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LOGARITHMIC STABILITY OF PARABOLIC CAUCHY PROBLEMS

MOURAD CHOULLI AND MASAHIRO YAMAMOTO

ABSTRACT. The uniqueness of parabolic Cauchy problems is nowadays a classical problem and since Hadamard [8] these kind of problems are known to be ill-posed and even severely ill-posed. Until now there are only few partial results concerning the quantification of the stability for parabolic Cauchy problems. We bring in the present work an answer to this issue for smooth solutions under the minimal condition that the domain is Lipschitz.

Mathematics subject classification : 35R25, 35K99, 58J35

Key words : Parabolic Cauchy problems, logarithmic stability, Carleman inequality, Hardy inequality.

1. INTRODUCTION

Throughout this article Ω is a bounded domain of \mathbb{R}^n with Lipschitz boundary Γ . Consider the parabolic operator

$$L = \operatorname{div}(A\nabla \cdot) - \partial_t.$$

Here $A = (a^{ij})$ is a symmetric matrix whose coefficients belong to $W^{1,\infty}(\Omega)$. Assume furthermore that there exists a constant $0 < \kappa \leq 1$ so that

$$(1.1) \quad A(x)\xi \cdot \xi \geq \kappa|\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n,$$

and

$$(1.2) \quad \|a^{ij}\|_{W^{1,\infty}(\Omega)} \leq \kappa^{-1}, \quad 1 \leq i, j \leq n.$$

Let $t_0 < t_1$ so that $t_1 - t_0 \leq T_0$, for some given $T_0 > 0$ and set $Q = \Omega \times (t_0, t_1)$. Recall the notation

$$H^{2,1}(Q) = L^2((t_0, t_1), H^2(\Omega)) \cap H^1((t_0, t_1), L^2(\Omega)).$$

If Γ_0 is a nonempty open subset of Γ then a classical result says that any $u \in H^{2,1}(Q)$ satisfying $Lu = 0$ in Q and $u = \nabla u = 0$ on $\Gamma_0 \times (t_0, t_1)$ must be identically equal to zero (see [2] and references therein). This result is known as the uniqueness of the Cauchy problem for the equation $Lu = 0$. Quantifying this uniqueness result consists in controlling a norm of a solution of $Lu = 0$ by a suitable function of the norm of $(u, \nabla u)|_{\Gamma_0 \times (t_0, t_1)}$ in some space.

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In the rest of this paper, $0 < \alpha < 1$ is fixed and, for simplicity's sake, we use the notations

$$\begin{aligned}\mathcal{X}(Q) &= C^{1+\alpha, (1+\alpha)/2}(\overline{Q}) \cap H^1((t_0, t_1), H^2(\Omega)), \\ \mathcal{Y}(Q) &= \{u \in \mathcal{X}(Q); \partial_t u \in \mathcal{X}(Q)\}, \\ \mathcal{Z}(Q) &= \mathcal{Y}(Q) \cap H^3((t_0, t_1), H^2(\Omega)).\end{aligned}$$

We endow $\mathcal{X}(Q)$, $\mathcal{Y}(Q)$ and $\mathcal{Z}(Q)$ with their natural norms

$$\begin{aligned}\|u\|_{\mathcal{X}(Q)} &= \|u\|_{C^{1+\alpha, (1+\alpha)/2}(\overline{Q})} + \|u\|_{H^1((t_0, t_1), H^2(\Omega))} \\ \|u\|_{\mathcal{Y}(Q)} &= \|u\|_{\mathcal{X}(Q)} + \|\partial_t u\|_{\mathcal{X}(Q)}. \\ \|u\|_{\mathcal{Z}(Q)} &= \|u\|_{\mathcal{Y}(Q)} + \|u\|_{H^3((t_0, t_1), H^2(\Omega))}\end{aligned}$$

Let $\varrho^* = e^{-e^e}$. Set then, for $\mu > 0$ and $\varrho_0 \leq \varrho^*$,

$$\Psi_{\varrho_0, \mu}(\varrho) = \begin{cases} 0 & \text{if } \varrho = 0, \\ (\ln \ln \ln |\ln \varrho|)^{-\mu} & \text{if } 0 < \varrho \leq \varrho_0, \\ \varrho & \text{if } \varrho \geq \varrho_0. \end{cases}$$

We are mainly concerned in the present work with the stability issue for the Cauchy problem associated to the parabolic operator L . Precisely, we are going to prove the following result.

Theorem 1.1. *Let Γ_0 be a nonempty open subset of Γ and $s \in (0, 1/2)$. Then there exist two constants $C > 0$ and $0 < \varrho_0 \leq \varrho^*$, depending on Ω , κ , T_0 , α , s and Γ_0 , so that, for any $u \in \mathcal{Z}(Q)$ satisfying $Lu = 0$ in Q , we have*

$$C\|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq \|u\|_{\mathcal{Z}(Q)} \Psi_{\varrho_0, \mu} \left(\frac{\mathcal{C}(u, \Gamma_0)}{\|u\|_{\mathcal{Z}(Q)}} \right),$$

with $\mu = \min(\alpha, s)/4$ and

$$\mathcal{C}(u, \Gamma_0) = \|u\|_{H^3((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{H^2((t_0, t_1), L^2(\Gamma_0))}.$$

It is straightforward to check that $\mathcal{C}(u, \Gamma_0)$ in the preceding theorem can be substituted by

$$\mathcal{C}(u, \Gamma_0) = \|u\|_{H^3((t_0, t_1), L^2(\Gamma_0)) \cap H^2((t_0, t_1), H^1(\Gamma_0))} + \|\partial_{\mathbf{n}} u\|_{H^2((t_0, t_1), L^2(\Gamma_0))}.$$

Here \mathbf{n} is the unit exterior normal field on Γ and $\partial_{\mathbf{n}} u = \nabla u \cdot \mathbf{n}$.

We observe that Theorem 1.1 remains valid if L is substituted by L plus an operator of first order in space variable whose coefficients are bounded.

Since the proofs are quite complicated we limited ourselves to the case $Lu = 0$. We believe that one can remove this condition by adding to $\mathcal{C}(u, \Gamma_0)$ the norm of Lu is a suitable space.

The second author [14, Theorem 5.1, page 24] proved a Hölder stability in a proper subdomain of Q depending on the part of the lateral boundary where the Cauchy data is given. In [13, Theorem 3.5.1, pages 45 and 46], Vessella establishes a local Hölder stability corresponding to the continuation of Cauchy data to an interior subdomain for solutions vanishing at the initial time. Recently, Bourgeois [1, Main theorem, page 2] proved a result similar to the one in Theorem 1.1, with a single logarithm function, in the case where $L = \Delta - \partial_t$, $\Omega = D \setminus O$, D and O are two domains of class C^2 , $O \Subset D$, and Γ_0 is either ∂D or ∂O . His result is based on a global Carleman estimate in which the weight function is built from the distance to the boundary of the space variable.

We discuss in the present work the Cauchy problem in all of its generality, that is without any restriction on the part of the boundary where the Cauchy data is given.

The proof of the main result is inspired by that used in the elliptic case by the first author in [3] (a substantial improvement of this result will appear in [4]). Note however that there is a great difference between the elliptic case and the parabolic case. The main difficulty in the parabolic case is due to the fact that the initial time and the final time data are wanting. So the proofs are more technical. The idea to overcome the fact that the initial time and the final time data are not known is to use a Hardy inequality with respect to time variable. This explains partially why we need to work with sufficiently smooth solutions. Another difference between the parabolic case and the elliptic case relies on the fact that in the elliptic case the main tool is a three-ball inequality with arbitrary radius. While in the parabolic case the method is based on a three-cylinder inequality (see Theorem 2.1) in which the radius depends on the distance to the boundary of the time variable. Roughly speaking, the radius becomes smaller and smaller as the time variable approaches the boundary. For this reason, contrary to the elliptic case where the stability is only of single logarithmic type, the stability is of multiple logarithmic type.

The three-cylinder inequality appears to be the right tool for continuing a solution of a parabolic equation. For this reason, we are not convinced that Theorem 1.1 can be improved by using a global method.

Although we used classical tools to establish our main result, the result itself is completely new and our proof is entirely self-contained. This is our modest contribution to the stability issue for parabolic Cauchy problems.

The most part in our analysis is build on a Carleman inequality (Theorem 2.2 below). We observe that Carleman inequalities are very useful tool in control theory and for establishing the unique continuation property for elliptic and parabolic partial differential equations. There is wide literature on this subject. We just quote here the few references [1, 5, 6, 10].

The rest of this article is organized as follows. Section 2 is devoted to a three-cylinder interpolation inequality for the $L_t^2(H_x^1)$ -norm. This inequality will be very useful for continuing the data on an interior subdomain to the lateral boundary data, and to continue the data from one subdomain to another subdomain. This is what we show in Section 3 and, as byproduct, we prove a stability estimate corresponding to the unique continuation from an interior data. The proof of Theorem 1.1 is completed in Section 4 by beforehand establishing a result that quantifies the stability from the Cauchy data to an interior subdomain.

2. THREE-CYLINDER INTERPOLATION INEQUALITY

We prove in this section

Theorem 2.1. *There exist $C > 0$ and $0 < \vartheta < 1$, only depending on κ , Ω and T_0 , so that, for any $0 < \epsilon < (t_1 - t_0)/2$, $u \in H^1((t_0, t_1), H^2(\Omega))$ satisfying $Lu = 0$ in Q , $y \in \Omega$ and $0 < r < r_y(\epsilon) = \min(\text{dist}(y, \Gamma)/3, \sqrt{\epsilon})$, we have*

$$(2.1) \quad r^3 \|u\|_{L^2((t_0+\epsilon, t_1-\epsilon), H^1(B(y, 2r)))} \leq C \|u\|_{L^2((t_0, t_1), H^1(B(y, r)))}^{\vartheta} \|u\|_{L^2((t_0, t_1), H^1(B(y, 3r)))}^{1-\vartheta}.$$

The proof of Theorem 2.1 is based on a Carleman inequality for a family of parabolic operators. To this end, let \mathcal{Z} be an arbitrary set and consider the family

of operators

$$L_z = \operatorname{div}(A_z \nabla \cdot) - \partial_t, \quad z \in \mathcal{Z},$$

where, for each $z \in \mathcal{Z}$, $A_z = (a_z^{ij})$ is a symmetric matrix with $W^{1,\infty}(\Omega)$ entries and there exists $0 < \kappa \leq 1$ so that

$$(2.2) \quad A_z(x) \xi \cdot \xi \geq \kappa |\xi|^2, \quad x \in \Omega, \xi \in \mathbb{R}^n \text{ and } z \in \mathcal{Z},$$

and

$$(2.3) \quad \|a_z^{ij}\|_{W^{1,\infty}(\Omega)} \leq \kappa^{-1}, \quad 1 \leq i, j \leq n, z \in \mathcal{Z}.$$

Pick $\psi \in C^2(\overline{\Omega})$ without critical points in $\overline{\Omega}$ and set $\Sigma = \Gamma \times (t_0, t_1)$. Let

$$g(t) = \frac{1}{(t - t_0)(t_1 - t)}$$

and

$$\begin{aligned} \varphi(x, t) &= g(t) \left(e^{4\lambda \|\psi\|_\infty} - e^{\lambda(2\|\psi\|_\infty + \psi(x))} \right), \\ \chi(x, t) &= g(t) e^{\lambda(2\|\psi\|_\infty + \psi(x))}. \end{aligned}$$

Theorem 2.2. (*Carleman inequality*) *There exist three positive constants C , λ_0 and τ_0 , only depending only on ψ , Ω , κ and T_0 , so that*

$$(2.4) \quad \begin{aligned} C \int_Q (\lambda^4 \tau^3 \chi^3 u^2 + \lambda^2 \tau \chi |\nabla u|^2) e^{-2\tau\varphi} dx dt \\ \leq \int_Q (L_z u)^2 e^{-2\tau\varphi} dx dt \\ + \int_\Sigma (\lambda^3 \tau^3 \chi^3 u^2 + \lambda \tau \chi |\nabla u|^2 + (\lambda \tau \chi)^{-1} (\partial_t u)^2) e^{-2\tau\varphi} d\sigma dt, \end{aligned}$$

for all $u \in H^1((t_0, t_1), H^2(\Omega))$, $z \in \mathcal{Z}$, $\lambda \geq \lambda_0$ and $\tau \geq \tau_0$.

