frac{1}{n}}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\theta_n,x_0})\right| < \infty$,
$\sup_{u\in\mathbb{R}}\sup_n \left|\frac{\sigma_n^2}{n^{1/\alpha}}\partial_\sigma p^{\beta_n}_{\frac{1}{n}}(x_0, \frac{u\sigma_n}{n^{1/\alpha}} + \varsigma_1^{n,\theta_n,x_0})\right| < \infty$.

Proof of A1. By the change of variable $y = \frac{u\sigma}{n^{1/\alpha}} + \varsigma_1^{n,\theta,X^\beta_{\frac{j-1}{n}}}$ proving **A1** is equivalent to show:

$$\frac{1}{n}\sum_{j=1}^n \int_{\mathbb{R}} \mathbb{E}\left[\left\{f_n(X^\beta_{\frac{j-1}{n}}, u) - g_n(X^\beta_{\frac{j-1}{n}}, u)\right\}^2\right] du \xrightarrow{n\to\infty} 0, \tag{4.2}$$

where

$$f_n(x,u) = \sqrt{\sigma} n^{1/2-1/2\alpha}\left[p^{\beta+u_nh}_{\frac{1}{n}}\left(x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)^{1/2} - p^{\beta}_{\frac{1}{n}}\left(x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)^{1/2}\right],$$
$$g_n(x,u) = \frac{\sqrt{\sigma}}{2} n^{1/2-1/2\alpha}(u_nh)^T\chi_n\left(\beta, x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right).$$

Following the proof of Proposition 8 in [5], the next three properties are sufficient to prove (4.2).

1. There exists a function f such that,
$$\begin{aligned} &\forall x,u, \quad f_n(x,u) \xrightarrow{n\to\infty} f(x,u), \\ &g_n(x,u) \xrightarrow{n\to\infty} f(x,u). \end{aligned} \tag{4.3}$$

2. We have for all x,
$$\begin{aligned} &\limsup_n \int_{\mathbb{R}} f_n(x,u)^2 du \le \int_{\mathbb{R}} f(x,u)^2 du, \\ &\limsup_n \int_{\mathbb{R}} g_n(x,u)^2 du \le \int_{\mathbb{R}} f(x,u)^2 du. \end{aligned} \tag{4.4}$$

3. We have
$$\begin{aligned} &\sup_{x,n} \int_{\mathbb{R}} f_n(x,u)^2 du < \infty, \\ &\sup_{x,n} \int_{\mathbb{R}} g_n(x,u)^2 du < \infty. \end{aligned} \tag{4.5}$$

We now need to check the validity of the conditions (4.3), (4.4) and (4.5).

We start with the proof of the condition (4.3).

We recall that $u_n = \begin{pmatrix} n^{\frac{1}{2}-\frac{1}{\alpha}} & 0 \\ 0 & n^{-\frac{1}{2}} \end{pmatrix}$ and $h = (h_1,h_2)^T \in \mathbb{R}^2$ then by a simple computation we have,

$$g_n(x,u) = \frac{\sqrt{\sigma}}{2} n^{1-\frac{3}{2\alpha}} h_1 \frac{\partial_\theta p^\beta_{\frac{1}{n}}\left(x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)}{p^\beta_{\frac{1}{n}}\left(x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)^{1/2}} + \frac{\sqrt{\sigma}}{2} n^{\frac{-1}{2\alpha}} h_2 \frac{\partial_\sigma p^\beta_{\frac{1}{n}}\left(x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)}{p^\beta_{\frac{1}{n}}\left(x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)^{1/2}}.$$

From Theorems 4.1 and 4.2, we see that

$$g_n(x,u) \xrightarrow{n\to\infty} f(x,u) := -\frac{h_1}{2\sigma}\partial_\theta b(x,\theta)\frac{\varphi'_\alpha(u)}{\varphi_\alpha(u)^{1/2}} - \frac{h_2}{2\sigma}\frac{(\varphi_\alpha(u)+u\varphi'_\alpha(u))}{\varphi_\alpha(u)^{1/2}}.$$

Let $m_t = p^{\beta+u_nht}_{\frac{1}{n}}\left(x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)$, $t\in[0,1]$, then we can rewrite $f_n(x,u)$ as

$$f_n(x,u) = \sqrt{\sigma}n^{1/2-1/2\alpha}\left[m_1^{1/2} - m_0^{1/2}\right].$$

Using the mean value theorem, we get

$f_n(x,u) = \frac{\sqrt{\sigma}}{2}n^{1/2-1/2\alpha}\frac{m'_s}{\sqrt{m_s}} = \frac{\sqrt{\sigma}}{2}n^{1/2-1/2\alpha}(u_nh)^T\frac{\nabla_\beta p^{\beta+u_nhs}_{\frac{1}{n}}\left(x,\frac{u\sigma}{n^{1/\alpha}}+\varsigma_1^{n,\theta,x}\right)}{p^{\beta+u_nhs}_{\frac{1}{n}}\left(x,\frac{u\sigma}{n^{1/\alpha}}+\varsigma_1^{n,\theta,x}\right)^{1/2}}$, for some $s\in(0,1)$.

From Theorems 4.1 and 4.2, we also get that $f_n(x,u) \xrightarrow{n\to\infty} f(x,u)$.

Now we prove the condition (4.4).

We have

$$\begin{aligned}\int_{\mathbb{R}} g_n(x,u)^2du = {}& \frac{\sigma h_1^2}{4}n^{2-3/\alpha}\int_{\mathbb{R}}\frac{\partial_\theta p^\beta_{\frac{1}{n}}\left(x,\frac{u\sigma}{n^{1/\alpha}}+\varsigma_1^{n,\theta,x}\right)^2}{p^\beta_{\frac{1}{n}}\left(x,\frac{u\sigma}{n^{1/\alpha}}+\varsigma_1^{n,\theta,x}\right)}du + \frac{\sigma h_2^2}{4}n^{-1/\alpha}\int_{\mathbb{R}}\frac{\partial_\sigma p^\beta_{\frac{1}{n}}\left(x,\frac{u\sigma}{n^{1/\alpha}}+\varsigma_1^{n,\theta,x}\right)^2}{p^\beta_{\frac{1}{n}}\left(x,\frac{u\sigma}{n^{1/\alpha}}+\varsigma_1^{n,\theta,x}\right)}du \\ &+ \frac{\sigma h_1h_2}{2}n^{1-2/\alpha}\int_{\mathbb{R}}\frac{\partial_\theta p^\beta_{\frac{1}{n}}\left(x,\frac{u\sigma}{n^{1/\alpha}}+\varsigma_1^{n,\theta,x}\right)}{p^\beta_{\frac{1}{n}}\left(x,\frac{u\sigma}{n^{1/\alpha}}+\varsigma_1^{n,\theta,x}\right)^{1/2}}\frac{\partial_\sigma p^\beta_{\frac{1}{n}}\left(x,\frac{u\sigma}{n^{1/\alpha}}+\varsigma_1^{n,\theta,x}\right)}{p^\beta_{\frac{1}{n}}\left(x,\frac{u\sigma}{n^{1/\alpha}}+\varsigma_1^{n,\theta,x}\right)^{1/2}}du.\end{aligned} \tag{4.6}$$

From Proposition 3.1, we get

$$\int_{\mathbb{R}} g_n(x,u)^2du \xrightarrow{n\to\infty} \int_{\mathbb{R}} f(x,u)^2du, \quad \forall x. \qquad . \tag{4.7}$$

Using

$$f_n(x,u) = \frac{\sqrt{\sigma}}{2}n^{1/2-1/2\alpha}\int_0^1 (u_nh)^T\frac{\nabla_\beta p^{\beta+u_nhs}_{\frac{1}{n}}\left(x,\frac{u\sigma}{n^{1/\alpha}}+\varsigma_1^{n,\theta,x}\right)}{p^{\beta+u_nhs}_{\frac{1}{n}}\left(x,\frac{u\sigma}{n^{1/\alpha}}+\varsigma_1^{n,\theta,x}\right)^{1/2}}ds.$$

We write:

$$\int_{\mathbb{R}} f_n(x,u)^2 du = ||f_n(x,.)||_2^2 = \frac{\sigma n^{1-1/\alpha}}{4}\left|\left| \int_0^1 (u_n h)^T \frac{\nabla_\beta p_{\frac{1}{n}}^{\beta+u_n hs}\left(x, \sigma\frac{\cdot}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)}{p_{\frac{1}{n}}^{\beta+u_n hs}\left(x, \sigma\frac{\cdot}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)^{1/2}} ds \right|\right|_2^2$$

$$\leq \frac{\sigma n^{1-1/\alpha}}{4}\left(\int_0^1 \left|\left| (u_n h)^T \frac{\nabla_\beta p_{\frac{1}{n}}^{\beta+u_n hs}\left(x, \sigma\frac{\cdot}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)}{p_{\frac{1}{n}}^{\beta+u_n hs}\left(x, \sigma\frac{\cdot}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)^{1/2}} \right|\right|_2 ds \right)^2$$

$$= \frac{\sigma n^{1-1/\alpha}}{4}\left(\int_0^1 \left|\left| n^{\frac{1}{2}-\frac{1}{\alpha}} h_1 \frac{\partial_\theta p_{\frac{1}{n}}^{\beta+u_n hs}\left(x, \sigma\frac{\cdot}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)}{p_{\frac{1}{n}}^{\beta+u_n hs}\left(x, \sigma\frac{\cdot}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)^{1/2}} + n^{\frac{-1}{2}} h_2 \frac{\partial_\sigma p_{\frac{1}{n}}^{\beta+u_n hs}\left(x, \sigma\frac{\cdot}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)}{p_{\frac{1}{n}}^{\beta+u_n hs}\left(x, \sigma\frac{\cdot}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)^{1/2}} \right|\right|_2 ds \right)^2$$

$$= \frac{\sigma n^{1-1/\alpha}}{4}\left[\int_0^1 \left(\int_{\mathbb{R}} n^{1-\frac{2}{\alpha}} h_1^2 \frac{\partial_\theta p_{\frac{1}{n}}^{\beta+su_n h}\left(x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)^2}{p_{\frac{1}{n}}^{\beta+su_n h}\left(x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)} du + \int_{\mathbb{R}} n^{-1} h_2^2 \frac{\partial_\sigma p_{\frac{1}{n}}^{\beta+su_n h}\left(x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)^2}{p_{\frac{1}{n}}^{\beta+su_n h}\left(x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)} du \right.\right.$$

$$\left.\left. + \int_{\mathbb{R}} n^{-\frac{1}{\alpha}} 2h_1 h_2 \frac{\partial_\theta p_{\frac{1}{n}}^{\beta+su_n h}\left(x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right) \partial_\sigma p_{\frac{1}{n}}^{\beta+su_n h}\left(x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)}{p_{\frac{1}{n}}^{\beta+su_n h}\left(x, \frac{u\sigma}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x}\right)} du \right)^{1/2} ds \right]^2$$

$$= \frac{n^{1-1/\alpha}}{4}\left[\int_0^1 \left(\int_{\mathbb{R}} n^{1-\frac{1}{\alpha}} h_1^2 \frac{\partial_\theta p_{\frac{1}{n}}^{\beta+su_n h}(x,v)^2}{p_{\frac{1}{n}}^{\beta+su_n h}(x,v)} dv + \int_{\mathbb{R}} n^{\frac{1}{\alpha}-1} h_2^2 \frac{\partial_\sigma p_{\frac{1}{n}}^{\beta+su_n h}(x,v)^2}{p_{\frac{1}{n}}^{\beta+su_n h}(x,v)} dv \right.\right.$$

$$\left.\left. + \int_{\mathbb{R}} 2h_1 h_2 \frac{\partial_\theta p_{\frac{1}{n}}^{\beta+su_n h}(x,v)\, \partial_\sigma p_{\frac{1}{n}}^{n,\beta+su_n h}(x,v)}{p_{\frac{1}{n}}^{\beta+su_n h}(x,v)} dv \right)^{1/2} ds \right]^2 \text{ by the change of variable } \frac{u\sigma}{n^{1/\alpha}} + \varsigma_1^{n,\theta,x} = v,$$

$$= \frac{1}{4}\left[\int_0^1 \left(n^{2-\frac{2}{\alpha}} h_1^2 \mathcal{I}_{11}^{n,\beta+su_n h,x} + h_2^2 \mathcal{I}_{22}^{n,\beta+su_n h,x} + n^{1-1/\alpha} 2h_1 h_2 \mathcal{I}_{12}^{n,\beta+su_n h,x} \right)^{1/2} ds \right]^2$$

$$\xrightarrow{n\to\infty} \frac{1}{\sigma^2}\frac{h_1^2}{4}(\partial_\theta b(x,\theta))^2 \int_{\mathbb{R}} \frac{(\varphi_\alpha'(u))^2}{\varphi_\alpha(u)} du + \frac{h_1 h_2}{2\sigma^2}\int_{\mathbb{R}} \frac{\varphi_\alpha'(u)\left[\varphi_\alpha(u) + u\varphi_\alpha'(u)\right]}{\varphi_\alpha(u)} du$$

$$+ \frac{1}{\sigma^2}\frac{h_2^2}{4}\int_{\mathbb{R}} \frac{(\varphi_\alpha(u) + u\varphi_\alpha'(u))^2}{\varphi_\alpha(u)} du = \int_{\mathbb{R}} f^2(x,u) du. \tag{4.8}$$

Where, in the last line, we have used Proposition 3.1 for the convergences of $\mathcal{I}_{11}^{n,\beta+su_n h,x}$, $\mathcal{I}_{22}^{n,\beta+su_n h,x}$, $\mathcal{I}_{12}^{n,\beta+su_n h,x}$, respectively and the application of the dominated convergence theorem. From (4.7) and (4.8) we get (4.4). (4.5) is deduced directly from Proposition 3.1. □

4.2 Proof of the conditions A2 and A3 (Theorem 2.2)

From the Markov property and (3.1) we have:

$$\mathcal{I}^{n,\beta,X^\beta_{i/n}} = \mathbb{E}\left[\xi_{i,n}(\beta)\xi_{i,n}(\beta)^T|\mathcal{G}_{i/n}\right] = \begin{pmatrix} \mathcal{I}_{11}^{n,\beta,X^\beta_{i/n}} & \mathcal{I}_{12}^{n,\beta,X^\beta_{i/n}} \\ \mathcal{I}_{12}^{n,\beta,X^\beta_{i/n}} & \mathcal{I}_{22}^{n,\beta,X^\beta_{i/n}} \end{pmatrix}.$$

From Proposition 3.1 we know that the quantities

$$\sup_{0\le i\le n-1}\left| nu_n\mathcal{I}^{n,\beta,X^\beta_{i/n}}u_n - \begin{pmatrix} \frac{1}{\sigma^2}\left[\partial_\theta b(X^\beta_{i/n})\right]^2\int_{\mathbb{R}}\frac{\varphi'(u)^2}{\varphi(u)}du & 0 \\ 0 & \frac{1}{\sigma^2}\int_{\mathbb{R}}\frac{[\varphi_\alpha(u)+u\varphi'_\alpha(u)]^2}{\varphi_\alpha(u)}du \end{pmatrix}\right|$$

converge to zero as $n\to\infty$. Then the convergence **A2** is a consequence of the convergence of a Riemann sum.

