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LOGARITHMIC STABILITY OF PARABOLIC CAUCHY PROBLEMS

MOURAD CHOULLI AND MASAHIRO YAMAMOTO†

ABSTRACT. The uniqueness of parabolic Cauchy problems is nowadays a classical problem and since Hadamard [6] these kind of problems are known to be ill-posed and even severely ill-posed. Until now there are only few partial results concerning the quantification of the stability for parabolic Cauchy problems. In the present article, we bring the complete answer to this issue, provided that the space domain has finite diameter with respect to the geodesic distance and assuming that solutions are sufficiently smooth.

Mathematics subject classification : 35R25, 35K99, 58J35

Key words : Parabolic Cauchy problems, logarithmic stability, Carleman inequality, Hardy inequality.

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1. INTRODUCTION

Throughout this article, Ω is a bounded domain of \mathbb{R}^n with Lipschitz boundary Γ . We denote the geodesic distance on Ω associated to the round metric by d_g . That is

$$d_g(x, y) = \inf \left\{ \int_0^1 |\dot{\psi}(t)| dt; \psi : [0, 1] \rightarrow \Omega \text{ is a piecewise } C^1 \text{ path joining } x \text{ to } y \right\}.$$

We denote by \mathbf{d}_g the diameter of Ω with respect to the distance d_g :

$$\mathbf{d}_g = \sup\{d_g(x, y); x, y \in \Omega\}.$$

We consider the parabolic operator

$$L = \operatorname{div}(A\nabla \cdot) - \partial_t.$$

Here $A = (a^{ij})$ is a symmetric matrix whose coefficients belong to $W^{1,\infty}(\Omega)$ and satisfy: there exist $\kappa > 0$ and $K > 0$ so that

$$(1.1) \quad A(x)\xi \cdot \xi \geq \kappa|\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n,$$

and

$$(1.2) \quad \|a^{ij}\|_{W^{1,\infty}(\Omega)} \leq K, \quad 1 \leq i, j \leq n.$$

We recall that Ω has the uniform exterior sphere property if there exists $\rho > 0$ so that, to any $\tilde{x} \in \Gamma$ corresponds $x_0 = x_0(\tilde{x}) \in \mathbb{R}^n \setminus \overline{\Omega}$ for which

$$B(x_0, \rho) \cap \Omega = \emptyset \quad \text{and} \quad \overline{B(x_0, \rho)} \cap \overline{\Omega} = \{\tilde{x}\}.$$

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Let $t_0 < t_1$, $Q = \Omega \times (t_0, t_1)$ and fix $0 < \alpha < 1$. For sake of simplicity, we use in the sequel the following notation:

$$\mathcal{X}^j = C^{1+\alpha, \frac{1+\alpha}{2}}(\overline{Q}) \cap H^j((t_0, t_1), H^2(\Omega)), \quad j = 1, 2.$$

We endow \mathcal{X}^j with its natural norm

$$\|u\|_{\mathcal{X}^j} = \|u\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\overline{Q})} + \|u\|_{H^j((t_0, t_1), H^2(\Omega))}.$$

From now on we assume that $t_1 - t_0 \leq T_0$, for some fixed $T_0 > 0$.

We are mainly concerned in the present work with the stability issue for the Cauchy problem associated to the parabolic operator L . Precisely we are going to prove

Theorem 1.1. *Assume that Ω possesses the uniform exterior sphere property and $\mathbf{d}_g < \infty$. Let Γ_0 be a nonempty open subset of Γ . Then there exist two constant $C > 0$ and $\gamma > 0$ depending on Ω , κ , K , T_0 , α and Γ_0 , so that, for any $u \in \mathcal{X}^2$ satisfying $Lu = 0$ in Q , we have*

$$C\|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq \|u\|_{\mathcal{X}^2} \left[\ln \left(\frac{\|u\|_{\mathcal{X}^2}}{\mathcal{C}(u, \Gamma_0)} \right) \right]^{-\gamma} + \mathcal{C}(u, \Gamma_0),$$

where

$$\mathcal{C}(u, \Gamma_0) = \|u\|_{H^2((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{H^1((t_0, t_1), L^2(\Gamma_0))}.$$

The stability estimate in Theorem 1.1 is of logarithmic type. Similarly to the elliptic case one can not reasonably expect Lipschitz or Hölder stability.

We observe that Theorem 1.1 remains valid if L is substituted by L plus an operator of first order in space variable whose coefficients are bounded.

The second author [10, Theorem 5.1, page 24] proved a Hölder stability in a proper subdomain of Q depending on the part of the lateral boundary where the Cauchy data is given. In [9, Theorem 3.5.1, pages 45 and 46], Vessella establishes a local Hölder stability corresponding to the continuation of Cauchy data to an interior subdomain for solutions vanishing at the initial time. Recently, Bourgeois [1, Main theorem, page 2] proved a result similar to the one in Theorem 1.1 in the case where $L = \Delta - \partial_t$, $\Omega = D \setminus O$, D and O are two domains of class C^2 , $O \Subset D$, and $\Gamma_0 = \partial D$ or $\Gamma_0 = \partial O$. His result is based on a global Carleman estimate in which the weight function is built from the distance to the boundary of the space variable.

The present work constitute an extension of the earlier result by Bourgeois [1], but for more regular solutions and some class of domains. The proof of the main result is inspired by that used in the elliptic case by the first author in [2]. Note however that there is a great difference between the elliptic and parabolic cases. The main difficulty in the parabolic case is that the initial time and the final time data are missing. So the proofs are more technical. The idea to overcome the fact that the initial time and the final time data are not known is to use a Hardy inequality with respect to time variable. This is explain why we need to work with sufficiently smooth solutions. The assumption that the domain must have finite diameter with respect to the geodesic distance is essential in our proof. We do not know whether this geometric assumption is necessary.

Althrought we used classical tools to establish our main result, the result itself is completely know. This is our modest contribution to stability issue for parabolic Cauchy problems.

A key tool in our analysis is a Carleman inequality (Theorem 2.2 below). We observe that Carleman inequalities are very useful tool in control theory or for establishing a unique continuation property for elliptic and parabolic partial differential equations. There is wide literature on the subject. We just quote here the few reference [1, 3, 4, 8].

The rest of this article is organized as follows. Section 2 is devoted to a three-cylinder interpolation inequality for the $H_x^1-L_t^2$ -norm. This inequality will be very useful for continuing the data on an interior subdomain to the lateral boundary data, and to continue the data from one subdomain to another subdomain. This is what we show in Section 3 and, as byproduct, we prove a logarithmic stability estimate corresponding

to the unique continuation from an interior data. The proof of Theorem 1.1 is completed in Section 4 by beforehand establishing a result which quantifies the stability from Cauchy data to an interior subdomain.

2. THREE-CYLINDER INTERPOLATION INEQUALITY

We prove in this section

Theorem 2.1. *There exist $C > 0$ and $0 < \vartheta < 1$ depending on κ , K , Ω and T_0 , so that, for any $0 < \epsilon < \frac{t_1 - t_0}{2}$, $u \in H^1((t_0, t_1), H^2(\Omega))$ satisfying $Lu = 0$ in Q , $y \in \Omega$ and $0 < r < r_y(\epsilon) = \min(\frac{1}{3} \text{dist}(y, \Gamma), \sqrt{\epsilon})$, we have*

$$(2.1) \quad r^3 \|u\|_{L^2((t_0+\epsilon, t_1-\epsilon), H^1(B(y, 2r)))} \leq C \|u\|_{L^2((t_0, t_1), H^1(B(y, r)))}^\vartheta \|u\|_{L^2((t_0, t_1), H^1(B(y, 3r)))}^{1-\vartheta}.$$

The proof of Theorem (2.1) is based on a Carleman inequality for a family of parabolic operators. To this end, let \mathcal{I} be an arbitrary set and consider the family of operators

$$L_s = \text{div}(A_s \nabla \cdot) - \partial_t, \quad s \in \mathcal{I},$$

where, for each $s \in \mathcal{I}$, $A_s = (a_s^{ij})$ is a symmetric matrix with $W^{1,\infty}(\Omega)$ entries so that the following assumptions hold: there exist $\kappa > 0$ and $K > 0$ so that

$$(2.2) \quad A_s(x) \xi \cdot \xi \geq \kappa |\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n \text{ and } s \in \mathcal{I},$$

and

$$(2.3) \quad \|a_s^{ij}\|_{W^{1,\infty}(\Omega)} \leq K, \quad 1 \leq i, j \leq n, \quad s \in \mathcal{I}.$$

Pick $\psi \in C^2(\overline{\Omega})$ without critical points in $\overline{\Omega}$ and set $\Sigma = \Gamma \times (t_0, t_1)$. Let

$$g(t) = \frac{1}{(t - t_0)(t_1 - t)}$$

and

$$\varphi(x, t) = g(t) \left(e^{4\lambda \|\psi\|_\infty} - e^{\lambda(2\|\psi\|_\infty + \psi(x))} \right), \quad \chi(x, t) = g(t) e^{\lambda(2\|\psi\|_\infty + \psi(x))}.$$

Theorem 2.2. (*Carleman inequality*) *There exist three positive constants C , λ_0 and τ_0 depending only on ψ , Ω , κ , K and T_0 , so that*

$$(2.4) \quad \begin{aligned} & C \int_Q (\lambda^4 \tau^3 \chi^3 u^2 + \lambda^2 \tau \chi |\nabla u|^2) e^{-2\tau \varphi} dx dt \\ & \leq \int_Q (L_s u)^2 e^{-2\tau \varphi} dx dt + \int_\Sigma (\lambda^3 \tau^3 \chi^3 u^2 + \lambda \tau \chi |\nabla u|^2 + (\lambda \tau \chi)^{-1} |\partial_t u|^2) e^{-2\tau \varphi} d\sigma dt, \end{aligned}$$

for all $u \in H^1((t_0, t_1), H^2(\Omega))$, $s \in \mathcal{I}$, $\lambda \geq \lambda_0$ and $\tau \geq \tau_0$.

Proof. Since the dependance of the constants will be uniform with respect to $s \in \mathcal{I}$, we drop for simplicity the subscript s in L_s and its coefficients. On the other hand since $C^\infty(\overline{Q})$ is dense in $H^1((t_0, t_1), H^2(\Omega))$ it is enough to prove (2.4) when $u \in C^\infty(\overline{Q})$.