Proof. Since the dependance of the constants will be uniform with respect to $z \in \mathcal{Z}$, we drop for simplicity the subscript z in L_z and its coefficients. On the other hand, as $C^\infty(\overline{Q})$ is dense in $H^1((t_0, t_1), H^2(\Omega))$, it is enough to prove (2.4) when $u \in C^\infty(\overline{Q})$.

Let $\Phi = e^{\tau\varphi}$, $u \in C^\infty(\overline{Q})$ and set $w = \Phi^{-1}u$ that we extend by continuity at $t = 0$ and $t = T$ by setting $w(\cdot, 0) = w(\cdot, T) = 0$. Then straightforward computations give

$$Pw = [\Phi^{-1}L\Phi]w = P_1w + P_2w + cw,$$

where

$$\begin{aligned} P_1w &= aw + \operatorname{div}(A\nabla w) - \tau \partial_t \varphi w, \\ P_2w &= B \cdot \nabla w + bw - \partial_t w, \end{aligned}$$

with

$$\begin{aligned} a &= a(x, t, \lambda, \tau) = \lambda^2 \tau^2 \chi^2 |\nabla \psi|_A^2, \\ B &= B(x, t, \lambda, \tau) = -2\lambda \tau \chi A \nabla \psi, \\ b &= b(x, t, \lambda, \tau) = -2\lambda^2 \tau \chi |\nabla \psi|_A^2, \\ c &= c(x, t, \lambda, \tau) = -\lambda \tau \chi \operatorname{div}(A \nabla \psi) + \lambda^2 \tau \chi |\nabla \psi|_A^2. \end{aligned}$$

Here

$$|\nabla \psi|_A = \sqrt{A \nabla \psi \cdot \nabla \psi} = |A^{1/2} \nabla \psi|.$$

We obtain by making integrations by parts

$$(2.5) \quad \begin{aligned} \int_Q aw(B \cdot \nabla w) dxdt &= \frac{1}{2} \int_Q a(B \cdot \nabla w^2) dxdt \\ &= -\frac{1}{2} \int_Q \operatorname{div}(aB)w^2 dxdt + \frac{1}{2} \int_\Sigma a(B \cdot \nu)w^2 d\sigma dt \end{aligned}$$

and

$$(2.6) \quad \begin{aligned} \int_Q \operatorname{div}(A\nabla w)B \cdot \nabla w dxdt &= - \int_Q A\nabla w \cdot \nabla(B \cdot \nabla w) dxdt + \int_\Sigma (B \cdot \nabla w)(A\nabla w \cdot \nu) d\sigma dt \\ &= - \int_Q B' \nabla w \cdot A\nabla w dxdt \\ &\quad - \int_Q \nabla^2 w B \cdot A\nabla w dxdt + \int_\Sigma (B \cdot \nabla w)(A\nabla w \cdot \nu) d\sigma dt. \end{aligned}$$

Here $B' = (\partial_j B_i)$ is the Jacobian matrix of B and $\nabla^2 w = (\partial_{ij}^2 w)$ is the Hessian matrix of w .

But

$$\begin{aligned} \int_Q B_j \partial_{ij}^2 w a^{ik} \partial_k w dxdt &= - \int_Q B_j a^{ik} \partial_{jk}^2 w \partial_i w dxdt \\ &\quad - \int_Q \partial_j [B_j a^{ik}] \partial_k w \partial_i w dxdt + \int_\Sigma B_j \nu_j a^{ik} \partial_k w \partial_i w d\sigma dt. \end{aligned}$$

Therefore

$$(2.7) \quad \begin{aligned} \int_Q \nabla^2 w B \cdot A\nabla w dxdt &= -\frac{1}{2} \int_Q ([\operatorname{div}(B)A + \tilde{A}] \nabla w) \cdot \nabla w dxdt \\ &\quad + \frac{1}{2} \int_\Sigma |\nabla w|_A^2 (B \cdot \nu) d\sigma dt, \end{aligned}$$

where $\tilde{A} = (\tilde{a}^{ij})$ with $\tilde{a}^{ij} = B \cdot \nabla a^{ij}$.

It follows from (2.6) and (2.7)

$$(2.8) \quad \begin{aligned} \int_Q \operatorname{div}(A\nabla w)B \cdot \nabla w dxdt &= \frac{1}{2} \int_Q (-2AB' + \operatorname{div}(B)A + \tilde{A}) \nabla w \cdot \nabla w dxdt \\ &\quad + \int_\Sigma (B \cdot \nabla w)(A\nabla w \cdot \nu) d\sigma dt - \frac{1}{2} \int_\Sigma |\nabla w|_A^2 (B \cdot \nu) d\sigma dt. \end{aligned}$$

A new integration by parts yields

$$\begin{aligned} \int_Q \operatorname{div}(A\nabla w)bw dxdt &= - \int_Q b|\nabla w|_A^2 dxdt - \int_Q w \nabla b \cdot A\nabla w dxdt \\ &\quad + \int_\Sigma bw A\nabla w \cdot \nu d\sigma dt. \end{aligned}$$

This and

$$- \int_Q w \nabla b \cdot A\nabla w dxdt \geq - \int_Q (\lambda^2 \chi)^{-1} |\nabla b|_A^2 w^2 dxdt - \int_Q \lambda^2 \chi |\nabla w|_A^2 dxdt$$

(where we used the inequality $|AX \cdot Y| \leq |X|_A^2 + |Y|_A^2$, for all $X, Y \in \mathbb{R}^n$) imply

$$(2.9) \quad \int_Q \operatorname{div}(A\nabla w)bw dxdt \geq - \int_Q (b + \lambda^2\chi)|\nabla w|_A^2 dxdt \\ - \int_Q (\lambda^2\chi)^{-1}|\nabla b|_A^2 w^2 dxdt + \int_\Gamma bw(A\nabla w \cdot \nu) d\sigma dt.$$

One more time, integrations by parts entail

$$(2.10) \quad \int_Q aw\partial_t w dxdt = \frac{1}{2} \int_Q a\partial_t w^2 dxdt = -\frac{1}{2} \int_Q \partial_t aw^2 dxdt,$$

$$(2.11) \quad \int_Q \partial_t \varphi w \partial_t w dxdt = \frac{1}{2} \int_Q \partial_t \varphi \partial_t w^2 dxdt = -\frac{1}{2} \int_Q \partial_t^2 \varphi w^2 dxdt,$$

$$(2.12) \quad \int_Q \partial_t \varphi w B \cdot \nabla w dxdt = \frac{1}{2} \int_Q \partial_t \varphi B \cdot \nabla w^2 dxdt \\ = -\frac{1}{2} \int_Q \operatorname{div}(\partial_t \varphi B) w^2 dxdt + \frac{1}{2} \int_\Sigma \partial_t \varphi (B \cdot \nu) w^2 d\sigma dt.$$

Also,

$$(2.13) \quad \int_Q \operatorname{div}(A\nabla w)\partial_t w dxdt = - \int_Q A\nabla w \cdot \nabla \partial_t w dxdt + \int_\Sigma (A\nabla w \cdot \nu)\partial_t w d\sigma dt.$$

But an integration by parts with respect to t gives

$$\int_Q A\nabla w \cdot \nabla \partial_t w dxdt = - \int_Q A\nabla \partial_t w \cdot \nabla w dxdt = - \int_Q \nabla \partial_t w \cdot A\nabla w dxdt,$$

where we used $w(\cdot, 0) = w(\cdot, T) = 0$.

Whence

$$\int_Q A\nabla w \cdot \nabla \partial_t w dxdt = 0.$$

This identity in (2.13) entails

$$(2.14) \quad \int_Q \operatorname{div}(A\nabla w)\partial_t w = \int_\Sigma (A\nabla w \cdot \nu)\partial_t w d\sigma dt.$$

Now a combination of (2.5), (2.8) to (2.12) and (2.14) gives

$$(2.15) \quad \int_Q P_1 w P_2 w dxdt - \int_Q c^2 w^2 dxdt \\ \geq \int_Q f w^2 dxdt + \int_Q F \nabla w \cdot \nabla w dxdt + \int_\Sigma g(w) d\sigma dt,$$

where

$$f = -\frac{1}{2} \operatorname{div}(aB) + ab - (\lambda^2\chi)^{-1}|\nabla b|_A^2 - c^2 + \frac{1}{2} \partial_t a - \frac{\tau}{2} \partial_t^2 \varphi + \frac{\tau}{2} \operatorname{div}(\partial_t \varphi B) - \tau b \partial_t \varphi,$$

$$F = -AB' + \frac{1}{2} (\operatorname{div}(B)A + \tilde{A}) - (b + \lambda^2\chi)A,$$

$$g(w) = \frac{1}{2} aw^2 (B \cdot \nu) - \frac{1}{2} |\nabla w|_A^2 (B \cdot \nu) + (B \cdot \nabla w)(A\nabla w \cdot \nu) \\ + bw(A\nabla w \cdot \nu) - \frac{\tau}{2} \partial_t \varphi (B \cdot \nu) w^2 - (A\nabla w \cdot \nu) \partial_t w.$$

We obtain, by using the elementary inequality $(\alpha - \beta)^2 \geq \alpha^2/2 - \beta^2$, $\alpha > 0$, $\beta > 0$,

$$\begin{aligned} \|Pw\|_2^2 &\geq (\|P_1w + P_2w\|_2 - \|cw\|_2)^2 \\ &\geq \frac{1}{2}\|P_1w + P_2w\|_2^2 - \|cw\|_2^2 \\ &\geq \int_{\Omega} P_1wP_2w dx - \int_{\Omega} c^2w^2 dx. \end{aligned}$$

Whence (2.15) implies

$$(2.16) \quad \|Pw\|_2^2 \geq \int_Q fw^2 dxdt + \int_Q F\nabla w \cdot \nabla w dxdt + \int_{\Sigma} g(w) d\sigma dt.$$

In light of the following inequalities, where C is a constant depending only on T_0 and ψ ,

$$\begin{aligned} |\partial_t \varphi| &\leq C\chi^2, \\ |\partial_t^2 \varphi|, |\nabla \partial_t \varphi| &\leq C\chi^3 \\ |(A\nabla w \cdot \nu) \partial_t w| &\leq \lambda\tau\chi |A\nabla w \cdot \nu|^2 + (\lambda\tau\chi)^{-1} (\partial_t w)^2, \end{aligned}$$

straightforward computations show that there exist four positive constants C_0, C_1, λ_0 and τ_0 , only depending only on ψ, Ω, T_0 and κ , such that, for all $\lambda \geq \lambda_0$ and $\tau \geq \tau_0$, so that

$$\begin{aligned} f &\geq C_0\lambda^4\tau^3\chi^3, \\ F\xi \cdot \xi &\geq C_0\lambda^2\tau\chi|\xi|^2, \text{ for any } \xi \in \mathbb{R}^n, \\ |g(w)| &\leq C_1(\lambda^3\tau^3\chi^3w^2 + \lambda\tau\chi|\nabla w|^2 + (\lambda\tau\chi)^{-1}(\partial_t w)^2). \end{aligned}$$

Hence

$$(2.17) \quad C \int_Q (\lambda^4\tau^3\chi^3w^2 + \lambda^2\tau\chi|\nabla w|^2) dxdt \leq \int_Q (Pw)^2 dxdt + \int_{\Sigma} (\lambda^3\tau^3\chi^3w^2 + \lambda\tau\chi|\nabla w|^2 + (\lambda\tau\chi)^{-1}(\partial_t w)^2) d\sigma dt.$$

As $\nabla w = \Phi^{-1}(\nabla u + \lambda\tau\chi u \nabla \psi)$, we obtain

$$|\nabla w|^2 = \Phi^{-2} (|\nabla u|^2 + \lambda^2\tau^2\chi^2|\nabla \psi|^2u^2 + 2\lambda\tau\chi u \nabla u \cdot \nabla \psi).$$

Therefore we find, by using an elementary inequality,

$$|\nabla w|^2 \geq \Phi^{-2} \left(|\nabla u|^2 + \lambda^2\tau^2|\nabla \psi|^2u^2 - 4\lambda^2\tau^2u^2|\nabla \psi|^2 - \frac{1}{2}|\nabla u|^2 \right)$$

and then

$$|\nabla w|^2 \geq \Phi^{-2} \left(\frac{1}{2}|\nabla u|^2 - 3\lambda^2\tau^2\chi^2u^2\|\nabla \psi\|_{\infty}^2 \right).$$

Consequently, modifying λ_0 if needed, we get

$$(2.18) \quad \lambda^2\tau\chi|\nabla w|^2 + \lambda^4\tau^3\chi^3w^2 \geq C\Phi^{-2} (\lambda^2\tau\chi|\nabla u|^2 + \lambda^4\tau^3\chi^3\tau^2u^2).$$

On the other hand, it is not hard to establish the inequality

$$(2.19) \quad (\partial_t w)^2 \leq \Phi^{-2} ((\partial_t u)^2 + C\tau^2\chi^2u^2).$$

The expected inequality follows then by combining (2.17), (2.18) and (2.19). \square

From the preceding proof it is obvious that Theorem 2.2 holds whenever L_z is replaced by L . That is we have

Theorem 2.3. (*Carleman inequality*) *There exist three positive constants C , λ_0 and τ_0 , only depending only on ψ , Ω , κ and T_0 , so that*

$$(2.20) \quad C \int_Q (\lambda^4 \tau^3 \chi^3 u^2 + \lambda^2 \tau \chi |\nabla u|^2) e^{-2\tau\varphi} dxdt \\ \leq \int_Q (Lu)^2 e^{-2\tau\varphi} dxdt \\ + \int_\Sigma (\lambda^3 \tau^3 \chi^3 u^2 + \lambda \tau \chi |\nabla u|^2 + (\lambda \tau \chi)^{-1} (\partial_t u)^2) e^{-2\tau\varphi} d\sigma dt,$$

for all $u \in H^1((t_0, t_1), H^2(\Omega))$, $\lambda \geq \lambda_0$ and $\tau \geq \tau_0$.