To prove **A3**, from the Markov property, we get: $\mathbb{E}\left[|u_n\xi_{i,n}(\beta)|^k|X^\beta_{i/n}=x\right] = \mathbb{E}\left[\left|u_n\frac{\nabla_\beta p^\beta_{\frac{1}{n}}(x,X^\beta_{\frac{1}{n}})}{p^\beta_{\frac{1}{n}}(x,X^\beta_{\frac{1}{n}})}\right|^k\right]$, for any $k\ge 1$. It then follows from Theorem 3.1 that,

$$\mathbb{E}\left[|u_n\xi_{i,n}(\beta)|^k|X^\beta_{i/n}=x\right] = \mathbb{E}\left[\left|u_n\mathbb{E}\left[\mathcal{H}_{\overline{Y}_1^{n,\beta,x}}(\nabla_\beta\overline{Y}_1^{n,\beta,x})|\overline{Y_1}^{n,\beta,x}\right]\right|^k\right] \le \mathbb{E}\left[\left|u_n\mathcal{H}_{\overline{Y}_1^{n,\beta,x}}(\nabla_\beta\overline{Y}_1^{n,\beta,x})\right|^k\right],$$

where we used the Jensen inequality in the last step. On the other hand, from the proof of the parts *ii)* - *iii)* of Proposition 3.1 the random variables $n^{1-1/\alpha}\mathcal{H}_{\overline{Y}_1^{n,\beta,x}}(\partial_\theta\overline{Y}_1^{n,\beta,x})$ and $\mathcal{H}_{\overline{Y}_1^{n,\beta,x}}(\partial_\sigma\overline{Y}_1^{n,\beta,x})$ are bounded in $\mathbf{L}^k$-norm independently of n and x. From this, we deduce

$$\sup_{0\le i\le n-1}\mathbb{E}\left[\left|\begin{pmatrix} n^{1-1/\alpha} & 0 \\ 0 & 1\end{pmatrix}\xi_{i,n}(\beta)\right|^k\right] \le C(k),\quad \forall k\ge 1,$$

where $C(k)$ is some finite constant. It can be classically checked that the previous control, for instance with $k=4$, is sufficient to imply the Lindeberg's condition **A3**.

5 Proof of Theorem 2.3 (Stable central limit theorem)

The aim in this section is to prove the stable convergence in law stated in Theorem 2.3. We first recall the following result established in [6] where $\overline{Y}_1^{n,\beta,x_0}$ is defined by (3.3) and is equal in law to $X^\beta_{\frac{1}{n}}$.

Lemma 5.1. *[Clément-Gloter-Nguyen [6]] Let $(\varsigma_t^{n,\theta,x_0})$ be the solution of the ordinary differential equation* (4.1)*, then*

$$n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0}) \xrightarrow[a.s.]{n\to\infty} \sigma L_1^\alpha, \tag{5.1}$$

and this convergence is uniform with respect to x_0.

Proof of Theorem 2.3. Since $u_n = \begin{pmatrix} n^{\frac{1}{2}-\frac{1}{\alpha}} & 0 \\ 0 & n^{-\frac{1}{2}} \end{pmatrix}$ we have

$$u_n \sum_{i=0}^{n-1} \xi_{i,n}(\beta) = \sum_{i=0}^{n-1} \begin{pmatrix} n^{\frac{1}{2}-\frac{1}{\alpha}} \frac{\partial_\theta p^\beta_{\frac{1}{n}}}{p^\beta_{\frac{1}{n}}} \left(X^\beta_{\frac{i}{n}}, X^\beta_{\frac{i+1}{n}} \right) \\ n^{-\frac{1}{2}} \frac{\partial_\sigma p^\beta_{\frac{1}{n}}}{p^\beta_{\frac{1}{n}}} \left(X^\beta_{\frac{i}{n}}, X^\beta_{\frac{i+1}{n}} \right) \end{pmatrix}.$$

Theorem 2.3 is an immediate consequence of Lemmas 5.2-5.3-5.4 below. □

Lemma 5.2. *We consider*

$$\varpi_{i,n} = n^{1-1/\alpha} \frac{\partial_\theta p^\beta_{\frac{1}{n}}}{p^\beta_{\frac{1}{n}}} (X^\beta_{\frac{i}{n}}, X^\beta_{\frac{i+1}{n}}) + \frac{1}{\sigma} \partial_\theta b(X^\beta_{\frac{i}{n}}, \theta) \frac{\varphi'_\alpha \left(n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)}{\varphi_\alpha \left(n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)}$$

then we have $n^{-1/2} \sum_{i=0}^{n-1} \varpi_{i,n} \xrightarrow[\mathbb{P}]{n\to\infty} 0$.

Proof. Using Lemma 9 in [7], it is sufficient to show that:

$$n^{-1/2} \sum_{i=0}^{n-1} |\mathbb{E}[\varpi_{i,n} | \mathcal{F}_{i/n}]| \xrightarrow[\mathbb{P}]{n\to\infty} 0, \tag{5.2}$$

$$n^{-1} \sum_{i=0}^{n-1} |\mathbb{E}[\varpi^2_{i,n} | \mathcal{F}_{i/n}]| \xrightarrow[\mathbb{P}]{n\to\infty} 0, \tag{5.3}$$

We start by the proof of (5.2). Since a score function has an expectation equal to zero, and $L_{\frac{i+1}{n}} - L_{\frac{i}{n}}$ is independent of $\mathcal{F}_{i/n}$, we deduce that

$$\mathbb{E}[\varpi_{i,n} | \mathcal{F}_{i/n}] = \frac{1}{\sigma} \partial_\theta b(X^\beta_{\frac{i}{n}}, \theta) \mathbb{E} \left[\frac{\varphi'_\alpha \left(n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)}{\varphi_\alpha \left(n^{1/\alpha} (L_{\frac{i+1}{n}} - L_{\frac{i}{n}}) \right)} \right].$$

Since $(L_t)_t$ has stationary increments, the law of $n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})$ is the same as the law of L^n_1. Moreover, we know from Lemma 3.1, that $\mathbb{P}(L^n_1 \neq L^\alpha_1) = O(1/n)$, thus

$$\mathbb{E}[\varpi_{i,n} | \mathcal{F}_{i/n}] = \frac{1}{\sigma} \partial_\theta b(X^\beta_{\frac{i}{n}}, \theta) \mathbb{E} \left[\frac{\varphi'_\alpha(L^\alpha_1)}{\varphi_\alpha(L^\alpha_1)} \right] + \left\| \frac{\varphi'_\alpha}{\varphi_\alpha} \right\|_\infty O(n^{-1}),$$

where we used the fact that $\frac{\varphi'_\alpha}{\varphi_\alpha}$ is bounded (see e.g. Theorem 7.3.2 in [18]). Using $\mathbb{E}\left[\frac{\varphi'_\alpha(L^\alpha_1)}{\varphi_\alpha(L^\alpha_1)} \right] = \int_{\mathbb{R}} \varphi'_\alpha(u) du = 0$, we deduce $|\mathbb{E}[\varpi_{i,n} | \mathcal{G}_{i/n}]| \leq C n^{-1}$ for some constant C and (5.2) follows.

We now prove (5.3). We have

$$\begin{aligned}\mathbb{E}[\varpi_{i,n}^2|\mathcal{F}_{i/n}] &= n^{2-2/\alpha}\mathcal{I}_{11}^{n,\beta,X^\beta_{\frac{i}{n}}} + \frac{1}{\sigma^2}\left[\partial_\theta b(X^\beta_{\frac{i}{n}},\theta)\right]^2 \mathbb{E}\left[\frac{\varphi'_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)^2}{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)^2}\right] \\ &\quad + 2\mathbb{E}\left[n^{1-1/\alpha}\frac{\partial_\theta p^\beta_{\frac{1}{n}}(X^\beta_{\frac{i}{n}},X^\beta_{\frac{i+1}{n}})}{p^\beta_{\frac{1}{n}}(X^\beta_{\frac{i}{n}},X^\beta_{\frac{i+1}{n}})}\frac{1}{\sigma}\partial_\theta b(X^\beta_{\frac{i}{n}},\theta)\frac{\varphi'_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)}{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)}\Big|\mathcal{F}_{\frac{i}{n}}\right].\end{aligned} \tag{5.4}$$

With a method analogous to the proof of (5.2), we can show that $\mathbb{E}\left[\frac{\varphi'_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)^2}{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)^2}\right] = \mathbb{E}\left[\frac{\varphi'_\alpha(L_1^\alpha)^2}{\varphi_\alpha(L_1^\alpha)^2}\right] + o(1)$. From Proposition 3.1, it appears that the first two terms in the right-hand side of (5.4) are asymptotically closed to the same quantities, and that (5.3) is proved as soon as we show that the following control holds, uniformly with respect to i,

$$\begin{aligned}&\mathbb{E}\left[n^{1-1/\alpha}\frac{\partial_\theta p^\beta_{\frac{1}{n}}(X^\beta_{\frac{i}{n}},X^\beta_{\frac{i+1}{n}})}{p^\beta_{\frac{1}{n}}(X^\beta_{\frac{i}{n}},X^\beta_{\frac{i+1}{n}})}\frac{1}{\sigma}\partial_\theta b(X^\beta_{\frac{i}{n}},\theta)\frac{\varphi'_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)}{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)}\Big|\mathcal{F}_{\frac{i}{n}}\right] \\ &= -\frac{1}{\sigma^2}\left[\partial_\theta b(X^\beta_{\frac{i}{n}},\theta)\right]^2\mathbb{E}\left[\frac{\varphi'_\alpha(L_1^\alpha)^2}{\varphi_\alpha(L_1^\alpha)^2}\right] + o(1).\end{aligned} \tag{5.5}$$

Using the notations of Section 3.2.1, we define $d^{n,\theta,x_0} = \mathbb{E}\left[n^{1-1/\alpha}\frac{\partial_\theta q^{n,\beta,x_0}(\overline{Y}_1^{n,\beta,x_0})}{q^{n,\beta,x_0}(\overline{Y}_1^{n,\beta,x_0})}\frac{1}{\sigma}\partial_\theta b(x_0,\theta)\frac{\varphi'_\alpha(L_1^n)}{\varphi_\alpha(L_1^n)}\right]$, so that the left-hand side of (5.5) reduces, from the Markov property, (3.4) and the fact that $\overline{Y}_1^{n,\beta,x_0}$ equals in law $X^\beta_{\frac{1}{n}}$, to $d^{n,\theta,X^\beta_{i/n}}$. On the other hand, we can rewrite d^{n,θ,x_0} as

$$\begin{aligned}d^{n,\theta,x_0} &= \mathbb{E}\left[n^{1-1/\alpha}\frac{\partial_\theta q^{n,\beta,x_0}(\overline{Y}_1^{n,\beta,x_0})}{q^{n,\beta,x_0}(\overline{Y}_1^{n,\beta,x_0})}\frac{1}{\sigma}\partial_\theta b(x_0,\theta)\frac{\varphi'_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})}{\sigma}\right)}{\varphi_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})}{\sigma}\right)}\right] \\ &\quad + \mathbb{E}\left[n^{1-1/\alpha}\frac{\partial_\theta q^{n,\beta,x_0}(\overline{Y}_1^{n,\beta,x_0})}{q^{n,\beta,x_0}(\overline{Y}_1^{n,\beta,x_0})}\frac{1}{\sigma}\partial_\theta b(x_0,\theta)\left(\frac{\varphi'_\alpha(L_1^n)}{\varphi_\alpha(L_1^n)} - \frac{\varphi'_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})}{\sigma}\right)}{\varphi_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})}{\sigma}\right)}\right)\right].\end{aligned} \tag{5.6}$$

Using the Cauchy-Schwarz inequality for the second term in the right-hand side of (5.6), we get

$$\begin{aligned}&\mathbb{E}\left[\left|n^{1-1/\alpha}\frac{\partial_\theta q^{n,\beta,x_0}(\overline{Y}_1^{n,\beta,x_0})}{q^{n,\beta,x_0}(\overline{Y}_1^{n,\beta,x_0})}\frac{1}{\sigma}\partial_\theta b(x_0,\theta)\left(\frac{\varphi'_\alpha(L_1^n)}{\varphi_\alpha(L_1^n)} - \frac{\varphi'_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})}{\sigma}\right)}{\varphi_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})}{\sigma}\right)}\right)\right|\right] \\ &\leq \left[\mathbb{E}\left(n^{1-1/\alpha}\frac{\partial_\theta q^{n,\beta,x_0}(\overline{Y}_1^{n,\beta,x_0})}{q^{n,\beta,x_0}(\overline{Y}_1^{n,\beta,x_0})}\frac{1}{\sigma}\partial_\theta b(x_0,\theta)\right)^2\right]^{1/2}\left[\mathbb{E}\left(\frac{\varphi'_\alpha(L_1^n)}{\varphi_\alpha(L_1^n)} - \frac{\varphi'_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})}{\sigma}\right)}{\varphi_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})}{\sigma}\right)}\right)^2\right]^{1/2}.\end{aligned} \tag{5.7}$$