Let $\Phi = e^{\tau \varphi}$ and $u \in C^\infty(\overline{Q})$, and set $w = \Phi^{-1} u$ extended by continuity at $t = 0$ and $t = T$ by setting $w(\cdot, 0) = w(\cdot, T) = 0$. Then straightforward computations give

$$Pw = [\Phi^{-1} L \Phi] w = P_1 w + P_2 w + cw,$$

where

$$\begin{aligned} P_1 w &= aw + \text{div}(A \nabla w) - \tau \partial_t \varphi w, \\ P_2 w &= B \cdot \nabla w + bw - \partial_t w, \end{aligned}$$

with

$$\begin{aligned} a &= a(x, t, \lambda, \tau) = \lambda^2 \tau^2 \chi^2 |\nabla \psi|_A^2, \\ B &= B(x, t, \lambda, \tau) = -2\lambda \tau \chi A \nabla \psi, \\ b &= b(x, t, \lambda, \tau) = -2\lambda^2 \tau \chi |\nabla \psi|_A^2, \\ c &= c(x, t, \lambda, \tau) = -\lambda \tau \chi \operatorname{div}(A \nabla \psi) + \lambda^2 \tau \chi |\nabla \psi|_A^2. \end{aligned}$$

Here

$$|\nabla \psi|_A = \sqrt{A \nabla \psi \cdot \nabla \psi} = \left| A^{\frac{1}{2}} \nabla \psi \right|.$$

We obtain by making integrations by parts

$$\begin{aligned} (2.5) \quad \int_Q a w B \cdot \nabla w dx dt &= \frac{1}{2} \int_Q a B \cdot \nabla w^2 dx dt \\ &= -\frac{1}{2} \int_Q \operatorname{div}(a B) w^2 dx dt + \frac{1}{2} \int_{\Sigma} a B \cdot \nu w^2 d\sigma dt \end{aligned}$$

and

$$\begin{aligned} (2.6) \quad \int_Q \operatorname{div}(A \nabla w) B \cdot \nabla w dx dt &= - \int_Q A \nabla w \cdot \nabla(B \cdot \nabla w) dx dt + \int_{\Sigma} B \cdot \nabla w A \nabla w \cdot \nu d\sigma dt \\ &= - \int_Q B' \nabla w \cdot A \nabla w dx dt \\ &\quad - \int_Q \nabla^2 w B \cdot A \nabla w dx dt + \int_{\Sigma} (B \cdot \nabla w)(A \nabla w \cdot \nu) d\sigma dt. \end{aligned}$$

Here $B' = (\partial_i B_j)$ is the Jacobian matrix of B and $\nabla^2 w = (\partial_{ij}^2 w)$ is the Hessian matrix of w .

But

$$\begin{aligned} \int_Q B_i \partial_{ij}^2 w a^{ik} \partial_k w dx dt &= - \int_Q B_i a^{ik} \partial_{ik}^2 w \partial_j w dx dt \\ &\quad - \int_Q \partial_i [B_i a^{ik}] \partial_k w \partial_j w dx dt + \int_{\Sigma} B_i \nu_i a^{jk} \partial_k w \partial_j w d\sigma dt. \end{aligned}$$

Therefore

$$\begin{aligned} (2.7) \quad \int_Q \nabla^2 w B \cdot A \nabla w dx dt &= -\frac{1}{2} \int_Q \left([\operatorname{div}(B)A + \tilde{A}] \nabla w \right) \cdot \nabla w dx dt \\ &\quad + \frac{1}{2} \int_{\Sigma} |\nabla w|_A^2 B \cdot \nu d\sigma dt, \end{aligned}$$

with $\tilde{A} = (\tilde{a}^{ij})$, $\tilde{a}^{ij} = B \cdot \nabla a^{ij}$.

It follows from (2.6) and (2.7)

$$\begin{aligned} (2.8) \quad \int_Q \operatorname{div}(A \nabla w) B \cdot \nabla w dx dt &= \frac{1}{2} \int_Q \left(-2AB' + \operatorname{div}(B)A + \tilde{A} \right) \nabla w \cdot \nabla w dx dt \\ &\quad + \int_{\Sigma} (B \cdot \nabla w)(A \nabla w \cdot \nu) d\sigma dt - \frac{1}{2} \int_{\Sigma} |\nabla w|_A^2 B \cdot \nu d\sigma dt. \end{aligned}$$

A new integration by parts yields

$$\int_Q \operatorname{div}(A \nabla w) b w dx dt = - \int_Q b |\nabla w|_A^2 dx dt - \int_Q w \nabla b \cdot A \nabla w dx dt + \int_{\Sigma} b w A \nabla w \cdot \nu d\sigma dt.$$

This and the following inequality

$$- \int_Q w \nabla b \cdot A \nabla w dx dt \geq - \int_Q (\lambda^2 \chi)^{-1} |\nabla b|_A^2 w^2 dx dt - \int_Q \lambda^2 \chi |\nabla w|_A^2 dx dt$$

imply

$$(2.9) \quad \int_Q \operatorname{div}(A\nabla w)bw dxdt \geq - \int_Q (b + \lambda^2\chi)|\nabla w|_A^2 dxdt - \int_Q (\lambda^2\chi)^{-1}|\nabla b|_A^2 w^2 dxdt \\ + \int_\Gamma bwA\nabla w \cdot \nu d\sigma dt.$$

One more time, integrations by parts entail

$$(2.10) \quad \int_Q aw\partial_t w dxdt = \frac{1}{2} \int_Q a\partial_t w^2 dxdt = -\frac{1}{2} \int_Q \partial_t aw^2 dxdt,$$

$$(2.11) \quad \int_Q \partial_t \varphi w \partial_t w dxdt = \frac{1}{2} \int_Q \partial_t \varphi \partial_t w^2 dxdt = -\frac{1}{2} \int_Q \partial_t^2 \varphi w^2 dxdt,$$

$$(2.12) \quad \int_Q \partial_t \varphi w B \cdot \nabla w dxdt = \frac{1}{2} \int_Q \partial_t \varphi B \cdot \nabla w^2 dxdt \\ = \frac{1}{2} \int_Q \operatorname{div}(\partial_t \varphi B) w^2 dxdt + \frac{1}{2} \int_\Sigma \partial_t \varphi B \cdot \nu w^2 d\sigma dt.$$

Also

$$(2.13) \quad \int_Q \operatorname{div}(A\nabla w)\partial_t w dxdt = - \int_Q A\nabla w \cdot \nabla \partial_t w dxdt + \int_\Sigma A\nabla w \cdot \nu \partial_t w d\sigma dt.$$

But

$$\int_Q A\nabla w \cdot \nabla \partial_t w dxdt = - \int_Q A\nabla \partial_t w \cdot \nabla w dxdt = - \int_Q \nabla \partial_t w \cdot A\nabla w dxdt.$$

Whence

$$\int_Q A\nabla w \cdot \nabla \partial_t w dxdt = 0.$$

This identity in (2.13) gives

$$(2.14) \quad \int_Q \operatorname{div}(A\nabla w)\partial_t w = \int_\Sigma A\nabla w \cdot \nu \partial_t w d\sigma dt.$$

Now a combination of (2.5), (2.8), (2.9)-(2.12) and (2.14) gives

$$(2.15) \quad \int_Q P_1 w P_2 w dxdt - \int_Q c^2 w^2 dxdt \geq \int_Q f w^2 dxdt + \int_Q F \nabla w \cdot \nabla w dxdt + \int_\Sigma g(w) d\sigma dt,$$

where

$$f = -\frac{1}{2} \operatorname{div}(aB) + ab - (\lambda^2\chi)^{-1}|\nabla b|_A^2 - c^2 + \frac{1}{2} \partial_t a - \frac{1}{2} \partial_t^2 \varphi - \frac{1}{2} \operatorname{div}(\partial_t \varphi B),$$

$$F = -AB' + \frac{1}{2} (\operatorname{div}(B)A + \tilde{A}) - (b + \lambda^2\chi)A,$$

$$g(w) = \frac{1}{2} aw^2 B \cdot \nu - \frac{1}{2} |\nabla w|_A^2 B \cdot \nu + B \cdot \nabla w A \nabla w \cdot \nu + bwA\nabla w \cdot \nu - \frac{1}{2} \partial_t \varphi B \cdot \nu w^2 - A\nabla w \cdot \nu \partial_t w.$$

We use the inequality $(\alpha - \beta)^2 \geq \alpha^2/2 - \beta^2$, $\alpha > 0$, $\beta > 0$, in order to derive

$$\|Pw\|_2^2 \geq (\|P_1 w + P_2 w\|_2 - \|cw\|_2)^2 \\ \geq \frac{1}{2} \|P_1 w + P_2 w\|_2^2 - \|cw\|_2^2 \\ \geq \int_\Omega P_1 w P_2 w dx - \int_\Omega c^2 w^2 dx.$$

Whence (2.15) implies

$$(2.16) \quad \|Pw\|_2^2 \geq \int_Q fw^2 dxdt + \int_Q F\nabla w \cdot \nabla w dxdt + \int_\Sigma g(w) d\sigma dt.$$

In light of the following inequalities, where c is a constant depending only on T_0 and ψ ,

$$\begin{aligned} |\partial_t \varphi| &\leq c\chi^2, \quad |\partial_t^2 \varphi|, |\nabla \partial_t \varphi| \leq c\chi^3 \\ |A\nabla w \cdot \nu \partial_t w| &\leq \lambda\tau\chi |A\nabla w \cdot \nu|^2 + (\lambda\tau\chi)^{-1} |\partial_t w|^2, \end{aligned}$$

straightforward computations show that there exist four positive constants C_0 , C_1 , λ_0 and τ_0 , that can depend only on ψ , Ω , T_0 , κ and K , such that for all $\lambda \geq \lambda_0$ and $\tau \geq \tau_0$,

$$\begin{aligned} f &\geq C_0 \lambda^4 \tau^3 \chi^3, \\ F\xi \cdot \xi &\geq C_0 \lambda^2 \tau \chi |\xi|^2, \quad \text{for any } \xi \in \mathbb{R}^n, \\ |g(w)| &\leq C_1 (\lambda^3 \tau^3 \chi^3 w^2 + \lambda\tau\chi |\nabla w|^2 + (\lambda\tau\chi)^{-1} |\partial_t w|^2). \end{aligned}$$

Hence

$$(2.17) \quad C \int_Q (\lambda^4 \tau^3 \chi^3 w^2 + \lambda^2 \tau \chi |\nabla w|^2) dxdt \leq \int_Q (Pw)^2 dxdt + \int_\Sigma (\lambda^3 \tau^3 \chi^3 w^2 + \lambda\tau\chi |\nabla w|^2 + (\lambda\tau\chi)^{-1} |\partial_t w|^2) d\sigma dt.$$

As $\nabla w = \Phi^{-1}(\nabla u + \lambda\tau u \nabla \psi)$,

$$|\nabla w|^2 = \Phi^{-2} (|\nabla u|^2 + \lambda^2 \tau^2 |\nabla \psi|^2 + 2\lambda\tau u \nabla u \cdot \nabla \psi).$$

Therefore, using an elementary inequality, we find

$$|\nabla w|^2 \geq \Phi^{-2} \left(|\nabla u|^2 + \lambda^2 \tau^2 |\nabla \psi|^2 u^2 - 4\lambda^2 \tau^2 u^2 |\nabla \psi|^2 - \frac{1}{2} |\nabla u|^2 \right)$$

and then

$$(2.18) \quad |\nabla w|^2 \geq \Phi^{-2} \left(\frac{1}{2} |\nabla u|^2 - 3\lambda^2 \tau^2 u^2 \|\nabla \psi\|_\infty^2 \right).$$

