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LOGARITHMIC STABILITY OF PARABOLIC CAUCHY PROBLEMS

MOURAD CHOULLI AND MASAHIRO YAMAMOTO

ABSTRACT. The uniqueness of parabolic Cauchy problems is nowadays a classical problem and since Hadamard [4] these kind of problems are known to be ill-posed and even severely ill-posed. Until now there are only few partial results concerning the quantification in the stability for parabolic Cauchy problems. In the present article, we bring the complete answer to this issue.

Mathematics subject classification : 35R25, 35K99, 58J35

Key words : Parabolic Cauchy problems, logarithmic stability, Carleman inequality, Hardy inequality.

1. INTRODUCTION

Throughout this article, Ω is a bounded domain of \mathbb{R}^n with Lipschitz boundary Γ and

$$L = \operatorname{div}(A\nabla \cdot) - \partial_t.$$

Here $A = (a^{ij})$ is a symmetric matrix whose coefficients belong to $W^{1,\infty}(\Omega)$ and satisfy: there exist $\kappa > 0$ and $K > 0$ so that

$$(1.1) \quad A(x)\xi \cdot \xi \geq \kappa|\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n,$$

and

$$(1.2) \quad \|a^{ij}\|_{W^{1,\infty}(\Omega)} \leq K, \quad 1 \leq i, j \leq n.$$

We recall that Ω has the uniform exterior sphere property if there exists $\rho > 0$ so that, to any $\tilde{x} \in \Gamma$ corresponds $x_0 = x_0(\tilde{x}) \in \mathbb{R}^n \setminus \overline{\Omega}$ for which

$$B(x_0, \rho) \cap \Omega = \emptyset \quad \text{and} \quad \overline{B(x_0, \rho)} \cap \overline{\Omega} = \{\tilde{x}\}.$$

Let $t_0 < t_1$, $Q = \Omega \times (t_0, t_1)$ and to simplify the notations we set, where $0 < \alpha < 1$,

$$\mathcal{X}^\alpha(Q) = C^{1+\alpha, (1+\alpha)/2}(\overline{Q}) \cap L^2((t_0, t_1), H^1(\Omega)).$$

We equip $\mathcal{X}^\alpha(Q)$ with its natural norm

$$\|u\|_{\mathcal{X}^\alpha(Q)} = \|u\|_{C^{1+\alpha, (1+\alpha)/2}(\overline{Q})} + \|u\|_{L^2((t_0, t_1), H^1(\Omega))}.$$

Let $\mathcal{Y}^\alpha(Q) = \mathcal{X}^\alpha(Q) \cap H^1((t_0, t_1), H^2(\Omega))$.

From now on we assume that $t_1 - t_0 \leq T_0$, for some fixed $T_0 > 0$.

We are mainly concerned in the present work with the stability issue for the Cauchy problem associated to the parabolic operator L . Precisely we are going to prove

Theorem 1.1. *Assume that Ω possesses the the uniform exterior sphere property. Let Γ_0 be a nonempty open subset of Γ and $0 < \alpha < 1$. Then there exist two constant $C > 0$ and $\gamma > 0$ depending on Ω , κ , K , T_0 , α and Γ_0 , so that, for any $u \in \mathcal{Y}^\alpha(Q)$ satisfying $Lu = 0$ in Q , we have*

$$C\|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq \|u\|_{\mathcal{X}^\alpha(Q)} \left[\ln \left(\frac{\|u\|_{\mathcal{X}^\alpha(Q)}}{\mathcal{C}(u, \Gamma_0)} \right) \right]^{-\gamma} + \mathcal{C}(u, \Gamma_0).$$

Here we set

$$\mathcal{C}(u, \Gamma_0) = \|u\|_{H^1((t_0, t_1), H^1(\Gamma_0))} + \|\nabla u\|_{L^2(\Gamma_0 \times (t_0, t_1))}.$$

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We will use several times the one dimension Hardy's inequality (see for instance [3, Lemma 1.5.1, page 25]) due to the fact that neither the initial data, nor the final data are assumed to be known.

Observe that Theorem 1.1 remains valid if L is replaced by an elliptic operator with lower-order terms whose coefficients are bounded in (x, t) .

The second author [7, Theorem 5.1, page 24] proved a Hölder stability in a proper subdomain of Q depending on the part of the lateral boundary where the Cauchy data is given. In [6, Theorem 3.5.1, pages 45 and 46], Vessella establishes a local Hölder stability corresponding to the continuation of Cauchy data to an interior subdomain for solutions vanishing at the initial time. Recently, Bourgeois [1, Main theorem, page 2] proved a result similar to the one in Theorem 1.1 in the case when $L = \Delta - \partial_t$, $\Omega = D \setminus O$, D and O two domains of class C^2 , $O \Subset D$, and $\Gamma_0 = \partial D$ or $\Gamma_0 = \partial O$. His result is based on a global Carleman estimate in which the weight function is built from the distance to the boundary of the space variable.

The rest of this article is organized as follows. Section 2 is devoted to a three-cylinder interpolation inequality for the H_x^1 - L_t^2 -norm. This inequality will be very useful for continuing the lateral boundary data to an interior subdomain and to continue the data from one subdomain to another subdomain. This is what we show in Section 3 and, as byproduct, we prove a logarithmic stability estimate corresponding to the unique continuation from interior data. The proof of Theorem 1.1 is completed in Section 4 by beforehand establishing a result which quantifies the stability from Cauchy data to an interior subdomain.

2. THREE-CYLINDER INTERPOLATION INEQUALITY

We prove in this section

Theorem 2.1. *There exist $C > 0$ and $0 < \vartheta < 1$ depending on κ , K , Ω and T_0 , so that, for any $0 < \epsilon < (t_1 - t_0)/2$, $u \in H^{2,1}(Q)$ satisfying $Lu = 0$ in Q , $y \in \Omega$ and $0 < r < r_y(\epsilon) = \min(\frac{1}{3}\text{dist}(y, \Gamma), \sqrt{\epsilon})$, we have*

$$(2.1) \quad r^3 \|u\|_{L^2((t_0+\epsilon, t_1-\epsilon), H^1(B(y, 2r)))} \leq C \|u\|_{L^2((t_0, t_1), H^1(B(y, r)))}^\vartheta \|u\|_{L^2((t_0, t_1), H^1(B(y, 3r)))}^{1-\vartheta}.$$

The proof of Theorem (2.1) is based on a Carleman inequality for a family of parabolic operators. To this end, let \mathcal{I} be an arbitrary set and consider the family of operators

$$L_s = \text{div}(A_s \nabla \cdot) - \partial_t, \quad s \in \mathcal{I},$$

where, for each $s \in \mathcal{I}$, $A_s = (a_s^{ij})$ is a symmetric matrix with $W^{1,\infty}(\Omega)$ entries so that the following assumptions hold: there exist $\kappa > 0$ and $K > 0$ so that

$$(2.2) \quad A_s(x) \xi \cdot \xi \geq \kappa |\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n \text{ and } s \in \mathcal{I},$$

and

$$(2.3) \quad \|a_s^{ij}\|_{W^{1,\infty}(\Omega)} \leq K, \quad 1 \leq i, j \leq n, \quad s \in \mathcal{I}.$$

Pick $\psi \in C^2(\overline{\Omega})$ without critical points in $\overline{\Omega}$ and set $\Sigma = \Gamma \times (t_0, t_1)$. Let $g(t) = \frac{1}{(t-t_0)(t_1-t)}$ and

$$\varphi(x, t) = g(t) \left(e^{4\lambda \|\psi\|_\infty} - e^{\lambda(2\|\psi\|_\infty + \psi(x))} \right), \quad \chi(x, t) = g(t) e^{\lambda(2\|\psi\|_\infty + \psi(x))}.$$

Recall the following notation by Lions and Magenes,

$$H^{2,1}(Q) = L^2((t_0, t_1), H^2(\Omega)) \cap H^1((t_0, t_1), L^2(\Omega)).$$

Theorem 2.2. *(Carleman inequality) There exist three positive constants C , λ_0 and τ_0 depending only on ψ , Ω , κ , K and T_0 , so that*

$$(2.4) \quad \begin{aligned} & C \int_Q (\lambda^4 \tau^3 \chi^3 u^2 + \lambda^2 \tau \chi |\nabla u|^2) e^{-2\tau \varphi} dx dt \\ & \leq \int_Q (L_s u)^2 e^{-2\tau \varphi} dx dt + \int_\Sigma (\lambda^3 \tau^3 \chi^3 u^2 + \lambda \tau \chi |\nabla u|^2 + (\lambda \tau \chi)^{-1} |\partial_t u|^2) e^{-2\tau \varphi} d\sigma, \end{aligned}$$

for all $u \in H^{2,1}(Q)$, $s \in \mathcal{I}$, $\lambda \geq \lambda_0$ and $\tau \geq \tau_0$.