Proof of Theorem 2.1. Let $u \in H^1((t_0, t_1), H^2(\Omega))$ satisfying $Lu = 0$ and set

$$Q(\mu) = B(0, \mu) \times (-1, 1), \quad \mu > 0.$$

Fix $(y, s) \in \Omega \times (t_0, t_1)$ and

$$0 < r < r_{(y,s)} = \min(\text{dist}(y, \Gamma)/3, \sqrt{s - t_0}, \sqrt{t_1 - s}) \leq r_0 = r_0(\text{diam}(\Omega), T_0).$$

Let

$$w(x, t) = u(rx + y, r^2 t + s), \quad (x, t) \in Q(3),$$

Then

$$L_r w = \text{div}(A_r \nabla w) - \partial_t w = 0 \quad \text{in } Q(3),$$

where $A_r(x) = (a^{ij}(rx + y))$.

Clearly, the family (A_r) satisfies (2.2) and (2.3) uniformly with respect to $r \in (0, r_{(y,s)})$.

Let $\chi \in C_0^\infty(U)$ satisfying $0 \leq \chi \leq 1$ and $\chi = 1$ in \mathcal{K} , with

$$U = \{x \in \mathbb{R}^n; 1/2 < |x| < 3\} \quad \text{and} \quad \mathcal{K} = \{x \in \mathbb{R}^n; 1 \leq |x| \leq 5/2\}.$$

Theorem 2.2 applied to χw when Ω is substituted by U and $g(t) = 1/(1 - t^2)$ gives, for $\lambda \geq \lambda_0$ and $\tau \geq \tau_0$,

$$(2.21) \quad C \int_{Q(2) \setminus Q(1)} (\lambda^4 \tau^3 \varphi^3 w^2 + \lambda^2 \tau \varphi |\nabla w|^2) e^{-2\tau\varphi} dxdt \\ \leq \int_{Q(3)} (L_r(\chi w))^2 e^{-2\tau\varphi} dxdt,$$

the constant C only depends on κ .

But

$$\text{supp}(L_r(\chi w)) \subset [\{1/2 \leq |x| \leq 1\} \cup \{5/2 \leq |x| \leq 3\}] \times (-1, 1)$$

and

$$(L_r(\chi w))^2 \leq \Lambda(w^2 + |\nabla w|^2),$$

where $\Lambda = \Lambda(r_0)$ is independent on r . Therefore, fixing λ and changing τ_0 if necessary, (2.21) implies, for $\tau \geq \tau_0$,

$$(2.22) \quad C \int_{Q(2)} (w^2 + |\nabla w|^2) e^{-2\tau\varphi} dxdt \leq \int_{Q(1)} (w^2 + |\nabla w|^2) e^{-2\tau\varphi} dxdt \\ + \int_{Q(3) \setminus Q(5/2)} (w^2 + |\nabla w|^2) e^{-2\tau\varphi} dxdt.$$

Let $0 < \rho < 1$ to be specified later and choose $\psi(x) = -|x|^2$ in (2.22) (which is without critical points in U). In that case

$$\varphi(x, t) = g(t) \left(e^{36\lambda} - e^{\lambda(18-|x|^2)} \right).$$

We have

$$\begin{aligned} \varphi(x, t) &\leq g(-1 + \rho) (e^{36\lambda} - e^{14\lambda}) \leq \frac{1}{\rho} (e^{36\lambda} - e^{14\lambda}) = \frac{\alpha}{\rho}, \\ &\quad (x, t) \in B(2) \times (-1 + \rho, 1 - \rho), \\ \varphi(x, t) &\geq g(0) (e^{36\lambda} - e^{18\lambda}) = (e^{36\lambda} - e^{18\lambda}) = \beta, \quad (x, t) \in Q(1), \\ \varphi(x, t) &\geq g(0) \left(e^{36\lambda} - e^{\frac{47}{4}\lambda} \right) = \left(e^{36\lambda} - e^{\frac{47}{4}\lambda} \right) = \gamma, \quad (x, t) \in Q(3) \setminus Q(5/2). \end{aligned}$$

As $\frac{\beta}{\alpha} < 1 < \frac{\gamma}{\alpha}$, we can fix $\theta \in (0, 1)$ so that

$$\frac{1}{\rho} := \theta \frac{\beta}{\alpha} + (1 - \theta) \frac{\gamma}{\alpha} > 1.$$

Set $a = 2(1 - \theta)(\gamma - \beta)$ and $b = 2\theta(\gamma - \beta)$ and $\tilde{Q}(2) = B(0, 2) \times (-1 + \rho, 1 - \rho)$. Then (2.22) yields

$$\begin{aligned} C \int_{\tilde{Q}(2)} (w^2 + |\nabla w|^2) dxdt \\ \leq e^{a\tau} \int_{Q(1)} (w^2 + |\nabla w|^2) dxdt + e^{-b\tau} \int_{Q(3)} (w^2 + |\nabla w|^2) dxdt. \end{aligned}$$

Similarly to the elliptic case [3, Theorem 2.17 and its proof, pages 19 to 21] (see also the proof of Proposition 4.1), we obtain from this inequality the following one

$$C \|w\|_{L^2((-1+\rho, 1-\rho), H^1(B(2)))} \leq \|w\|_{L^2((-1, 1), H^1(B(1)))} \|w\|_{L^2((-1, 1), H^1(B(3)))}^{1-\vartheta},$$

with $\vartheta = \frac{a}{a+b}$.

We get by making a change of variable, where $\tau = 1 - \rho$,

$$\begin{aligned} (2.23) \quad r \|u\|_{L^2((s-\tau r^2, s+\tau r^2), H^1(B(y, 2r)))} \\ \leq C \|u\|_{L^2((s-r^2, s+r^2), H^1(B(y, r)))} \|u\|_{L^2((s-r^2, s+r^2), H^1(B(y, 3r)))}^{1-\vartheta}. \end{aligned}$$

Here and until the end of this proof, the generic constant C only depends on Ω , κ and T_0 .

Fix $0 < \epsilon < (t_1 - t_0)/2$. Let $s_0 = t_0 + \epsilon$ and $s_k = s_{k-1} + 2\tau r^2$, $k \geq 1$, in such a way that

$$(s_{k-1}, s_k) = ((s_{k-1} + \tau r^2) - \tau r^2, (s_{k-1} + \tau r^2) + \tau r^2).$$

We consider q the smallest integer so that $(t_1 - \epsilon) - s_{q-1} \leq 2\tau r^2$ or equivalently $(t_1 - \epsilon) - s_{q-2} > 2\tau r^2$. Whence

$$(2.24) \quad q < \frac{t_1 - t_0 - 2\epsilon}{2\tau r^2} + 3 < \frac{\delta}{2\tau r^2} + \frac{3\text{diam}(\Omega)^2}{r^2} = \left(\frac{T_0}{2\tau} + 3\text{diam}(\Omega)^2 \right) \frac{1}{r^2}.$$

Let $r < r_y(\epsilon) = \min(\text{dist}(y, \Gamma)/3, \sqrt{\epsilon})$. It follows from (2.23) that

$$\begin{aligned} r \|u\|_{L^2((s_{k-1}, s_k), H^1(B(y, 2r)))} \\ \leq C \|u\|_{L^2((t_0, t_1), H^1(B(y, r)))} \|u\|_{L^2((t_0, t_1), H^1(B(y, 3r)))}^{1-\vartheta}, \end{aligned}$$

with $s_q = t_1 - \epsilon$.

Thus

$$\begin{aligned} & r \sum_{k=1}^q \|u\|_{L^2((s_{k-1}, s_k), H^1(B(y, 2r)))} \\ & \leq Cq \|u\|_{L^2((t_0, t_1), H^1(B(y, r)))}^\vartheta \|u\|_{L^2((t_0, t_1), H^1(B(y, 3r)))}^{1-\vartheta}. \end{aligned}$$

In consequence

$$(2.25) \quad \begin{aligned} r \|u\|_{L^2((t_0+\epsilon, t_1-\epsilon), H^1(B(y, 2r)))} \\ \leq Cq \|u\|_{L^2((t_0, t_1), H^1(B(y, r)))}^\vartheta \|u\|_{L^2((t_0, t_1), H^1(B(y, 3r)))}^{1-\vartheta}. \end{aligned}$$

Estimate (2.24) in (2.25) yields

$$\begin{aligned} Cr^3 \|u\|_{L^2((t_0+\epsilon, t_1-\epsilon), H^1(B(y, 2r)))} \\ \leq \|u\|_{L^2((t_0, t_1), H^1(B(y, r)))}^\vartheta \|u\|_{L^2((t_0, t_1), H^1(B(y, 3r)))}^{1-\vartheta}, \quad r < r_y(\epsilon). \end{aligned}$$

The proof is then complete. \square

3. QUANTIFYING THE UNIQUENESS OF CONTINUATION FROM AN INTERIOR DATA

We start with a Hardy inequality for vector valued functions.

Lemma 3.1. *Let X be a Banach space with norm $\|\cdot\|$ and $s \in (0, 1/2)$. There exists a constant $c > 0$ so that, for any $u \in H^s((t_0, t_1), X)$, we have*

$$\left\| \frac{u}{\delta^s} \right\|_{L^2((t_0, t_1), X)} \leq c \|u\|_{H^s((t_0, t_1), X)}.$$

Here $\delta = \delta(t) = \min\{|t - t_0|, |t - t_1|\}$.

Proof. Let $u \in H^s((t_0, t_1), X)$. From the usual Hardy's inequality in dimension one (see for instance [7]) we have

$$(3.1) \quad \int_{t_0}^{t_1} \frac{\|u(t)\|^2}{\delta^{2s}(t)} dt \leq c \|u(\cdot)\|_{H^s((t_0, t_1))}^2.$$

But

$$\begin{aligned} \|u(\cdot)\|_{H^s((t_0, t_1))}^2 &= \|u\|_{L^2((t_0, t_1), X)}^2 + \int_{t_0}^{t_1} \int_{t_0}^{t_1} \frac{\|u(\tau) - u(t)\|^2}{|\tau - t|^{1+2s}} dt d\tau \\ &\leq \|u\|_{L^2((t_0, t_1), X)}^2 + \int_{t_0}^{t_1} \int_{t_0}^{t_1} \frac{\|u(\tau) - u(t)\|^2}{|\tau - t|^{1+2s}} dt d\tau = \|u\|_{H^s((t_0, t_1), X)}^2. \end{aligned}$$

Whence the result follows. \square

In the rest of this paper we shall often apply Hardy's inequality in Lemma 3.1 to functions from $H^k((t_0, t_1), H)$, where $k \geq 1$ is an integer and H is a Hilbert space. This is made possible by [11, Remark 9.5, page 46] saying that $H^s((t_0, t_1), H)$, $0 < s < 1$, can be seen as an interpolated space between $L^2((t_0, t_1), H)$ and $H^1((t_0, t_1), H)$. Precisely, we have

$$H^s((t_0, t_1), H) = [L^2((t_0, t_1), H), H^1((t_0, t_1), H)]_{1-s}, \quad 0 < s < 1.$$

We readily obtain from Lemma 3.1 the following corollary.