(2.18) implies in a straightforward manner that

$$\|\nabla \psi\|_\infty^{-2} |\nabla w|^2 \geq \Phi^{-2} \left(\frac{\|\nabla \psi\|_\infty^{-2}}{12} |\nabla u|^2 - \frac{1}{2} \lambda^2 \tau^2 u^2 \right).$$

Consequently

$$(2.19) \quad \|\nabla \psi\|_\infty^{-2} \lambda^2 \tau |\nabla w|^2 + \lambda^4 \tau^3 w^2 \geq \Phi^{-2} \left(\frac{\|\nabla \psi\|_\infty^{-2}}{12} |\nabla u|^2 + \frac{1}{2} \lambda^2 \tau^2 u^2 \right).$$

On the other hand, it is not hard to establish the inequality

$$(2.20) \quad |\partial_t w|^2 \leq \Phi^{-2} (|\partial_t u|^2 + c\tau^2 \chi^2 u^2).$$

The expected inequality follows by combining (2.17), (2.19) and (2.20). □

Proof of Theorem 2.1. Let $u \in H^1((t_0, t_1), H^2(\Omega))$ satisfying $Lu = 0$, set

$$Q(\mu) = B(0, \mu) \times (-1, 1), \quad \mu > 0.$$

Fix $(y, s) \in \Omega \times (t_0, t_1)$ and

$$0 < r < r_{(y,s)} = \min \left(\frac{1}{3} \text{dist}(y, \Gamma), \sqrt{s - t_0}, \sqrt{t_1 - s} \right) \leq r_0 = r_0(\text{diam}(\Omega), T_0).$$

Let

$$w(x, t) = u(rx + y, r^2 t + s), \quad (x, t) \in Q(3),$$

Then $L_r w = \text{div}(A_r \nabla w) - \partial_t w = 0$ in $Q(3)$, where $A_r(x) = (a^{ij}(rx + y))$.

Clearly, the family (A_r) satisfies (2.2) and (2.3), uniformly with respect to $r \in (0, r_{(y,s)})$.

Let $\chi \in C_0^\infty(U)$ satisfying $0 \leq \chi \leq 1$ and $\chi = 1$ in \mathcal{K} , with

$$U = \left\{ x \in \mathbb{R}^n; \frac{1}{2} < |x| < 3 \right\}, \quad \mathcal{K} = \left\{ x \in \mathbb{R}^n; 1 \leq |x| \leq \frac{5}{2} \right\}.$$

Theorem 2.2 applied to χw , when Ω is substituted by U , gives, for $\lambda \geq \lambda_0$ and $\tau \geq \tau_0$,

$$(2.21) \quad C \int_{Q(2) \setminus Q(1)} (\lambda^4 \tau^3 \varphi^3 w^2 + \lambda^2 \tau \varphi |\nabla w|^2) e^{-2\tau\varphi} dxdt \leq \int_{Q(3)} (L_r(\chi w))^2 e^{-2\tau\varphi} dxdt.$$

But

$$\text{supp}(L_r(\chi w)) \subset \left[\left\{ \frac{1}{2} \leq |x| \leq 1 \right\} \cup \left\{ \frac{5}{2} \leq |x| \leq 3 \right\} \right] \times (-1, 1)$$

and

$$(L_r(\chi w))^2 \leq \Lambda(w^2 + |\nabla w|^2),$$

where $\Lambda = \Lambda(r_0)$ is independent on r . Therefore, fixing λ and changing τ_0 if necessary, (2.21) implies, for $\tau \geq \tau_0$,

$$(2.22) \quad C \int_{Q(2)} (w^2 + |\nabla w|^2) e^{-2\tau\varphi} dxdt \leq \int_{Q(1)} (w^2 + |\nabla w|^2) e^{-2\tau\varphi} dxdt + \int_{Q(3) \setminus Q(5/2)} (w^2 + |\nabla w|^2) e^{-2\tau\varphi} dxdt.$$

Let $s' \in (-1, 0)$ and choose $\psi(x) = -|x|^2$ in (2.22) (without critical points in U). In that case

$$\varphi(x, t) = g(t) \left(e^{36\lambda} - e^{\lambda(18-|x|^2)} \right).$$

We have, where $s' = -1 + \varrho$ for some $0 < \varrho < 1$,

$$\begin{aligned} \varphi(x, t) &\leq g(s') (e^{36\lambda} - e^{14\lambda}) \leq \frac{1}{\varrho} (e^{36\lambda} - e^{14\lambda}) = \frac{\alpha}{\varrho}, \quad (x, t) \in B(2) \times (s', -s'), \\ \varphi(x, t) &\geq g(t_m) (e^{36\lambda} - e^{18\lambda}) = (e^{36\lambda} - e^{18\lambda}) = \beta, \quad (x, t) \in Q(1) \\ \varphi(x, t) &\geq g(t_m) \left(e^{36\lambda} - e^{\frac{47}{4}\lambda} \right) = \left(e^{36\lambda} - e^{\frac{47}{4}\lambda} \right) = \gamma, \quad (x, t) \in Q(3) \setminus Q\left(\frac{5}{2}\right). \end{aligned}$$

As $\frac{\beta}{\alpha} < 1 < \frac{\gamma}{\alpha}$, we can fix $\theta \in (0, 1)$ so that

$$\frac{1}{\varrho} = \theta \frac{\beta}{\alpha} + (1 - \theta) \frac{\gamma}{\alpha} > 1.$$

Set $a = 2(1 - \theta)(\gamma - \beta)$ and $b = 2\theta(\gamma - \beta)$ and $\tilde{Q}(2) = B(0, 2) \times (-1 + \epsilon, 1 - \epsilon)$. Then (2.22) yields

$$C \int_{\tilde{Q}(2)} (w^2 + |\nabla w|^2) dxdt \leq e^{a\tau} \int_{Q(1)} (w^2 + |\nabla w|^2) dxdt + e^{-b\tau} \int_{Q(3)} (w^2 + |\nabla w|^2) dxdt.$$

As in the elliptic case (see [2, Theorem 2.17 and its proof, pages 19 to 21]), we derive from this inequality the following one

$$C \|w\|_{L^2((-1+\varrho, 1-\varrho), H^1(B(2)))} \leq \|w\|_{L^2((-1, 1), H^1(B(1)))}^\vartheta \|w\|_{L^2((-1, 1), H^1(B(3)))}^{1-\vartheta},$$

with $\vartheta = \frac{a}{a+b}$.

Making a change of variable we get, with $\tau = 1 - \varrho$,

$$(2.23) \quad r \|u\|_{L^2((s-\tau r^2, s+\tau r^2), H^1(B(y, 2r)))} \leq C \|u\|_{L^2((s-r^2, s+r^2), H^1(B(y, r)))}^\vartheta \|u\|_{L^2((s-r^2, s+r^2), H^1(B(y, 3r)))}^{1-\vartheta}.$$

We fix $0 < \epsilon < \frac{t_1 - t_0}{2}$ and let $s_0 = t_0 + \epsilon$ and $s_k = s_{k-1} + 2\tau r^2$. We consider q the smallest integer so that $(t_1 - \epsilon) - s_{q-1} \leq 2\tau r^2$ or equivalently $(t_1 - \epsilon) - s_{q-2} > 2\tau r^2$. Whence

$$(2.24) \quad q < \frac{t_1 - t_0 - 2\epsilon}{2\tau r^2} + 3 < \frac{\delta}{2\tau r^2} + \frac{3\text{diam}(\Omega)^2}{r^2} = \left(\frac{T_0}{2\tau} + 3\text{diam}(\Omega)^2 \right) \frac{1}{r^2}.$$

In light of (2.23) we get, for $r < r_y(\epsilon) = \min\left(\frac{1}{3}\text{dist}(y, \Gamma), \sqrt{\epsilon}\right)$,

$$(2.25) \quad r\|u\|_{L^2((t_0+\epsilon, t_1-\epsilon), H^1(B(y, 2r)))} \leq Cq\|u\|_{L^2((t_0, t_1), H^1(B(y, r)))} \|u\|_{L^2((t_0, t_1), H^1(B(y, 3r)))}^{1-\vartheta}.$$

Estimate (2.24) in (2.25) yields

$$(2.26) \quad Cr^3\|u\|_{L^2((t_0+\epsilon, t_1-\epsilon), H^1(B(y, 2r)))} \leq \|u\|_{L^2((t_0, t_1), H^1(B(y, r)))} \|u\|_{L^2((t_0, t_1), H^1(B(y, 3r)))}^{1-\vartheta}, \quad r < r_y(\epsilon).$$

The proof is then complete. \square

3. QUANTIFYING THE UNIQUENESS OF CONTINUATION FROM AN INTERIOR DATA

In the rest of this text, we will often use the following Hardy's inequality for vector valued function.

Lemma 3.1. *Let X be a Banach space and $s \in (0, \frac{1}{2})$. There exists a constant $c > 0$ so that for any $u \in H^s((t_0, t_1), X)$, we have*

$$\left\| \frac{u}{\delta^s} \right\|_{L^2((t_0, t_1), X)} \leq c\|u\|_{H^s((t_0, t_1), X)}.$$

Here $\delta = \delta(t) = \min\{|t - t_0|, |t - t_1|\}$.

Proof. For simplicity, we use the same symbol $\|\cdot\|$ for the norm of X and its dual space dual X^* . The duality pairing between X and X^* is denoted by $\langle \cdot, \cdot \rangle$. Let $u \in H^s((t_0, t_1), X)$ and pick $x^* \in X^*$ with $\|x^*\| = 1$. From the usual Hardy's inequality in one dimension (see for instance [5]), we have

$$(3.1) \quad \int_{t_0}^{t_1} \frac{|\langle x^*, u(t) \rangle|^2}{\delta^{2s}(t)} dt \leq c\|x^*, u(\cdot)\|_{H^s((t_0, t_1))}^2.$$

But

$$\begin{aligned} \|\langle x^*, u(\cdot) \rangle\|_{H^s((t_0, t_1))}^2 &= \|\langle x^*, u(\cdot) \rangle\|_{L^2((t_0, t_1))}^2 + \int_{t_0}^{t_1} \int_{t_0}^{t_1} \frac{|\langle x^*, u(\tau) \rangle - \langle x^*, u(t) \rangle|^2}{|\tau - t|^{n+2s}} dt d\tau \\ &\leq \|u\|_{L^2((t_0, t_1), X)}^2 + \int_{t_0}^{t_1} \int_{t_0}^{t_1} \frac{\|u(\tau) - u(t)\|^2}{|\tau - t|^{n+2s}} dt d\tau = \|u\|_{H^s((t_0, t_1), X)}^2. \end{aligned}$$

Whence

$$\left\| \frac{u}{\delta^s} \right\|_{L^2((t_0, t_1), X)}^2 = \int_{t_0}^{t_1} \frac{\sup_{\|x^*\|=1} |\langle x^*, u(t) \rangle|^2}{\delta^{2s}(t)} dt \leq c\|u\|_{H^s((t_0, t_1), X)}^2.$$

\square

Next, we prove

Proposition 3.1. *There exist $\omega \Subset \Omega$ and four constants $\mu > 0$, $c > 0$, $C > 0$ and $0 < \sigma_0 < 1$, that can depend on Ω , κ , K , T_0 and α , so that, for any $u \in \mathcal{X}^1$ satisfying $Lu = 0$ in Q and $0 < \sigma < \sigma_0$, we have*

$$(3.2) \quad \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma)^n)} \leq C(\sigma^\mu \|u\|_{\mathcal{X}^1} + e^{\frac{\sigma}{\mu}} \|u\|_{L^2((t_0, t_1), H^1(\omega))}).$$

Proof. Since Ω is Lipschitz, it has the uniform interior cone property (see for instance [7]). That is there exist $R > 0$ and $\theta \in]0, \frac{\pi}{2}[$ so that, to any $\tilde{x} \in \Gamma$ corresponds $\xi = \xi(\tilde{x}) \in \mathbb{S}^{n-1}$ for which

$$\mathcal{C}(\tilde{x}) = \{x \in \mathbb{R}^n; |x - \tilde{x}| < R, (x - \tilde{x}) \cdot \xi > |x - \tilde{x}| \cos \theta\} \subset \Omega.$$

We fix $\tilde{x} \in \Gamma$ and let $\xi = \xi(\tilde{x})$ be as in the definition above. Let $x_0 = x_0(\tilde{x}) = \tilde{x} + \frac{R}{2}\xi$, $d_0 = |x_0 - \tilde{x}|$ and $\rho_0 = (d_0/3) \sin \theta$.