Furthermore, $\forall \epsilon > 0$ we have

$$
\begin{aligned}
&\mathbb{E}\left(\frac{\varphi'_\alpha\left(L_1^n\right)}{\varphi_\alpha\left(L_1^n\right)} - \frac{\varphi'_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})}{\sigma}\right)}{\varphi_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})}{\sigma}\right)}\right)^2 \\
&= \mathbb{E}\left(\frac{\varphi'_\alpha\left(L_1^n\right)}{\varphi_\alpha\left(L_1^n\right)} - \frac{\varphi'_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})}{\sigma}\right)}{\varphi_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})}{\sigma}\right)}\right)^2 1_{\left\{\left|\frac{n^{1/\alpha}(\overline{Y}_1^{n,\theta,x_0} - \varsigma_1^{n,\theta,x_0})}{\sigma} - L_1^n\right| \leq \epsilon\right\}} \\
&+ \mathbb{E}\left(\frac{\varphi'_\alpha\left(L_1^n\right)}{\varphi_\alpha\left(L_1^n\right)} - \frac{\varphi'_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})}{\sigma}\right)}{\varphi_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})}{\sigma}\right)}\right)^2 1_{\left\{\left|\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})}{\sigma} - L_1^n\right| > \epsilon\right\}} \\
&\leq C_1\epsilon^2 + 2C_2\mathbb{E}\left[1_{\left\{\left|\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})}{\sigma} - L_1^n\right| > \epsilon\right\}}\right] \\
&= C_1\epsilon^2 + 2C_2\mathbb{P}\left[\left|\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})}{\sigma} - L_1^n\right| > \epsilon\right] \xrightarrow{n\to\infty} C_1\epsilon^2,
\end{aligned}
$$

where we used the fact that $\frac{\varphi'_\alpha}{\varphi_\alpha}$ is bounded with a bounded derivative (see e.g. Theorem 7.3.2 in [18]) and Lemma 5.1. From Proposition 3.1 *ii*), and the estimates above we deduce that (5.7) converges to zero as $n \to \infty$. Then,

$$
d^{n,\theta,x_0} = \mathbb{E}\left[n^{1-1/\alpha}\frac{\partial_\theta q^{n,\beta,x_0}(\overline{Y}_1^{n,\beta,x_0})}{q^{n,\beta,x_0}(\overline{Y}_1^{n,\beta,x_0})}\frac{1}{\sigma}\partial_\theta b(x_0,\theta)\frac{\varphi'_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})}{\sigma}\right)}{\varphi_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\theta,x_0} - \varsigma_1^{n,\theta,x_0})}{\sigma}\right)}\right] + o(1), \tag{5.8}
$$

where the $o(1)$ term is uniform with respect to x_0. Now, using Theorem 3.1, we get

$$
d^{n,\theta,x_0} = \mathbb{E}\left[n^{1-1/\alpha}\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta,x_0})\frac{1}{\sigma}\partial_\theta b(x_0,\theta)\frac{\varphi'_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})}{\sigma}\right)}{\varphi_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})}{\sigma}\right)}\right] + o(1).
$$

From Lemma 3.3, we also have

$$
\sup_{x_0}\left|d^{n,\theta,x_0} - \frac{1}{\sigma^2}\partial_\theta b(x_0,\theta)^2\mathbb{E}\left[\mathcal{H}_{L^\alpha}(1)\frac{\varphi'_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})}{\sigma}\right)}{\varphi_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})}{\sigma}\right)}\right]\right| \xrightarrow{n\to\infty} 0.
$$

From Lemma 5.1 we can deduce that

$$
d^{n,\theta,x_0} \xrightarrow{n\to\infty} \frac{1}{\sigma^2}\left[\partial_\theta b(x_0,\theta)\right]^2\mathbb{E}\left[\mathcal{H}_{L^\alpha}(1)\frac{\varphi'_\alpha\left(L_1^\alpha\right)}{\varphi_\alpha\left(L_1^\alpha\right)}\right],
$$

uniformly with respect to x_0. Then, the relation (3.27) enables to rewrite this convergence as,

$$d^{n,\theta,x_0} \xrightarrow{n\to\infty} -\frac{1}{\sigma^2}\left[\partial_\theta b(x_0,\theta)\right]^2 \mathbb{E}\left[\frac{\varphi'_\alpha\left(L_1^\alpha\right)^2}{\varphi_\alpha\left(L_1^\alpha\right)^2}\right], \text{ uniformly with respect to } x_0.$$

This result implies (5.5) and hence (5.3). □

Lemma 5.3. *We consider*

$$\varrho_{i,n} = \frac{\partial_\sigma p^\beta_{\frac{1}{n}}}{p^\beta_{\frac{1}{n}}}(X^\beta_{\frac{i}{n}}, X^\beta_{\frac{i+1}{n}}) + \frac{1}{\sigma}\frac{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})\right) + n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})\varphi'_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})\right)}{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})\right)}$$

then we have $n^{-1/2}\sum_{i=0}^{n-1}\eta_{i,n} \xrightarrow[\mathbb{P}]{n\to\infty} 0$.

Proof. We proceed as in the proof of Lemma 5.2 and check that

$$n^{-1/2}\sum_{i=0}^{n-1}\left|\mathbb{E}[\varrho_{i,n}|\mathcal{F}_{i/n}]\right| \xrightarrow[\mathbb{P}]{n\to\infty} 0, \tag{5.9}$$

$$n^{-1}\sum_{i=0}^{n-1}\left|\mathbb{E}[\varrho^2_{i,n}|\mathcal{F}_{i/n}]\right| \xrightarrow[\mathbb{P}]{n\to\infty} 0. \tag{5.10}$$

We start by the proof of (5.9). Since a score function has an expectation equal to zero, and $L_{\frac{i+1}{n}} - L_{\frac{i}{n}}$ is independent of $\mathcal{F}_{i/n}$, we deduce that

$$\mathbb{E}[\varrho_{i,n}|\mathcal{F}_{i/n}] = \frac{1}{\sigma}\mathbb{E}\left[\frac{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})\right) + n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})\varphi'_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})\right)}{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})\right)}\right].$$

Since $(L_t)_t$ has stationary increments, the law of $n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})$ is the same as the law of L_1^n. Moreover, we know from Lemma 3.1, that $\mathbb{P}(L_1^n \neq L_1^\alpha) = O(1/n)$, thus

$$\mathbb{E}[\varrho_{i,n}|\mathcal{F}_{i/n}] = \frac{1}{\sigma}\mathbb{E}\left[\frac{\varphi_\alpha(L_1^\alpha) + L_1^\alpha\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)}\right] + \left\|\frac{\varphi_\alpha(u) + u\varphi'_\alpha(u)}{\varphi_\alpha(u)}\right\|_\infty O(n^{-1}),$$

where we used the fact that $\frac{u\varphi'_\alpha(u)}{\varphi_\alpha(u)}$ is bounded (see e.g. Theorem 7.3.2 in [18]).
Using $\mathbb{E}\left[\frac{L_1^\alpha\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)}\right] = \int_{\mathbb{R}} u\varphi'_\alpha(u)du = -1$, we deduce $\left|\mathbb{E}[\varrho_{i,n}|\mathcal{F}_{i/n}]\right| \leq Cn^{-1}$ for some constant C and (5.9) follows.

We now prove (5.10). We have

$$\mathbb{E}[\varrho^2_{i,n}|\mathcal{F}_{i/n}] = \mathcal{I}_{22}^{n,\beta,X^\beta_{\frac{i}{n}}} + \frac{1}{\sigma^2}\mathbb{E}\left[\frac{\left[\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})\right) + n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})\varphi'_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})\right)\right]^2}{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})\right)^2}\right]$$
$$+ 2\mathbb{E}\left[\frac{\partial_\sigma p^\beta_{\frac{1}{n}}(X^\beta_{\frac{i}{n}}, X^\beta_{\frac{i+1}{n}})}{p^\beta_{\frac{1}{n}}(X^\beta_{\frac{i}{n}}, X^\beta_{\frac{i+1}{n}})}\frac{1}{\sigma}\frac{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})\right) + n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})\varphi'_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})\right)}{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}} - L_{\frac{i}{n}})\right)}\middle|\mathcal{F}_{\frac{i}{n}}\right]. \tag{5.11}$$

With a method analogous to the proof of (5.9), we can show that

$$\mathbb{E}\left[\frac{\left[\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)+n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\varphi'_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)\right]^2}{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)^2}\right]=\mathbb{E}\left[\frac{\left[\varphi_\alpha(L_1^\alpha)+L_1^\alpha\varphi'_\alpha(L_1^\alpha)\right]^2}{\varphi_\alpha(L_1^\alpha)^2}\right]+o(1).$$

Proceeding as in the proof of (5.4), then (5.9) is proved as soon as we show that the following control holds uniformly with respect to i,

$$\mathbb{E}\left[\frac{\partial_\sigma p^\beta_{\frac{1}{n}}(X^\beta_{\frac{i}{n}},X^\beta_{\frac{i+1}{n}})}{p^\beta_{\frac{1}{n}}(X^\beta_{\frac{i}{n}},X^\beta_{\frac{i+1}{n}})}\frac{1}{\sigma}\frac{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)+n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\varphi'_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)}{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)}\middle|\mathcal{F}_{\frac{i}{n}}\right]$$
$$=-\frac{1}{\sigma^2}\mathbb{E}\left[\frac{(\varphi_\alpha(L_1^\alpha)+L_1^\alpha\varphi'_\alpha(L_1^\alpha))^2}{\varphi_\alpha(L_1^\alpha)^2}\right]+o(1). \quad (5.12)$$

We define $d_1^{n,\sigma,x_0}=\mathbb{E}\left[\frac{\partial_\sigma q(\overline{Y}_1^{n,\beta,x_0})}{q(\overline{Y}_1^{n,\beta,x_0})}\frac{1}{\sigma}\frac{\varphi_\alpha(L_1^n)+L_1^n\varphi'_\alpha(L_1^n)}{\varphi_\alpha(L_1^n)}\right]$, so that the left-hand side of (5.12) reduces, from the Markov property, to $d_1^{n,\sigma,X^\beta_{i/n}}$.

Proceeding as in the proof of (5.8), noting that $\frac{u\varphi'_\alpha(u)}{\varphi_\alpha(u)}$ is bounded with a bounded derivative (see e.g. Theorem 7.3.2 in [18]), then we also get that

$$d_1^{n,\sigma,x_0}=\mathbb{E}\left[\frac{\partial_\sigma q(\overline{Y}_1^{n,\beta,x_0})}{q(\overline{Y}_1^{n,\beta,x_0})}\frac{1}{\sigma}\right]+\mathbb{E}\left[\frac{\partial_\sigma q(\overline{Y}_1^{n,\beta,x_0})}{q(\overline{Y}_1^{n,\beta,x_0})}\frac{1}{\sigma}\frac{\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})}{\sigma}\varphi'_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})}{\sigma}\right)}{\varphi_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})}{\sigma}\right)}\right]+o(1)$$

where the $o(1)$ term is uniform with respect to x_0. Now, using Theorem 3.1, we get

$$d_1^{n,\sigma,x_0}=\mathbb{E}\left[\frac{\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\partial_\sigma\overline{Y}_1^{n,\beta,x_0})}{\sigma}\right]$$
$$+\mathbb{E}\left[\frac{\mathcal{H}_{\overline{Y}_1^{n,\beta,x_0}}(\partial_\sigma\overline{Y}_1^{n,\beta,x_0})}{\sigma}\frac{\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})}{\sigma}\varphi'_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})}{\sigma}\right)}{\varphi_\alpha\left(\frac{n^{1/\alpha}(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})}{\sigma}\right)}\right]+o(1).$$

From Lemma 5.1 and the convergence result (3.25) we can deduce that

$$d_1^{n,\sigma,x_0}\xrightarrow{n\to\infty}\frac{1}{\sigma^2}\mathbb{E}\left[(L_1^\alpha\mathcal{H}_{L^\alpha}(1)-1)\frac{\varphi_\alpha(L_1^\alpha)+L_1^\alpha\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)}\right],$$

uniformly with respect to x_0. Then, the relation (3.31) enables to rewrite this convergence as,

$$d_1^{n,\sigma,x_0}\xrightarrow{n\to\infty}-\frac{1}{\sigma^2}\mathbb{E}\left[\frac{(\varphi_\alpha(L_1^\alpha)+L_1^\alpha\varphi'_\alpha(L_1^\alpha))^2}{\varphi_\alpha(L_1^\alpha)^2}\right].$$

This result implies (5.12) and hence (5.10). □

Lemma 5.4. *We have the convergence in law,*

$$\begin{pmatrix} -n^{-1/2}\sum_{i=0}^{n-1} \frac{\varphi'_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)}{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)}\frac{1}{\sigma}\partial_\theta b(X^\beta_{\frac{i}{n}},\theta) \\ -n^{-1/2}\sum_{i=0}^{n-1}\frac{1}{\sigma}\frac{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)+n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\varphi'_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)}{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)} \end{pmatrix} \Rightarrow \mathcal{N}(0,\mathcal{I}(\beta)) \qquad (5.13)$$

where the convergence is stable with respect to the σ-field $\sigma(L_s, s\le 1)$.

Proof. We define the following processes:

$$Z^n_t = \sum_{i=0}^{\lfloor nt\rfloor}\left(L_{\frac{i+1}{n}}-L_{\frac{i}{n}}\right),$$

$$\Gamma^n_t = \begin{pmatrix}\Gamma^{n,1}_t\\ \Gamma^{n,2}_t\end{pmatrix} = n^{-1/2}\begin{pmatrix} \sum_{i=0}^{\lfloor nt\rfloor}\frac{\varphi'_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)}{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)}\frac{1}{\sigma}\partial_\theta b(X^\beta_{\frac{i}{n}},\theta) \\ \sum_{i=0}^{\lfloor nt\rfloor}\frac{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)+n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\varphi'_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)}{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)}\frac{1}{\sigma}\end{pmatrix},$$

$$\Gamma'^n_t = \begin{pmatrix}\Gamma'^{n,1}_t\\ \Gamma'^{n,2}_t\end{pmatrix} = n^{-1/2}\begin{pmatrix} \sum_{i=0}^{[nt]}\frac{\varphi'_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)}{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)} \\ \sum_{i=0}^{[nt]}\frac{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)+n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\varphi'_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)}{\varphi_\alpha\left(n^{1/\alpha}(L_{\frac{i+1}{n}}-L_{\frac{i}{n}})\right)}\end{pmatrix}.$$

We will apply Lemma 2.8 in [12] to prove (5.13). Indeed, we will show that there exists a Gaussian random vector γ with $var(\gamma) = \begin{pmatrix}\mathbb{E}\left[\frac{\varphi'_\alpha(L^\alpha_1)^2}{\varphi_\alpha(L^\alpha_1)^2}\right] & 0\\ 0 & \mathbb{E}\left[\frac{(\varphi_\alpha(L^\alpha_1)+L^\alpha_1\varphi'_\alpha(L^\alpha_1))^2}{\varphi_\alpha(L^\alpha_1)^2}\right]\end{pmatrix}$, independent of L_1 such that one has the convergence in law

$$(\Gamma'^n_1, Z^n_1)\Rightarrow(\gamma, L_1). \qquad (5.14)$$

Then, by application of Lemma 2.8 in [12], there exists a bi-dimensional Brownian motion $(\Gamma'_t)_t$ independent of $(L_t)_t$ such that one has the convergence in law $(Z^n,\Gamma^n,\Gamma'^n)\Rightarrow(L,\Gamma,\Gamma')$ where $\Gamma_t = \int_0^t\begin{pmatrix}\frac{1}{\sigma}\partial_\theta b(X^\beta_s,\theta) & 0\\ 0 & \frac{1}{\sigma}\end{pmatrix}d\Gamma'_s$ and $var(\Gamma'_1) = var(\gamma)$.