Corollary 3.1. *Let H be a Hilbert space and $s \in (0, 1/2)$. There exists a constant $c > 0$ so that, for any $u \in H^1((t_0, t_1), H)$, we have*

$$\left\| \frac{u}{\delta^s} \right\|_{L^2((t_0, t_1), H)} \leq c \|u\|_{H^1((t_0, t_1), H)},$$

where δ is as in Lemma 3.1.

Next, we prove

Proposition 3.1. *Let $s \in (0, 1/2)$. There exist $\omega \Subset \Omega$, only depending on Ω , and three constants $c > 0$, $C > 0$ and $\sigma_0 > 0$, only depending on Ω , κ , T_0 , s and α , so that, for any $u \in \mathcal{X}(Q)$ satisfying $Lu = 0$ in Q and $0 < \sigma < \sigma_0$, we have*

$$(3.2) \quad \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))} \leq C \left(\sigma^{\min(\alpha, s)/2} \|u\|_{\mathcal{X}(Q)} + e^{e/\sqrt{\sigma}} \|u\|_{L^2((t_0, t_1), H^1(\omega))} \right).$$

Proof. Since Ω is Lipschitz, it has the uniform interior cone property (see for instance [9]). That is there exist $R > 0$ and $\theta \in]0, \frac{\pi}{2}[$ so that, for any $\tilde{x} \in \Gamma$, we may find $\xi = \xi(\tilde{x}) \in \mathbb{S}^{n-1}$ for which

$$\mathcal{C}(\tilde{x}) = \{x \in \mathbb{R}^n; |x - \tilde{x}| < R, (x - \tilde{x}) \cdot \xi > |x - \tilde{x}| \cos \theta\} \subset \Omega.$$

Fix $\tilde{x} \in \Gamma$ and let $\xi = \xi(\tilde{x})$ be as in the definition above.

Let $0 < \epsilon \leq \epsilon_0 < \min((3R/(2 \sin \theta))^2, (t_1 - t_0)/4)$, set $y_0 = y_0(\tilde{x}) = \tilde{x} + (R/2)\xi$ and $\rho = \sqrt{\epsilon} \sin \theta/3$. Let $N = [R/(2\rho)]$, the integer part of $R/(2\rho)$, if $R/(2\rho) \notin \mathbb{N}$ and $N = R/(2\rho) - 1$ if $R/(2\rho) \in \mathbb{N}$. Define then

$$x_0 = \tilde{x} + (R/2 - N\rho)\xi.$$

Furthermore, consider the sequence

$$y_j = \tilde{x} + (R/2 - j\rho)\xi, \quad 0 \leq j \leq N.$$

By construction, $B(y_j, 3\rho) \subset \mathcal{C}(\tilde{x})$, $0 \leq j \leq N$ and, as $|y_{j+1} - y_j| = \rho$, we have

$$B(y_{j+1}, \rho) \subset B(y_j, 2\rho), \quad 0 \leq j \leq N-1.$$

Let $u \in \mathcal{X}(Q)$. We use in the sequel the temporary notation

$$M = M(u) = \|u\|_{\mathcal{X}(Q)}.$$

Set $I_j = (t_0^j, t_1^j)$, where $t_i^j = t_i + (-1)^i j\epsilon$, with $i = 0, 1$ and $0 \leq j \leq N$. Note that

$$N\epsilon \leq (R/(2\rho))\epsilon = (3R/(2 \sin \theta))\sqrt{\epsilon}.$$

Then $I_N \neq \emptyset$ if $(3R/\sin \theta)\sqrt{\epsilon} < t_1 - t_0$. This condition always holds provided that we substitute ϵ_0 by $\min(\epsilon_0, (t_1 - t_0)^2 \sin^2 \theta / (9R^2))$.

In the rest of this proof C is a generic constant only depending on Ω , κ , α , s and T_0 .

Using that $I_{j+1} = (t_0^j + \epsilon, t_1^j - \epsilon)$ and noting that $\rho < \sqrt{\epsilon}$, we get from (2.1)

$$\begin{aligned} & \rho^3 \|u\|_{L^2(I_{j+1}, H^1(B(y_j, \rho)))} \\ & \leq CM^{1-\vartheta} \|u\|_{L^2(I_j, H^1(B(x_j, \rho)))}, \quad 0 \leq j \leq N-1. \end{aligned}$$

the constant ϑ , only depending on Ω , κ and T_0 , satisfies $0 < \vartheta < 1$.

Whence

$$\|u\|_{L^2(I_N, H^1(B(y_N, \rho)))} \leq (C\rho^{-3})^{(1-\beta)/(1-\vartheta)} M^{1-\beta} \|u\|_{L^2(I_0, H^1(B(y_0, \rho)))},$$

with $\beta = \vartheta^{N+1}$.

In this inequality, modifying C if necessary, we may assume that $C\rho^{-3} \geq 1$. Thus

$$\|u\|_{L^2(I_N, H^1(B(y_N, \rho)))} \leq C\rho^{-3/(1-\vartheta)} M^{1-\beta} \|u\|_{L^2(I_0, H^1(B(y_0, \rho)))}^\beta,$$

Let $J = I_N$. Since $B(y_0, \rho) \subset B(y_0, R \sin \theta/6) \subset \mathcal{C}(\tilde{x})$ and $y_N = x_0$, the last inequality entails

$$(3.3) \quad \|u\|_{L^2(J, H^1(B(x_0, \rho)))} \leq C\rho^{-3/(1-\vartheta)} M^{1-\beta} \|u\|_{L^2(I_0, H^1(B(y_0, R \sin \theta/6)))}^\beta.$$

Define

$$\omega = \bigcup_{\tilde{y} \in \Gamma} B(y_0(\tilde{y}), R \sin \theta/6).$$

It is worth mentioning that ω only depends on Ω .

We get from (3.27)

$$(3.4) \quad \|u\|_{L^2(J, H^1(B(x_0, \rho)))} \leq C\rho^{-3/(1-\vartheta)} M^{1-\beta} \|u\|_{L^2(I_0, H^1(\omega))}^\beta.$$

Now, since u is Hölder continuous, we have

$$|u(\tilde{x}, t)| \leq [u]_\alpha |\tilde{x} - x|^\alpha + |u(x, t)|, \quad x \in B(x_0, \rho), \quad t \in J.$$

Here and henceforth

$$[w]_\alpha = \sup_{\substack{(x_1, t_1), (x_2, t_2) \in \bar{Q} \\ (x_1, t_1) \neq (x_2, t_2)}} \frac{|w(x_1, t_1) - w(x_2, t_2)|}{|x_1 - x_2|^\alpha + |t_1 - t_2|^{\alpha/2}}, \quad w \in C^{\alpha, \alpha/2}(\bar{Q}).$$

Whence

$$(3.5) \quad |\mathbb{S}^{n-1}| \rho^n \int_J |u(\tilde{x}, t)|^2 dt \leq 2n[u]_\alpha^2 \int_{B(x_0, \rho) \times J} |\tilde{x} - x|^{2\alpha} dx dt \\ + 2n \int_{B(x_0, \rho) \times J} |u(x, t)|^2 dx dt,$$

Similarly, where $1 \leq i \leq n$,

$$(3.6) \quad |\mathbb{S}^{n-1}| \rho^n \int_J |\partial_i u(\tilde{x}, t)|^2 dt \leq 2n[\partial_i u]_\alpha^2 \int_{B(x_0, \rho) \times J} |\tilde{x} - x|^{2\alpha} dx dt \\ + 2n \int_{B(x_0, \rho) \times J} |\partial_i u(x, t)|^2 dx dt.$$

We have

$$(3.7) \quad |\tilde{x} - x| \leq |\tilde{x} - x_0| + |x_0 - x| \leq \rho + (R/2 - N\rho) \leq 2\rho.$$

Therefore, we have as a consequence of a combination of (3.5), (3.6) and (3.7)

$$\int_J |u(\tilde{x}, t)|^2 dt + \int_J |\nabla u(\tilde{x}, t)|^2 dt \leq C (M^2 \rho^{2\alpha} + \rho^{-n} \|u\|_{L^2(J, H^1(B(x_0, \rho)))})$$

which, in light of (3.28), yields

$$\int_J |u(\tilde{x}, t)|^2 dt + \int_J |\nabla u(\tilde{x}, t)|^2 dt \leq \\ C \left(M^2 \rho^{2\alpha} + \rho^{-n-6/(1-\vartheta)} M^{2(1-\beta)} \|u\|_{L^2(I_0, H^1(\omega))}^{2\beta} \right).$$

Integrating over Γ both sides of this inequality with respect to \tilde{x} , we find

$$(3.8) \quad \begin{aligned} & \|u\|_{L^2(\Gamma \times J)} + \|\nabla u\|_{L^2(\Gamma \times J)} \leq \\ & C \left(M\rho^\alpha + \rho_0^{-n/2-3/(1-\vartheta)} M^{1-\beta} \|u\|_{L^2(I_0, H^1(\omega))}^\beta \right). \end{aligned}$$

Bearing in mind that $J = (t_0 + N\epsilon, t_1 + N\epsilon)$, we get by applying Corollary 3.1, for some fixed $s \in (0, 1/2)$,

$$\begin{aligned} & \|u\|_{L^2((t_0, t_0+N\epsilon), L^2(\Gamma))}, \|u\|_{L^2((t_1-N\epsilon, t_1), L^2(\Gamma))} \leq \\ & \quad c(N\epsilon)^s \|u\|_{H^1((t_0, t_1), L^2(\Gamma))}, \\ & \|\nabla u\|_{L^2((t_0, t_0+N\epsilon), L^2(\Gamma))}, \|\nabla u\|_{L^2((t_1-N\epsilon, t_1), L^2(\Gamma))} \leq \\ & \quad c(N\epsilon)^s \|\nabla u\|_{H^1((t_0, t_1), L^2(\Gamma))}. \end{aligned}$$

Therefore, as the trace operator

$$u \in H^1((t_0, t_1), H^2(\Omega)) \rightarrow (u, \nabla u)|_\Sigma \in H^1((t_0, t_1), L^2(\Gamma))^{n+1},$$

is bounded, we obtain

$$\begin{aligned} & \|u\|_{L^2((t_0, t_0+N\epsilon), L^2(\Gamma))}, \|u\|_{L^2((t_1-N\epsilon, t_1), L^2(\Gamma))} \leq \\ & \quad c(N\epsilon)^s \|u\|_{H^1((t_0, t_1), H^2(\Omega))}, \\ & \|\nabla u\|_{L^2((t_0, t_0+N\epsilon), L^2(\Gamma))}, \|\nabla u\|_{L^2((t_1-N\epsilon, t_1), L^2(\Gamma))} \leq \\ & \quad c(N\epsilon)^s \|\nabla u\|_{H^1((t_0, t_1), H^2(\Omega))}. \end{aligned}$$

These inequalities together with (3.8) give

$$\begin{aligned} C \left(\|u\|_{L^2(\Sigma)} + \|\nabla u\|_{L^2(\Sigma)} \right) & \leq (\rho^\alpha + (N\epsilon)^s)M \\ & + \rho^{-n/2-3/(1-\vartheta)} M^{1-\beta} \|u\|_{L^2(I_0, H^1(\omega))}^\beta. \end{aligned}$$

We obtain by applying Young's inequality to the last term

$$(3.9) \quad \begin{aligned} C \left(\|u\|_{L^2(\Sigma)} + \|\nabla u\|_{L^2(\Sigma)} \right) & \leq (\rho^\alpha + (N\epsilon)^s + \epsilon^{\alpha/2})M \\ & + \rho^{-(n/2+3/(1-\vartheta))/\beta} \epsilon^{-(1-\beta)\alpha/(2\beta)} \|u\|_{L^2(I_0, H^1(\omega))}. \end{aligned}$$

Next, we have

$$(3.10) \quad \rho^\alpha + (N\epsilon)^s + \epsilon^{\alpha/2} \leq C\epsilon^{\min(\alpha, s)/2}$$

and, as $\beta = \theta^{N+1}$, we have $\beta = O(e^{-c/\sqrt{\epsilon}})$, from which we deduce in a straightforward manner that

$$(3.11) \quad \rho^{-(n/2+3/(1-\vartheta))/\beta} \epsilon^{-(1-\beta)\alpha/(2\beta)} \leq C e^{e^{c\sqrt{\epsilon}}}.$$

We end up by observing that (3.10) and (3.11) in (3.9) give the expected inequality. \square

The a priori estimate in the following lemma is well adapted to our purpose. It does not involve neither the initial time data nor the final time data.