Proof. Since the dependance of the constants will be uniform with respect to $s \in \mathcal{I}$, we drop for simplicity the subscript s in L_s and its coefficients. On the other hand since $C^\infty(\overline{Q})$ is dense in $H^{2,1}(Q)$ it is enough to prove (2.4) when $u \in C^\infty(\overline{Q})$.

Let $\Phi = e^{\tau\varphi}$ and $u \in C^\infty(\overline{Q})$, and set $w = \Phi^{-1}u$ extended by continuity at $t = 0$ and $t = T$ by setting $w(\cdot, 0) = w(\cdot, T) = 0$. Then straightforward computations give

$$Pw = [\Phi^{-1}L\Phi]w = P_1w + P_2w + cw,$$

where

$$\begin{aligned} P_1w &= aw + \operatorname{div}(A\nabla w) - \tau\partial_t\varphi w, \\ P_2w &= B \cdot \nabla w + bw - \partial_t w, \end{aligned}$$

with

$$\begin{aligned} a &= a(x, t, \lambda, \tau) = \lambda^2\tau^2\chi^2|\nabla\psi|_A^2, \\ B &= B(x, t, \lambda, \tau) = -2\lambda\tau\chi A\nabla\psi, \\ b &= b(x, t, \lambda, \tau) = -2\lambda^2\tau\chi|\nabla\psi|_A^2, \\ c &= c(x, t, \lambda, \tau) = -\lambda\tau\chi\operatorname{div}(A\nabla\psi) + \lambda^2\tau\chi|\nabla\psi|_A^2. \end{aligned}$$

Here

$$|\nabla\psi|_A = \sqrt{A\nabla\psi \cdot \nabla\psi} = |A^{1/2}\nabla\psi|.$$

We obtain by making integrations by parts

$$\begin{aligned} \int_Q awB \cdot \nabla w dx dt &= \frac{1}{2} \int_Q aB \cdot \nabla w^2 dx dt \\ (2.5) \qquad \qquad \qquad &= -\frac{1}{2} \int_Q \operatorname{div}(aB)w^2 dx + \frac{1}{2} \int_\Sigma aB \cdot \nu w^2 d\sigma \end{aligned}$$

and

$$\begin{aligned} \int_Q \operatorname{div}(A\nabla w)B \cdot \nabla w dx dt &= - \int_Q A\nabla w \cdot \nabla(B \cdot \nabla w) dx dt + \int_\Sigma B \cdot \nabla w A\nabla w \cdot \nu d\sigma dt \\ (2.6) \qquad \qquad \qquad &= - \int_Q B' \nabla w \cdot A\nabla w dx dt \\ &\quad - \int_Q \nabla^2 w B \cdot A\nabla w dx + \int_\Sigma (B \cdot \nabla w)(A\nabla w \cdot \nu) d\sigma dt. \end{aligned}$$

Here $B' = (\partial_i B_j)$ is the Jacobian matrix of B and $\nabla^2 w = (\partial_{ij}^2 w)$ is the Hessian matrix of w .

But

$$\begin{aligned} \int_Q B_i \partial_{ij}^2 w a^{ik} \partial_k w dx dt &= - \int_Q B_i a^{ik} \partial_{ik}^2 w \partial_j w dx dt \\ &\quad - \int_Q \partial_i [B_i a^{ik}] \partial_k w \partial_j w dx dt + \int_\Sigma B_i \nu_i a^{jk} \partial_k w \partial_j w d\sigma dt. \end{aligned}$$

Therefore

$$\begin{aligned} \int_Q \nabla^2 w B \cdot A\nabla w dx dt &= -\frac{1}{2} \int_Q \left([\operatorname{div}(B)A + \tilde{A}] \nabla w \right) \cdot \nabla w dx dt \\ (2.7) \qquad \qquad \qquad &\quad + \frac{1}{2} \int_\Sigma |\nabla w|_A^2 B \cdot \nu d\sigma dt, \end{aligned}$$

with $\tilde{A} = (\tilde{a}^{ij})$, $\tilde{a}^{ij} = B \cdot \nabla a^{ij}$.

It follows from (2.6) and (2.7)

$$(2.8) \quad \begin{aligned} \int_Q \operatorname{div}(A\nabla w)B \cdot \nabla w dxdt &= \frac{1}{2} \int_Q \left(-2AB' + \operatorname{div}(B)A + \tilde{A} \right) \nabla w \cdot \nabla w dxdt \\ &+ \int_{\Sigma} (B \cdot \nabla w) (A\nabla w \cdot \nu) d\sigma dt - \frac{1}{2} \int_{\Sigma} |\nabla w|_A^2 B \cdot \nu d\sigma dt. \end{aligned}$$

A new integration by parts yields

$$\int_Q \operatorname{div}(A\nabla w)bw dxdt = - \int_Q b|\nabla w|_A^2 dxdt - \int_Q w\nabla b \cdot A\nabla w dxdt + \int_{\Sigma} bwA\nabla w \cdot \nu d\sigma dt.$$

This and the following inequality

$$- \int_Q w\nabla b \cdot A\nabla w dxdt \geq - \int_Q (\lambda^2 \chi)^{-1} |\nabla b|_A^2 w^2 dxdt - \int_Q \lambda^2 \chi |\nabla w|_A^2 dxdt$$

imply

$$(2.9) \quad \begin{aligned} \int_Q \operatorname{div}(A\nabla w)bw dxdt &\geq - \int_Q (b + \lambda^2 \chi) |\nabla w|_A^2 dxdt - \int_Q (\lambda^2 \chi)^{-1} |\nabla b|_A^2 w^2 dxdt \\ &+ \int_{\Gamma} bwA\nabla w \cdot \nu d\sigma. \end{aligned}$$

One more time, integrations by parts entail

$$(2.10) \quad \int_Q aw\partial_t w dxdt = \frac{1}{2} \int_Q a\partial_t w^2 dxdt = -\frac{1}{2} \int_Q \partial_t a w^2 dxdt,$$

$$(2.11) \quad \int_Q \partial_t \varphi w \partial_t w dxdt = \frac{1}{2} \int_Q \partial_t \varphi \partial_t w^2 dxdt = -\frac{1}{2} \int_Q \partial_t^2 \varphi w^2 dxdt,$$

$$(2.12) \quad \begin{aligned} \int_Q \partial_t \varphi w B \cdot \nabla w dxdt &= \frac{1}{2} \int_Q \partial_t \varphi B \cdot \nabla w^2 dxdt \\ &= \frac{1}{2} \int_Q \operatorname{div}(\partial_t \varphi B) w^2 dxdt + \frac{1}{2} \int_{\Sigma} \partial_t \varphi B \cdot \nu w^2 d\sigma dt. \end{aligned}$$