Let us focus on the derivation of the convergence (5.14). For $(u,v,w)\in\mathbb{R}^3$, we set

$$\mathcal{X}_n(u,v,w) = \mathbb{E}\left[exp\left(i\frac{u}{n^{1/2}}\frac{\varphi'_\alpha(n^{1/\alpha}L_{\frac{1}{n}})}{\varphi_\alpha(n^{1/\alpha}L_{\frac{1}{n}})}+i\frac{v}{n^{1/2}}\frac{\left(\varphi_\alpha(n^{1/\alpha}L_{\frac{1}{n}})+n^{1/\alpha}L_{\frac{1}{n}}\varphi'_\alpha(n^{1/\alpha}L_{\frac{1}{n}})\right)}{\varphi_\alpha(n^{1/\alpha}L_{\frac{1}{n}})}+iwL_{\frac{1}{n}}\right)\right].$$

Using the i.i.d. structure of the increments of the Lévy process L, we easily get the following expression about the characteristic function of $(\Gamma_1'^n, Z_1^n)$

$$\log \mathbb{E}\left[\exp(iu\Gamma_1'^{n,1} + iv\Gamma_1'^{n,2} + iwZ_1^n)\right] = n \log \mathcal{X}_n(u,v,w). \tag{5.15}$$

Let us study the asymptotic behavior of $\mathcal{X}_n(u,v,w)$. Using the expansion of the exponential function near zero and that $\frac{\varphi_\alpha'}{\varphi_\alpha}$ and $\frac{x\varphi_\alpha'(x)}{\varphi_\alpha(x)}$ are bounded we get

$$\begin{aligned}
\mathcal{X}_n(u,v,w) = \mathbb{E}\Bigg\{ \left[e^{iwL_{\frac{1}{n}}}\right] \Bigg[1 + \left(\frac{iu}{n^{1/2}} \frac{\varphi_\alpha'(n^{1/\alpha}L_{\frac{1}{n}})}{\varphi_\alpha(n^{1/\alpha}L_{\frac{1}{n}})} + \frac{iv}{n^{1/2}} \frac{\left(\varphi_\alpha(n^{1/\alpha}L_{\frac{1}{n}}) + n^{1/\alpha}L_{\frac{1}{n}}\varphi_\alpha'(n^{1/\alpha}L_{\frac{1}{n}})\right)}{\varphi_\alpha(n^{1/\alpha}L_{\frac{1}{n}})} \right) \\
+ \frac{1}{2}\left(\frac{iu}{n^{1/2}} \frac{\left(\varphi_\alpha'(n^{1/\alpha}L_{\frac{1}{n}})\right)}{\varphi_\alpha(n^{1/\alpha}L_{\frac{1}{n}})} + \frac{iv}{n^{1/2}} \frac{\left(\varphi_\alpha(n^{1/\alpha}L_{\frac{1}{n}}) + n^{1/\alpha}L_{\frac{1}{n}}\varphi_\alpha'(n^{1/\alpha}L_{\frac{1}{n}})\right)}{\varphi_\alpha(n^{1/\alpha}L_{\frac{1}{n}})} \right)^2 \Bigg] \Bigg\} + O(n^{-3/2})
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}\left[e^{iwL_{\frac{1}{n}}}\right] + \frac{iu}{n^{1/2}}\mathbb{E}\left[\frac{\varphi_\alpha'(n^{1/\alpha}L_{\frac{1}{n}})}{\varphi_\alpha(n^{1/\alpha}L_{\frac{1}{n}})} e^{iwL_{\frac{1}{n}}}\right] + \frac{iv}{n^{1/2}}\mathbb{E}\left[\frac{\left(\varphi_\alpha(n^{1/\alpha}L_{\frac{1}{n}}) + n^{1/\alpha}L_{\frac{1}{n}}\varphi_\alpha'(n^{1/\alpha}L_{\frac{1}{n}})\right)}{\varphi_\alpha(n^{1/\alpha}L_{\frac{1}{n}})} e^{iwL_{\frac{1}{n}}}\right] \\
&- \frac{u^2}{2n}\mathbb{E}\left[\left(\frac{\varphi_\alpha'(n^{1/\alpha}L_{\frac{1}{n}})}{\varphi_\alpha(n^{1/\alpha}L_{\frac{1}{n}})}\right)^2 e^{iwL_{\frac{1}{n}}}\right] - \frac{uv}{n}\mathbb{E}\left[\frac{\varphi_\alpha'(n^{1/\alpha}L_{\frac{1}{n}})}{\varphi_\alpha(n^{1/\alpha}L_{\frac{1}{n}})} \frac{\left(\varphi_\alpha(n^{1/\alpha}L_{\frac{1}{n}}) + n^{1/\alpha}L_{\frac{1}{n}}\varphi_\alpha'(n^{1/\alpha}L_{\frac{1}{n}})\right)}{\varphi_\alpha(n^{1/\alpha}L_{\frac{1}{n}})} e^{iwL_{\frac{1}{n}}}\right] \\
&- \frac{v^2}{2n}\mathbb{E}\left[\left(\frac{\varphi_\alpha(n^{1/\alpha}L_{\frac{1}{n}}) + n^{1/\alpha}L_{\frac{1}{n}}\varphi_\alpha'(n^{1/\alpha}L_{\frac{1}{n}})}{\varphi_\alpha(n^{1/\alpha}L_{\frac{1}{n}})}\right)^2 e^{iwL_{\frac{1}{n}}}\right] + O(n^{-3/2})
\end{aligned}$$

$$\begin{aligned}
&:= \mathcal{X}_n^{(1)}(u,v,w) + \frac{iu}{n^{1/2}}\mathcal{X}_n^{(2)}(u,v,w) + \frac{iv}{n^{1/2}}\mathcal{X}_n^{(3)}(u,v,w) - \frac{u^2}{2n}\mathcal{X}_n^{(4)}(u,v,w) - \frac{uv}{n}\mathcal{X}_n^{(5)}(u,v,w) \\
&\qquad - \frac{v^2}{2n}\mathcal{X}_n^{(6)}(u,v,w) + O(n^{-3/2}).
\end{aligned} \tag{5.16}$$

First, we have

$$\mathcal{X}_n^{(1)}(u,v,w) = e^{\psi(w)/n} = 1 + \psi(w)/n + O(n^{-2}) \tag{5.17}$$

where $\psi(w)$ is the Lévy Khintchine exponent of L_1.

We now focus on the term $\mathcal{X}_n^{(2)}(u,v,w)$. Using the results of Lemma 3.1, and the fact that $n^{1/\alpha}L_{1/n}$

has the same law as L_1^n, we get

$$\begin{aligned}
\mathcal{X}_n^{(2)}(u,v,w) &= \mathbb{E}\left[\frac{\varphi'_\alpha}{\varphi_\alpha}(L_1^\alpha)e^{i\frac{wL_1^\alpha}{n^{1/\alpha}}}\right] + O(n^{-1}) \\
&= \int_{\mathbb{R}} \varphi'_\alpha(s)e^{i\frac{ws}{n^{1/\alpha}}}ds + O(n^{-1}) \\
&= -\frac{iw}{n^{1/\alpha}}\int_{\mathbb{R}} \varphi_\alpha(s)e^{i\frac{ws}{n^{1/\alpha}}}ds + O(n^{-1}) \quad \text{using integration by parts formula} \\
&= O(\max(n^{-1/\alpha}, n^{-1})).
\end{aligned} \tag{5.18}$$

For the term, $\mathcal{X}_n^{(3)}(u,v,w)$, using Lemma 3.1 again, it is easy see that

$$\begin{aligned}
\mathcal{X}_n^{(3)}(u,v,w) &= \mathbb{E}\left[\left(\frac{\varphi_\alpha(L_1^\alpha) + L_1^\alpha\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)}\right)e^{i\frac{wL_1^\alpha}{n^{1/\alpha}}}\right] + O(n^{-1}) \\
&= \int_{\mathbb{R}} \varphi_\alpha(s)e^{i\frac{ws}{n^{1/\alpha}}}ds + \int_{\mathbb{R}} s\varphi'_\alpha(s)e^{i\frac{ws}{n^{1/\alpha}}}ds + O(n^{-1}).
\end{aligned} \tag{5.19}$$

Using integration by parts formula, we have

$$\mathbb{E}(e^{iwL_{\frac{1}{n}}^\alpha}) = \int_{\mathbb{R}} \varphi_\alpha(s)e^{\frac{iws}{n^{1/\alpha}}}ds = -\frac{n^{1/\alpha}}{iw}\int_{\mathbb{R}} \varphi'_\alpha(s)e^{\frac{iws}{n^{1/\alpha}}}ds.$$

Then, we deduce that

$$\int_{\mathbb{R}} \varphi'_\alpha(s)e^{\frac{iws}{n^{1/\alpha}}}ds = -\frac{iw}{n^{1/\alpha}}\mathbb{E}(e^{iwL_{\frac{1}{n}}^\alpha}) = -\frac{iw}{n^{1/\alpha}}\mathbb{E}(e^{\frac{iw}{n^{1/\alpha}}L_1^\alpha}). \tag{5.20}$$

Since L_1^α is a symmetric α-stable process then we have for some constant $C(\alpha) > 0$

$$\mathbb{E}(e^{\frac{iw}{n^{1/\alpha}}L_1^\alpha}) = e^{-C(\alpha)\left|\frac{w}{n^{1/\alpha}}\right|^\alpha}. \tag{5.21}$$

Combining (5.20) with (5.21), we have

$$\int_{\mathbb{R}} \varphi'_\alpha(s)e^{\frac{iws}{n^{1/\alpha}}}ds = -\frac{iw}{n^{1/\alpha}}e^{-C(\alpha)\left|\frac{w}{n^{1/\alpha}}\right|^\alpha}. \tag{5.22}$$

Taking the derivative with respect to w of the both sides of (5.22), we get

$$\begin{aligned}
\int_{\mathbb{R}} s\varphi'_\alpha(s)e^{\frac{iws}{n^{1/\alpha}}}ds &= -e^{-C(\alpha)\left|\frac{w}{n^{1/\alpha}}\right|^\alpha} + \alpha C(\alpha)e^{-C(\alpha)\left|\frac{w}{n^{1/\alpha}}\right|^\alpha}\frac{|w|^\alpha}{n} \\
&= -\int_{\mathbb{R}} \varphi_\alpha(s)e^{\frac{iws}{n^{1/\alpha}}}ds + \alpha C(\alpha)e^{-C(\alpha)\left|\frac{w}{n^{1/\alpha}}\right|^\alpha}\frac{|w|^\alpha}{n}.
\end{aligned} \tag{5.23}$$

From (5.19) and (5.23) we can deduce that

$$\mathcal{X}_n^{(3)}(u,v,w) = O(n^{-1}). \tag{5.24}$$

For the term $\mathcal{X}_n^{(4)}(u,v,w)$, using Lemma 3.1 again, we have

$$\begin{aligned}\mathcal{X}_n^{(4)}(u,v,w) &= \mathbb{E}\left[\frac{\varphi'_\alpha}{\varphi_\alpha}(L_1^\alpha)^2 e^{i\frac{wL_1^\alpha}{n^{1/\alpha}}}\right] + O(n^{-1}) \\ &\xrightarrow{n\to\infty} \mathbb{E}\left[\frac{\varphi'_\alpha}{\varphi_\alpha}(L_1^\alpha)^2\right].\end{aligned} \tag{5.25}$$

For the term $\mathcal{X}_n^{(5)}(u,v,w)$ we have

$$\begin{aligned}\mathcal{X}_n^{(5)}(u,v,w) &= \mathbb{E}\left[\frac{\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)}\frac{(\varphi_\alpha(L_1^\alpha)+L_1^\alpha\varphi'_\alpha(L_1^\alpha))}{\varphi_\alpha(L_1^\alpha)}e^{\frac{iwL_1^\alpha}{n^{1/\alpha}}}\right] + O(n^{-1}) \\ &\xrightarrow{n\to\infty} \mathbb{E}\left[\frac{\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)}\frac{(\varphi_\alpha(L_1^\alpha)+L_1^\alpha\varphi'_\alpha(L_1^\alpha))}{\varphi_\alpha(L_1^\alpha)}\right] = 0 \quad \text{from (3.34).}\end{aligned} \tag{5.26}$$

For the term, $\mathcal{X}_n^{(6)}(u,v,w)$ it is easy to see that

$$\begin{aligned}\mathcal{X}_n^{(6)}(u,v,w) &= \mathbb{E}\left[\left(\frac{\varphi'_\alpha(L_1^\alpha)+L_1^\alpha\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)}\right)^2 e^{\frac{iwL_1^\alpha}{n^{1/\alpha}}}\right] + O(n^{-1}) \\ &\xrightarrow{n\to\infty} \mathbb{E}\left[\left(\frac{\varphi'_\alpha(L_1^\alpha)+L_1^\alpha\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)}\right)^2\right].\end{aligned} \tag{5.27}$$

Collecting the convergence of $(\mathcal{X}_n^{(i)}(u,v,w))_{1\le i\le 6}$, we deduce the convergence

$$\log\mathbb{E}\left[\exp(iu\Gamma_1'^{n,1}+iv\Gamma_1'^{n,2}+iwZ_1^n)\right] \xrightarrow{n\to\infty} \psi(v) - \frac{u^2}{2}\mathbb{E}\left[\frac{\varphi'_\alpha}{\varphi_\alpha}(L_1^\alpha)^2\right] - \frac{v^2}{2}\mathbb{E}\left[\left(\frac{\varphi'_\alpha(L_1^\alpha)+L_1^\alpha\varphi'_\alpha(L_1^\alpha)}{\varphi_\alpha(L_1^\alpha)}\right)^2\right]$$

and thus the convergence in law of this lemma. □

6 Appendix 1: Proofs of Lemmas 3.3 - 3.5

In order to prove Lemmas 3.4 - 3.5, we will establish a continuity property with respect to the conditioning variable in a conditional expectation. We first recall some technical results on Malliavin weight given in [6].