Lemma 3.2. *There exists a constant $C > 0$, only depending on Ω , κ and T_0 , so that, for any $u \in H^1((t_0, t_1), H^2(\Omega))$ satisfying $Lu = 0$ in Q , we have*

$$(3.12) \quad C \|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq \|u\|_{H^1((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))}.$$

Proof. In this proof C is a generic constant that can only depend on Ω , κ and T_0 .

Let $u \in H^1((t_0, t_1), H^2(\Omega))$ satisfying $Lu = 0$ in Q and set $v = e^{-t}u$. Then v solves the following equation

$$(3.13) \quad \operatorname{div}(A\nabla v) - v - \partial_t v = 0 \quad \text{in } Q.$$

Let $0 < \epsilon < (t_1 - t_0)/2$ and choose $\chi \in C_0^\infty((t_0, t_1))$ satisfying $0 \leq \chi \leq 1$, $\chi = 1$ in $(t_0 + \epsilon, t_1 - \epsilon)$ and, for some universal constant c , $|\chi'| \leq c/\epsilon$.

We multiply (3.13) by χv and integrate over Q . We then get by making an integration by parts

$$- \int_Q \chi A \nabla v \cdot \nabla v dx dt - \int_Q \chi v^2 dx dt + \int_\Sigma \chi v A \nabla v \cdot \nu d\sigma dt + \frac{1}{2} \int_Q v^2 \chi' dx dt = 0,$$

from which we deduce in a straightforward manner

$$(3.14) \quad \int_Q \chi A \nabla u \cdot \nabla u dx dt + \int_Q \chi u^2 dx dt \leq e^{2(t_1 - t_0)} \left(C \int_\Sigma (u^2 + |\nabla u|^2) d\sigma dx + \frac{1}{2} \int_Q u^2 |\chi'| dx dt \right).$$

On the other hand, as $\operatorname{supp}(\chi') \subset (t_0, t_1) \setminus (t_0 + \epsilon, t_1 - \epsilon)$, we have

$$\mathcal{J}_\epsilon^2 = \int_Q u^2 |\chi'| dx dt \leq \frac{c}{\epsilon} \int_{t_0}^{t_0 - \epsilon} \int_\Omega u^2 dx dt + \frac{c}{\epsilon} \int_{t_1 - \epsilon}^{t_1} \int_\Omega u^2 dx dt.$$

Therefore

$$(3.15) \quad \limsup_{\epsilon \rightarrow 0} \mathcal{J}_\epsilon^2 \leq \int_\Omega u^2(x, t_0) dx + \int_\Omega u^2(x, t_1) dx.$$

We rewrite (3.14) in the form

$$(3.16) \quad C \|u\|_{L^2((t_0 + \epsilon, t_1 - \epsilon), H^1(\Omega))} \leq \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \mathcal{J}_\epsilon.$$

We apply Hardy's inequality in Corollary 3.1. We obtain

$$\|u\|_{L^2((t_0, t_0 + \epsilon), H^1(\Omega))}, \|u\|_{L^2((t_1 - \epsilon, t_1), H^1(\Omega))} \leq C \epsilon^s \|u\|_{H^1((t_0, t_1), H^1(\Omega))}.$$

This and (3.16) produce

$$C \|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \epsilon^s \|u\|_{H^1((t_0, t_1), H^1(\Omega))} + \mathcal{J}_\epsilon.$$

Making $\epsilon \rightarrow 0$, we get by using (3.15)

$$C \|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|u(\cdot, t_0)\|_{L^2(\Omega)} + \|u(\cdot, t_1)\|_{L^2(\Omega)}.$$

We complete the proof by using the following inequality

$$C (\|u(\cdot, t_0)\|_{L^2(\Omega)} + \|u(\cdot, t_1)\|_{L^2(\Omega)}) \leq \|u\|_{H^1((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))}.$$

To prove this inequality we proceed similarly to the proof of observability inequalities for parabolic equation. First, if $s_0 = (3t_0 + t_1)/4$ and $s_1 = (t_0 + 3t_1)/4$, we get as a straightforward consequence of the Carleman inequality in Theorem 2.3,

$$(3.17) \quad C \|u\|_{L^2((s_0, s_1) \times \Omega)} \leq \|u\|_{H^1((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))}.$$

Next pick $\psi \in C^\infty([t_0, t_1])$ so that $0 \leq \psi \leq 1$, $\psi = 0$ in $[t_0, s_0]$ and $\psi = 1$ in $[s_1, t_1]$. Then $v = \psi u$ is the solution of the IBVP

$$\begin{cases} -\operatorname{div}(A\nabla v) + \partial_t v = \psi' u & \text{in } Q, \\ v = u & \text{on } \Sigma, \\ v(\cdot, t_0) = 0. \end{cases}$$

Hence

$$\int_Q A\nabla v \cdot \nabla v dx dt + \int_\Sigma v(A\nabla v \cdot \nu) + \frac{1}{2} \int_Q \partial_t v^2 dx dt = \int_Q \psi' u v dx dt.$$

But

$$\int_Q \partial_t v^2 dx dt = \int_\Omega v^2(x, t_1) dx.$$

That is we have

$$\int_Q A\nabla v \cdot \nabla v dx dt + \int_\Sigma v A\nabla v \cdot \nu + \frac{1}{2} \int_\Omega v^2(x, t_1) dx = \int_Q \psi' u v dx dt.$$

We deduce from this identity

$$(3.18) \quad C \left(\|v(\cdot, t_1)\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(Q)}^2 \right) \leq \|u\|_{L^2((t_0, t_1), L^2(\Gamma))}^2 + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))}^2 + \|\psi' u\|_{L^2(Q)} \|v\|_{L^2(Q)}.$$

Noting that

$$w \rightarrow \left(\int_\Omega |\nabla w|^2 dx + \int_\Gamma w^2(x) d\sigma(x) \right)^{\frac{1}{2}}$$

defines an equivalent norm on $H^1(\Omega)$, we get

$$\|v\|_{L^2(Q)}^2 \leq c_\Omega \left(\|\nabla v\|_{L^2(Q)}^2 + \|u\|_{L^2((t_0, t_1), L^2(\Gamma))}^2 \right).$$

We obtain then from Young's inequality

$$\|\psi' u\|_{L^2(Q)} \|v\|_{L^2(Q)} \leq \frac{1}{2\epsilon} \|\psi' u\|_{L^2(Q)}^2 + \frac{c_\Omega \epsilon}{2} \|\nabla v\|_{L^2(Q)}^2 + \frac{c_\Omega \epsilon}{2} \|u\|_{L^2((t_0, t_1), L^2(\Gamma))}^2.$$

This inequality in (3.18), with ϵ sufficiently small, yields

$$(3.19) \quad C \|u(\cdot, t_1)\|_{L^2(\Omega)} = C \|v(\cdot, t_1)\|_{L^2(\Omega)} \leq \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\psi' u\|_{L^2(Q)}.$$

Bearing in mind that $\operatorname{supp}(\psi') \subset [s_0, s_1]$, we deduce from (3.17)

$$\begin{aligned} \|\psi' u\|_{L^2(Q)} &\leq C \|u\|_{L^2(\Omega \times (s_0, s_1))} \\ &\leq \|u\|_{H^1((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))}. \end{aligned}$$

This in (3.19) yields

$$C \|u(\cdot, t_1)\|_{L^2(\Omega)} \leq \|u\|_{H^1((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))}.$$

As the Carleman estimate in Theorem 2.3 still holds for the backward parabolic equation $\operatorname{div}(A\nabla u) + \partial_t u = 0$, we have similarly

$$C \|u(\cdot, t_0)\|_{L^2(\Omega)} \leq \|u\|_{H^1((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))}.$$

The proof is then complete. \square

If $u \in \mathcal{Y}(Q)$ satisfies $Lu = 0$ in Q then $\partial_t u \in \mathcal{X}(Q)$ and $L\partial_t u = 0$ in Q . Proposition 3.1 applied to both u and $\partial_t u$ together with Lemma 3.2 produce the following result.

Corollary 3.2. *Let $s \in (0, 1/2)$. There exist $\omega \Subset \Omega$, only depending on Ω , and three constants $c > 0$, $C > 0$ and $\sigma_0 > 0$, only depending on Ω , κ , T_0 , s and α , so that, for any $u \in \mathcal{Y}(Q)$ satisfying $Lu = 0$ in Q and $0 < \sigma < \sigma_0$, we have*

$$(3.20) \quad \|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq C \left(\sigma^{\min(\alpha, s)/2} \|u\|_{\mathcal{Y}(Q)} + e^{e^{c/\sqrt{\sigma}}} \|u\|_{H^1((t_0, t_1), H^1(\omega))} \right).$$

We now quantify the uniqueness of continuation from an interior subdomain to an another interior subdomain. Prior to that, we define the geometric distance d_g^D on a bounded domain D of \mathbb{R}^n by

$$d_g^D(x, y) = \inf \{ \ell(\psi); \psi : [0, 1] \rightarrow D \text{ Lipschitz path joining } x \text{ to } y \},$$

where

$$\ell(\psi) = \int_0^1 |\dot{\psi}(t)| dt$$

is the length of ψ .

Observe that, according to Rademacher's theorem, any Lipschitz continuous function $\psi : [0, 1] \rightarrow D$ is almost everywhere differentiable with $|\dot{\psi}(t)| \leq k$ a.e. $t \in [0, 1]$, where k is the Lipschitz constant of ψ . In particular, $\ell(\psi)$ is well defined.

The following lemma will be used to prove the next proposition. We provide its proof in Appendix A.

Lemma 3.3. *Let D be a bounded Lipschitz domain of \mathbb{R}^n . Then $d_g^D \in L^\infty(D \times D)$.*

Proposition 3.2. *Let $\omega \Subset \Omega$, $\tilde{\omega} \Subset \Omega$ and $s \in (0, 1/2)$. There exist three constants $\gamma > 0$, $C > 0$ and $\epsilon_0 > 0$, only depending on Ω , κ , T_0 , s , ω and $\tilde{\omega}$, so that, for any $u \in H^2((t_0, t_1), H^1(\Omega))$ satisfying $Lu = 0$ in Q and $0 < \epsilon < \epsilon_0$, we have*

$$(3.21) \quad C \|u\|_{H^1((t_0, t_1), H^1(\tilde{\omega}))} \leq \epsilon^s \|u\|_{H^2((t_0, t_1), H^1(\Omega))} + e^{\gamma/\epsilon} \|u\|_{H^1((t_0, t_1), H^1(\omega))}.$$

Proof. Pick Ω_0 a Lipschitz domain so that $\Omega_0 \Subset \Omega$, $\omega \Subset \Omega_0$ and $\tilde{\omega} \Subset \Omega_0$. Set then $d_0 = \text{dist}(\overline{\Omega_0}, \Gamma)$. Fix $0 < \epsilon < \epsilon_0 := \min(d_0^2/9, 1)$ and let $0 < \delta < \sqrt{\epsilon}$. Let $x_0 \in \omega$, $x \in \tilde{\omega}$ and let $\psi : [0, 1] \rightarrow \Omega_0$ be a Lipschitz path joining x_0 to x so that $\ell(\psi) \leq d_g^{\Omega_0}(x_0, x) + 1$. For simplicity's sake, we use in this proof the notation

$$\mathbf{d} = \|d_g^{\Omega_0}\|_{L^\infty(\Omega_0 \times \Omega_0)}.$$

Let $\tau_0 = 0$ and $\tau_{k+1} = \inf\{\tau \in [\tau_k, 1]; \psi(\tau) \notin B(\psi(\tau_k), \delta)\}$, $k \geq 0$. We claim that there exists an integer $N \geq 1$ so that $\psi(1) \in B(\psi(\tau_N), \delta/2)$. If not, we would have $\psi(1) \notin B(\psi(\tau_k), \delta/2)$, for any $k \geq 0$. As the sequence (τ_k) is non decreasing and bounded from above by 1, it converges to $\hat{\tau} \leq 1$. In particular, there exists an integer $k_0 \geq 1$ so that $\psi(\tau_k) \in B(\psi(\hat{\tau}), \delta/2)$, $k \geq k_0$. But this contradicts the fact that $|\psi(\tau_{k+1}) - \psi(\tau_k)| = \delta$, for any $k \geq 0$.