Also

$$(2.13) \quad \int_Q \operatorname{div}(A\nabla w)\partial_t w = - \int_Q A\nabla w \cdot \nabla \partial_t w dxdt + \int_{\Sigma} A\nabla w \cdot \nu \partial_t w d\sigma dt.$$

But

$$\int_Q A\nabla w \cdot \nabla \partial_t w dxdt = - \int_Q A\nabla \partial_t w \cdot \nabla w dxdt = - \int_Q \nabla \partial_t w \cdot A\nabla w dxdt.$$

Whence

$$\int_Q A\nabla w \cdot \nabla \partial_t w dxdt = 0.$$

This identity in (2.13) gives

$$(2.14) \quad \int_Q \operatorname{div}(A\nabla w)\partial_t w = \int_{\Sigma} A\nabla w \cdot \nu \partial_t w d\sigma dt.$$

Now a combination of (2.5), (2.8), (2.9)-(2.12) and (2.14) gives

$$(2.15) \quad \int_Q P_1 w P_2 w dxdt - \int_Q c^2 w^2 dxdt \geq \int_Q f w^2 dxdt + \int_Q F \nabla w \cdot \nabla w dxdt + \int_{\Sigma} g(w) d\sigma dt,$$

where

$$\begin{aligned} f &= -\frac{1}{2}\operatorname{div}(aB) + ab - (\lambda^2\chi)^{-1}|\nabla b|_A^2 - c^2 + \frac{1}{2}\partial_t a - \frac{1}{2}\partial_t^2\varphi - \frac{1}{2}\operatorname{div}(\partial_t\varphi B), \\ F &= -AB' + \frac{1}{2}\left(\operatorname{div}(B)A + \tilde{A}\right) - (b + \lambda^2\chi)A, \\ g(w) &= \frac{1}{2}aw^2B \cdot \nu - \frac{1}{2}|\nabla w|_A^2B \cdot \nu + B \cdot \nabla w A \nabla w \cdot \nu + bw A \nabla w \cdot \nu - \frac{1}{2}\partial_t\varphi B \cdot \nu w^2 - A \nabla w \cdot \nu \partial_t w. \end{aligned}$$

We use the elementary inequality $(\alpha - \beta)^2 \geq \alpha^2/2 - \beta^2$, $\alpha > 0$, $\beta > 0$, in order to derive

$$\begin{aligned} \|Pw\|_2^2 &\geq (\|P_1w + P_2w\|_2 - \|cw\|_2)^2 \\ &\geq \frac{1}{2}\|P_1w + P_2w\|_2^2 - \|cw\|_2^2 \\ &\geq \int_{\Omega} P_1wP_2w dx - \int_{\Omega} c^2w^2 dx. \end{aligned}$$

Whence (2.15) implies

$$(2.16) \quad \|Pw\|_2^2 \geq \int_Q fw^2 dxdt + \int_Q F \nabla w \cdot \nabla w dxdt + \int_{\Sigma} g(w) d\sigma dt.$$

In light of the following inequalities, where c is a constant depending only on T_0 and ψ ,

$$\begin{aligned} |\partial_t\varphi| &\leq c\chi^2, \quad |\partial_t^2\varphi|, \quad |\nabla\partial_t\varphi| \leq c\chi^3 \\ |A \nabla w \cdot \nu \partial_t w| &\leq \lambda\tau\chi|A \nabla w \cdot \nu|^2 + (\lambda\tau\chi)^{-1}|\partial_t w|^2, \end{aligned}$$

straightforward computations show that there exist four positive constants C_0 , C_1 , λ_0 and τ_0 , that can depend only on ψ , Ω , T_0 , κ and K , such that for all $\lambda \geq \lambda_0$ and $\tau \geq \tau_0$,

$$\begin{aligned} f &\geq C_0\lambda^4\tau^3\chi^3, \\ F\xi \cdot \xi &\geq C_0\lambda^2\tau\chi|\xi|^2, \quad \text{for any } \xi \in \mathbb{R}^n, \\ |g(w)| &\leq C_1(\lambda^3\tau^3\chi^3w^2 + \lambda\tau\chi|\nabla w|^2 + (\lambda\tau\chi)^{-1}|\partial_t w|^2). \end{aligned}$$

Hence

$$(2.17) \quad C \int_Q (\lambda^4\tau^3\chi^3w^2 + \lambda^2\tau\chi|\nabla w|^2) dxdt \leq \int_Q (Pw)^2 dxdt + \int_{\Sigma} (\lambda^3\tau^3\chi^3w^2 + \lambda\tau\chi|\nabla w|^2 + (\lambda\tau\chi)^{-1}|\partial_t w|^2) d\sigma dt.$$

As $\nabla w = \Phi^{-1}(\nabla u + \lambda\tau u \nabla \psi)$,

$$|\nabla w|^2 = \Phi^{-2}(|\nabla u|^2 + \lambda^2\tau^2|\nabla \psi|^2 + 2\lambda\tau u \nabla u \cdot \nabla \psi).$$

Therefore, using an elementary inequality, we find

$$|\nabla w|^2 \geq \Phi^{-2} \left(|\nabla u|^2 + \lambda^2\tau^2|\nabla \psi|^2 u^2 - 4\lambda^2\tau^2 u^2 |\nabla \psi|^2 - \frac{1}{2}|\nabla u|^2 \right)$$

and then

$$(2.18) \quad |\nabla w|^2 \geq \Phi^{-2} \left(\frac{1}{2}|\nabla u|^2 - 3\lambda^2\tau^2 u^2 \|\nabla \psi\|_{\infty}^2 \right).$$

(2.18) implies in a straightforward manner that

$$\|\nabla \psi\|_{\infty}^{-2} |\nabla w|^2 \geq \Phi^{-2} \left(\frac{\|\nabla \psi\|_{\infty}^{-2}}{12} |\nabla u|^2 - \frac{1}{2}\lambda^2\tau^2 u^2 \right).$$

Consequently

$$(2.19) \quad \|\nabla \psi\|_{\infty}^{-2} \lambda^2\tau |\nabla w|^2 + \lambda^4\tau^3w^2 \geq \Phi^{-2} \left(\frac{\|\nabla \psi\|_{\infty}^{-2}}{12} |\nabla u|^2 + \frac{1}{2}\lambda^2\tau^2 u^2 \right).$$

On the other hand, it is not hard to establish the inequality

$$(2.20) \quad |\partial_t w|^2 \leq \Phi^{-2} (|\partial_t u|^2 + c\tau^2 \chi^2 u^2).$$

The expected inequality follows by combining (2.17), (2.19) and (2.20). \square

Proof of Theorem 2.1. Let $u \in H^{2,1}(Q)$ satisfying $Lu = 0$, set $Q(\mu) = B(0, \mu) \times (-1, 1)$, $\mu > 0$. Fix $(y, s) \in \Omega \times (t_0, t_1)$ and $0 < r < r_{(y,s)} = \min(\frac{1}{3}\text{dist}(y, \Gamma), \sqrt{s-t_0}, \sqrt{t_1-s}) \leq r_0 = r_0(\text{diam}(\Omega), T_0)$. Let

$$w(x, t) = u(rx + y, r^2 t - s), \quad (x, t) \in Q(3),$$

Then $L_r w = \text{div}(A_r \nabla w) - \partial_t w = 0$ in $Q(3)$, where $A_r(x) = (a^{ij}(rx + y))$.

Clearly, the family (A_r) satisfies (2.2) and (2.3), uniformly with respect to $r \in (0, r_{(y,s)})$.