Let $0 < \epsilon < \epsilon_0$, where $\epsilon_0 < \frac{t_1 - t_0}{4}$ is chosen in such a way that

$$(3.3) \quad 0 < \delta = \delta(\epsilon) = \frac{\sqrt{\epsilon}}{4\rho_0} < 1, \quad 0 < \epsilon < \epsilon_0.$$

By induction on k , we construct a sequence of balls $(B(x_k, 3\delta\rho_k))$ as follows

$$\begin{cases} x_{k+1} = x_k - \alpha_k \xi, \\ \rho_{k+1} = \mu\rho_k, \\ d_{k+1} = \mu d_k, \end{cases}$$

where

$$d_k = |x_k - \tilde{x}|, \quad \rho_k = cd_k, \quad \alpha_k = (1 - \mu)d_k,$$

with

$$c = \frac{\sin \theta}{3}, \quad \mu = \frac{3 - 2 \sin \theta}{3 - \sin \theta}.$$

This construction guaranties that, for each k , $B(x_k, 3\delta\rho_k) \subset \mathcal{C}(\tilde{x})$ and

$$(3.4) \quad B(x_{k+1}, \delta\rho_{k+1}) \subset B(x_k, 2\delta\rho_k).$$

Let $u \in \mathcal{X}^1$. In the sequel

$$M = M(u) = \|u\|_{\mathcal{X}^1}.$$

From (2.1) we have

$$(3.5) \quad (\delta\rho_0)^3 \|u\|_{L^2((t_0+\epsilon, t_1-\epsilon), H^1(B(x_0, 2\delta\rho_0)))} \leq CM^{1-\vartheta} \|u\|_{L^2((t_0, t_1), H^1(B(x_0, \delta\rho_0)))}^\vartheta.$$

Since $B(x_0, 2\delta\rho_0) \supset B(x_1, \delta\rho_1)$, (3.5) implies

$$(3.6) \quad (\delta\rho_0)^3 \|u\|_{L^2((t_0+\epsilon, t_1-\epsilon), H^1(B(x_1, \delta\rho_1)))} \leq CM^{1-\vartheta} \|u\|_{L^2((t_0, t_1), H^1(B(x_0, \delta\rho_0)))}^\vartheta.$$

Set

$$I_0 = (t_0, t_1), \quad I_k = (t_0 + \epsilon\eta_k, t_1 - \epsilon\eta_k) \quad \text{with} \quad \eta_k = \sum_{j=0}^{k-1} \frac{1}{2^j}, \quad k \geq 1.$$

With these notations (3.6) takes the form

$$(3.7) \quad (\delta\rho_0)^3 \|u\|_{L^2(I_1, H^1(B(x_1, \delta\rho_1)))} \leq CM^{1-\vartheta} \|u\|_{L^2(I_0, H^1(B(x_0, \delta\rho_0)))}^\vartheta.$$

By induction in k , we see that

$$(3.8) \quad (\delta\rho_k)^3 \|u\|_{L^2(I_{k+1}, H^1(B(x_{k+1}, \delta\rho_{k+1})))} \leq CM^{1-\vartheta} \|u\|_{L^2(I_k, H^1(B(x_k, \delta\rho_k)))}^\vartheta.$$

Let $\Lambda_k = \|u\|_{L^2(I_k, H^1(B(x_k, \delta\rho_k)))}$. Then (3.8) is rewritten as follows

$$(\delta\rho_k)^3 \Lambda_{k+1} \leq CM^{1-\vartheta} \Lambda_k^\vartheta.$$

Again, by induction in k , we obtain

$$\Lambda_k \leq \frac{C^{1+\vartheta+\dots+\vartheta^{k-1}}}{(\delta\rho_{k-1})^3 (\delta\rho_{k-2})^{3\vartheta} \dots (\delta\rho_0)^{3\vartheta^{k-1}}} M^{(1-\vartheta)(1+\vartheta+\dots+\vartheta^{k-1})} \Lambda_0^{\vartheta^k}.$$

From the following inequality, where we used $0 < \mu < 1$,

$$\begin{aligned} (\delta\rho_{k-1})^3 (\delta\rho_{k-2})^{3\vartheta} \dots (\delta\rho_0)^{\vartheta^{3(k-1)}} &= (\mu^{3(k-1)} (\delta\rho_0)^3) (\mu^{3(k-2)} (\delta\rho_0)^3)^\vartheta \dots (\mu^3 \rho_0^3)^{\vartheta^{k-2}} \rho_0^{\vartheta^{k-1}} \\ &\geq (\mu^{3(k-1)} (\delta\rho_0)^3)^{1+\vartheta+\dots+\vartheta^{k-1}} = (\mu^{3(k-1)} (\delta\rho_0)^3)^{\frac{1-\vartheta^k}{1-\vartheta}}, \end{aligned}$$

it follows

$$(3.9) \quad \Lambda_k \leq \left[\left(\frac{C}{\mu^{3(k-1)} (\delta\rho_0)^3} \right)^{\frac{1}{1-\vartheta}} M \right]^{1-\vartheta^k} \Lambda_0^{\vartheta^k}.$$

Applying Young's inequality we obtain, for any $\varrho > 0$,

$$\begin{aligned}
\Lambda_k &\leq (1 - \vartheta^k) \varrho^{\frac{1}{1-\vartheta^k}} \left(\frac{C}{\mu^{3(k-1)}(\delta\rho_0)^3} \right)^{\frac{1}{1-\vartheta}} M + \vartheta^k \varrho^{-\frac{1}{\vartheta^k}} \Lambda_0 \\
(3.10) \quad &\leq \varrho^{\frac{1}{1-\vartheta^k}} \left(\frac{C}{\mu^{3(k-1)}(\delta\rho_0)^3} \right)^{\frac{1}{1-\vartheta}} M + \varrho^{-\frac{1}{\vartheta^k}} \Lambda_0.
\end{aligned}$$

Set $\tilde{\Lambda}_k = \|u\|_{L^2((t_0+2\epsilon, t_1-2\epsilon), H^1(B(x_k, \delta\rho_k)))}$. As $I_k \supset J = (t_0 - 2\epsilon, t_1 - 2\epsilon)$ for any $k \geq 1$, (3.10) implies

$$(3.11) \quad \tilde{\Lambda}_k \leq \varrho^{\frac{1}{1-\vartheta^k}} \left(\frac{C}{\mu^{3(k-1)}(\delta\rho_0)^3} \right)^{\frac{1}{1-\vartheta}} M + \varrho^{-\frac{1}{\vartheta^k}} \Lambda_0.$$

Changing C if necessary, we can always assume that $C\rho_0^{-3} \geq 1$. In that case (3.11) gives

$$(3.12) \quad \tilde{\Lambda}_k \leq C\delta^{-\frac{3}{1-\vartheta}} \left(\varrho^{\frac{1}{1-\vartheta^k}} \mu^{-\frac{3(k-1)}{1-\vartheta}} M + \varrho^{-\frac{1}{\vartheta^k}} \Lambda_0 \right).$$

On the other hand, since u is Hölder continuous,

$$|u(\tilde{x}, t)| \leq [u]_\alpha |\tilde{x} - x|^\alpha + |u(x, t)|, \quad x \in B(x_k, \delta\rho_k), \quad t \in J.$$

Here and henceforth,

$$[w]_\alpha = \sup_{\substack{(x_1, t_1), (x_2, t_2) \in \bar{Q} \\ (x_1, t_1) \neq (x_2, t_2)}} \frac{|w(x_1, t_1) - w(x_2, t_2)|}{|x_1 - x_2|^\alpha + |t_1 - t_2|^{\frac{\alpha}{2}}}, \quad w \in C^{\alpha, \frac{\alpha}{2}}(\bar{Q}).$$

Whence

$$(3.13) \quad |\mathbb{S}^{n-1}| \delta^n \rho_k^n \int_J |u(\tilde{x}, t)|^2 dt \leq 2[u]_\alpha^2 \int_{B(x_k, \delta\rho_k) \times J} |\tilde{x} - x|^{2\alpha} dx dt + 2 \int_{B(x_k, \delta\rho_k) \times J} |u(x, t)|^2 dx dt,$$

Similarly, where $1 \leq i \leq n$,

$$(3.14) \quad |\mathbb{S}^{n-1}| \delta^n \rho_k^n \int_J |\partial_i u(\tilde{x}, t)|^2 dt \leq 2[\partial_i u]_\alpha^2 \int_{B(x_k, \delta\rho_k) \times J} |\tilde{x} - x|^{2\alpha} dx dt + 2 \int_{B(x_k, \delta\rho_k) \times J} |\partial_i u(x, t)|^2 dx dt,$$

A simple computation shows that $d_k = \mu^k d_0$ yielding

$$(3.15) \quad |\tilde{x} - x| \leq |\tilde{x} - x_k| + |x_k - x| \leq d_k + \delta\rho_k = (1 + c)d_k = (1 + c)\mu^k d_0.$$

Therefore, as a consequence of a combination of (3.13), (3.14) and (3.15),

$$\int_J |\partial_i u(\tilde{x}, t)|^2 dt + \int_J |\nabla u(\tilde{x}, t)|^2 dt \leq C\delta^{-n} \left(M^2 \mu^{2\alpha k} + \mu^{-nk} \tilde{\Lambda}_k^2 \right).$$

This and (3.12) imply, where $\varkappa = \max\left(n, \frac{3}{1-\vartheta}\right)$,

$$(3.16) \quad \int_J |u(\tilde{x}, t)|^2 dt + \int_J |\nabla u(\tilde{x}, t)|^2 dt \leq C\delta^{-\varkappa} \left(M^2 \mu^{2\alpha k} + \mu^{-nk} \left[\varrho^{\frac{2}{1-\vartheta^k}} \mu^{-\frac{6(k-1)}{1-\vartheta}} M^2 + \varrho^{-\frac{2}{\vartheta^k}} \Lambda_0^2 \right] \right).$$

We choose $\varrho > 0$ so that

$$\varrho^{\frac{2}{1-\vartheta^k}} \mu^{-\frac{6(k-1)}{1-\vartheta} - nk} = \mu^{2\alpha k}.$$

That is to say $\varrho = \mu^{\phi(k)}$, with

$$\phi(k) = \frac{[(2\alpha + n)(1 - \vartheta)k + 6(k - 1)](1 - \vartheta^k)}{2(1 - \vartheta)}.$$

Consequently, we have from (3.17)

$$(3.17) \quad \int_J |u(\tilde{x}, t)|^2 dt + \int_J |\nabla u(\tilde{x}, t)|^2 dt \leq C\delta^{-\varkappa} \left(M^2 \mu^{2\alpha k} + \mu^{-\psi(k)} \Lambda_0^2 \right),$$

where $\psi(k) = nk + \phi(k)/\vartheta^k$.