Let us check that $N \leq N_0$, where N_0 depends only on \mathbf{d} and δ . Pick $1 \leq j \leq n$ so that

$$\max_{1 \leq i \leq n} |\psi_i(\tau_{k+1}) - \psi_i(\tau_k)| = |\psi_j(\tau_{k+1}) - \psi_j(\tau_k)|.$$

Then

$$\delta \leq n |\psi_j(\tau_{k+1}) - \psi_j(\tau_k)| = n \left| \int_{\tau_k}^{\tau_{k+1}} \dot{\psi}_j(t) dt \right| \leq n \int_{\tau_k}^{\tau_{k+1}} |\dot{\psi}(t)| dt.$$

In consequence, where $\tau_{N+1} = 1$,

$$(N+1)\delta \leq n \sum_{k=0}^N \int_{\tau_k}^{\tau_{k+1}} |\dot{\psi}(\tau)| d\tau = n\ell(\psi) \leq n(\mathbf{d}+1).$$

Therefore

$$N \leq N_0 = \left[\frac{n(\mathbf{d}+1)}{\delta} \right].$$

Here $[n(\mathbf{d}+1)/\delta]$ is the integer part of $n(\mathbf{d}+1)/\delta$.

Let $x_k = \psi(t_k)$, $0 \leq k \leq N$. If $|z - x_{k+1}| < \delta$ then

$$|z - x_k| \leq |z - x_{k+1}| + |x_{k+1} - x_k| < 2\delta.$$

In other words, $B(x_{k+1}, \delta) \subset B(x_k, 2\delta)$.

Remark that we have also $B(x_k, 3\delta) \subset \Omega$, for each k .

Define

$$I_k = (t_0 + k\epsilon, t_1 - k\epsilon) \quad k \geq 0.$$

If $t_0^k = t_0 + k\epsilon$ and $t_1^k = t_1 - k\epsilon$ then $I_k = (t_0^k, t_1^k)$ and $I_{k+1} = (t_0^k + \epsilon, t_1^k - \epsilon)$.

Let $u \in H^2((t_0, t_1), H^1(\Omega))$ satisfying $Lu = 0$ in Q . In this proof we use the following temporary notation

$$M = \|u\|_{H^1((t_0, t_1), H^1(\Omega))}.$$

Taking into account that $\delta < \sqrt{\epsilon}$, we have from the three-cylinder inequality (2.1)

$$\|u\|_{L^2(I_{k+1}, H^1(B(x_{k+1}, \delta)))} \leq C_0 \delta^{-3} M^{1-\vartheta} \|u\|_{L^2(I_k, H^1(B(x_k, \delta)))},$$

the constants C_0 and ϑ , $0 < \vartheta < 1$, only depend on Ω , κ and T_0 .

Set $\Lambda_k = \|u\|_{L^2(I_k, H^1(B(x_k, \delta)))}$, $0 \leq k \leq N$, and $\Lambda_{N+1} = \|u\|_{L^2(I_{N+1}, H^1(B(x, \delta/2)))}$. We can then rewrite this inequality in the form

$$(3.22) \quad \Lambda_{k+1} \leq C_0 \delta^{-3} M^{1-\vartheta} \Lambda_k^\vartheta.$$

Let $\beta = \vartheta^{N+1}$. We get in a straightforward manner from (3.22)

$$\Lambda_{N+1} \leq (C_0 \delta^{-3})^{\frac{1-\vartheta^{N+1}}{1-\vartheta}} M^{1-\beta} \Lambda_0^\beta.$$

Substituting if necessary C_0 by $\max(C_0, 1)$, we may assume that $C_0 \geq 1$. Then the last inequality gives

$$\Lambda_{N+1} \leq C \delta^{\frac{-3}{1-\vartheta}} M^{1-\beta} \Lambda_0^\beta.$$

From here and until the end of the proof, C is a generic constant, depending only on Ω , κ , ω , $\tilde{\omega}$, s and T_0 .

Young's inequality then leads, for $\sigma > 0$,

$$\begin{aligned} \Lambda_{N+1} &\leq C \delta^{\frac{-3}{1-\vartheta}} ((1-\beta)\sigma^{\frac{\beta}{1-\beta}} M + \beta\sigma^{-1}\Lambda_0) \\ &\leq C \delta^{\frac{-3}{1-\vartheta}} (\sigma^{\frac{\beta}{1-\beta}} M + \sigma^{-1}\Lambda_0). \end{aligned}$$

If δ is sufficiently small $B(x_0, \delta) \subset \omega$. On the other hand, $\tilde{\omega}$ can be recovered by $O(\delta^{-n})$ balls of radius $\frac{\delta}{2}$. Whence, bearing in mind that

$$I_{N+1} \supset J = (t_0 + (N_0 + 1)\epsilon, t_1 - (N_0 + 1)\epsilon),$$

we have

$$(3.23) \quad \|u\|_{L^2(J, H^1(\tilde{\omega}))} \leq \delta^{-\frac{3}{1-\vartheta} - n} \left(\sigma^{\frac{\beta}{1-\beta}} M + \sigma^{-1} \|u\|_{L^2((t_0, t_1), H^1(\omega))} \right).$$

Then we take σ in (3.23) in order to satisfy

$$\delta^{-\frac{3}{1-\vartheta}-n} \sigma^{\frac{\beta}{1-\beta}} = \delta.$$

In that case

$$\sigma^{-1} \leq \delta^{-\frac{m}{\beta}},$$

with $m = (4 - \vartheta)/(1 - \vartheta) + n$.

But

$$\beta = \vartheta^{N+1} = e^{-(N+1)|\ln \vartheta|} \geq \vartheta e^{-N_0 |\ln \vartheta|} \geq \vartheta e^{-\frac{n |\ln \vartheta| (\mathbf{d}+1)}{\delta}}.$$

Hence

$$\delta^{-\frac{3}{1-\vartheta}-n} \sigma^{-1} \leq e^{\gamma/\delta}.$$

This inequality in (3.23) yields

$$(3.24) \quad C \|u\|_{L^2(J, H^1(\bar{\omega}))} \leq \delta M + e^{\gamma/\delta} \|u\|_{L^2((t_0, t_1), H^1(\omega))}.$$

Let $\aleph = 2n(\mathbf{d}+1) + \sqrt{\epsilon_0}$. Substituting ϵ_0 by $\min(\epsilon_0, (t_1 - t_0)/(3\aleph))$, we may assume that $J_0 = (t_0 - \aleph\epsilon, t_1 + \aleph\epsilon) \neq \emptyset$. Taking $\delta = \sqrt{\epsilon}/2$, we get in straightforward manner that $J \supset J_0$. Hence (3.24) yields

$$(3.25) \quad C \|u\|_{L^2(J_0, H^1(\bar{\omega}))} \leq \sqrt{\epsilon} M + e^{2\gamma/\sqrt{\epsilon}} \|u\|_{L^2((t_0, t_1), H^1(\omega))}.$$

We get by applying again Hardy's inequality in Lemma 3.1, for some fixed $s \in (0, 1/2)$,

$$(3.26) \quad \|u\|_{L^2((t_0, t_0 + \aleph\epsilon), H^1(\bar{\omega}))}, \|u\|_{L^2((t_1 - \aleph\epsilon, t_1), H^1(\bar{\omega}))} \leq c \aleph^s \epsilon^s M.$$

Then (3.25) and (3.26) entail

$$C \|u\|_{L^2((t_0, t_1), H^1(\bar{\omega}))} \leq \epsilon^{s/2} M + e^{2\gamma/\sqrt{\epsilon}} \|u\|_{L^2((t_0, t_1), H^1(\omega))}.$$

Substituting ϵ by ϵ^2 and 2γ by γ , we obtain

$$(3.27) \quad C \|u\|_{L^2((t_0, t_1), H^1(\bar{\omega}))} \leq \epsilon^s \|u\|_{H^1((t_0, t_1), H^1(\Omega))} + e^{\gamma/\epsilon} \|u\|_{L^2((t_0, t_1), H^1(\omega))}.$$

As $\partial_t u \in H^1((t_0, t_1), H^1(\Omega))$ satisfies $L\partial_t u = 0$ in Q , (3.27) is applicable with u substituted by $\partial_t u$. That is we have

$$(3.28) \quad C \|\partial_t u\|_{L^2((t_0, t_1), H^1(\bar{\omega}))} \leq \epsilon^s \|\partial_t u\|_{H^1((t_0, t_1), H^1(\Omega))} + e^{\gamma/\epsilon} \|\partial_t u\|_{L^2((t_0, t_1), H^1(\omega))}.$$

Putting together (3.27) and (3.28) to obtain the expected inequality. \square

We are now ready to prove the result quantifying the uniqueness of continuation from an interior data. Prior to do that, we need to introduce a definition. Set $\varrho_* = e^{-e}$ and, for $\mu > 0$ and $\varrho_0 \leq \varrho_*$,

$$\Phi_{\varrho_0, \mu}(\varrho) = \begin{cases} 0 & \text{if } \varrho = 0, \\ (\ln \ln |\ln \varrho|)^{-\mu} & \text{if } 0 < \varrho \leq \varrho_0, \\ \varrho & \text{if } \varrho \geq \varrho_0. \end{cases}$$

Theorem 3.1. *Let $\omega \Subset \Omega$ and $s \in (0, 1/2)$. There exist two constants $C > 0$ and $0 < \varrho_0 \leq \varrho_*$, only depending on $\Omega, \kappa, \omega, \alpha, s$ and T_0 , so that, for any $u \in \mathcal{Y}(Q)$ satisfying $Lu = 0$ in Q , we have*

$$C \|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq \|u\|_{\mathcal{Y}(Q)} \Phi_{\varrho_0, \mu} \left(\frac{\mathcal{I}(u, \omega)}{\|u\|_{\mathcal{Y}(Q)}} \right).$$

Here $\mu = \min(s, \alpha)/4$ and $\mathcal{I}(u, \omega) = \|u\|_{H^1((t_0, t_1), H^1(\omega))}$.

Proof. From Corollary 3.2, there exist $\tilde{\omega} \Subset \Omega$ and three constants $c > 0$, $C > 0$ and $\sigma_0 > 0$ so that, for any $u \in \mathcal{Y}(Q)$ satisfying $Lu = 0$ in Q and $0 < \sigma < \sigma_0$, we have

$$C\|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq \sigma^\mu \|u\|_{\mathcal{Y}(Q)} + e^{e^{c/\sqrt{\sigma}}} \|u\|_{H^1((t_0, t_1), H^1(\tilde{\omega}))}.$$

Here $\mu = \min(s, \alpha)/2$.

But according to Proposition 3.2, there exist three constants $\gamma > 0$, $C > 0$ and $\epsilon_0 > 0$ so that, for any $u \in \mathcal{X}^1(Q)$ satisfying $Lu = 0$ in Q and $0 < \epsilon < \epsilon_0$, we have

$$C\|u\|_{H^1((t_0, t_1), H^1(\tilde{\omega}))} \leq \epsilon^s \|u\|_{\mathcal{Y}(Q)} + e^{\gamma/\epsilon} \|u\|_{H^1((t_0, t_1), H^1(\omega))}.$$

The last two inequalities yield

$$(3.29) \quad C\|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq (\sigma^\mu + \epsilon^s e^{e^{c/\sqrt{\sigma}}}) \|u\|_{\mathcal{Y}(Q)} + e^{\gamma/\epsilon} e^{e^{c/\sqrt{\sigma}}} \|u\|_{H^1((0, T), H^1(\omega))},$$

for any $u \in \mathcal{Y}(Q)$ satisfying $Lu = 0$ in Q , $0 < \sigma < \sigma_0$ and $0 < \epsilon < \epsilon_0$.

We assume, by reducing σ_0 if needed, that $\sigma_0^\mu e^{-e^{c/\sqrt{\sigma_0}}} < \epsilon_0^s$. We get, by taking ϵ so that $\epsilon^s = \sigma^\mu e^{-e^{c/\sqrt{\sigma}}}$,

$$(3.30) \quad C\|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq \sigma^\mu \|u\|_{\mathcal{Y}(Q)} + e^{e^{e^{c/\sqrt{\sigma}}}} \|u\|_{H^1((t_0, t_1), H^1(\omega))},$$

for any $u \in \mathcal{Y}(Q)$ satisfying $Lu = 0$ in Q , and $0 < \sigma < \sigma_0$.