Let $\chi \in C_c^\infty(U)$ satisfying $0 \leq \chi \leq 1$ and $\chi = 1$ in \mathcal{K} , with

$$U = \{x \in \mathbb{R}^n; 1/2 < |x| < 3\}, \quad \mathcal{K} = \{x \in \mathbb{R}^n; 1 \leq |x| \leq 5/2\}.$$

Theorem 2.2 applied to χw , when Ω is substituted by U , gives, for $\lambda \geq \lambda_0$ and $\tau \geq \tau_0$,

$$(2.21) \quad \begin{aligned} C \int_{Q(2) \setminus Q(1)} (\lambda^4 \tau^3 \varphi^3 w^2 + \lambda^2 \tau \varphi |\nabla w|^2) e^{-2\tau\varphi} dx dt \\ \leq \int_{Q(3)} (L_r(\chi w))^2 e^{-2\tau\varphi} dx dt. \end{aligned}$$

But

$$\text{supp}(L_r(\chi w)) \subset [\{1/2 \leq |x| \leq 1\} \cup \{5/2 \leq |x| \leq 3\}] \times (-1, 1)$$

and

$$(L_r(\chi w))^2 \leq \Lambda(w^2 + |\nabla w|^2),$$

where $\Lambda = \Lambda(r_0)$ is independent on r . Therefore, fixing λ and changing τ_0 if necessary, (2.21) implies, for $\tau \geq \tau_0$,

$$(2.22) \quad \begin{aligned} C \int_{Q(2)} (w^2 + |\nabla w|^2) e^{-2\tau\varphi} dx dt \leq \int_{Q(1)} (w^2 + |\nabla w|^2) e^{-2\tau\varphi} dx dt \\ + \int_{Q(3) \setminus Q(5/2)} (w^2 + |\nabla w|^2) e^{-2\tau\varphi} dx dt. \end{aligned}$$

Let $s' \in (-1, 0)$ and choose $\psi(x) = -|x|^2$ in (2.22) (without critical points in U). In that case

$$\varphi(x, t) = g(t) \left(e^{36\lambda} - e^{\lambda(18-|x|^2)} \right).$$

We have, where $s' = -1 + \epsilon$ for some $0 < \epsilon < 1$,

$$\begin{aligned} \varphi(x, t) &\leq g(s') (e^{36\lambda} - e^{14\lambda}) \leq \frac{1}{\epsilon} (e^{36\lambda} - e^{14\lambda}) = \frac{\alpha}{\epsilon}, \quad (x, t) \in B(2) \times (s', -s'), \\ \varphi(x, t) &\geq g(t_m) (e^{36\lambda} - e^{18\lambda}) = (e^{36\lambda} - e^{18\lambda}) = \beta, \quad (x, t) \in Q(1) \\ \varphi(x, t) &\geq g(t_m) \left(e^{36\lambda} - e^{\frac{47}{4}\lambda} \right) = \left(e^{36\lambda} - e^{\frac{47}{4}\lambda} \right) = \gamma, \quad (x, t) \in Q(3) \setminus Q(5/2). \end{aligned}$$

As $\frac{\beta}{\alpha} < 1 < \frac{\gamma}{\alpha}$, we can fix $\theta \in (0, 1)$ so that

$$\frac{1}{\epsilon} = \theta \frac{\beta}{\alpha} + (1 - \theta) \frac{\gamma}{\alpha} > 1.$$

Set $a = 2(1 - \theta)(\gamma - \beta)$ and $b = 2\theta(\gamma - \beta)$ and $\tilde{Q}(2) = B(0, 2) \times (-1 + \epsilon, 1 - \epsilon)$. Then (2.22) yields

$$C \int_{\tilde{Q}(2)} (w^2 + |\nabla w|^2) dx dt \leq e^{a\tau} \int_{Q(1)} (w^2 + |\nabla w|^2) dx dt + e^{-b\tau} \int_{Q(3)} (w^2 + |\nabla w|^2) dx dt.$$

As in the elliptic case (see [2, Theorem 2.17 and its proof, pages 19 to 21]), we derive from this inequality the following one

$$C \|w\|_{L^2((-1+\epsilon, 1-\epsilon), H^1(B(2)))} \leq \|w\|_{L^2((-1, 1), H^1(B(1)))}^\vartheta \|w\|_{L^2((-1, 1), H^1(B(3)))}^{1-\vartheta},$$

with $\vartheta = \frac{a}{a+b}$.

Making a change of variable we get, with $\tau = 1 - \epsilon$,

$$(2.23) \quad r \|u\|_{L^2((s-\tau r^2, s+\tau r^2), H^1(B(y, 2r)))} \leq C \|u\|_{L^2((s-r^2, s+r^2), H^1(B(y, r)))} \|u\|_{L^2((s-r^2, s+r^2), H^1(B(y, 3r)))}^{1-\vartheta}.$$

Fix $0 < \epsilon < (t_1 - t_0)/2$ and let $s_0 = t_0 + \epsilon$ and $s_k = s_{k-1} + 2\tau r^2$. Consider q the smallest integer so that $(t_1 - \epsilon) - s_{q-1} \leq 2\tau r^2$ or equivalently $(t_1 - \epsilon) - s_{q-2} > 2\tau r^2$. Whence

$$(2.24) \quad q < \frac{t_1 - t_0 - 2\epsilon}{2\tau r^2} + 3 < \frac{\delta}{2\tau r^2} + \frac{3\text{diam}(\Omega)^2}{r^2} = \left(\frac{T_0}{2\tau} + 3\text{diam}(\Omega)^2 \right) \frac{1}{r^2}.$$

In light of (2.23) we get, for $r < r_y(\epsilon) = \min\left(\frac{1}{3}\text{dist}(y, \Gamma), \sqrt{\epsilon}\right)$,

$$(2.25) \quad r \|u\|_{L^2((t_0+\epsilon, t_1-\epsilon), H^1(B(y, 2r)))} \leq Cq \|u\|_{L^2((t_0, t_1), H^1(B(y, r)))} \|u\|_{L^2((t_0, t_1), H^1(B(y, 3r)))}^{1-\vartheta}.$$

Estimate (2.24) in (2.25) yields

$$r^3 \|u\|_{L^2((t_0+\epsilon, t_1-\epsilon), H^1(B(y, 2r)))} \leq C \|u\|_{L^2((t_0, t_1), H^1(B(y, r)))} \|u\|_{L^2((t_0, t_1), H^1(B(y, 3r)))}^{1-\vartheta}, \quad r < r_y(\epsilon).$$

The proof is then complete. \square

3. QUANTIFYING THE UNIQUENESS OF CONTINUATION FROM AN INTERIOR DATA

We firstly prove

Proposition 3.1. *Let $0 < \alpha < 1$. There exist $\omega \Subset \Omega$ and three constants $\gamma > 0$, $c > 0$ and $C > 0$, that can depend on Ω , κ , K , T_0 and α , so that, for any $u \in \mathcal{Y}^\alpha(Q)$ satisfying $Lu = 0$ in Q and $0 < \sigma < 1$, we have*

$$(3.1) \quad \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma)^n)} \leq C (\sigma^\gamma \|u\|_{\mathcal{X}^\alpha(Q)} + e^{\frac{c}{\sigma}} \|u\|_{L^2((t_0, t_1), H^1(\omega))}).$$

Proof. Since Ω is Lipschitz, it has the uniform interior cone property. That is there exist $R > 0$ and $\theta \in]0, \pi/2[$ so that, to any $\tilde{x} \in \Gamma$ corresponds $\xi = \xi(\tilde{x}) \in \mathbb{S}^{n-1}$ for which

$$\mathcal{C}(\tilde{x}) = \{x \in \mathbb{R}^n; |x - \tilde{x}| < R, (x - \tilde{x}) \cdot \xi > |x - \tilde{x}| \cos \theta\} \subset \Omega.$$

Fix $\tilde{x} \in \Gamma$ and let $\xi = \xi(\tilde{x})$ be as in the definition above. Let $x_0 = x_0(\tilde{x}) = \tilde{x} + (R/2)\xi$, $d_0 = |x_0 - \tilde{x}|$ and $\rho_0 = (d_0/3) \sin \theta$.