Let $t > 0$ and k be the integer so that $k \leq t < k + 1$. Using that

$$\psi(k) \leq \left[n + \frac{(2\alpha + n) + 6}{2(1 - \vartheta)} \right] \frac{k}{\vartheta^k} \leq \left[n + \frac{(2\alpha + n) + 6}{2(1 - \vartheta)} \right] e^{k(1 + |\ln \vartheta|)} \leq \left[n + \frac{(2\alpha + n) + 6}{2(1 - \vartheta)} \right] e^{t(1 + |\ln \vartheta|)},$$

we find

$$(3.18) \quad \int_J |u(\tilde{x}, t)|^2 dt + \int_J |\nabla u(\tilde{x}, t)|^2 dt \leq C\delta^{-\varkappa} \left(M^2 \mu^{2\alpha t} + \mu^{-\kappa} e^{ct} \Lambda_0^2 \right),$$

where $\kappa = n + \frac{(2\alpha + n) + 6}{2(1 - \vartheta)}$ and $c = 1 + |\ln \vartheta|$.

Thus, for $\frac{1}{\sigma} = e^{ct} (> 1)$,

$$(3.19) \quad \int_J |u(\tilde{x}, t)|^2 dt + \int_J |\nabla u(\tilde{x}, t)|^2 dt \leq C\delta^{-\varkappa} \left(M^2 \sigma^{2\gamma} + e^{\frac{\xi}{\sigma}} \Lambda_0^2 \right),$$

with $\gamma = \frac{\alpha |\ln \mu|}{c}$ and $\xi = |\ln \mu|^\kappa$.

Let $\omega = \cup_{\tilde{x} \in \Gamma} B(x_0(\tilde{x}), \rho_0)$ (independent on u) and $N = \|u\|_{L^2((0, T); H^1(\omega))}$. Then (3.19) entails

$$(3.20) \quad \int_J |u(\tilde{x}, t)|^2 dt + \int_J |\nabla u(\tilde{x}, t)|^2 dt \leq C\delta^{-\varkappa} \left(M^2 \sigma^{2\gamma} + e^{\frac{\xi}{\sigma}} N^2 \right).$$

Integrating over Γ , we obtain in a straightforward manner

$$(3.21) \quad \|u\|_{L^2(J, L^2(\Gamma))} + \|\nabla u\|_{L^2(J, L^2(\Gamma)^n)} \leq C\delta^{-\frac{\varkappa}{2}} \left(M\sigma^\gamma + e^{\frac{\xi}{\sigma}} N \right).$$

In light of (3.3), this inequality yields

$$(3.22) \quad \|u\|_{L^2(J, L^2(\Gamma))} + \|\nabla u\|_{L^2(J, L^2(\Gamma)^n)} \leq C\epsilon^{-\frac{\varkappa}{2}} \left(M\sigma^\gamma + e^{\frac{\xi}{\sigma}} N \right).$$

Applying Lemma 3.1 for some fixed $s \in (0, \frac{1}{2})$, we obtain

$$\begin{aligned} \|u\|_{L^2((t_0, t_0 + 2\epsilon), L^2(\Gamma))}, \|u\|_{L^2((t_1 - 2\epsilon, t_1), L^2(\Gamma))} &\leq 2^{1+s} C\epsilon^s \|u\|_{H^s((t_0, t_1), L^2(\Gamma))}, \\ \|\nabla u\|_{L^2((t_0, t_0 + 2\epsilon), L^2(\Gamma)^n)}, \|\nabla u\|_{L^2((t_1 - 2\epsilon, t_1), L^2(\Gamma)^n)} &\leq 2^{1+s} C\epsilon^s \|\nabla u\|_{H^s((t_0, t_1), L^2(\Gamma)^n)}. \end{aligned}$$

These estimates in (3.22) yields

$$\begin{aligned} C \left(\|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma)^n)} \right) &\leq \epsilon^{-\frac{\varkappa}{2}} \left(M\sigma^\gamma + e^{\frac{\xi}{\sigma}} N \right) \\ &\quad + \epsilon^s \left(\|u\|_{H^s((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{H^s((t_0, t_1), L^2(\Gamma)^n)} \right). \end{aligned}$$

We take $\epsilon = \sigma^{\frac{\gamma}{\varkappa}}$ in order to get

$$\begin{aligned} C \left(\|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma)^n)} \right) &\leq M\sigma^{\frac{\gamma}{2}} + e^{\frac{\xi}{\sigma}} N \\ &\quad + \sigma^{\frac{s\gamma}{\varkappa}} \left(\|u\|_{H^s((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{H^s((t_0, t_1), L^2(\Gamma)^n)} \right), \end{aligned}$$

provided that $0 < \sigma < \sigma_0 = \epsilon_0^{\frac{\varkappa}{\gamma}} (< 1)$. Letting $\mu = \min\left(\frac{\gamma}{2}, \frac{s\gamma}{\varkappa}\right)$, we get

$$C \left(\|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma)^n)} \right) \leq \sigma^\mu M + e^{\frac{\xi}{\sigma}} N,$$

which is the expected inequality. \square

The a priori estimate in the following lemma is well adapted to our purpose. It does not involve either the initial time data, nor the final time data.

Lemma 3.2. *There exists a constant $C > 0$, depending only on Ω , κ , K and T_0 , so that, for any $u \in H^1((t_0, t_1), H^2(\Omega))$ satisfying $Lu = 0$ in Q , we have*

$$(3.23) \quad C\|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))}.$$

Proof. Let $u \in H^1((t_0, t_1), H^2(\Omega))$ satisfying $Lu = 0$ in Q and set $v = e^{-t}u$. Then v solves the following equation

$$(3.24) \quad \operatorname{div}(A\nabla v) - v - \partial_t v = 0 \quad \text{in } Q.$$

Choose $\chi \in C_0^\infty(t_0, t_1)$ satisfying $0 \leq \chi \leq 1$, $\chi = 1$ in $(t_0 + \epsilon, t_1 - \epsilon)$ and, for some universal constant c , $|\chi'| \leq c\epsilon$.

We multiply (3.24) by χv and integrate over Q . We make then an integrate by parts in order to get

$$- \int_Q \chi A\nabla v \cdot \nabla v dxdt - \int_Q \chi v^2 dxdt + \int_\Sigma \chi u A\nabla u \cdot \nu d\sigma dt + \frac{1}{2} \int_Q v^2 \chi' dxdt,$$

from which we deduce in a straightforward manner

$$(3.25) \quad \int_Q \chi A\nabla u \cdot \nabla u dxdt + \int_Q \chi u^2 dxdt \leq e^{2(t_1-t_0)} \left(C \int_\Sigma (u^2 + |\nabla u|^2) d\sigma dx + \frac{1}{2} \int_Q u^2 |\chi'| dxdt \right).$$

On the other hand, as $\operatorname{supp}(\chi') \subset (t_0, t_1) \setminus (t_0 + \epsilon, t_1 - \epsilon)$, we have

$$\mathcal{J}_\epsilon^2 = \int_Q u^2 |\chi'| dxdt \leq \frac{c}{\epsilon} \int_{t_0}^{t_0-\epsilon} \int_\Omega u^2 dxdt + \int_{t_1-\epsilon}^{t_1} \int_\Omega u^2 dxdt.$$

Therefore

$$(3.26) \quad \limsup_{\epsilon \rightarrow 0} \mathcal{J}_\epsilon^2 \leq \int_\Omega u^2(x, t_0) dx + \int_\Omega u^2(x, t_1) dx.$$

We rewrite (3.25) in the form

$$(3.27) \quad C \|u\|_{L^2((t_0+\epsilon, t_1-\epsilon), H^1(\Omega))} \leq \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \mathcal{J}_\epsilon.$$

We apply Hardy's inequality in lemma 3.1. We obtain

$$\|u\|_{L^2((t_0, t_0+\epsilon), H^1(\Omega))}, \|u\|_{L^2((t_1-\epsilon, t_1), H^1(\Omega))} \leq c\epsilon^s \|u\|_{H^s((t_0, t_1), H^1(\Omega))}.$$

This and (3.27) produce

$$C \|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \epsilon^s \|u\|_{H^s((t_0, t_1), H^1(\Omega))} + \mathcal{J}_\epsilon.$$

Making $\epsilon \rightarrow 0$, we get by using (3.26)

$$C \|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|u(\cdot, t_0)\|_{L^2(\Omega)} + \|u(\cdot, t_1)\|_{L^2(\Omega)}.$$

We complete the proof by using the following inequality

$$C (\|u(\cdot, t_0)\|_{L^2(\Omega)} + \|u(\cdot, t_1)\|_{L^2(\Omega)}) \leq \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))}.$$

To prove this inequality, we proceed as in the proof of observability inequalities for parabolic equation. First, if $s_0 = \frac{3t_0+t_1}{4}$ and $s_1 = \frac{t_0+3t_1}{4}$, we get as a straightforward consequence of the Carleman inequality in Theorem 2.2

$$(3.28) \quad C \|u\|_{L^2((s_0, s_1) \times \Omega)} \leq \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))}.$$

Next, pick $\psi \in C^\infty([t_0, t_1])$ so that $0 \leq \psi \leq 1$, $\psi = 0$ in $[t_0, s_0]$ and $\psi = 1$ in $[s_1, t_1]$. Then $v = \psi u$ is the solution of the IBVP

$$\begin{cases} \operatorname{div}(A\nabla v) - \partial_t v = -\psi' u & \text{in } Q, \\ v = u & \text{in } \Sigma, \\ v(\cdot, t_0) = 0. \end{cases}$$

Hence

$$\int_Q A\nabla v \cdot \nabla v dxdt + \int_\Sigma v A\nabla v \cdot \nu - \frac{1}{2} \int_Q \partial_t v^2 dxdt = - \int_Q \psi' u v dxdt.$$