6.1 Preliminary lemmas

We recall the Malliavin weight $\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\nabla_\beta \overline{Y}_1^{n,\beta_n,x_0})$ defined in Theorem 3.1. From (3.19), (3.20), we can rewrite $\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\nabla_\beta \overline{Y}_1^{n,\beta_n,x_0})$ as,

$$\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\nabla_\beta \overline{Y}_1^{n,\beta_n,x_0}) = \begin{pmatrix} \frac{1}{\sigma_n} n^{1/\alpha}\partial_\theta \overline{Y}_1^{n,\beta_n,x_0}\widehat{\mathcal{H}}_{\beta_n}^n(1) \\ \frac{1}{\sigma_n}\left(n^{1/\alpha}\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0}\widehat{\mathcal{H}}_{\beta_n}^n(1) - 1\right)\end{pmatrix} + \mathcal{R}_{\beta_n}^n(\nabla_\beta \overline{Y}_1^{n,\beta_n,x_0}) \tag{6.1}$$

where $\widehat{\mathcal{H}}^n_\beta(1)$ is given by (3.21) and

$$\mathcal{R}^n_{\beta_n}(\nabla_\beta \overline{Y}_1^{n,\beta_n,x_0}) = \begin{pmatrix} \partial_\theta \overline{Y}_1^{n,\beta_n,x_0} \\ \partial_\sigma \overline{Y}_1^{n,\beta_n,x_0} \end{pmatrix} \left[\mathcal{R}^n_{1,\beta_n}(1) + \mathcal{R}^n_{2,\beta_n}(1) + \mathcal{R}^n_{3,\beta_n}(1)\right] - \begin{pmatrix} \frac{V_1^{n,\theta_n}}{U_1^{n,\beta_n}} \\ \frac{V_1^{n,\sigma_n}}{U_1^{n,\beta_n}} - \frac{1}{\sigma_n} \end{pmatrix} \tag{6.2}$$

with $U_1^{n,\beta} = \Gamma(\overline{Y}_1^{n,\beta,x_0}, \overline{Y}_1^{n,\beta,x_0})$, $V_1^{n,\theta} = \Gamma(\overline{Y}_1^{n,\beta,x_0}, \partial_\theta \overline{Y}_1^{n,\beta,x_0})$ and $V_1^{n,\sigma} = \Gamma(\overline{Y}_1^{n,\beta,x_0}, \partial_\sigma \overline{Y}_1^{n,\beta,x_0})$ given by

$$U_1^{n,\beta} = \frac{(\epsilon_1^n)^2\sigma^2}{n^{2/\alpha}} \int_0^1 \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \mu^{(n)}(ds, dz), \tag{6.3}$$

$$V_1^{n,\theta} = \frac{1}{n}(\epsilon_1^n)^2 \int_0^1 (\epsilon_s^n)^{-2} \left(U_s^{n,\beta} \left[(\partial_\theta b)'(\overline{Y}_s^{n,\beta,x_0}, \theta) + b''(\overline{Y}_s^{n,\beta,x_0}, \theta) \partial_\theta \overline{Y}_s^{n,\beta,x_0} \right] \right) ds, \tag{6.4}$$

$$V_1^{n,\sigma} = \frac{1}{n}(\epsilon_1^n)^2 \int_0^1 (\epsilon_s^n)^{-2} \left(b''(\overline{Y}_s^{n,\beta,x_0}, \theta) \partial_\sigma \overline{Y}_s^{n,\beta,x_0} U_s^{n,\beta} \right) ds + \frac{\sigma}{n^{2/\alpha}} (\epsilon_1^n)^2 \int_0^t \int_{\mathbb{R}} (\epsilon_s^n)^{-2} \rho^n(z) \mu^{(n)}(ds, dz) \tag{6.5}$$

and $(\epsilon_s^n)_{s\in[0,1]}$ given by (3.22). Now we recall two technical lemmas given in [6] useful to study the convergence of the Malliavin weight $\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\nabla_\beta \overline{Y}_1^{n,\beta_n,x_0})$ in the proof of Lemmas 3.4 - 3.5 later.

Lemma 6.1. *[Clément-Gloter-Nguyen [6]]*

For all $p \geq 1$, we have

i) $n\partial_\theta \overline{Y}_1^{n,\beta,x_0} \widehat{\mathcal{H}}^n_\beta(1) \xrightarrow[\boldsymbol{L}^p]{n\to\infty} \partial_\theta b(x_0,\theta) \mathcal{H}_{L^\alpha}(1)$,

ii) $n^{1/\alpha} \partial_\sigma \overline{Y}_1^{n,\beta,x_0} \widehat{\mathcal{H}}^n_\beta(1) \xrightarrow[\boldsymbol{L}^p]{n\to\infty} L_1^\alpha \mathcal{H}_{L^\alpha}(1)$,
where $\widehat{\mathcal{H}}^n_\beta(1)$ is given by (3.21) and $\mathcal{H}_{L^\alpha}(1)$ is given by (3.24).

Lemma 6.2. *[Clément-Gloter-Nguyen [6]]*

The following estimates hold:

i) $\left| \frac{V_1^{n,\theta}}{U_1^{n,\beta}} \right| \leq \frac{C}{n}$,

ii) $\frac{V_1^{n,\sigma}}{U_1^{n,\beta}} = \frac{1}{\sigma} + \mathcal{R}^n_{9,\beta}(1)$,

where C is some deterministic constant and $\mathcal{R}^n_{9,\beta}(1)$ is a remainder term which converges to zero as $n \to \infty$ in $\boldsymbol{L}^p, \forall p \geq 2$.

6.2 Regularity of the conditional expectation

In this section, we prove a regularity property of the conditional expectation with respect to the conditioning variable.

Proposition 6.1. *Let $\mathcal{H}$ be a random variable such that $\mathbb{E}(\mathcal{H})^2 < \infty$. We assume that there exists a sequence of random variables $(\mathcal{H}^n)_{n\geq 1}$ with $\mathbb{E}(\mathcal{H}^n)^2 < \infty$ and such that $\mathcal{H}^n \xrightarrow[\boldsymbol{L^2}]{n\to\infty} \mathcal{H}$ and $\sup_n ||\Gamma(\mathcal{H}^n, \mathcal{H}^n)||_2 < \infty$. Then,*

$$\mathbb{E}\left[\mathbb{E}[\mathcal{H}|\overline{Y}_1^{n,\beta_n,x_0}]^2\right] - \mathbb{E}\left[\mathbb{E}[\mathcal{H}|L_1^\alpha]^2\right] \xrightarrow{n\to\infty} 0 \tag{6.6}$$

and this convergence is uniform with respect to x_0.

Remark 6.1. *Note that if the random variable $\mathcal{H}$ depends on all the measure μ then the Malliavin calculus of Section 3.2.1 is not defined. So we need to introduce the sequence of random variables $(\mathcal{H}^n)$, for which the Malliavin calculus of Section 3.2.1 is defined, such that $\Gamma(\mathcal{H}^n, \mathcal{H}^n)$ is also well defined . It is the case, for instance, if $\mathcal{H}^n$ is a simple functional of $\mu^{(n)}$.*

Proof of Proposition 6.1 . First we reduce the situation to the case where the random variable in the expectation is bounded. Let $K > 1$ and denote by $x \mapsto \mathcal{X}_K(x)$ a smooth truncation function with

$$\begin{cases} \mathcal{X}_K(x) = 0 & \text{for} \quad |x| > K \\ \mathcal{X}_K(x) = 1 & \text{for} \quad |x| \leq K/2 \\ 0 \leq \mathcal{X}_K(x) \leq 1 & \text{for} \quad K/2 \leq |x| \leq K. \end{cases} \tag{6.7}$$

For all $\epsilon > 0$, we can choose K large enough such that $||\mathcal{H} - \mathcal{H}\mathcal{X}_K(\mathcal{H})||_2^2 < \epsilon$ and then, one can see that (6.6) is implied by the following convergence, $\forall K > 2$

$$\sup_{x_0} \left| \mathbb{E}\left[\mathbb{E}[\mathcal{H}\mathcal{X}_K(\mathcal{H})|\overline{Y}_1^{n,\beta_n,x_0}]^2\right] - \mathbb{E}\left[\mathbb{E}[\mathcal{H}\mathcal{X}_K(\mathcal{H})|L_1^\alpha]^2\right] \right| \xrightarrow{n\to\infty} 0.$$

Now since $\mathbb{E}\left(\mathcal{H}^n - \mathcal{H}\right)^2 \xrightarrow{n\to\infty} 0$, it is sufficient to prove that, $\forall K > 2$

$$\sup_{x_0} \left| \mathbb{E}\left[\mathbb{E}[\mathcal{H}^n\mathcal{X}_K(\mathcal{H}^n)|\overline{Y}_1^{n,\beta_n,x_0}]^2\right] - \mathbb{E}\left[\mathbb{E}[\mathcal{H}^n\mathcal{X}_K(\mathcal{H}^n)|L_1^\alpha]^2\right] \right| \xrightarrow{n\to\infty} 0. \tag{6.8}$$

We now prove (6.8). First, we define $\eta^{\mathcal{H}^{n,K}}$ and $\overline{\eta}^{\mathcal{H}^{n,K}}$ as follows

$$\begin{aligned} &\eta^{\mathcal{H}^{n,K}}(\overline{Y}_1^{n,\beta_n,x_0} - \varsigma_1^{n,\theta_n,x_0}) = \mathbb{E}\left[\mathcal{H}^n\mathcal{X}_K(\mathcal{H}^n)|\overline{Y}_1^{n,\beta_n,x_0} - \varsigma_1^{n,\theta_n,x_0}\right] = \mathbb{E}\left[\mathcal{H}^n\mathcal{X}_K(\mathcal{H}^n)|\overline{Y}_1^{n,\beta_n,x_0}\right], \\ &\overline{\eta}^{\mathcal{H}^{n,K}}(\frac{\sigma L_1^\alpha}{n^{1/\alpha}}) = \mathbb{E}\left[\mathcal{H}^n\mathcal{X}_K(\mathcal{H}^n)|\frac{\sigma L_1^\alpha}{n^{1/\alpha}}\right] = \mathbb{E}\left[\mathcal{H}^n\mathcal{X}_K(\mathcal{H}^n)|L_1^\alpha\right]. \end{aligned} \tag{6.9}$$

With these notations, we can rewrite (6.8) as

$$\sup_{x_0}\left|\mathbb{E}\left[\eta^{\mathcal{H}^{n,K}}(\overline{Y}_1^{n,\beta_n,x_0}-\varsigma_1^{n,\theta_n,x_0})^2\right]-\mathbb{E}\left[\overline{\eta}^{\mathcal{H}^{n,K}}(\frac{\sigma L_1^\alpha}{n^{1/\alpha}})^2\right]\right|\xrightarrow{n\to\infty}0. \tag{6.10}$$

Using Lemma 7.2 in Appendix 2, we know that:

$$\sup_{x_0}\mathbb{E}\left[\left|\eta^{\mathcal{H}^{n,K}}(\overline{Y}_1^{n,\beta_n,x_0}-\varsigma_1^{n,\theta_n,x_0})-\overline{\eta}^{\mathcal{H}^{n,K}}(\overline{Y}_1^{n,\beta_n,x_0}-\varsigma_1^{n,\theta_n,x_0})\right|\right]\xrightarrow{n\to\infty}0$$

and since $|\eta^{\mathcal{H}^{n,K}}|$ and $|\overline{\eta}^{\mathcal{H}^{n,K}}|$ are bounded by the constant K, we deduce

$$\sup_{x_0}\left|\mathbb{E}[\eta^{\mathcal{H}^{n,K}}(\overline{Y}_1^{n,\beta_n,x_0}-\varsigma_1^{n,\theta_n,x_0})^2]-\mathbb{E}[\overline{\eta}^{\mathcal{H}^{n,K}}(\overline{Y}_1^{n,\beta_n,x_0}-\varsigma_1^{n,\theta_n,x_0})^2]\right|\xrightarrow{n\to\infty}0.$$

Now, applying Lemma 7.1 in Appendix 2, with the choice $\mathcal{H}^n=1$ with the bounded function $(\overline{\eta}^{\mathcal{H}^{n,K}})^2$ we get (6.10) and the proposition is proved. □

We can now prove Lemma 3.3, Lemma 3.4 and Lemma 3.5.

6.3 Proofs of Lemma 3.3, Lemma 3.4 and Lemma 3.5

Proof of Lemma 3.3. The proof of this lemma is based on the results in [6] recalled at the beginning of Section 3.2.1 and Section 6.1. From (6.2), Lemma 3.2, Lemma 6.2 and (3.23), we easily deduce that $\mathcal{R}^n_{\beta_n}(\nabla_\beta\overline{Y}_1^{n,\beta_n,x_0})$ converges to zero in $\mathbf{L}^p, \forall p\geq 2$.
From Lemma 6.1 $i-ii$) and (6.1) we can deduce the result of this lemma. □

Proof of Lemma 3.4. From Jensen's inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\mathbb{E}[n^{1-1/\alpha}\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta\overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]-\mathbb{E}[\frac{1}{\sigma}\partial_\theta b(x_0,\theta)\mathcal{H}_{L^\alpha}(1)|\overline{Y}_1^{n,\beta_n,x_0}]\right|^2\right]\\
\leq&\mathbb{E}\left[\mathbb{E}\left[\left|n^{1-1/\alpha}\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta\overline{Y}_1^{n,\beta_n,x_0})-\frac{1}{\sigma}\partial_\theta b(x_0,\theta)\mathcal{H}_{L^\alpha}(1)\right|^2|\overline{Y}_1^{n,\beta_n,x_0}\right]\right]\\
=&\mathbb{E}\left[\left|n^{1-1/\alpha}\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta\overline{Y}_1^{n,\beta_n,x_0})-\frac{1}{\sigma}\partial_\theta b(x_0,\theta)\mathcal{H}_{L^\alpha}(1)\right|^2\right]
\end{aligned}$$

From Lemma 3.3, the last term converges to zero uniformly with respect to x_0. In turn, it gives the uniform convergence

$$\sup_{x_0}\left|n^{2-2/\alpha}\mathbb{E}\left[\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta\overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]^2\right]-\frac{1}{\sigma^2}\partial_\theta b(x_0,\theta)^2\mathbb{E}\left[\mathbb{E}[\mathcal{H}_{L^\alpha}(1)|\overline{Y}_1^{n,\beta_n,x_0}]^2\right]\right|\xrightarrow{n\to\infty}0.$$

By the same method as above, we also get the uniform convergence

$$\sup_{x_0}\left|\mathbb{E}\left[\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma\overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]^2\right]-\frac{1}{\sigma^2}\mathbb{E}\left[\mathbb{E}[(L_1^\alpha\mathcal{H}_{L^\alpha}(1)-1)\,|\overline{Y}_1^{n,\beta_n,x_0}]^2\right]\right|\xrightarrow{n\to\infty}0.$$