Fix $u \in \mathcal{Y}(Q)$, non identically equal to zero, satisfying $Lu = 0$ in Q . Let

$$M = \frac{\|u\|_{L^2((t_0, t_1), H^1(\Omega))}}{\|u\|_{\mathcal{Y}(Q)}} \quad \text{and} \quad N = \frac{\|u\|_{H^1((t_0, t_1), H^1(\omega))}}{\|u\|_{\mathcal{Y}(Q)}}.$$

Then (3.30) can be rewritten as

$$(3.31) \quad CM \leq \sigma^\mu + e^{e^{e^{c/\sqrt{\sigma}}}} N, \quad 0 < \sigma < \sigma_0.$$

Define the function ℓ by $\ell(\sigma) = \sigma^\mu e^{-e^{e^{c/\sqrt{\sigma}}}}$. If $N < \min(\ell(\sigma_0), \varrho_*) = \varrho_0$ then there exists $\bar{\sigma}$ so that $\ell(\bar{\sigma}) = N$. Changing c if necessary, we have

$$\frac{1}{N} \leq e^{e^{e^{c/\sqrt{\bar{\sigma}}}}}.$$

Or equivalently

$$\bar{\sigma} \leq [\ln \ln |\ln N|]^{-1/2}$$

It follows readily by taking $\sigma = \bar{\sigma}$ in (3.31) that

$$(3.32) \quad CM \leq [\ln \ln |\ln N|]^{-\mu/2}.$$

If $N \geq \varrho_0$ then obviously we have

$$(3.33) \quad M \leq 1 \leq \frac{N}{\varrho_0}.$$

The expected inequality follows then from (3.32) and (3.33). \square

4. STABILITY OF PARABOLIC CAUCHY PROBLEMS

An additional step is necessary to prove our stability estimate for the Cauchy problem. It consists in quantifying the uniqueness of continuation from the Cauchy data to an interior subdomain.

Proposition 4.1. *Let $\nu \in (0, 1/2)$. There exist $\omega \Subset \Omega$, only depending on Ω and Γ_0 , and two constants $C > 0$ and $c > 0$, only depending on Ω , κ , Γ_0 , ν and T_0 , so that, for any $u \in H^2((t_0, t_1), H^2(\Omega))$ satisfying $Lu = 0$ in Q and $0 < \epsilon < (t_1 - t_0)/2$, we have*

$$(4.1) \quad C \|u\|_{H^1((t_0, t_1), L^2(\omega))} \leq \epsilon^\nu \|u\|_{H^2((t_0, t_1), H^1(\Omega))} \\ + e^{c/\epsilon^2} (\|u\|_{H^2((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{H^1((t_0, t_1), L^2(\Gamma_0))}).$$

Proof. Pick $0 < \epsilon < (t_1 - t_0)/2$, $0 < \eta < \epsilon$ and let $s \in [t_0 + \epsilon, t_1 - \epsilon]$. Let $\tilde{x} \in \Gamma_0$ be arbitrarily fixed and let $R > 0$ so that $B(\tilde{x}, R) \cap \Gamma \subset \Gamma_0$. Take x_0 in the interior of $\mathbb{R}^n \setminus \overline{\Omega}$ sufficiently close to \tilde{x} is such a way that $\rho = \text{dist}(x_0, K) < R$, where $K = \overline{B(\tilde{x}, R)} \cap \Gamma_0$ (think to the fact that Ω is on one side of its boundary). Fix then $r > 0$ in order to satisfy $B(x_0, \rho + r) \cap \Gamma \subset \Gamma_0$ and $B(x_0, \rho + \theta r) \cap \Omega \neq \emptyset$, for some $0 < \theta < 1$.

Let $\phi \in C_0^\infty(B(\tilde{x}, \rho + r))$ satisfying $\phi = 1$ on $B(\tilde{x}, \rho + (\theta + 1)r/2)$. Set, where $0 < \delta < 1$ is a constant to be specified in the sequel,

$$Q_0 = [B(x_0, \rho + r) \cap \Omega] \times (s - \eta, s + \eta), \\ Q_1 = [B(x_0, \rho + \theta r) \cap \Omega] \times (s - \delta\eta, s + \delta\eta), \\ Q_2 = \{[B(x_0, \rho + r) \setminus B(x_0, \rho + (\theta + 1)r/2)] \cap \Omega\} \times (s - \eta, s + \eta), \\ \Sigma_0 = [B(x_0, \rho + r) \cap \Gamma] \times (s - \eta, s + \eta).$$

We apply Theorem 2.3, with Q substituted by Q_0 , $\psi = (\rho + r)^2 - |x - x_0|^2$, $g(t) = 1/[(t - s + \eta)(s + \eta - t)]$ and λ fixed, to ϕu so that $u \in H^1((t_0, t_1), H^2(\Omega))$ satisfies $Lu = 0$ in Q in order to obtain

$$(4.2) \quad C \int_{Q_1} u^2 e^{-2\tau\varphi} dx dt \leq \int_{Q_0} (L(\phi u))^2 e^{-2\tau\varphi} dx dt \\ + \int_{\Sigma_0} (u^2 + |\nabla u|^2 + (\partial_t u)^2) e^{-2\tau\varphi} dx dt.$$

Here and henceforth, C is a generic constant that can only depend on Ω , κ , ν , Γ_0 and T_0 .

But

$$L(\phi u) = L\phi u + 2A\nabla\phi \cdot \nabla u.$$

Whence $\text{supp}(L(\phi u)) \cap Q_0 \subset Q_2$ together with (4.2) yield

$$(4.3) \quad C \int_{Q_1} u^2 e^{-2\tau\varphi} dx dt \leq \int_{Q_2} (u^2 + |\nabla u|^2) e^{-2\tau\varphi} dx dt \\ + \int_{\Sigma_0} (u^2 + |\nabla u|^2 + (\partial_t u)^2) e^{-2\tau\varphi} dx dt.$$

Let

$$\begin{aligned}\alpha &= \eta^{-2} \left[e^{4\lambda(\rho+r)^2} - e^{\lambda(2(\rho+r)^2 - (\rho+\theta r)^2)} \right] := \eta^{-2} \tilde{\alpha}, \\ \beta &= \eta^{-2} \left[e^{4\lambda(\rho+r)^2} - e^{2\lambda(\rho+r)^2} \right] := \eta^{-2} \tilde{\beta}, \\ \gamma &= \eta^{-2} \left[e^{4\lambda(\rho+r)^2} - e^{\lambda(2(\rho+r)^2 - (\rho+(\theta+1)r/2)^2)} \right] := \eta^{-2} \tilde{\gamma}.\end{aligned}$$

Then it is straightforward to check that

$$\begin{aligned}\varphi(x, t) &\leq \frac{\alpha}{1-\delta} \quad \text{in } Q_1, \\ \varphi(x, t) &\geq \beta \quad \text{in } \Sigma_0, \\ \varphi(x, t) &\geq \gamma \quad \text{in } Q_2.\end{aligned}$$

Noting that

$$\frac{\beta}{\alpha} = \frac{\tilde{\beta}}{\tilde{\alpha}} < 1 < \frac{\gamma}{\alpha} = \frac{\tilde{\gamma}}{\tilde{\alpha}},$$

we can choose $0 < \varkappa < 1$ so that

$$\frac{1}{1-\delta} := \varkappa \frac{\tilde{\beta}}{\tilde{\alpha}} + (1-\varkappa) \frac{\tilde{\gamma}}{\tilde{\alpha}} > 1.$$

With this choice of δ , (4.3) yields

$$(4.4) \quad \begin{aligned}C \int_{Q_1} u^2 dxdt &\leq e^{-4b\eta^{-2}\tau} \int_{Q_2} (u^2 + |\nabla u|^2) dxdt \\ &\quad + e^{4a\eta^{-2}\tau} \int_{\Sigma_0} (u^2 + |\nabla u|^2 + (\partial_t u)^2) dxdt.\end{aligned}$$

Here $a = (1-\varkappa)(\tilde{\gamma} - \tilde{\beta})/2$ and $b = \varkappa(\tilde{\gamma} - \tilde{\beta})/2$.

Let $\eta = \epsilon/2$, $s_0 = t_0 + \epsilon - \delta\epsilon/2$, $s_1 = s_0 + \delta\epsilon/2 \dots s_k = s_0 + k\delta\epsilon/2$. Let $K = K(\epsilon)$ so that

$$\bigcup_{k=0}^K (s_k - \delta\epsilon/2, s_k + \delta\epsilon/2) \supset [t_0 + \epsilon, t_1 - \epsilon].$$

If Q_j^k (resp. Σ_0^k) denotes Q_j (resp. Σ_0), $j = 1, 2$, when s is substituted by s_k , then it follows from (4.4)

$$\begin{aligned}C \sum_{k=0}^K \int_{Q_1^k} u^2 dxdt &\leq e^{a\epsilon^{-2}\tau} \sum_{k=0}^K \int_{Q_2^k} (u^2 + |\nabla u|^2) dxdt \\ &\quad + e^{-b\epsilon^{-2}\tau} \sum_{k=0}^K \int_{\Sigma_0^k} (u^2 + |\nabla u|^2 + (\partial_t u)^2) dxdt.\end{aligned}$$

Note that the intervals $(s_k - \frac{\epsilon}{2}, s_k + \frac{\epsilon}{2})$ overlap, but their union can cover at most two times a subdomain of (t_0, t_1) . Whence

$$(4.5) \quad CI \leq e^{a\epsilon^{-2}\tau} N + e^{-b\epsilon^{-2}\tau} M, \quad \tau \geq \tau_0,$$

where we used the temporary notations

$$\begin{aligned}I &= \|u\|_{L^2([B(x_0, \rho+\theta r)] \times (t_0+\epsilon, t_1-\epsilon))}, \\ M &= \|u\|_{L^2((t_0, t_1), H^1(\Omega))}, \\ N &= \|u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma_0))}.\end{aligned}$$

In (4.5), we get by substituting τ by $\epsilon^2\tau$

$$(4.6) \quad CI \leq e^{a\tau}M + e^{-b\tau}N, \quad \tau \geq \tau_0/\epsilon^2.$$

Set

$$\tau_1 = \frac{\ln(N/M)}{a+b}.$$

If $\tau_1 \geq \tau_0/\epsilon^2$ then $\tau = \tau_1$ in (4.6) yields

$$(4.7) \quad CI \leq M^\vartheta N^{1-\vartheta},$$

with $\vartheta = \frac{b}{a+b}$.

When $\tau_1 < \tau_0/\epsilon^2$, we have

$$M < e^{(a+b)/\epsilon^2\tau_0}N.$$

This inequality entails

$$(4.8) \quad I \leq M = M^\vartheta M^{1-\vartheta} \leq M^\vartheta e^{(1-\vartheta)(a+b)\tau_0/\epsilon^2} N^{1-\vartheta}.$$

So, in any case, one of estimates (4.7) and (4.8) holds. In other words, we proved

$$\begin{aligned} & e^{-c/\epsilon^2} \|u\|_{L^2([B(x_0, \rho + \theta r) \cap \Omega]) \times (t_0 + \epsilon, t_1 - \epsilon)} \\ & \leq \|u\|_{L^2((t_0, t_1), H^1(\Omega))} \left(\|u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma_0))} \right)^{1-\vartheta}. \end{aligned}$$

Fix $\omega \Subset B(x_0, \rho + \theta r) \cap \Omega$. Then the last inequality implies

$$\begin{aligned} & e^{-c/\epsilon^2} \|u\|_{L^2(\omega \times (t_0 + \epsilon, t_1 - \epsilon))} \\ & \leq \|u\|_{L^2((t_0, t_1), H^1(\Omega))} \left(\|u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma_0))} \right)^{1-\vartheta}. \end{aligned}$$

Hence

$$(4.9) \quad \begin{aligned} C \|u\|_{L^2(\omega \times (t_0 + \epsilon, t_1 - \epsilon))} \\ \leq \sigma^\gamma e^{c/\epsilon^2} M + \sigma^{-1} e^{c/\epsilon^2} \left(\|u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma_0))} \right), \end{aligned}$$

for $\sigma > 0$, where $\gamma = \frac{1-\vartheta}{\vartheta}$.

We get, once again from Hardy's inequality in Lemma 3.1,

$$\|u\|_{L^2(\omega \times (t_0, t_0 + \epsilon))}, \|u\|_{L^2(\omega \times (t_1 - \epsilon, t_1))} \leq C\epsilon^\nu M_1(u),$$

where $M_1(u) = \|u\|_{H^1((t_0, t_1), H^1(\Omega))}$.