But

$$\int_Q \partial_t v^2 dxdt = \int_\Omega v^2(x, t_1) dx.$$

That is we have

$$\int_Q A\nabla v \cdot \nabla v dx dt + \int_\Sigma v A\nabla v \cdot \nu + \frac{1}{2} \int_\Omega v^2(x, t_1) dx. = - \int_Q \psi' u v dx dt.$$

We derive from this identity

$$(3.29) \quad C \left(\|v(\cdot, t_1)\|_{L^2(\Omega)}^2 + \|\nabla v\|_{L^2(Q)}^2 \right) \leq \|u\|_{L^2((t_0, t_1), L^2(\Gamma))}^2 + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))}^2 + \|\psi' u\|_{L^2(Q)} \|v\|_{L^2(Q)}$$

Noting that

$$w \rightarrow \left(\int_\Omega |\nabla w|^2 dx + \int_\Gamma w^2(x) d\sigma(x) \right)^{\frac{1}{2}}$$

defines an equivalent norm on $H^1(\Omega)$, we get

$$\|v\|_{L^2(Q)}^2 \leq c_\Omega \left(\|\nabla v\|_{L^2(Q)}^2 + \|u\|_{L^2((t_0, t_1), L^2(\Gamma))}^2 \right).$$

Applying Young's inequality, we obtain

$$\|\psi' u\|_{L^2(Q)} \|v\|_{L^2(Q)} \leq \frac{1}{2\epsilon} \|\psi' u\|_{L^2(Q)}^2 + \frac{c_\Omega \epsilon}{2} \|\nabla v\|_{L^2(Q)}^2 + \frac{c_\Omega \epsilon}{2} \|u\|_{L^2((t_0, t_1), L^2(\Gamma))}^2.$$

This inequality in (3.29), with ϵ sufficiently small, yields

$$(3.30) \quad C \|u(\cdot, t_1)\|_{L^2(\Omega)} = C \|v(\cdot, t_1)\|_{L^2(\Omega)} \leq \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\psi' u\|_{L^2(Q)}.$$

Bearing in mind that $\text{supp}(\psi') \subset [s_0, s_1]$, we derive from (3.28)

$$\begin{aligned} \|\psi' u\|_{L^2(Q)} &\leq C \|u\|_{L^2(\Omega \times (s_0, s_1))} \\ &\leq \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))}. \end{aligned}$$

This in (3.30) yields

$$C \|u(\cdot, t_1)\|_{L^2(\Omega)} = C \|v(\cdot, t_1)\|_{L^2(\Omega)} \leq \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))}.$$

As the Carleman estimate in Theorem 2.2 still holds for the backward parabolic equation $\text{div}(A\nabla u) + \partial_t u = 0$, we have similarly

$$C \|u(\cdot, t_0)\|_{L^2(\Omega)} \leq \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))}.$$

The proof is then complete. \square

In light of Proposition 3.1 and Lemma 3.2, we get

Corollary 3.1. *There exist $\omega \Subset \Omega$ and four constants $\mu > 0$, $c > 0$, $C > 0$ and $0 < \sigma_0 < 1$, that can depend on Ω , κ , K , T_0 and α , so that, for any $u \in \mathcal{X}^1$ satisfying $Lu = 0$ in Q and $0 < \sigma < \sigma_0$, we have*

$$(3.31) \quad \|u\|_{L^2((t_0, t_1), H^1(\omega))} \leq C \left(\sigma^\mu \|u\|_{\mathcal{X}^1} + e^{\frac{c}{\sigma}} \|u\|_{L^2((t_0, t_1), H^1(\omega))} \right).$$

We now establish a result quantifying the uniqueness of continuation from a subdomain to an another subdomain. Precisely, we prove

Proposition 3.2. *Let $\omega \Subset \Omega$ and $\tilde{\omega} \Subset \Omega$ and $s \in (0, \frac{1}{2})$. There exist three constants $\gamma > 0$, $C > 0$ and $0 < \epsilon_0 < 1$, depending only on Ω , κ , K , T_0 , s , ω and $\tilde{\omega}$, so that, for any $u \in H^1((t_0, t_1), H^2(\Omega))$ satisfying $Lu = 0$ in Q and $0 < \epsilon < \epsilon_0$, we have*

$$(3.32) \quad C \|u\|_{L^2((t_0, t_1), H^1(\tilde{\omega}))} \leq \epsilon^{2s} \|u\|_{H^1((t_0, t_1), H^1(\Omega))} + e^{\frac{c}{\epsilon}} \|u\|_{L^2((t_0, t_1), H^1(\omega))}.$$

Proof. Set $d_0 = \text{dist}(\overline{\omega} \cup \overline{\tilde{\omega}}, \Gamma)$. Fix $0 < \epsilon < \epsilon_0 := \min\left(\frac{d_0^2}{9}, 1\right)$ and let $0 < \delta < \sqrt{\epsilon}$. Let $x_0 \in \omega$, $x \in \overline{\tilde{\omega}}$ and $\psi : [0, 1] \rightarrow \Omega$ be a C^1 -piecewise smooth path joining x_0 to x so that $\ell(\psi) \leq d_g(x_0, x) + 1$. Let $\tau_0 = 0$ and $\tau_{k+1} = \inf\{\tau \in [\tau_k, 1]; \psi(\tau) \notin B(\psi(\tau_k), \delta)\}$, $k \geq 0$. We claim that there exists an integer $N \geq 1$ so that $\psi(1) \in B(\psi(\tau_N), \frac{\delta}{2})$. If not, we would have $\psi(1) \notin B(\psi(\tau_k), \frac{\delta}{2})$ for any $k \geq 0$. As the sequence (τ_k) is non decreasing and bounded from above by 1, it converges to $\hat{\tau} \leq 1$. In particular, there exists an integer $k_0 \geq 1$ so that $\psi(\tau_k) \in B(\psi(\hat{\tau}), \frac{\delta}{2})$, $k \geq k_0$. But this contradicts the fact that $|\psi(\tau_{k+1}) - \psi(\tau_k)| = \delta$, $k \geq 0$.

Let us check that $N \leq N_0$ where N_0 depends only on \mathbf{d}_g and δ . Pick $1 \leq j \leq n$ so that

$$\max_{1 \leq i \leq n} |\psi_i(\tau_{k+1}) - \psi_i(\tau_k)| = |\psi_j(\tau_{k+1}) - \psi_j(\tau_k)|.$$

Then

$$\delta \leq n |\psi_j(\tau_{k+1}) - \psi_j(\tau_k)| = n \left| \int_{\tau_k}^{\tau_{k+1}} \dot{\psi}_j(t) dt \right| \leq n \int_{\tau_k}^{\tau_{k+1}} |\dot{\psi}(t)| dt.$$

Consequently, where $\tau_{N+1} = 1$,

$$(N+1)\delta \leq n \sum_{k=0}^N \int_{\tau_k}^{\tau_{k+1}} |\dot{\psi}(\tau)| d\tau = n\ell(\psi) \leq n(\mathbf{d}_g + 1).$$

Therefore

$$N \leq N_0 = \left\lceil \frac{n(\mathbf{d}_g + 1)}{\delta} \right\rceil.$$

Let $x_k = \psi(t_k)$, $0 \leq k \leq N$. If $|z - x_{k+1}| < \delta$, then $|z - x_k| \leq |z - x_{k+1}| + |x_{k+1} - x_k| < 2\delta$. In other words $B(x_{k+1}, \delta) \subset B(x_k, 2\delta)$.

As above,

$$I_0 = (t_0, t_1), \quad I_k = (t_0 + \epsilon\eta_k, t_1 - \epsilon\eta_k) \quad \text{with} \quad \eta_k = \sum_{j=0}^{k-1} \frac{1}{2^j}, \quad k \geq 1.$$

Let $u \in H^1((t_0, t_1), H^2(\Omega))$ satisfying $Lu = 0$ in Q . In the rest of this proof

$$M = M(u) = \|u\|_{H^1((t_0, t_1), H^1(\Omega))}.$$

From the three-cylinder inequality (2.1), we have

$$\|u\|_{L^2(I_{k+1}, H^1(B(x_{k+1}, \delta)))} \leq C_0 \delta^{-3} M^{1-\vartheta} \|u\|_{L^2(I_k, H^1(B(x_k, \delta)))}^{\vartheta}.$$

Setting $\Lambda_k = \|u\|_{L^2(I_k, H^1(B(x_k, \delta)))}$, $0 \leq k \leq N$ and $\Lambda_{N+1} = \|u\|_{L^2(I_{N+1}, H^1(B(x, \frac{\delta}{2})))}$, we can rewrite this inequality in the form

$$(3.33) \quad \Lambda_{k+1} \leq C_0 M^{1-\vartheta} \Lambda_k^{\vartheta}.$$

Let $\beta = \vartheta^{N+1}$. We get in a straightforward manner from (3.33)

$$\Lambda_{N+1} \leq (C_0 \delta^{-3})^{\frac{1-\vartheta^{N+2}}{1-\vartheta}} M^{1-\beta} \Lambda_0^{\beta}.$$

Since we can always assume, by changing C_0 , that $C_0 \delta^{-3} \geq 1$, the last inequality gives

$$\Lambda_{N+1} \leq C \delta^{\frac{-3}{1-\vartheta}} M^{1-\beta} \Lambda_0^{\beta}.$$

This and Young's inequality lead, for $\sigma > 0$,

$$\Lambda_{N+1} \leq C \delta^{\frac{-3}{1-\vartheta}} ((1-\beta)\sigma^{\frac{\beta}{1-\beta}} M + \beta\sigma^{-1}\Lambda_0) \leq C_1 (\sigma^{\frac{\beta}{1-\beta}} M + \sigma^{-1}\Lambda_0).$$

That is

$$\Lambda_{N+1} \leq C \delta^{\frac{-3}{1-\vartheta}} (\sigma^{\frac{\beta}{1-\beta}} M + \sigma^{-1}\Lambda_0).$$

If δ is sufficiently small $B(x_0, \delta) \subset \omega$. On the other hand $\bar{\omega}$ can be recovered by $O(\delta^{-n})$ balls of radius $\frac{\delta}{2}$. Whence, bearing in mind that $I_{N+1} \supset (t_0 + 2\epsilon, t_1 - 2\epsilon)$,

$$(3.34) \quad \|u\|_{L^2((t_0+2\epsilon, t_1-2\epsilon), H^1(\bar{\omega}))} \leq \delta^{-\frac{3}{1-\vartheta}-n} \left(\sigma^{\frac{\beta}{1-\beta}} M + \sigma^{-1} \|u\|_{L^2((t_0, t_1), H^1(\omega))} \right).$$

Reducing ϵ_0 if necessary, we can assume that $\delta < 1$. Then we take σ in (3.34) in order that

$$\delta^{-\frac{3}{1-\vartheta}-n} \sigma^{\frac{\beta}{1-\beta}} = \delta.$$

In that case

$$\sigma^{-1} \leq \delta^{-\frac{m}{\beta}},$$

with $m = \frac{4-\vartheta}{1-\vartheta} + n$.