Hence, this lemma will be proved as soon as we show that

$$\sup_{x_0}\left|\mathbb{E}\left[\mathbb{E}[\mathcal{H}_{L^\alpha}(1)|\overline{Y}_1^{n,\theta_n,x_0}]^2\right]-\mathbb{E}\left[\mathbb{E}[\mathcal{H}_{L^\alpha}(1)|L_1^\alpha]^2\right]\right|\xrightarrow{n\to\infty}0, \tag{6.11}$$

$$\sup_{x_0}\left|\mathbb{E}\left[\mathbb{E}[(L_1^\alpha\mathcal{H}_{L^\alpha}(1)-1)\,|\overline{Y}_1^{n,\beta_n,x_0}]^2\right]-\mathbb{E}\left[\mathbb{E}[(L_1^\alpha\mathcal{H}_{L^\alpha}(1)-1)\,|L_1^\alpha]^2\right]\right|\xrightarrow{n\to\infty}0. \tag{6.12}$$

To prove (6.11), we apply Proposition 6.1 with the choice $\mathcal{H}=\mathcal{H}_{L^\alpha}(1)$ and

$$\mathcal{H}^n=\frac{\int_0^1\int_{\mathbb{R}}\rho^n(z)(\rho^n)'(z)\mu^n(ds,dz)}{\left[\int_0^1\int_{\mathbb{R}}\rho^n(z)\mu^n(ds,dz)\right]^2}-\frac{\int_0^1\int_{\mathbb{R}}\left[(\rho^n)'(z)-\frac{1+\alpha}{z}\rho^n(z)\right]\mu^n(ds,dz)}{\int_0^1\int_{\mathbb{R}}\rho^n(z)\mu^n(ds,dz)}:=\tilde{\mathcal{H}}^n. \tag{6.13}$$

From Remark 3.3 *ii*) we get that $\tilde{\mathcal{H}}^n=\mathcal{H}_{L_1^n}(1)-\frac{1}{n^{1/\alpha}}\mathcal{R}^n_{1,\beta}$, moreover $\mathbb{E}(\mathcal{H}_{L^\alpha}(1))^2<\infty$ and $\tilde{\mathcal{H}}^n\xrightarrow[\mathbf{L}^2]{n\to\infty}\mathcal{H}_{L^\alpha}(1)$. The computation of $\Gamma(\tilde{\mathcal{H}}^n,\tilde{\mathcal{H}}^n)$ is omitted but reduces to the computation of the Γ-bracket between simple functionals. After some calculus (similar to those in the proof of Theorem 3.2 in [6]) we get that $\Gamma(\tilde{\mathcal{H}}^n,\tilde{\mathcal{H}}^n)$ is bounded by a random variable independent of n and belonging to $\cap_{p\geq1}\mathbf{L}^p$. Turning to 6.12, we proceed similarly with the choices $\mathcal{H}=L_1^\alpha\mathcal{H}_{L^\alpha}(1)$ and $\mathcal{H}^n=L_1^n\tilde{\mathcal{H}}^n$. Note that using Lemma 6.1 *ii*) with $b(x,\theta)=0$ and $\sigma=1$ we deduce that $L_1^n\tilde{\mathcal{H}}^n\xrightarrow[\mathbf{L}^2]{n\to\infty}L_1^\alpha\mathcal{H}_{L^\alpha}(1)$ moreover we can prove that $\sup_n||\Gamma(L_1^n\tilde{\mathcal{H}}^n,L_1^n\tilde{\mathcal{H}}^n)||_2<\infty$. □

Proof of Lemma 3.5. It is easy to see that

$$\begin{aligned}
&\mathbb{E}\Bigg[\Bigg|\, n^{1-1/\alpha}\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta\overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma\overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]\\
&\qquad\qquad-\mathbb{E}\left[\frac{1}{\sigma}\,(L_1^\alpha\mathcal{H}_{L^\alpha}(1)-1)\,|\overline{Y}_1^{n,\beta_n,x_0}\right]\mathbb{E}\left[\frac{1}{\sigma}\partial_\theta b(x_0,\theta)\mathcal{H}_{L^\alpha}(1)|\overline{Y}_1^{n,\beta_n,x_0}\right]\Bigg|\,\Bigg]\\
&\leq\mathbb{E}\Bigg[\Bigg|\, n^{1-1/\alpha}\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta\overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]\\
&\qquad\left(\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma\overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]-\mathbb{E}\left[\frac{(L_1^\alpha\mathcal{H}_{L^\alpha}(1)-1)}{\sigma}|\overline{Y}_1^{n,\beta_n,x_0}\right]\right)\Bigg|\,\Bigg]\\
&+\mathbb{E}\Bigg[\Bigg|\,\mathbb{E}\left[\frac{(L_1^\alpha\mathcal{H}_{L^\alpha}(1)-1)}{\sigma}|\overline{Y}_1^{n,\beta_n,x_0}\right]\\
&\qquad\left(n^{1-1/\alpha}\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta\overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]-\mathbb{E}\left[\frac{\partial_\theta b(x_0,\theta)\mathcal{H}_{L^\alpha}(1)}{\sigma}|\overline{Y}_1^{n,\beta_n,x_0}\right]\right)\Bigg|\,\Bigg]. \qquad (6.14)
\end{aligned}$$

Then using Cauchy-Schwarz inequality

$$\mathbb{E}\Bigg[\Bigg| n^{1-1/\alpha}\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]\times \\ \times\Bigg(\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]-\mathbb{E}\left[\frac{(L_1^\alpha \mathcal{H}_{L^\alpha}(1)-1)}{\sigma}|\overline{Y}_1^{n,\beta_n,x_0}\right]\Bigg)\Bigg|\Bigg]$$

$$\leq \left[\mathbb{E}\left(\left(n^{1-1/\alpha}\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]\right)^2\right)\right]^{1/2}\times \\ \times\left[\mathbb{E}\left(\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]-\mathbb{E}\left[\frac{(L_1^\alpha \mathcal{H}_{L^\alpha}(1)-1)}{\sigma}|\overline{Y}_1^{n,\beta_n,x_0}\right]\right)^2\right]^{1/2}$$

$$\leq \left[\mathbb{E}\left(n^{2-2/\alpha}\mathbb{E}\left[\left|\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta_n,x_0})\right|^2|\overline{Y}_1^{n,\beta_n,x_0}\right]\right)\right]^{1/2}\times \\ \times\left[\mathbb{E}\left(\mathbb{E}\left[\left|\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0})-\frac{(L_1^\alpha \mathcal{H}_{L^\alpha}(1)-1)}{\sigma}\right|^2|\overline{Y}_1^{n,\beta_n,x_0}\right]\right)\right]^{1/2}$$

$$=\left[\mathbb{E}\left(n^{2-2/\alpha}\left|\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta_n,x_0})\right|^2\right)\right]^{1/2}\left[\mathbb{E}\left(\left|\mathcal{H}^n_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0})-\frac{(L_1^\alpha \mathcal{H}_{L^\alpha}(1)-1)}{\sigma}\right|^2\right)\right]^{1/2}. \tag{6.15}$$

Furthermore, from (3.25) we easily deduce that (6.15) converges to zero uniformly with respect to x_0. Similarly, we also get that
$\mathbb{E}\left[\left|\mathbb{E}\left[\frac{(L_1^\alpha \mathcal{H}_{L^\alpha}(1)-1)}{\sigma}|\overline{Y}_1^{n,\beta_n,x_0}\right]\left(n^{1-1/\alpha}\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]-\mathbb{E}\left[\frac{\partial_\theta b(x_0,\theta)\mathcal{H}_{L^\alpha}(1)}{\sigma}|\overline{Y}_1^{n,\beta_n,x_0}\right]\right)\right|\right]$
tends to zero uniformly with respect to x_0. And then, we can conclude that (6.14) converges to zero uniformly with respect to x_0. In turn, it gives the uniform convergence

$$\sup_{x_0}\Bigg|\mathbb{E}\left[n^{1-1/\alpha}\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\theta \overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]\mathbb{E}[\mathcal{H}_{\overline{Y}_1^{n,\beta_n,x_0}}(\partial_\sigma \overline{Y}_1^{n,\beta_n,x_0})|\overline{Y}_1^{n,\beta_n,x_0}]\right]- \\ -\mathbb{E}\left[\mathbb{E}\left[\frac{1}{\sigma}\left(L_1^\alpha \mathcal{H}_{L^\alpha}(1)-1\right)|\overline{Y}_1^{n,\beta_n,x_0}\right]\mathbb{E}\left[\frac{1}{\sigma}\partial_\theta b(x_0,\theta)\mathcal{H}_{L^\alpha}(1)|\overline{Y}_1^{n,\beta_n,x_0}\right]\right]\Bigg|\xrightarrow{n\to\infty}0. \tag{6.16}$$

On the other hand, we can rewrite

$$\mathbb{E}\left[\mathbb{E}\left[\frac{1}{\sigma}\left(L_1^\alpha \mathcal{H}_{L^\alpha}(1)-1\right)|\overline{Y}_1^{n,\beta_n,x_0}\right]\mathbb{E}\left[\frac{1}{\sigma}\partial_\theta b(x_0,\theta)\mathcal{H}_{L^\alpha}(1)|\overline{Y}_1^{n,\beta_n,x_0}\right]\right]$$
$$=\frac{1}{4}\mathbb{E}\left\{\left[\mathbb{E}\left(\frac{1}{\sigma}\left(L_1^\alpha \mathcal{H}_{L^\alpha}(1)-1\right)+\frac{1}{\sigma}\partial_\theta b(x_0,\theta)\mathcal{H}_{L^\alpha}(1)|\overline{Y}_1^{n,\beta_n,x_0}\right)\right]^2\right.$$
$$\left.-\left[\mathbb{E}\left(\frac{1}{\sigma}\left(L_1^\alpha \mathcal{H}_{L^\alpha}(1)-1\right)-\frac{1}{\sigma}\partial_\theta b(x_0,\theta)\mathcal{H}_{L^\alpha}(1)|\overline{Y}_1^{n,\beta_n,x_0}\right)\right]^2\right\}. \quad (6.17)$$

Then, the lemma will be proved as soon as we show that

$$\mathbb{E}\left\{\left[\mathbb{E}\left(\frac{1}{\sigma}\left(L_1^\alpha \mathcal{H}_{L^\alpha}(1)-1\right)+\frac{1}{\sigma}\partial_\theta b(x_0,\theta)\mathcal{H}_{L^\alpha}(1)\,\Big|\,\overline{Y}_1^{n,\beta_n,x_0}\right)\right]^2\right.$$
$$\left.-\left[\mathbb{E}\left(\frac{1}{\sigma}\left(L_1^\alpha \mathcal{H}_{L^\alpha}(1)-1\right)-\frac{1}{\sigma}\partial_\theta b(x_0,\theta)\mathcal{H}_{L^\alpha}(1)\,\Big|\,\overline{Y}_1^{n,\beta_n,x_0}\right)\right]^2\right\} \quad (6.18)$$

is uniformly convergent with with respect to x_0 to

$$\mathbb{E}\left\{\left[\mathbb{E}\left(\frac{1}{\sigma}\left(L_1^\alpha \mathcal{H}_{L^\alpha}(1)-1\right)+\frac{1}{\sigma}\partial_\theta b(x_0,\theta)\mathcal{H}_{L^\alpha}(1)\,\Big|\,L_1^\alpha\right)\right]^2\right.$$
$$\left.-\left[\mathbb{E}\left(\frac{1}{\sigma}\left(L_1^\alpha \mathcal{H}_{L^\alpha}(1)-1\right)-\frac{1}{\sigma}\partial_\theta b(x_0,\theta)\mathcal{H}_{L^\alpha}(1)\,\Big|\,L_1^\alpha\right)\right]^2\right\}. \quad (6.19)$$

We end the proof by using Proposition 6.1 with $\mathcal{H}=L_1^\alpha\mathcal{H}_{L^\alpha}(1)\pm\partial_\theta b(x_0,\theta)\mathcal{H}_{L^\alpha}(1)$ and $\mathcal{H}^n=L_1^n\tilde{\mathcal{H}}^n\pm\partial_\theta b(x_0,\theta)\tilde{\mathcal{H}}^n$ where $\tilde{\mathcal{H}}^n$ given by (6.13). □

7 Appendix 2

The aim of this section is to show that the functions $\eta^{\mathcal{H}^{n,K}}$ and $\overline{\eta}^{\mathcal{H}^{n,K}}$ defined by (6.9) are close in some sense. The idea is mainly based on [5, Proposition 9, p.2348], however we need a more technical study since $\alpha\in(0,2)$ and the function b is not assumed to be bounded. Our first result is the following.

Lemma 7.1. *Under the assumptions of Proposition 6.1, for all bounded function h, $\forall K>2$, there exists a constant $C_K>0$ such that*

$$|\mathbb{E}[\mathcal{H}^n\mathcal{X}_K(\mathcal{H}^n)h(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})]-E[\mathcal{H}^n\mathcal{X}_K(\mathcal{H}^n)h(\frac{\sigma L_1^\alpha}{n^{1/\alpha}})]\leq\frac{C_K}{n}||h||_\infty$$

and the above estimate is uniform with respect to $x_0\in\mathbb{R}$ and $\beta\in Q$, for any compact set $Q\subset\mathbb{R}\times(0,\infty)$.

Proof. Since $\mathcal{H}^n\mathcal{X}_K(\mathcal{H}^n)$ is bounded and $\mathbb{P}(L_1^n \neq L_1^\alpha) \leq \frac{C}{n}$ [see Lemma 3.1] it is sufficient to show that

$$\left|\mathbb{E}[\mathcal{H}^n\mathcal{X}_K(\mathcal{H}^n)h(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})] - \mathbb{E}[\mathcal{H}^n\mathcal{X}_K(\mathcal{H}^n)h(\frac{\sigma L_1^n}{n^{1/\alpha}})]\right| \leq \frac{C_K}{n}||h||_\infty. \tag{7.1}$$

We now prove (7.1).