By induction on k , we construct a sequence of balls $(B(x_k, 3\rho_k))$ as follows

$$\begin{cases} x_{k+1} = x_k - \alpha_k \xi, \\ \rho_{k+1} = \mu \rho_k, \\ d_{k+1} = \mu d_k, \end{cases}$$

where

$$d_k = |x_k - \tilde{x}|, \quad \rho_k = c d_k, \quad \alpha_k = (1 - \mu) d_k,$$

with

$$c = \frac{\sin \theta}{3}, \quad \mu = \frac{3 - 2 \sin \theta}{3 - \sin \theta}.$$

This construction guaranties that, for each k , $B(x_k, 3\rho_k) \subset \mathcal{C}(\tilde{x})$ and

$$(3.2) \quad B(x_{k+1}, \rho_{k+1}) \subset B(x_k, 2\rho_k).$$

Let $u \in \mathcal{X}^\alpha(Q)$ and set $M_0 = \|u\|_{L^2((t_0, t_1), H^1(\Omega))}$. Let $0 < \epsilon < (t_1 - t_0)/4$. From (2.1) we have

$$(3.3) \quad \rho_0^3 \|u\|_{L^2((t_0+\epsilon, t_1-\epsilon), H^1(B(x_0, 2\rho_0)))} \leq C M_0^{1-\vartheta} \|u\|_{L^2((t_0, t_1), H^1(B(x_0, \rho_0)))}^\vartheta.$$

Since $B(x_0, 2\rho_0) \supset B(x_1, \rho_1)$, (3.3) implies

$$(3.4) \quad \rho_0^3 \|u\|_{L^2((t_0+\epsilon, t_1-\epsilon), H^1(B(x_1, \rho_1)))} \leq C M_0^{1-\vartheta} \|u\|_{L^2((t_0, t_1), H^1(B(x_0, \rho_0)))}^\vartheta.$$

Set

$$I_0 = (t_0, t_1), \quad I_k = (t_0 + \epsilon \eta_k, t_1 - \epsilon \eta_k) \quad \text{with} \quad \eta_k = \sum_{j=0}^{k-1} \frac{1}{2^j}, \quad k \geq 1.$$

With these notations (3.4) takes the form

$$(3.5) \quad \rho_0^3 \|u\|_{L^2(I_1, H^1(B(x_1, \rho_1)))} \leq CM_0^{1-\vartheta} \|u\|_{L^2(I_0, H^1(B(x_0, \rho_0)))}.$$

By induction in k , we see that

$$(3.6) \quad \rho_k^3 \|u\|_{H^1(L^2(I_{k+1}, B(x_{k+1}, \rho_{k+1})))} \leq CM_0^{1-\vartheta} \|u\|_{L^2(I_k, H^1(B(x_k, \rho_k)))}.$$

Let $\Lambda_k = \|u\|_{L^2(I_k, H^1(B(x_k, \rho_k)))}$. Then (3.6) is rewritten as follows

$$\rho_k^3 \Lambda_{k+1} \leq CM_0^{1-\vartheta} \Lambda_k.$$

An induction on k yields

$$\Lambda_k \leq \frac{C^{1+\vartheta+\dots+\vartheta^{k-1}}}{\rho_{k-1}^3 \rho_{k-2}^{3\vartheta} \dots \rho_0^{3\vartheta^{(k-1)}}} M_0^{(1-\vartheta)(1+\vartheta+\dots+\vartheta^{k-1})} \Lambda_0^{\vartheta^k}.$$

From the following inequality, where we used $0 < \mu < 1$,

$$\begin{aligned} \rho_{k-1}^3 \rho_{k-2}^{3\vartheta} \dots \rho_0^{3\vartheta^{(k-1)}} &= (\mu^{3(k-1)} \rho_0^3) (\mu^{3(k-2)} \rho_0^3)^{\vartheta} \dots (\mu^3 \rho_0^3)^{\vartheta^{k-2}} \rho_0^{\vartheta^{k-1}} \\ &\geq (\mu^{3(k-1)} \rho_0^3)^{1+\vartheta+\dots+\vartheta^{k-1}} = (\mu^{3(k-1)} \rho_0^3)^{\frac{1-\vartheta^k}{1-\vartheta}}, \end{aligned}$$

it follows

$$(3.7) \quad \Lambda_k \leq \left[\left(\frac{C}{\mu^{3(k-1)} \rho_0^3} \right)^{\frac{1}{1-\vartheta}} M_0 \right]^{1-\vartheta^k} \Lambda_0^{\vartheta^k}.$$

Applying Young's inequality we obtain, for any $\varrho > 0$,

$$\begin{aligned} \Lambda_k &\leq (1 - \vartheta^k) \varrho^{\frac{1}{1-\vartheta^k}} \left(\frac{C}{\mu^{3(k-1)} \rho_0^3} \right)^{\frac{1}{1-\vartheta}} M_0 + \vartheta^k \varrho^{-\frac{1}{\vartheta^k}} \Lambda_0 \\ (3.8) \quad &\leq \varrho^{\frac{1}{1-\vartheta^k}} \left(\frac{C}{\mu^{3(k-1)} \rho_0^3} \right)^{\frac{1}{1-\vartheta}} M_0 + \varrho^{-\frac{1}{\vartheta^k}} \Lambda_0. \end{aligned}$$

Set $\tilde{\Lambda}_k = \|u\|_{L^2((2\epsilon, T-2\epsilon), H^1(B(x_k, \rho_k)))}$. As $I_k \supset J = (2\epsilon, T-2\epsilon)$ for any $k \geq 1$, (3.8) implies

$$(3.9) \quad \tilde{\Lambda}_k \leq \varrho^{\frac{1}{1-\vartheta^k}} \left(\frac{C}{\mu^{3(k-1)} \rho_0^3} \right)^{\frac{1}{1-\vartheta}} M_0 + \varrho^{-\frac{1}{\vartheta^k}} \Lambda_0.$$

Changing if necessary C , we can always assume that $C\rho_0^{-3} \geq 1$. In that case (3.9) gives

$$(3.10) \quad \tilde{\Lambda}_k \leq C \left(\varrho^{\frac{1}{1-\vartheta^k}} \mu^{\frac{-3(k-1)}{1-\vartheta}} M_0 + \varrho^{-\frac{1}{\vartheta^k}} \Lambda_0 \right).$$

On the other hand, since u is Hölder continuous,

$$|u(\tilde{x}, t)| \leq [u]_{\alpha} |\tilde{x} - x|^{\alpha} + |u(x, t)|, \quad x \in B(x_k, \rho_k), \quad t \in J.$$

Here and henceforth,

$$[w]_{\alpha} = \sup_{\substack{(x_1, t_1), (x_2, t_2) \in \bar{Q} \\ (x_1, t_1) \neq (x_2, t_2)}} \frac{|w(x_1, t_1) - w(x_2, t_2)|}{|x_1 - x_2|^{\alpha} + |t_1 - t_2|^{\alpha/2}}, \quad w \in C^{\alpha, \alpha/2}(\bar{Q}).$$

Whence

$$(3.11) \quad |\mathbb{S}^{n-1}| \rho_k^n \int_J |u(\tilde{x}, t)|^2 dt \leq 2[u]_{\alpha}^2 \int_{B(x_k, \rho_k) \times J} |\tilde{x} - x|^{2\alpha} dx dt + 2 \int_{B(x_k, \rho_k) \times J} |u(x, t)|^2 dx dt,$$

Similarly, where $1 \leq i \leq n$,

$$(3.12) \quad |\mathbb{S}^{n-1}| \rho_k^n \int_J |\partial_i u(\tilde{x}, t)|^2 dt \leq 2[\partial_i u]_{\alpha}^2 \int_{B(x_k, \rho_k) \times J} |\tilde{x} - x|^{2\alpha} dx dt + 2 \int_{B(x_k, \rho_k) \times J} |\partial_i u(x, t)|^2 dx dt,$$