But

$$\beta = \vartheta^{N+1} = e^{-(N+1)|\ln \vartheta|} \geq \vartheta e^{-N_0 |\ln \vartheta|} \geq \vartheta e^{-\frac{n|\ln \vartheta|(\mathbf{d}_g+1)}{\delta}}.$$

Hence

$$\delta^{-\frac{3}{1-\vartheta}-n} \sigma^{-1} \leq e^{\frac{\kappa}{\delta}}.$$

This inequality in (3.34) yields

$$(3.35) \quad C \|u\|_{L^2((t_0+2\epsilon, t_1-2\epsilon), H^1(\tilde{\omega}))} \leq \delta M + e^{\frac{\kappa}{\delta}} \|u\|_{L^2((t_0, t_1), H^1(\omega))}.$$

One more time Hardy's inequality in Lemma 3.1, for some fixed $s \in (0, \frac{1}{2})$, yields

$$(3.36) \quad \|u\|_{L^2((t_0, t_0+2\epsilon), H^1(\tilde{\omega}))}, \|u\|_{L^2((t_1-2\epsilon, t_1), H^1(\tilde{\omega}))} \leq 2^s c \epsilon^s M.$$

If $0 < \epsilon \leq \epsilon_0$, for some sufficiently small ϵ_0 , we can take $\delta = \frac{\sqrt{\epsilon}}{2}$, then (3.35) and (3.36) entail

$$C \|u\|_{L^2((t_0, t_1), H^1(\tilde{\omega}))} \leq \epsilon^s M + e^{\frac{\kappa}{\sqrt{\epsilon}}} \|u\|_{L^2((t_0, t_1), H^1(\omega))}.$$

Substituting ϵ by ϵ^2 , we obtain

$$C \|u\|_{L^2((t_0, t_1), H^1(\tilde{\omega}))} \leq \epsilon^{2s} M + e^{\frac{\kappa}{\epsilon}} \|u\|_{L^2((t_0, t_1), H^1(\omega))}.$$

This is the expected inequality. □

We are now ready to prove the result showing how one can quantify the uniqueness of continuation from an interior data

Theorem 3.1. *Let $\omega \Subset \Omega$. There exist two constants $C > 0$ and $\gamma > 0$, depending only on Ω , κ , K , ω and T_0 , so that, for any $u \in \mathcal{X}^1$ satisfying $Lu = 0$ in Q , we have*

$$C \|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq \|u\|_{\mathcal{X}^1} \left| \ln \left(\frac{\|u\|_{\mathcal{X}^1}}{\|u\|_{L^2((t_0, t_1), H^1(\omega))}} \right) \right|^{-\gamma} + \|u\|_{L^2((t_0, t_1), H^1(\omega))}.$$

Proof. Let $u \in \mathcal{X}^1$ satisfying $Lu = 0$ in Q . From Corollary 3.1, there exist $\tilde{\omega} \Subset \Omega$ and four constants $\mu > 0$, $c > 0$, $C > 0$ and $0 < \sigma_0 < 1$ so that, for any $0 < \sigma < \sigma_0$, we have

$$C \|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq \sigma^\mu \|u\|_{\mathcal{X}^1} + e^{\frac{c}{\sigma}} \|u\|_{L^2((t_0, t_1), H^1(\tilde{\omega}))}.$$

But according to Proposition 3.2, with fixed $s \in (0, \frac{1}{2})$, there exist three constants $\gamma > 0$, $C > 0$ and $0 < \epsilon_0 < 1$ so that, for any $0 < \epsilon < \epsilon_0$,

$$C \|u\|_{L^2((t_0, t_1), H^1(\tilde{\omega}))} \leq \epsilon^{2s} \|u\|_{\mathcal{X}^1} + e^{\frac{c}{\epsilon}} \|u\|_{L^2((t_0, t_1), H^1(\omega))}.$$

The last two inequalities yield

$$(3.37) \quad C \|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq (\sigma^\mu + \epsilon^{2s} e^{\frac{c}{\sigma}}) \|u\|_{\mathcal{X}^1} + e^{\frac{c}{\epsilon}} e^{\frac{c}{\sigma}} \|u\|_{L^2((0, T), H^1(\omega))}, \quad 0 < \sigma < \sigma_0, \quad 0 < \epsilon < \epsilon_0.$$

Reducing σ_0 if necessary, we assume that $\sigma_0^\mu e^{-\frac{c}{\sigma_0}} < \epsilon_0^{\frac{1}{2s}}$. In that case we can take in (3.37) ϵ so that $\epsilon^{2s} e^{\frac{c}{\sigma}} = \sigma^\mu$. We get, by using that $\epsilon \geq \sigma^p$, with $p = \frac{\mu}{2s}$,

$$C \|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq \sigma^\mu \|u\|_{\mathcal{X}^1} + e^{\frac{c}{\sigma^p}} e^{\frac{c}{\sigma}} \|u\|_{L^2((t_0, t_1), H^1(\omega))}, \quad 0 < \sigma < \sigma_0.$$

Whence, where $q = \max(1, p)$,

$$(3.38) \quad C \|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq \sigma^\mu \|u\|_{\mathcal{X}^1} + e^{\frac{c}{\sigma^q}} \|u\|_{L^2((t_0, t_1), H^1(\omega))}, \quad 0 < \sigma < \sigma_0.$$

We end up getting the expected inequality by minimizing with respect to σ . □

4. STABILITY OF PARABOLIC CAUCHY PROBLEMS

In order to prove our stability estimate for the Cauchy problem, one additional step is necessary. It consists in quantifying the uniqueness of continuation from the Cauchy data to an interior subdomain. To do that we start with the following proposition.

Proposition 4.1. *Let Γ_0 be a nonempty open subset of Γ and $\nu \in (0, \frac{1}{2})$. There exist $\omega \Subset \Omega$ and two constants $C > 0$ and $c > 0$, depending only on Ω , κ , K , Γ_0 , s and T_0 , so that, for any $u \in H^1((t_0, t_1), H^2(\Omega))$ satisfying $Lu = 0$ in Q and $0 < \epsilon < \frac{t_1 - t_0}{2}$, we have*

$$(4.1) \quad C \|u\|_{L^2(\omega \times (t_0, t_1))} \leq \epsilon^\nu \|u\|_{H^1((t_0, t_1), H^1(\Omega))} + e^{\frac{c}{\epsilon^2}} (\|u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma_0))}).$$

Proof. Pick $0 < \epsilon < \frac{t_1 - t_0}{2}$, $0 < \eta < \epsilon$ and let $s \in [t_0 + \epsilon, t_1 - \epsilon]$. Let $\tilde{x} \in \Gamma_0$ and $x_0 = x_0(\tilde{x})$ and ρ be as in the definition of the uniform exterior sphere property. Fix $r > 0$ in such a way that $B(x_0, \rho + r) \cap \Gamma \subset \Gamma_0$.

Let $\phi \in C_0^\infty(B(\tilde{x}, \rho + r))$, $\phi = 1$ on $B(\tilde{x}, \rho + \frac{3r}{4})$ and $|\partial^\alpha \phi| \leq cr^{-|\alpha|}$, $|\alpha| \leq 2$, where c is a constant independent on r . Set, where $0 < \delta < 1$ is to be determined in the sequel,

$$\begin{aligned} Q_0 &= [B(x_0, \rho + r) \cap \Omega] \times (s - \eta, s + \eta), \\ Q_1 &= [B(x_0, \rho + r/2) \cap \Omega] \times (s - \delta\eta, s + \delta\eta), \\ Q_2 &= \{[B(x_0, \rho + r) \setminus B(x_0, \rho + 3r/4)] \cap \Omega\} \times (s - \eta, s + \eta), \\ \Sigma_0 &= [B(x_0, \rho + r) \cap \Gamma] \times (s - \eta, s + \eta). \end{aligned}$$

We apply Theorem 2.2, with $L_s = L$, λ fixed and $\psi = (\rho + r)^2 - |x - x_0|^2$, to ϕu so that $u \in H^1((t_0, t_1), H^2(\Omega))$ satisfies $Lu = 0$ in Q in order to obtain

$$(4.2) \quad C \int_{Q_1} u^2 e^{-2\tau\varphi} dxdt \leq \int_{Q_0} (L[\phi u])^2 e^{-2\tau\varphi} dxdt + \int_{\Sigma_0} (u^2 + |\nabla u|^2 + (\partial_t u)^2) e^{-2\tau\varphi} dxdt.$$

But

$$L[\phi u] = L\phi u + 2A\nabla\phi \cdot \nabla u.$$

Whence, $\text{supp}(L[\phi u]) \cap Q_0 \subset Q_2$ together with (4.2) yield

$$(4.3) \quad C \int_{Q_1} u^2 e^{-2\tau\varphi} dxdt \leq \int_{Q_2} (u^2 + |\nabla u|^2) e^{-2\tau\varphi} dxdt + \int_{\Sigma_0} (u^2 + |\nabla u|^2 + (\partial_t u)^2) e^{-2\tau\varphi} dxdt.$$

Let

$$\begin{aligned} \alpha &= \eta^{-2} \left[e^{4\lambda(\rho+r)^2} - e^{\lambda(2(\rho+r)^2 - (\rho+r/2)^2)} \right] := \eta^{-2} \tilde{\alpha}, \\ \beta &= \eta^{-2} \left[e^{4\lambda(\rho+r)^2} - e^{2\lambda(\rho+r)^2} \right] := \eta^{-2} \tilde{\beta}, \\ \gamma &= \eta^{-2} \left[e^{4\lambda(\rho+r)^2} - e^{\lambda(2(\rho+r)^2 - (\rho+3r/4)^2)} \right] := \eta^{-2} \tilde{\gamma}. \end{aligned}$$

Then it is straightforward to check that

$$\begin{aligned} \varphi(x, t) &\leq \frac{\alpha}{1 - \delta} \text{ in } Q_1, \\ \varphi(x, t) &\geq \beta \text{ in } \Sigma_0, \\ \varphi(x, t) &\geq \gamma \text{ in } Q_2. \end{aligned}$$

Noting $\frac{\beta}{\alpha} < 1 < \frac{\gamma}{\alpha}$, we can choose $0 < \theta < 1$ so that

$$\frac{1}{1 - \delta} = \theta \frac{\beta}{\alpha} + (1 - \theta) \frac{\gamma}{\alpha} = \theta \frac{\tilde{\beta}}{\tilde{\alpha}} + (1 - \theta) \frac{\tilde{\gamma}}{\tilde{\alpha}} > 1.$$

With this choice of δ , (4.3) yields

$$(4.4) \quad C \int_{Q_1} u^2 dxdt \leq e^{-4b\eta^{-2}\tau} \int_{Q_2} (u^2 + |\nabla u|^2) dxdt + e^{4a\eta^{-2}\tau} \int_{\Sigma_0} (u^2 + |\nabla u|^2 + (\partial_t u)^2) dxdt.$$

Here $a = \frac{1}{2}(1 - \theta)(\tilde{\gamma} - \tilde{\beta})$ and $b = \frac{1}{2}\theta(\tilde{\gamma} - \tilde{\beta})$.