Let us denote $\mathcal{H}^{n,K} = \mathcal{H}^n\mathcal{X}_K(\mathcal{H}^n)$ and H any primitive function of h. Using the integration by parts formula (3.14), we have

$$\mathbb{E}\left[h(\frac{\sigma L_1^n}{n^{1/\alpha}})\mathcal{H}^{n,K}\right] = \mathbb{E}\left[H(\frac{\sigma L_1^n}{n^{1/\alpha}})\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})\right] \tag{7.2}$$

where $\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})$ is given by (3.17), namely here

$$\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K}) = L\left(\frac{\mathcal{H}^{n,K}}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}})}\right)\frac{\sigma L_1^n}{n^{1/\alpha}} - \frac{L(\frac{\sigma L_1^n}{n^{1/\alpha}})\mathcal{H}^{n,K}}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}})} - L\left(\frac{\mathcal{H}^{n,K}\frac{\sigma L_1^n}{n^{1/\alpha}}}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}})}\right).$$

On the other hand, we have

$$\left|\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0} - \frac{\sigma L_1^n}{n^{1/\alpha}}\right| = \left|\frac{1}{n}\int_0^1 \left[b(\overline{Y}_s^{n,\beta,x_0}, \theta) - b(\varsigma_1^{n,\theta,x_0}, \theta)\right] ds\right|$$

$$\leq \frac{1}{n}\int_0^1 ||b'||_\infty|\overline{Y}_s^{n,\beta,x_0} - \varsigma_s^{n,\theta,x_0}|ds \leq \frac{1}{n}\int_0^1 ||b'||_\infty|\overline{Y}_s^{n,\beta,x_0} - \varsigma_s^{n,\theta,x_0} - \frac{\sigma L_s^n}{n^{1/\alpha}}|ds + \frac{1}{n^{1+1/\alpha}}\int_0^1 |\sigma L_s^n|ds$$

Applying the Gronwall's inequality, for C a positive constant, independent of n and K,

$$\left|\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0} - \frac{\sigma L_1^n}{n^{1/\alpha}}\right| \leq \frac{C}{n^{1+1/\alpha}}\int_0^1 |\sigma L_s^n|ds. \tag{7.3}$$

Using that the function H is globally Lipschitz with a Lipschitz constant $||h||_\infty$, we deduce from (7.2) that

$$\begin{aligned}&\left|\mathbb{E}\left[\mathcal{H}^{n,K}h(\frac{\sigma L_1^n}{n^{1/\alpha}})\right] - \mathbb{E}\left[H(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})\right]\right| \\ &\leq \left|\frac{\sigma C}{n^{1+1/\alpha}}\right| ||h||_\infty \mathbb{E}\left[\left|\int_0^1 |L_s^n|ds\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})\right|\right].\end{aligned} \tag{7.4}$$

Now we compute $\mathbb{E}[H(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})]$ using successively the self-adjoint property of

the operator L, (3.10) and (3.17), to obtain an I.B.P formula in a reverse direction:

$$\begin{aligned}
&\mathbb{E}[H(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})]\\
&=\mathbb{E}\left[H(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})\left\{L\left(\frac{\mathcal{H}^{n,K}}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}},\frac{\sigma L_1^n}{n^{1/\alpha}})}\right)\frac{\sigma L_1^n}{n^{1/\alpha}}-\frac{L(\frac{\sigma L_1^n}{n^{1/\alpha}})\mathcal{H}^{n,K}}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}},\frac{\sigma L_1^n}{n^{1/\alpha}})}-L\left(\frac{\mathcal{H}^{n,K}\frac{\sigma L_1^n}{n^{1/\alpha}}}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}},\frac{\sigma L_1^n}{n^{1/\alpha}})}\right)\right\}\right]\\
&=\mathbb{E}\left[\frac{\{L(H(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})\frac{\sigma L_1^n}{n^{1/\alpha}})-H(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})L(\frac{\sigma L_1^n}{n^{1/\alpha}})-L(H(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0}))\frac{\sigma L_1^n}{n^{1/\alpha}})\}\mathcal{H}^{n,K}}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}},\frac{\sigma L_1^n}{n^{1/\alpha}})}\right]\\
&=\mathbb{E}\left[\mathcal{H}^{n,K}\frac{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}},H(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0}))}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}},\frac{\sigma L_1^n}{n^{1/\alpha}})}\right]=\mathbb{E}\left[\mathcal{H}^{n,K}h(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})\frac{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}},\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}},\frac{\sigma L_1^n}{n^{1/\alpha}})}\right].
\end{aligned}\tag{7.5}$$

Putting together (7.4) and (7.5) we deduce,

$$\begin{aligned}
&\left|\mathbb{E}[\mathcal{H}^{n,K}h(\frac{\sigma L_1^n}{n^{1/\alpha}})]-\mathbb{E}[\mathcal{H}^{n,K}h(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})]\right|\\
&\leq(\frac{\sigma C}{n^{1+1/\alpha}})||h||_\infty\left|\left|\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})\int_0^1|L_s^n|ds\right|\right|_1\\
&\qquad+\mathbb{E}\left[\left|\mathcal{H}^{n,K}h(\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})\left\{\frac{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}},\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}},\frac{\sigma L_1^n}{n^{1/\alpha}})}-1\right\}\right|\right]\\
&\leq(\frac{\sigma C}{n^{1+1/\alpha}})||h||_\infty\left|\left|\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})\int_0^1|L_s^n|ds\right|\right|_1+||\mathcal{H}^{n,K}||_\infty||h||_\infty\left|\left|\frac{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}},\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}},\frac{\sigma L_1^n}{n^{1/\alpha}})}-1\right|\right|_1\\
&\leq\frac{C}{n}||h||_\infty\int_0^1\underbrace{\left|\left|\frac{\sigma}{n^{1/\alpha}}\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})|L_s^n|\right|\right|_1}_{I_1^{(n,s)}}ds+||\mathcal{H}^{n,K}||_\infty||h||_\infty\underbrace{\left|\left|\frac{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}},\overline{Y}_1^{n,\beta,x_0}-\varsigma_1^{n,\theta,x_0})}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}},\frac{\sigma L_1^n}{n^{1/\alpha}})}-1\right|\right|_1}_{I_2^n}.
\end{aligned}\tag{7.6}$$

Hence the proposition will be proved if we show that $\sup_n\sup_{s\in[0,1]}I_1^{(n,s)}<\infty$ and $\sup_n nI_2^{(n)}<\infty$.

Step 1: we show that $\sup_n\sup_{s\in[0,1]}I_1^{(n,s)}<\infty$.

We can write from (3.16)

$$\begin{aligned}
\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})&=\frac{-2L(\frac{\sigma L_1^n}{n^{1/\alpha}})\mathcal{H}^{n,K}}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}},\frac{\sigma L_1^n}{n^{1/\alpha}})}+\frac{\mathcal{H}^{n,K}}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}},\frac{\sigma L_1^n}{n^{1/\alpha}})^2}\Gamma\left(\frac{\sigma L_1^n}{n^{1/\alpha}},\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}},\frac{\sigma L_1^n}{n^{1/\alpha}})\right)-\frac{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}},\mathcal{H}^{n,K})}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}},\frac{\sigma L_1^n}{n^{1/\alpha}})}\\
&=\frac{n^{1/\alpha}}{\sigma}\left[\frac{-2L(L_1^n)\mathcal{H}^{n,K}}{\Gamma(L_1^n,L_1^n)}+\frac{\mathcal{H}^{n,K}}{\Gamma(L_1^n,L_1^n)^2}\Gamma(L_1^n,\Gamma(L_1^n,L_1^n))-\frac{\Gamma(L_1^n,\mathcal{H}^{n,K})}{\Gamma(L_1^n,L_1^n)}\right].
\end{aligned}\tag{7.7}$$

Now, let us recall that from (3.2)

$$L_s^n=\int_0^s\int_{|z|\leq1}z\tilde{\mu}^{(n)}(dt,dz)+\int_0^s\int_{|z|>1}z\mu^{(n)}(dt,dz),$$

then,

$$\left|\frac{\sigma}{n^{1/\alpha}}\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})|L_s^n|\right| \leq \left|\frac{\sigma}{n^{1/\alpha}}\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})\int_0^s\int_{|z|\leq 1} z\tilde{\mu}^{(n)}(dt,dz)\right| \tag{7.8}$$

$$+\left|\frac{\sigma}{n^{1/\alpha}}\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})\int_0^1\int_{|z|>1} |z|\mu^{(n)}(dt,dz)\right| \tag{7.9}$$

First, we consider the expectation of the right-hand side term in (7.8), we have:

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{\sigma}{n^{1/\alpha}}\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})\int_0^s\int_{|z|\leq 1} z\tilde{\mu}^{(n)}(dt,dz)\right|\right] \\
&\leq \left[\mathbb{E}\left(\int_0^s\int_{|z|\leq 1} z\tilde{\mu}^{(n)}(dt,dz)\right)^2\right]^{1/2}\left[\mathbb{E}\left(\frac{\sigma}{n^{1/\alpha}}\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})\right)^2\right]^{1/2} \text{ by Cauchy-Schwarz inequality,} \\
&= \left[\int_0^1\int_{|z|\leq 1} z^2\frac{1}{|z|^{1+\alpha}}dzdt\right]^{1/2}\left[\mathbb{E}\left(\frac{\sigma}{n^{1/\alpha}}\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})\right)^2\right]^{1/2} \leq M\left[\mathbb{E}\left(\frac{\sigma}{n^{1/\alpha}}\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})\right)^2\right]^{1/2}
\end{aligned} \tag{7.10}$$

where M is a deterministic constant.

Furthermore, from (3.13) we have $\left|\Gamma(L_1^n,\mathcal{H}^{n,K})\right| \leq \Gamma(\mathcal{H}^{n,K},\mathcal{H}^{n,K})^{1/2}\Gamma(L_1^n,L_1^n)^{1/2}$ and from (7.7), convexity inequality, we get

$$\left[\left|\frac{\sigma}{n^{1/\alpha}}\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})\right|\right]^2 \leq 2\left[\left(\frac{-2L(L_1^n)\mathcal{H}^{n,K}}{\Gamma(L_1^n,L_1^n)}+\frac{\mathcal{H}^{n,K}}{\Gamma(L_1^n,L_1^n)^2}\Gamma(L_1^n,\Gamma(L_1^n,L_1^n))\right)^2+\frac{\Gamma(\mathcal{H}^{n,K},\mathcal{H}^{n,K})}{\Gamma(L_1^n,L_1^n)}\right]. \tag{7.11}$$

Then, we can deduce that

$$\begin{aligned}
&\mathbb{E}\left(\left|\frac{\sigma}{n^{1/\alpha}}\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})\right|\right)^2 \\
&\leq 2\underbrace{\mathbb{E}\left[\left(\frac{-2L(L_1^n)\mathcal{H}^{n,K}}{\Gamma(L_1^n,L_1^n)}+\frac{\mathcal{H}^{n,K}}{\Gamma(L_1^n,L_1^n)^2}\Gamma(L_1^n,\Gamma(L_1^n,L_1^n))\right)^2\right]}_{I_{1.1}^n}+2\underbrace{\mathbb{E}\left[\frac{\Gamma(\mathcal{H}^{n,K},\mathcal{H}^{n,K})}{\Gamma(L_1^n,L_1^n)}\right]}_{I_{1.2}^n}.
\end{aligned} \tag{7.12}$$

Our aim is to prove that $I_{1.1}^n$ and $I_{1.2}^n$ are bounded independently of n.

For $I_{1.1}^n$, we see from (3.15) and Remark 3.3,

$$\frac{\Gamma(L_1^n,\Gamma(L_1^n,L_1^n))}{\Gamma(L_1^n,L_1^n)^2}-\frac{2L(L_1^n)}{\Gamma(L_1^n,L_1^n)} = \mathcal{H}_{L_1^n}(1) = \widehat{\mathcal{H}}_\beta^n(1)+\frac{1}{n^{1/\alpha}}\mathcal{R}_{1,\beta}^n. \tag{7.13}$$

From the crucial fact $||\mathcal{H}^{n,K}||_\infty \leq K$ and from (3.20), (3.23), Remark 3.3 we can deduce that $I_{1.1}^n$ is bounded by a random variable independent of n (but depending of K).

For $I_{1.2}^n$, from (3.8) and (6.3), we have

$$I_{1.2}^n = \mathbb{E}\left(\frac{\Gamma(\mathcal{H}^{n,K},\mathcal{H}^{n,K})}{\Gamma(L_1^n,L_1^n)}\right) = \mathbb{E}\left(\frac{\Gamma(\mathcal{H}^{n,K},\mathcal{H}^{n,K})}{\int_0^1\int_{\mathbb{R}}\rho^n(z)\mu^{(n)}(dt,dz)}\right) \leq \mathbb{E}\left(\frac{\Gamma(\mathcal{H}^{n,K},\mathcal{H}^{n,K})}{\int_0^1\int_{|z|<1} z^4\mu(dt,dz)}\right).$$

Now since $\mathcal{H}^{n,K}$ is a smooth Malliavin functional, using the chain rule property (3.11) we have

$$\Gamma(\mathcal{H}^{n,K},\mathcal{H}^{n,K}) \leq c_K^2 \Gamma(\mathcal{H}^n,\mathcal{H}^n) \tag{7.14}$$

where c_K is any upper bound of the derivative of $x \mapsto x\mathcal{X}_K(x)$.

Then we deduce that

$$I_{1.2}^n \leq c_K^2 \mathbb{E}\left(\frac{\Gamma(\mathcal{H}^n,\mathcal{H}^n)}{\int_0^1 \int_{|z|<1} z^4 \mu(dt,dz)}\right).$$

From the assumption on $\Gamma(\mathcal{H}^n,\mathcal{H}^n)$ in Proposition 6.1 and since $\left(\int_0^1 \int_{|z|<1} z^4 \mu(dt,dz)\right)^{-1}$ belongs to $\cap_{p\geq 1}\mathbf{L}^p$ [see [5, the proof of Theorem 4], we can deduce that $I_{1.2}^n$ is bounded independently of n. Thus, we get that the expectation of the right-hand side term in (7.8) is finite.