A simple computation shows that $d_k = \mu^k d_0$ yielding

$$(3.13) \quad |\tilde{x} - x| \leq |\tilde{x} - x_k| + |x_k - x| \leq d_k + \rho_k = (1+c)d_k = (1+c)\mu^k d_0.$$

Let

$$M = M_0 + \|u\|_{C^{1+\alpha, (1+\alpha)/2}(\overline{Q})} = \|u\|_{L^2((0,T), H^1(\Omega))} + \|u\|_{C^{1+\alpha, (1+\alpha)/2}(\overline{Q})} = \|u\|_{\mathcal{X}^\alpha(Q)}.$$

Therefore, as a consequence of a combination of (3.11), (3.12) and (3.13),

$$\int_J |\partial_i u(\tilde{x}, t)|^2 dt + \int_J |\nabla u(\tilde{x}, t)|^2 dt \leq C \left(M^2 \mu^{2\alpha k} + \mu^{-nk} \tilde{\Lambda}_k^2 \right).$$

This and (3.10) imply

$$(3.14) \quad \int_J |u(\tilde{x}, t)|^2 dt + \int_J |\nabla u(\tilde{x}, t)|^2 dt \leq C \left(M^2 \mu^{2\alpha k} + \mu^{-nk} \left[\varrho^{\frac{2}{1-\vartheta^k}} \mu^{-\frac{6(k-1)}{1-\vartheta}} M_1^2 + \varrho^{-\frac{2}{\vartheta^k}} \Lambda_0^2 \right] \right).$$

We choose $\varrho > 0$ so that

$$\varrho^{\frac{2}{1-\vartheta^k}} \mu^{-\frac{6(k-1)}{1-\vartheta} - nk} = \mu^{2\alpha k}.$$

That is to say $\varrho = \mu^{\phi(k)}$, with

$$\phi(k) = \frac{[(2\alpha + n)(1 - \vartheta)k + 6(k - 1)](1 - \vartheta^k)}{2(1 - \vartheta)}.$$

Consequently, we have from (3.15)

$$(3.15) \quad \int_J |u(\tilde{x}, t)|^2 dt + \int_J |\nabla u(\tilde{x}, t)|^2 dt \leq C \left(M^2 \mu^{2\alpha k} + \mu^{-\psi(k)} \Lambda_0^2 \right),$$

where $\psi(k) = nk + \phi(k)/\vartheta^k$.

Let $t > 0$ and k be the integer so that $k \leq t < k + 1$. Using that

$$\psi(k) \leq \left[n + \frac{(2\alpha + n) + 6}{2(1 - \vartheta)} \right] \frac{k}{\vartheta^k} \leq \left[n + \frac{(2\alpha + n) + 6}{2(1 - \vartheta)} \right] e^{k(1 + |\ln \vartheta|)} \leq \left[n + \frac{(2\alpha + n) + 6}{2(1 - \vartheta)} \right] e^{t(1 + |\ln \vartheta|)},$$

we find

$$(3.16) \quad \int_J |u(\tilde{x}, t)|^2 dt + \int_J |\nabla u(\tilde{x}, t)|^2 dt \leq C \left(M^2 \mu^{2\alpha t} + \mu^{-\kappa e^{ct}} \Lambda_0^2 \right),$$

where $\kappa = n + \frac{(2\alpha + n) + 6}{2(1 - \vartheta)}$ and $c = 1 + |\ln \vartheta|$.

Thus, for $\frac{1}{\sigma} = e^{ct} (> 1)$,

$$(3.17) \quad \int_J |u(\tilde{x}, t)|^2 dt + \int_J |\nabla u(\tilde{x}, t)|^2 dt \leq C \left(M^2 \sigma^{2\gamma} + e^{\frac{\xi}{\sigma}} \Lambda_0^2 \right),$$

with $\gamma = \frac{\alpha |\ln \mu|}{c}$ and $\xi = |\ln \mu|^\kappa$.

Let $\omega = \cup_{\tilde{x} \in \Gamma} B(x_0(\tilde{x}), \rho_0)$ (independent on u) and $N = \|u\|_{L^2((0,T); H^1(\omega))}$. Then (3.17) entails

$$(3.18) \quad \int_J |u(\tilde{x}, t)|^2 dt + \int_J |\nabla u(\tilde{x}, t)|^2 dt \leq C \left(M^2 \sigma^{2\gamma} + e^{\frac{\xi}{\sigma}} N^2 \right).$$

Integration over Γ , we obtain in a straightforward manner

$$(3.19) \quad \|u\|_{L^2(J, L^2(\Gamma))} + \|\nabla u\|_{L^2(J, L^2(\Gamma)^n)} \leq C \left(M \sigma^\gamma + e^{\frac{\xi}{\sigma}} N \right).$$

Applying the one dimension Hardy's inequality, we obtain

$$\begin{aligned} \|u\|_{L^2((0, 2\epsilon), L^2(\Gamma))}, \|u\|_{L^2((T-2\epsilon, T), L^2(\Gamma))} &\leq 4\epsilon \|u\|_{H^1((0, T), L^2(\Gamma))}, \\ \|\nabla u\|_{L^2((0, 2\epsilon), L^2(\Gamma)^n)}, \|\nabla u\|_{L^2((T-2\epsilon, T), L^2(\Gamma)^n)} &\leq 4\epsilon \|\nabla u\|_{H^1((0, T), L^2(\Gamma)^n)}. \end{aligned}$$

These estimate in (3.19) yields

$$\begin{aligned} \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma)^n)} &\leq C (M\sigma^\gamma + e^{\frac{\epsilon}{\sigma}} N) \\ &\quad + 4\epsilon (\|u\|_{H^1((0, T), L^2(\Gamma))} + \|\nabla u\|_{H^1((0, T), L^2(\Gamma)^n)}). \end{aligned}$$

The expected inequality follows by making $\epsilon \rightarrow 0$. That is we have

$$\|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma)^n)} \leq C (M\sigma^\gamma + e^{\frac{\epsilon}{\sigma}} N).$$

□

Lemma 3.1. *There exists a constant $C > 0$, depending only on Ω , κ , K and T_0 , so that, for any $u \in H^{2,1}(Q) \cap H^1((t_0, t_1), H^1(\Omega))$ satisfying $Lu = 0$ in Q , we have*

$$(3.20) \quad C\|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))}.$$

Proof. Let $u \in H^{2,1}(Q)$ satisfying $Lu = 0$ in Q and set $v = e^{-t}u$. Then v solves the following equation

$$(3.21) \quad \operatorname{div}(A\nabla v) - v - \partial_t v = 0 \text{ in } Q.$$

Choose $\chi \in C_0^\infty(t_0, t_1)$ satisfying $0 \leq \chi \leq 1$, $\chi = 1$ in $(t_0 + \epsilon, t_1 - \epsilon)$ and $|\chi'| \leq c\epsilon^{-1}$, for some universal constant c .