Combined with (4.9) this inequality yields

$$\begin{aligned} C \|u\|_{L^2(\omega \times (t_0, t_1))} \\ \leq \left(\sigma^\gamma e^{c/\epsilon^2} + \epsilon^\nu \right) M_1 + \sigma^{-1} e^{c/\epsilon^2} \left(\|u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma_0))} \right). \end{aligned}$$

In this inequality, we take σ so that $\sigma^\gamma = \epsilon^\nu e^{-c/\epsilon^2}$. Noting that $\sigma^{-1} \leq \epsilon^{-\nu/\gamma}$, we find

$$(4.10) \quad \begin{aligned} C \|u\|_{L^2(\omega \times (t_0, t_1))} & \leq \epsilon^\nu M_1(u) \\ & + e^{c/\epsilon^2} \left(\|u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma_0))} \right). \end{aligned}$$

As we have seen in the preceding proof, inequality (4.10) still holds when u is substituted by $\partial_t u$. That is we have

$$(4.11) \quad \begin{aligned} C \|\partial_t u\|_{L^2(\omega \times (t_0, t_1))} & \leq \epsilon^\nu M_1(\partial_t u) \\ & + e^{c/\epsilon^2} \left(\|u\|_{H^2((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} \right). \end{aligned}$$

We add side by side (4.10) and (4.11) in order to obtain the expected inequality. \square

Remark 4.1. In the preceding proof we assumed that $u \in H^2((t_0, t_1), H^2(\Omega))$. But a quick inspection of the proof of this proposition shows that in fact the result can be extended to functions from $H^2((t_0, t_1), H^1(\Omega)) \cap H^1((t_0, t_1), H^2(\Omega))$.

Proposition 4.1 gives only an estimate in $H^1((t_0, t_1), L^2(\omega))$. But we can obtain from it an estimate in $H^1((t_0, t_1), H^1(\omega))$ by using the following Caccioppoli type inequality for the parabolic equation $Lu = 0$.

Lemma 4.1. *Let $\omega_0 \Subset \omega_1 \Subset \Omega$. There exists a constant $C > 0$, only depending on Ω , κ , T_0 , ω_0 and ω_1 , so that, for any $u \in H^2((t_0, t_1), H^2(\Omega))$ satisfying $Lu = 0$ in Q , we have*

$$(4.12) \quad C \|u\|_{H^1((t_0, t_1), H^1(\omega_0))} \leq \|u\|_{H^2((t_0, t_1), L^2(\omega_1))}.$$

Proof. Let $u \in H^2((t_0, t_1), H^2(\Omega))$ satisfying $Lu = 0$ in Q . As we have done in the preceding proofs, it is sufficient to prove

$$C \|u\|_{L^2((t_0, t_1), H^1(\omega_0))} \leq \|u\|_{H^1((t_0, t_1), L^2(\omega_1))},$$

because this inequality holds for both u and $\partial_t u$.

By Green's formula, for any $v \in L^2((t_0, t_1), H_0^1(\Omega))$, we have

$$(4.13) \quad \int_{t_0}^{t_1} \int_{\Omega} A \nabla u \cdot \nabla v dx dt - \int_{t_0}^{t_1} \int_{\Omega} \partial_t u v dx dt = 0.$$

Let $\phi \in C_0^\infty(\omega_1)$ satisfying $0 \leq \phi \leq 1$ and $\phi = 1$ in ω_0 .

Taking $v = \phi^2 u$ in (4.13) we get in straightforward manner

$$\int_{t_0}^{t_1} \int_{\omega_1} \phi^2 A \nabla u \cdot \nabla u dx dt = -2 \int_{t_0}^{t_1} \int_{\omega_1} (\phi \nabla u) \cdot (u A \nabla \phi) dx dt + \int_{t_0}^{t_1} \int_{\omega_1} \phi^2 \partial_t u u dx dt.$$

But

$$\int_{t_0}^{t_1} \int_{\omega_1} \phi^2 A \nabla u \cdot \nabla u dx dt \geq \kappa \int_{t_0}^{t_1} \int_{\omega_1} \phi^2 |\nabla u|^2 dx dt.$$

Therefore

$$(4.14) \quad \kappa \int_{t_0}^{t_1} \int_{\omega_1} \phi^2 |\nabla u|^2 dx dt \leq -2 \int_{t_0}^{t_1} \int_{\omega_1} (\phi \nabla u) \cdot (u A \nabla \phi) dx dt + \int_{t_0}^{t_1} \int_{\omega_1} \phi^2 \partial_t u u dx dt.$$

An elementary convexity inequality yields

$$(4.15) \quad 2 \left| \int_{t_0}^{t_1} \int_{\omega_1} (\phi \nabla u) \cdot (u A \nabla \phi) dx dt \right| \leq \frac{\kappa}{2} \int_{t_0}^{t_1} \int_{\omega_1} \phi^2 |\nabla u|^2 dx dt + C \int_{t_0}^{t_1} \int_{\omega_1} u^2 dx dt.$$

On the other hand, we have

$$(4.16) \quad \left| \int_{t_0}^{t_1} \int_{\omega_1} \phi^2 \partial_t u u dx dt \right| \leq \int_{t_0}^{t_1} \int_{\Omega} \phi^2 u^2 dx dt + \int_{t_0}^{t_1} \int_{\Omega} \phi^2 (\partial_t u)^2 dx dt.$$

Combining (4.14), (4.15) and (4.16), we end up getting

$$C \|\nabla u\|_{L^2((t_0, t_1), L^2(\omega_0))} \leq \|u\|_{L^2(\omega_1 \times (t_0, t_1))} + \|\partial_t u\|_{L^2(\omega_1 \times (t_0, t_1))}.$$

Or equivalently

$$C\|u\|_{L^2((t_0, t_1), H^1(\omega_0))} \leq \|u\|_{H^1((t_0, t_1), L^2(\omega_1))}$$

as expected. \square

An immediate consequence of Caccioppoli's inequality (4.12) and Proposition 4.1 (applied both to u and $\partial_t u$), we have

Corollary 4.1. *Let $\nu \in (0, 1/2)$. There exist $\omega \Subset \Omega$, only depending on Ω and Γ_0 , and two constants $C > 0$ and $c > 0$, only depending on Ω , κ , T_0 , ν and Γ_0 , so that, for any $u \in H^3((t_0, t_1), H^2(\Omega))$ satisfying $Lu = 0$ and $0 < \epsilon < (t_1 - t_0)/2$, we have*

$$(4.17) \quad C\|u\|_{H^1((t_0, t_1), H^1(\omega))} \leq \epsilon^\nu \|u\|_{H^3((t_0, t_1), H^1(\Omega))} \\ + e^{c/\epsilon^2} \left(\|u\|_{H^3((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{H^2((t_0, t_1), L^2(\Gamma_0))} \right).$$

We are now in position to complete the proof of Theorem 1.1. We recall that

$$\mathcal{C}(u, \Gamma_0) = \|u\|_{H^3((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{H^2((t_0, t_1), L^2(\Gamma_0))}.$$

If $M = \|u\|_{\mathcal{Z}(Q)}$ then, in light of inequality (3.30) in the end of the proof of Theorem 3.1 and inequality (4.17), we get, for $0 < \epsilon < (t_1 - t_0)/2$ and $0 < \sigma < \sigma_0$,

$$C\|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq \left(\sigma^{\min(\nu, \alpha)/2} + e^{e^{c/\sqrt{\sigma}}} \epsilon^\nu \right) M + e^{e^{c/\sqrt{\sigma}}} e^{c/\epsilon^2} \mathcal{C}(u, \Gamma_0),$$

the constants $C > 0$, $c > 0$ and $\sigma_0 > 0$ only depend on Ω , κ , T_0 , ν and Γ_0 .

The rest of the proof is quite similar to that of Theorem 3.1.

As we have noted above, we have $L\partial_t^j u = 0$ in Q as soon as $Lu = 0$ in Q , for any integer $j \geq 0$. This observation enables us to state the following variante of Theorem 1.1, where $\mathcal{Z}^j(Q) = \mathcal{Y}(Q) \cap H^{3+j}((t_0, t_1), H^2(\Omega))$ is endowed with its natural norm

$$\|u\|_{\mathcal{Z}^j(Q)} = \|u\|_{\mathcal{Y}(Q)} + \|u\|_{H^{3+j}((t_0, t_1), H^2(\Omega))}$$

Theorem 4.1. *Let Γ_0 be a nonempty open subset of Γ and $s \in (0, 1/2)$. Then there exist two constants $C > 0$ and $0 < \varrho_0 \leq \varrho^*$, depending on Ω , κ , T_0 , α , s and Γ_0 , so that, for any integer $j \geq 0$ and $u \in \mathcal{Z}^j(Q)$ satisfying $Lu = 0$ in Q , we have*

$$C\|u\|_{H^j((t_0, t_1), H^1(\Omega))} \leq (j+1)\|u\|_{\mathcal{Z}^j(Q)} \Psi_{\varrho_0, \mu} \left(\frac{\mathcal{C}(u, \Gamma_0)}{\|u\|_{\mathcal{Z}^j(Q)}} \right),$$

with $\mu = \min(\alpha, s)/4$ and

$$\mathcal{C}(u, \Gamma_0) = \|u\|_{H^{j+3}((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{H^{j+2}((t_0, t_1), L^2(\Gamma_0))}.$$

APPENDIX A

We are grateful to Tom ter Elst [12] for having communicated to us the proofs of Lemma 3.3 and Corollary A.1 below. We reproduce in this appendix these proofs.

Proof of Lemma 3.3. Let D be a Lipschitz domain of \mathbb{R}^n and introduce the notations

$$Q = \{x = (x', x_n) \in \mathbb{R}^n; |x'| < 1, -1 < x_n < 1\}, \\ Q_- = \{x \in E; x_n < 0\}, \\ Q_0 = \{x \in E; x_n = 0\}.$$

As D is Lipschitz, if $\tilde{x} \in \Gamma$ then there exist a neighborhood U of \tilde{x} in \mathbb{R}^n and a bijective map $\phi : Q \rightarrow U$ so that $\phi : \overline{Q} \rightarrow \overline{U}$ and $\phi^{-1} : \overline{U} \rightarrow \overline{Q}$ are Lipschitz continuous, and

$$\phi(Q_-) = \Omega \cap U \quad \phi(Q_0) = U \cap \Gamma.$$

For $x, y \in D \cap U$ define

$$\psi(t) = \phi(\phi^{-1}(x) + t[\phi^{-1}(y) - \phi^{-1}(x)]), \quad t \in [0, 1].$$

Clearly, noting that Q_- is convex, ψ is a Lipschitz path in D joining x to y . We have in addition, where k_+ and k_- are the respective Lipschitz constants of ψ and ψ^{-1} ,

$$|\psi(t) - \psi(s)| \leq k_+|t - s| |\phi^{-1}(y) - \phi^{-1}(x)| \leq k_+k_-|t - s||x - y|,$$

$t, s \in [0, 1]$.

Therefore

$$(A.1) \quad \int_0^1 |\dot{\psi}(t)| dt \leq k|x - y| \leq k \text{diam}(D).$$

Here $k = k_+k_-$.

A compactness argument shows that \overline{D} can be recovered by finite number of open subsets U_j with U_j is either a ball or an open subset of the form U . As D is a domain then necessarily any U_j intersect at least U_ℓ for some $\ell \neq j$. In consequence, any arbitrary two points $x, y \in D$ can be joined by a Lipschitz path consisting on finite number (independent on x and y) of line segments and paths of the form ψ . Whence, in light of (A.1), we may find a constant $C > 0$ depending only on D so that $d_g^D(x, y) \leq C$ for any $x, y \in D$. \square

It is worth mentioning the following consequence of Lemma 3.3.

Corollary A.1. *Let D be a Lipschitz bounded domain of \mathbb{R}^n . Then there exists a constant $\varkappa > 0$ so that*

$$d_g^D(x, y) \leq \varkappa|x - y| \quad \text{for any } x, y \in D.$$

Proof. We proceed by contradiction. Assume then that there is no $\varkappa > 0$ so that $d_g^D(x, y) \leq \varkappa|x - y|$, for any $x, y \in D$. In particular, for any positive integer i , we may find two sequences (x_i) and (y_i) in D so that $x_i \neq y_i$ and $d_g^D(x_i, y_i) > i|x_i - y_i|$, for each i . Thus

$$(A.2) \quad |x_i - y_i| \leq \frac{1}{i} \|d_g^D\|_{L^\infty(D \times D)}, \quad \text{for each } i.$$

Subtracting if necessary a subsequence, we may assume that x_i converges to some $x \in \overline{D}$. Using (A.2) we see that y_i converges also to x . Fix j so that $x \in U_j$, where U_j is as in the preceding proof. According to (A.1), we have

$$d_g^D(y, z) \leq K|y - z|, \quad \text{for any } y, z \in U_j,$$

for some constant $K > 0$.

On the other hand, there exists a positive integer i_0 so that $x_i, y_i \in U_j$, for $i \geq i_0$. Hence, for any $i \geq \max(i_0, K)$, we get

$$i|x_i - y_i| \geq K|x_i - y_i| \geq d_g^D(x_i, y_i) > i|x_i - y_i|.$$

This leads to the expected contradiction. \square

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