Turning to the expectation of (7.9), we have:

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{\sigma}{n^{1/\alpha}}\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})\int_0^1\int_{|z|>1}|z|\mu^{(n)}(dt,dz)\right|\right]\\
&= \mathbb{E}\left[\left|\frac{\sigma}{n^{1/\alpha}}\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})\left(\int_0^1\int_{1\leq|z|\leq 2}|z|\mu^{(n)}(dt,dz)+\int_0^1\int_{|z|>2}|z|\mu^{(n)}(dt,dz)\right)\right|\right]\\
&\leq \mathbb{E}\left[\left|\frac{\sigma}{n^{1/\alpha}}\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})\int_0^1\int_{1\leq|z|\leq 2}|z|\mu^{(n)}(dt,dz)\right|\right]+\mathbb{E}\left[\left|\frac{\sigma}{n^{1/\alpha}}\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})\int_0^1\int_{|z|>2}|z|\mu^{(n)}(dt,dz)\right|\right].
\end{aligned}$$

By a similar estimation technique as for the bound of (7.8), we get that

$$\sup_n \mathbb{E}\left[\left|\frac{\sigma}{n^{1/\alpha}}\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})\int_0^1\int_{1\leq|z|\leq 2}|z|\mu^{(n)}(dt,dz)\right|\right] \leq C_K < +\infty.$$

We now show that

$$\sup_n \mathbb{E}\left[\left|\frac{\sigma}{n^{1/\alpha}}\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})\int_0^1\int_{|z|>2}|z|\mu^{(n)}(dt,dz)\right|\right] \leq C_K < +\infty. \tag{7.15}$$

In fact, from (7.7) we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{\sigma}{n^{1/\alpha}}\mathcal{H}_{(\frac{\sigma L_1^n}{n^{1/\alpha}})}(\mathcal{H}^{n,K})\int_0^1\int_{|z|>2}|z|\mu^{(n)}(dt,dz)\right|\right]\\
&\leq \mathbb{E}\left[\left[\left|\frac{-2L(L_1^n)\mathcal{H}^{n,K}}{\Gamma(L_1^n,L_1^n)}+\frac{\mathcal{H}^{n,K}}{\Gamma(L_1^n,L_1^n)^2}\Gamma(L_1^n,\Gamma(L_1^n,L_1^n))\right|+\left|\frac{\Gamma(L_1^n,\mathcal{H}^{n,K})}{\Gamma(L_1^n,L_1^n)}\right|\right]\times\int_0^1\int_{|z|>2}|z|\mu^{(n)}(dt,dz)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \underbrace{\mathbb{E}\left[\left|\frac{-2L(L_1^n)\mathcal{H}^{n,K}}{\Gamma(L_1^n,L_1^n)} + \frac{\mathcal{H}^{n,K}}{\Gamma(L_1^n,L_1^n)^2}\Gamma(L_1^n,\Gamma(L_1^n,L_1^n))\right|\int_0^1\int_{|z|>2}|z|\mu^{(n)}(dt,dz)\right]}_{I_{1.3}^n} \\
&+ \underbrace{\mathbb{E}\left[\left|\frac{\Gamma(L_1^n,\mathcal{H}^{n,K})}{\Gamma(L_1^n,L_1^n)}\right|\int_0^1\int_{|z|>2}|z|\mu^{(n)}(dt,dz)\right]}_{I_{1.4}^n}
\end{aligned}
\tag{7.16}
$$

To prove (7.15), we just have to show that $I_{1.3}^n$ and $I_{1.4}^n$ are bounded independently of n. For $I_{1.3}^n$, we recall here (7.13),

$$
\frac{\Gamma(L_1^n,\Gamma(L_1^n,L_1^n))}{\Gamma(L_1^n,L_1^n)^2} - \frac{2L(L_1^n)}{\Gamma(L_1^n,L_1^n)} = \widehat{\mathcal{H}}_\beta^n(1) + \frac{1}{n^{1/\alpha}}\mathcal{R}_{1,\beta}^n(1).
$$

Then from the boundedness of $\mathcal{H}^{n,K}$, we get

$$
I_{1.3}^n \le K\mathbb{E}\left[\left|\widehat{\mathcal{H}}_\beta^n(1)\right|\int_0^1\int_{|z|>2}|z|\mu^{(n)}(dt,dz)\right] + K\mathbb{E}\left[\left|\mathcal{R}_{1,\beta}^n(1)\right|\frac{1}{n^{1/\alpha}}\int_0^1\int_{|z|>2}|z|\mu^{(n)}(dt,dz)\right]
$$

From the proof of Lemma 3.6 in [6], we can deduce that $\left|\widehat{\mathcal{H}}_\beta^n(1)\int_0^1\int_{|z|>2}|z|\mu^{(n)}(dt,dz)\right|$ is bounded by a random variable independent of n and belonging to $\cap_{p\ge1}\mathbf{L}^p$.

Using Cauchy-Schwarz inequality and (3.23), we get

$$
\mathbb{E}\left[\left|\mathcal{R}_{1,\beta}^n(1)\right|\frac{1}{n^{1/\alpha}}\int_0^1\int_{|z|>2}|z|\mu^{(n)}(dt,dz)\right] \le \frac{C}{\sqrt{n}}\left(\mathbb{E}\left[\int_0^1\int_{|z|>2}\frac{|z|}{n^{1/\alpha}}\mu^{(n)}(dt,dz)\right]^2\right)^{1/2} \tag{7.17}
$$

Now from $\mu^{(n)}(ds,dz) = \tilde{\mu}^{(n)}(ds,dz) + \upsilon^{(n)}(ds,dz)$, by convexity inequality, we have

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^1\int_{|z|>2}\frac{|z|}{n^{1/\alpha}}\mu^{(n)}(ds,dz)\right]^2 \le 2\mathbb{E}\left[\int_0^1\int_{|z|>2}\frac{|z|}{n^{1/\alpha}}\tilde{\mu}^{(n)}(ds,dz)\right]^2 + 2\left[\int_0^1\int_{|z|>2}\frac{|z|}{n^{1/\alpha}}\upsilon^{(n)}(ds,dz)\right]^2 \\
&\le 2\mathbb{E}\left[\int_0^1\int_{2<|z|\le 2n^{1/\alpha}}\frac{|z|^2}{n^{2/\alpha}}\frac{1}{|z|^{1+\alpha}}dzds\right] + 2\left[\int_0^1\int_{2<|z|\le 2n^{1/\alpha}}\frac{|z|}{n^{1/\alpha}}\frac{1}{|z|^{1+\alpha}}dzds\right]^2 \\
&\le \frac{C}{n^{2/\alpha}}\left(\frac{n^{2/\alpha}}{n} + \frac{n^{1/\alpha}}{n}\right) \quad \text{where } C \text{ is a deterministic constant.}
\end{aligned}
\tag{7.18}
$$

From (7.17) and (7.18), we deduce that the left-hand side of (7.17) is bounded by $\frac{C}{n}$. Then we get that $\sup_n I_{1.3}^n < +\infty$.

For $I_{1.4}^n$, from the boundedness of $\mathcal{H}^{n,K}$, (6.3), and the fact that

$\Gamma(L_1^n, \mathcal{H}^{n,K}) \leq \Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K})^{1/2}\Gamma(L_1^n, L_1^n)^{1/2}$ we have:

$$
\begin{aligned}
I_{1.4}^n &\leq E\left[\left(\left|\frac{\Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K})}{\Gamma(L_1^n, L_1^n)}\right|\right)^{1/2} \int_0^1 \int_{|z|>2} |z|\mu^{(n)}(dt, dz)\right] \\
&= E\left[\frac{\left|\Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K})\right|^{1/2}}{\left|\int_0^1 \int_{\mathbb{R}} \rho^n(z)\mu^{(n)}(ds, dz)\right|^{1/2}} \int_0^1 \int_{|z|>2} |z|\mu^{(n)}(dt, dz)\right] \\
&\leq \mathbb{E}\left[\frac{\left|\Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K})\right|^{1/2}}{\left|\int_0^1 \int_{|z|>2} z^2\mu^{(n)}(ds, dz)\right|^{1/2}} \int_0^1 \int_{|z|>2} |z|\mu^{(n)}(dt, dz)\right].
\end{aligned}
$$

Applying the Cauchy-Schwarz inequality, we have

$$
\int_0^1 \int_{|z|>2} \mu^n(dt, dz) \times \int_0^1 \int_{|z|>2} z^2\mu^{(n)}(ds, dz) \geq \left(\int_0^1 \int_{|z|>2} |z|\mu^n(dt, dz)\right)^2. \tag{7.19}
$$

We deduce

$$
\begin{aligned}
&\mathbb{E}\left[\frac{\left|\Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K})\right|^{1/2}}{\left|\int_0^1 \int_{|z|>2} z^2\mu^{(n)}(ds, dz)\right|^{1/2}} \int_0^1 \int_{|z|>2} |z|\mu^{(n)}(dt, dz)\right] \\
&\leq \mathbb{E}\left[\left|\Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K})\right|^{1/2} \left\{\int_0^1 \int_{|z|>2} \mu^{(n)}(dt, dz)\right\}^{1/2}\right] \qquad (7.20) \\
&\leq \left\{\mathbb{E}\left[\left|\Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K})\right|\right]\right\}^{1/2} \left\{\mathbb{E}\left[\int_0^1 \int_{|z|>2} \mu^{(n)}(dt, dz)\right]\right\}^{1/2}.
\end{aligned}
$$

From $\left\{\mathbb{E}\left[\int_0^1 \int_{|z|>2} \mu^{(n)}(dt, dz)\right]\right\}^{1/2} \leq C_4$ where C_4 is a constant and the fact that $\Gamma(\mathcal{H}^{n,K}, \mathcal{H}^{n,K})$ admits finite moment, independently of n (but depending of K). Then, (7.20) is bounded independently of n and $I_{1.4}^n$ is proved. Hence (7.15) follows.

Step 2: We now prove $\sup_n nI_2^n < \mathrm{C}^*$ where C^* is a positive constant.

We have

$$
\begin{aligned}
&\left|\frac{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}})} - 1\right| = \frac{1}{n}\left|\int_0^1 \frac{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, b(\overline{Y}_s^{n,\beta,x_0}, \theta) - b(\varsigma_s^{n,\theta,x_0}, \theta))}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}})} ds\right| \\
&\leq \frac{1}{n} \frac{||b'||_\infty}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}})} \int_0^1 \left|\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \overline{Y}_s^{n,\beta,x_0})\right| ds.
\end{aligned}
$$

Using that

$$
\begin{cases}
\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \overline{Y}_s^{n,\beta,x_0}) \leq \Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}})^{1/2} \Gamma(\overline{Y}_s^{n,\beta,x_0}, \overline{Y}_s^{n,\beta,x_0})^{1/2} \\
\Gamma(\overline{Y}_s^{n,\beta,x_0}, \overline{Y}_s^{n,\beta,x_0})^{1/2} \leq M\Gamma(\overline{Y}_1^{n,\beta,x_0}, \overline{Y}_1^{n,\beta,x_0})^{1/2} \\
\Gamma(\overline{Y}_1^{n,\beta,x_0}, \overline{Y}_1^{n,\beta,x_0})^{1/2} \leq M_1\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}})^{1/2}
\end{cases}
$$

for some constants M and M_1, we can easily deduce that

$$\left|\frac{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})}{\Gamma(\frac{\sigma L_1^n}{n^{1/\alpha}}, \frac{\sigma L_1^n}{n^{1/\alpha}})} - 1\right| \leq \frac{C^*}{n},$$

for a positive constant C^*. □

Lemma 7.2. *Under the assumptions of Proposition 6.1, for any compact set $Q \subset \mathbb{R}\times(0,\infty)$, $\forall K > 2$, there exists a constant $c_K > 0$ such that*

$$\sup_{x_0,\beta\in Q} \left|\left|\eta^{\mathcal{H}^{n,K}}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0}) - \overline{\eta}^{\mathcal{H}^{n,K}}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})\right|\right|_1 \leq \frac{c_K}{n}. \tag{7.21}$$

Proof. We estimate the $\mathbf{L}^1$- norm appearing in (7.21) by duality. Let $\overline{\beta} : \mathbb{R} \to [-1,1]$ be a measurable function, we evaluate:

$$\begin{aligned}
&\left|\mathbb{E}\left[(\eta^{\mathcal{H}^{n,K}}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0}) - \overline{\eta}^{\mathcal{H}^{n,K}}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})\overline{\beta}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})\right]\right| \\
&\leq \left|\mathbb{E}\left[(\eta^{\mathcal{H}^{n,K}}((\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})\overline{\beta}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})\right] - \mathbb{E}\left[\overline{\eta}^{\mathcal{H}^{n,K}}(\frac{\sigma L_1^\alpha}{n^{1/\alpha}})\overline{\beta}(\frac{\sigma L_1^\alpha}{n^{1/\alpha}})\right]\right| \\
&+ \left|\mathbb{E}\left[\overline{\eta}^{\mathcal{H}^{n,K}}(\frac{\sigma L_1^\alpha}{n^{1/\alpha}})\overline{\beta}(\frac{\sigma L_1^\alpha}{n^{1/\alpha}})\right] - \mathbb{E}\left[\overline{\eta}^{\mathcal{H}^{n,K}}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})\overline{\beta}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})\right]\right| \\
&\leq \left|\mathbb{E}\left[(\eta^{\mathcal{H}^{n,K}}((\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})\overline{\beta}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})\right] - \mathbb{E}\left[\overline{\eta}^{\mathcal{H}^{n,K}}(\frac{\sigma L_1^\alpha}{n^{1/\alpha}})\overline{\beta}(\frac{\sigma L_1^\alpha}{n^{1/\alpha}})\right]\right| + \frac{C_K}{n}K
\end{aligned}$$

where we have used Lemma 7.1 with the choice $\mathcal{H}^n = 1, K > 2$ and the choice $h = \overline{\eta}^{\mathcal{H}^{n,K}}\overline{\beta}$, recalling that $||\overline{\eta}^{\mathcal{H}^{n,K}}||_\infty \leq K$. From the definition of $\overline{\eta}^{\mathcal{H}^{n,K}}(\frac{\sigma L_1^\alpha}{n^{1/\alpha}})$ and $\eta^{\mathcal{H}^{n,K}}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})$ as conditional expectations, we have:

$$\begin{aligned}
&\left|\mathbb{E}\left[(\eta^{\mathcal{H}^{n,K}}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})\overline{\beta}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})\right] - E\left[\overline{\eta}^{\mathcal{H}^{n,K}}(\frac{\sigma L_1^\alpha}{n^{1/\alpha}})\overline{\beta}(\frac{\sigma L_1^\alpha}{n^{1/\alpha}})\right]\right| \\
&= \mathbb{E}\left[\mathcal{H}^n\mathcal{X}_K(\mathcal{H}^n)\overline{\beta}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})\right] - \mathbb{E}\left[\mathcal{H}^n\mathcal{X}_K(\mathcal{H}^n)\overline{\beta}(\frac{L_1^\alpha}{n^{1/\alpha}})\right] \leq \frac{C_K}{n}
\end{aligned}$$

where we used Lemma 7.1. This gives,

$$\sup_{||\overline{\beta}||_\infty\leq 1}\left|\mathbb{E}\left[(\eta^{\mathcal{H}^{n,K}}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0}) - \overline{\eta}^{\mathcal{H}^{n,K}}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0}))\overline{\beta}(\overline{Y}_1^{n,\beta,x_0} - \varsigma_1^{n,\theta,x_0})\right]\right| \leq (1+K)\frac{C_K}{n}$$

and we deduce the result of this lemma. □

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