Multiply (3.21) by χv , integrate over Q and integrate by parts in order to get

$$-\int_Q \chi A\nabla v \cdot \nabla v dxdt - \int_Q \chi v^2 dxdt + \int_\Sigma \chi u A\nabla u \cdot \nu d\sigma dt + \frac{1}{2} \int_Q v^2 \chi' dxdt,$$

from which we deduce in a straightforward manner

$$(3.22) \quad \int_Q \chi A\nabla u \cdot \nabla u dxdt + \int_Q \chi u^2 dxdt \leq e^{2(t_1 - t_0)} \left(C \int_\Sigma (u^2 + |\nabla u|^2) d\sigma dx + \frac{1}{2} \int_Q u^2 |\chi'| dxdt \right).$$

On the other hand, as $\operatorname{supp}(\chi') \subset (t_0, t_1) \setminus (t_0 + \epsilon, t_1 - \epsilon)$, we have

$$\int_Q u^2 |\chi'| dxdt \leq c\epsilon^{-1} \left(\int_{t_0}^{t_0 - \epsilon} \int_\Omega u^2 dxdt + \int_{t_1 - \epsilon}^{t_1} \int_\Omega u^2 dxdt \right).$$

Combined with Hardy's inequality, this estimate implies

$$(3.23) \quad \int_Q u^2 |\chi'| dxdt \leq C\epsilon \int_Q (\partial_t u)^2 dxdt.$$

(3.22) and (3.23) produce the following estimate

$$C\|u\|_{L^2((t_0 + \epsilon, t_1 - \epsilon), H^1(\Omega))} \leq \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \sqrt{\epsilon} \|u\|_{H^1((t_0, t_1), L^2(\Omega))}.$$

Again, according to Hardy's inequality,

$$\|u\|_{L^2((t_0, t_0 + \epsilon), H^1(\Omega))}, \|u\|_{L^2((t_1 - \epsilon, t_1), H^1(\Omega))} \leq 2\epsilon \|u\|_{H^1((t_0, t_1), H^1(\Omega))}.$$

Whence

$$C\|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq \|u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma))} + \sqrt{\epsilon} \|u\|_{H^1((t_0, t_1), H^1(\Omega))}.$$

The expected inequality follows by making $\epsilon \rightarrow 0$. □

In light of Proposition 3.1 and Lemma 3.1, we get

Corollary 3.1. *Let $0 < \alpha < 1$. There exist $\omega \Subset \Omega$ and three constants $\gamma > 0$, $c > 0$ and $C > 0$, depending only on Ω , κ , K , T_0 and α , so that, for any $u \in \mathcal{B}^\alpha(Q)$ satisfying $Lu = 0$ in Q and $0 < \sigma < 1$, we have*

$$(3.24) \quad \|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq C (\sigma^\gamma \|u\|_{\mathcal{B}^\alpha(Q)} + e^{\frac{\epsilon}{\sigma}} \|u\|_{L^2((t_0, t_1), H^1(\omega))}).$$

Next, we establish a result quantifying the uniqueness of continuation from a subdomain to an another subdomain. Precisely, we prove

Proposition 3.2. *Let ω and $\tilde{\omega}$ be two non empty open subsets of Ω . There exist two constants $\gamma > 0$ and $C > 0$, depending only on $\Omega, \kappa, K, T_0, \omega$ and $\tilde{\omega}$, so that, for any $u \in H^{2,1}(Q) \cap H^1((t_0, t_1), H^1(\Omega))$ satisfying $Lu = 0$ in Q and $\sigma > 0$, we have*

$$(3.25) \quad \|u\|_{L^2((t_0, t_1), H^1(\tilde{\omega}))} \leq C (\sigma^\gamma \|u\|_{L^2((t_0, t_1), H^1(\Omega))} + \sigma^{-1} \|u\|_{L^2((t_0, t_1), H^1(\omega))}).$$

Proof. Given $x_0 \in \omega$ and $x \in \tilde{\omega}$, there exists a sequence of balls $B(x_j, r)$, $r > 0$, $j = 0, \dots, N$, such that

$$\begin{cases} B(x_0, r) \subset \omega, \\ B(x_{j+1}, r) \subset B(x_j, 2r), \quad j = 0, \dots, N-1, \\ x \in B(x_N, r), \\ B(x_j, 3r) \subset \Omega, \quad j = 0, \dots, N. \end{cases}$$

The construction of such sequence of balls can be found in [2, end of the proof of Proposition 2.28 in page 29].

As above,

$$I_0 = (t_0, t_1), \quad I_k = (t_0 + \epsilon \eta_k, t_1 - \epsilon \eta_k) \quad \text{with} \quad \eta_k = \sum_{j=0}^{k-1} \frac{1}{2^j}, \quad k \geq 1.$$

In the sequel, r is fixed and C_i denote generic constants that can depend on r .

Let $u \in H^{2,1}(Q) \cap H^1((t_0, t_1), H^1(\Omega))$ satisfying $Lu = 0$ in Q . From the three-cylinder inequality (2.1), we have

$$\|u\|_{H^1(L^2(I_{k+1}, B(x_{k+1}, r)))} \leq C_0 M_0^{1-\vartheta} \|u\|_{L^2(I_k, H^1(B(x_k, r)))},$$

where $M_0 = \|u\|_{L^2((0, T), H^1(\Omega))}$.

Setting $I_k = \|u\|_{L^2(I_k, H^1(B(x_k, r)))}$, we can rewrite this inequality in the form

$$(3.26) \quad I_{k+1} \leq C_0 M^{1-\vartheta} I_k^\vartheta.$$

Let $C_1 = C_0^{1+s+\dots+s^N}$ and $\beta = \vartheta^N$. We get in a straightforward manner from (3.26)

$$I_N \leq C_1 M_0^{1-\beta} I_0^\beta.$$

This and Young's inequality lead, for $\sigma > 0$,

$$I_N \leq C_1 ((1-\beta)\sigma^{\frac{\beta}{1-\beta}} M_0 + \beta\sigma^{-1} I_0) \leq C_1 (\sigma^{\frac{\beta}{1-\beta}} M + \sigma^{-1} I_0).$$

That is, where $\gamma = \frac{\beta}{1-\beta}$,

$$(3.27) \quad \|u\|_{L^2(J_N, H^1(B(x_N, r)))} \leq C_1 (\sigma^\gamma M_0 + \sigma^{-1} \|u\|_{L^2((0, T), H^1(\omega))}).$$

Since $J_N \supset (2\epsilon, T - 2\epsilon)$, (3.27) entails

$$(3.28) \quad \|u\|_{L^2((2\epsilon, T-2\epsilon), H^1(B(x_N, r)))} \leq C_1 (\sigma^\gamma M_0 + \sigma^{-1} \|u\|_{L^2((0, T), H^1(\omega))}).$$

Using Hardy's inequality again, we obtain

$$\|u\|_{L^2((0, 2\epsilon), H^1(B(x_N, r)))}, \|u\|_{L^2((T-2\epsilon, T), H^1(B(x_N, r)))} \leq 4\epsilon \|u\|_{H^1((0, T), H^1(\Omega))}.$$

This inequality in (3.28) produces

$$(3.29) \quad \|u\|_{L^2((0, T), H^1(B(x_N, r)))} \leq C_2 (\sigma^\gamma M_0 + \epsilon M + \sigma^{-1} \|u\|_{L^2((0, T), H^1(\omega))}),$$

with $M = \|u\|_{H^1((0, T), H^1(\Omega))}$. Making $\epsilon \rightarrow 0$, (3.29) gives

$$(3.30) \quad \|u\|_{L^2((0, T), H^1(B(x_N, r)))} \leq C_3 (\sigma^\gamma M_0 + \sigma^{-1} \|u\|_{L^2((0, T), H^1(\omega))}),$$

By compactness, $\tilde{\omega}$ can be recovered by a finite number of balls $B(x_N, r)$. If (φ_k) is a partition of unity subordinate to this covering then

$$\|u\|_{L^2((0, T), H^1(\tilde{\omega}))} = \left\| \sum \varphi_k u \right\|_{L^2((0, T), H^1(\tilde{\omega}))} \leq C_2 \sum \|u\|_{L^2((0, T), H^1(B(x_N, r)))}.$$

Whence (3.30) implies

$$(3.31) \quad \|u\|_{L^2((0,T),H^1(\tilde{\omega}))} \leq C_3 \left(\sigma^\gamma M_0 + \sigma^{-1} \|u\|_{L^2((0,T),H^1(\omega))} \right).$$