Let $\eta = \frac{\epsilon}{2}$, $s_0 = t_0 + \epsilon - \frac{\epsilon}{2}$, $s_1 = s_0 + \delta\frac{\epsilon}{2} \dots s_k = s_0 + k\delta\frac{\epsilon}{2}$. Let $N = N(\epsilon)$ so that

$$\bigcup_{k=0}^N \left[s_k - \delta\frac{\epsilon}{2}, s_k + \delta\frac{\epsilon}{2} \right] \supset [t_0 + \epsilon, t_1 - \epsilon].$$

If Q_j^k (resp. Σ_0^k) denotes Q_j (resp. Σ_0), $j = 1, 2$, when s is substituted by s_k , then it follows from (4.3)

$$\sum_{k=0}^N C \int_{Q_1^k} u^2 dxdt \leq e^{a\epsilon^{-2}\tau} \sum_{k=0}^N \int_{Q_2^k} (u^2 + |\nabla u|^2) dxdt + e^{-b\epsilon^{-2}\tau} \sum_{k=0}^N \int_{\Sigma_0^k} (u^2 + |\nabla u|^2 + (\partial_t u)^2) dxdt.$$

Note that the intervals $[s_k - \frac{\epsilon}{2}, s_k + \frac{\epsilon}{2}]$ can overlap, but their union can cover at most two times a subdomain of (t_0, t_1) . Whence

$$(4.5) \quad CI \leq e^{a\epsilon^{-2}\tau} N + e^{-b\epsilon^{-2}\tau} M, \quad \tau \geq \tau_0,$$

where we used the following temporary notation

$$\begin{aligned} I &= \|u\|_{L^2([B(x_0, \rho+r/2) \cap \Omega] \times (t_0 + \epsilon, t_1 - \epsilon))}, \\ M &= \|u\|_{L^2((t_0, t_1), H^1(\Omega))}, \\ N &= \|u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma_0))}. \end{aligned}$$

In (4.5), substituting τ by $\epsilon^2\tau$, we get

$$(4.6) \quad CI \leq e^{a\tau} M + e^{-b\tau} N, \quad \tau \geq \epsilon^{-2}\tau_0.$$

Set

$$\tau_1 = \frac{\ln \frac{N}{M}}{a+b}.$$

If $\tau_1 \geq \epsilon^{-2}\tau_0$, then $\tau = \tau_1$ in (4.6) yields

$$(4.7) \quad CI \leq M^\vartheta N^{1-\vartheta},$$

with $\vartheta = \frac{b}{a+b}$.

When $\tau_1 < \epsilon^{-2}\tau_0$,

$$M < e^{(a+b)\epsilon^{-2}\tau_0} N.$$

This inequality entails

$$(4.8) \quad I \leq M = M^\vartheta M^{1-\vartheta} \leq M^\vartheta e^{(1-\vartheta)(a+b)\epsilon^{-2}\tau_0} N^{1-\vartheta}.$$

So in any case one of estimates (4.7) and (4.8) holds. In other words, we proved

$$\begin{aligned} e^{-\frac{\sigma}{\epsilon^2}} \|u\|_{L^2([B(x_0, \rho+r/2) \cap \Omega] \times (t_0 + \epsilon, t_1 - \epsilon))} \\ \leq \|u\|_{L^2((t_0, t_1), H^1(\Omega))}^\vartheta \left(\|u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma_0))} \right)^{1-\vartheta}. \end{aligned}$$

Fix $\omega \in B(x_0, \rho + r/2) \cap \Omega$. Then the last inequality implies

$$(4.9) \quad e^{-\frac{\sigma}{\epsilon^2}} \|u\|_{L^2(\omega \times (t_0 + \epsilon, t_1 - \epsilon))} \leq \|u\|_{L^2((t_0, t_1), H^1(\Omega))}^\vartheta \left(\|u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma_0))} \right)^{1-\vartheta},$$

for $\sigma > 0$, where $\gamma = \frac{1-\vartheta}{\vartheta}$.

Once again from Hardy's inequality in Lemma 3.1, where $M_1 = \|u\|_{H^1((t_0, t_1), H^1(\Omega))}$,

$$\|u\|_{L^2(\omega \times (t_0, t_0 + \epsilon))}, \|u\|_{L^2(\omega \times (t_1 - \epsilon, t_1))} \leq c' \epsilon^\nu M_1.$$

Combined with (4.9), this inequality yields

$$C \|u\|_{L^2(\omega \times (t_0, t_1))} \leq \left(\sigma^\gamma e^{\frac{\sigma}{\epsilon^2}} + \epsilon^\nu \right) M_1 + \sigma^{-1} e^{\frac{\sigma}{\epsilon^2}} \left(\|u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma_0))} \right).$$

In this inequality we take σ so that $\sigma^\gamma = \epsilon^\nu e^{\frac{\kappa}{2\sigma}}$. Noting that $\sigma^{-1} \leq \epsilon^{-\frac{\kappa}{\gamma}}$, we find

$$C\|u\|_{L^2(\omega \times (t_0, t_1))} \leq \epsilon^\nu M_1 + e^{\frac{\kappa}{2\sigma}} (\|u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma_0))}).$$

The proof is then complete. \square

Proposition 4.1 gives an estimate only in $L^2(\omega \times (t_0, t_1))$. But we can derive from it an estimate in $L^2((t_0, t_1), H^1(\omega))$ by using the following Caccioppoli type inequality for the parabolic equation $Lu = 0$.

Lemma 4.1. *Let $\omega_0 \Subset \omega_1 \Subset \Omega$. There exist a constant $C > 0$, depending only on Ω , κ , K , T_0 , ω_0 and ω_1 , so that, for $0 < \epsilon < \frac{t_1 - t_0}{4}$ and $u \in H^1((t_0, t_1), H^2(\Omega))$ satisfying $Lu = 0$ in Q , we have*

$$(4.10) \quad C\|u\|_{L^2((t_0, t_1), H^1(\omega_0))} \leq \|u\|_{H^1((t_0, t_1), L^2(\omega_1))}.$$

Proof. Let $u \in H^1((0, T), H^2(\Omega))$ satisfying $Lu = 0$ in Q . By Green's formula, for any $v \in L^2((t_0, t_1), H_0^1(\Omega))$, we have

$$(4.11) \quad \int_{t_0}^{t_1} \int_{\Omega} A \nabla u \cdot \nabla v dx dt - \int_{t_0}^{t_1} \int_{\Omega} \partial_t u v dx dt = 0.$$

Let $\phi \in C_0^\infty(\omega_1)$ satisfying $0 \leq \phi \leq 1$ and $\phi = 1$ in ω_0 .

Taking $v = \phi^2 u$ in (4.11), we get in straightforward manner

$$\int_{t_0}^{t_1} \int_{\omega_1} \phi^2 A \nabla u \cdot \nabla u dx dt = -2 \int_{t_0}^{t_1} \int_{\omega_1} (\phi \nabla u) \cdot (u A \nabla \phi) dx dt + \int_{t_0}^{t_1} \int_{\omega_1} \phi^2 \partial_t u u dx dt.$$

But

$$\int_{t_0}^{t_1} \int_{\omega_1} \phi^2 A \nabla u \cdot \nabla u dx dt \geq \kappa \int_{t_0}^{t_1} \int_{\omega_1} \phi^2 |\nabla u|^2 dx dt.$$

Therefore

$$(4.12) \quad \kappa \int_{t_0}^{t_1} \int_{\omega_1} \phi^2 |\nabla u|^2 dx dt \leq -2 \int_{t_0}^{t_1} \int_{\omega_1} (\phi \nabla u) \cdot (u A \nabla \phi) dx dt + \int_{t_0}^{t_1} \int_{\omega_1} \phi^2 \partial_t u u dx dt.$$

An elementary convexity inequality yields

$$(4.13) \quad 2 \left| \int_{t_0}^{t_1} \int_{\omega_1} (\phi \nabla u) \cdot (u A \nabla \phi) dx dt \right| \leq \frac{\kappa}{2} \int_{t_0}^{t_1} \int_{\omega_1} \phi^2 |\nabla u|^2 dx dt + C \int_{t_0}^{t_1} \int_{\omega_1} u^2 dx dt.$$

On the other hand, we have

$$(4.14) \quad \left| \int_{t_0}^{t_1} \int_{\omega_1} \phi^2 \partial_t u u dx dt \right| \leq \int_{t_0}^{t_1} \int_{\Omega} \phi^2 u^2 dx dt + \int_{t_0}^{t_1} \int_{\Omega} \phi^2 (\partial_t u)^2 dx dt$$

We combine (4.12), (4.13) and (4.14) in order to get

$$C\|\nabla u\|_{L^2((t_0, t_1), L^2(\omega_0))} \leq \|u\|_{L^2(\omega_1 \times (t_0, t_1))} + \|\partial_t u\|_{L^2(\omega_1 \times (t_0, t_1))}.$$

Or equivalently

$$C\|u\|_{L^2((t_0, t_1), H^1(\omega_0))} \leq \|u\|_{H^1((t_0, t_1), L^2(\omega_1))}.$$

\square

We observe that if $u \in H^2((t_0, t_1), H^2(\Omega))$ satisfies $Lu = 0$, then $\partial_t u \in H^1((t_0, t_1), H^2(\Omega))$ satisfies $L\partial_t u = 0$. Therefore, as an immediate consequence of Caccioppoli's type inequality (4.10) and Proposition 4.1 (applied both to u and $\partial_t u$), we have

Corollary 4.1. *Let Γ_0 be a nonempty open subset of Γ and $\nu \in (0, \frac{1}{2})$. There exist $\omega \Subset \Omega$ and two constants $C > 0$ and $c > 0$, that can depend on Ω , κ , K , T_0 , s and Γ_0 so that, for any $u \in H^2((t_0, t_1), H^2(\Omega))$ satisfying $Lu = 0$ and $\sigma > 0$, we have*

$$(4.15) \quad C\|u\|_{L^2((t_0, t_1), H^1(\omega))} \leq \epsilon^\nu \|u\|_{H^2((t_0, t_1), H^1(\Omega))} + e^{\frac{\kappa}{2\sigma}} (\|u\|_{H^2((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{H^1((t_0, t_1), L^2(\Gamma_0))}).$$

We are now in position to complete the proof of Theorem 1.1. We recall that

$$\mathcal{C}(u, \Gamma_0) = \|u\|_{H^2((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{H^1((t_0, t_1), L^2(\Gamma_0))}.$$

In light of inequality (3.38) in the end of the proof of Theorem 3.1 and inequality (4.15), we get

$$C\|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq (\sigma^\mu + e^{\frac{c}{\sigma^q}} \epsilon^\nu) M + e^{\frac{c}{\sigma^q}} e^{\frac{c'}{\epsilon^2}} \mathcal{C}(u, \Gamma_0).$$

The rest of the proof is quite similar to that of the end of Theorem 3.1.

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