□

We are now ready to prove the result showing how one can quantify the uniqueness of continuation from an interior data

Theorem 3.1. *Let $\omega \Subset \Omega$ and $0 < \alpha < 1$. There exist two constants $C > 0$ and $\gamma > 0$, depending only on $\Omega, \kappa, K, \omega$ and T_0 , so that, for any $u \in \mathcal{B}^\alpha(Q)$ satisfying $Lu = 0$ in Q , we have*

$$C \|u\|_{L^2((t_0,t_1),H^1(\Omega))} \leq \|u\|_{\mathcal{B}^\alpha(Q)} \left| \ln \left(\frac{\|u\|_{\mathcal{B}^\alpha(Q)}}{\|u\|_{L^2((t_0,t_1),H^1(\omega))}} \right) \right|^{-\gamma} + \|u\|_{L^2((t_0,t_1),H^1(\omega))}.$$

Proof. Let $u \in \mathcal{B}^\alpha(Q)$ satisfying $Lu = 0$ in Q . From Corollary 3.1, there exist $\tilde{\omega} \Subset \Omega$ and three constants $\gamma > 0, c > 0$ and $C > 0$ so that, for any $0 < \sigma < 1$, we have

$$\|u\|_{L^2((t_0,t_1),H^1(\Omega))} \leq C \left(\sigma^\gamma \|u\|_{\mathcal{B}^\alpha(Q)} + e^{\frac{c}{\sigma}} \|u\|_{L^2((t_0,t_1),H^1(\tilde{\omega}))} \right).$$

But according to Proposition 3.2, for any $\epsilon > 0$,

$$\|u\|_{L^2((t_0,t_1),H^1(\tilde{\omega}))} \leq C' \left(\epsilon^\delta \|u\|_{L^2((0,T),H^1(\Omega))} + \epsilon^{-1} \|u\|_{L^2((0,T),H^1(\omega))} \right),$$

for some constants $C' > 0$ and $\delta > 0$.

These last two inequalities yield

$$C \|u\|_{L^2((t_0,t_1),H^1(\Omega))} \leq (\sigma^\gamma + \epsilon^\delta e^{\frac{c}{\sigma}}) \|u\|_{\mathcal{B}^\alpha(Q)} + \epsilon^{-1} e^{\frac{c}{\sigma}} \|u\|_{L^2((0,T),H^1(\omega))}, \quad 0 < \sigma < 1, \quad \epsilon > 0.$$

We choose ϵ so that $\epsilon^\delta e^{\frac{c}{\sigma}} = \sigma^\gamma$. Whence, changing the constants C and c if necessary, we get

$$(3.32) \quad \|u\|_{L^2((t_0,t_1),H^1(\Omega))} \leq C \left(\sigma^\gamma \|u\|_{\mathcal{B}^\alpha(Q)} + e^{\frac{c}{\sigma}} \|u\|_{L^2((t_0,t_1),H^1(\omega))} \right), \quad 0 < \sigma < 1,$$

from which we deduce in a usual way

$$C \|u\|_{L^2((t_0,t_1),H^1(\Omega))} \leq \|u\|_{\mathcal{B}^\alpha(Q)} \left| \ln \left(\frac{\|u\|_{\mathcal{B}^\alpha(Q)}}{\|u\|_{L^2((t_0,t_1),H^1(\omega))}} \right) \right|^{-\gamma} + \|u\|_{L^2((t_0,t_1),H^1(\omega))}.$$

□

4. STABILITY OF PARABOLIC CAUCHY PROBLEMS

Additionally to the results of the preceding section we need to quantify the uniqueness of continuation from a Cauchy data to an interior subdomain. To do that we start with the following proposition.

Proposition 4.1. *Let Γ_0 be a nonempty open subset of Γ . There exist $\omega \Subset \Omega$ and two constants $C > 0$ and $\gamma > 0$, depending only on $\Omega, \kappa, K, \Gamma_0$ and T_0 , so that, for any $u \in H^{2,1}(Q) \cap H^1((t_0, t_1), H^1(\Omega))$ satisfying $Lu = 0$ in Q , $\sigma > 0$ and $0 < \epsilon < (t_1 - t_0)/2$, we have*

$$(4.1) \quad C \|u\|_{L^2(\omega \times (t_0 + \epsilon, t_1 - \epsilon))} \leq \sigma^\gamma \|u\|_{L^2((t_0,t_1),H^1(\Omega))} + \sigma^{-1} \left(\|u\|_{H^1((t_0,t_1),L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0,t_1),L^2(\Gamma_0))} \right).$$

Proof. Pick $0 < \epsilon < (t_1 - t_0)/2, 0 < \eta < \epsilon$ and let $s \in [t_0 + \epsilon, t_1 - \epsilon]$. Let $\tilde{x} \in \Gamma_0$ and $x_0 = x_0(\tilde{x})$ and ρ be as in the definition of the uniform exterior sphere property. Fix $r > 0$ in such a way that $B(x_0, \rho + r) \cap \Gamma \subset \Gamma_0$.

Let $\phi \in C_0^\infty(B(\tilde{x}, \rho + r))$, $\phi = 1$ on $B(\tilde{x}, \rho + \frac{3r}{4})$ and $|\partial^\alpha \phi| \leq cr^{-|\alpha|}$, $|\alpha| \leq 2$, where c is a constant independent on r . Set, where $0 < \tau < 1$ is to be determined in the sequel,

$$\begin{aligned} Q_0 &= [B(x_0, \rho + r) \cap \Omega] \times (s - \eta, s + \eta), \\ Q_1 &= [B(x_0, \rho + r/2) \cap \Omega] \times (s - \tau\eta, s + \tau\eta), \\ Q_2 &= \{[B(x_0, \rho + r) \setminus B(x_0, \rho + 3r/4)] \cap \Omega\} \times (s - \eta, s + \eta), \\ \Sigma_0 &= [B(x_0, \rho + r) \cap \Gamma] \times (s - \eta, s + \eta). \end{aligned}$$

Apply Theorem 2.2, with $L_s = L$, λ fixed and $\psi = (\rho + r)^2 - |x - x_0|^2$, to ϕu so that $u \in H^2(Q)$ satisfies $Lu = 0$ in Q in order to obtain

$$(4.2) \quad C \int_{Q_1} u^2 e^{-2\tau\varphi} dxdt \leq \int_{Q_0} (L[\phi u])^2 e^{-2\tau\varphi} dxdt + \int_{\Sigma_0} (u^2 + |\nabla u|^2 + (\partial_t u)^2) e^{-2\tau\varphi} dxdt.$$

But

$$L[\phi u] = L\phi u + 2A\nabla\phi \cdot \nabla u.$$

Whence, $\text{supp}(L[\phi u]) \cap Q_0 \subset Q_2$ together with (4.2) yield

$$(4.3) \quad C \int_{Q_1} u^2 e^{-2\tau\varphi} dxdt \leq \int_{Q_2} (u^2 + |\nabla u|^2) e^{-2\tau\varphi} dxdt + \int_{\Sigma_0} (u^2 + |\nabla u|^2 + (\partial_t u)^2) e^{-2\tau\varphi} dxdt.$$

Let

$$\begin{aligned} \alpha &= \eta^{-2} \left[e^{4\lambda(\rho+r)^2} - e^{\lambda(2(\rho+r)^2 - (\rho+r/2)^2)} \right], \\ \beta &= \eta^{-2} \left[e^{4\lambda(\rho+r)^2} - e^{2\lambda(\rho+r)^2} \right], \\ \gamma &= \eta^{-2} \left[e^{4\lambda(\rho+r)^2} - e^{\lambda(2(\rho+r)^2 - (\rho+3r/4)^2)} \right]. \end{aligned}$$

Then it is straightforward to check that

$$\begin{aligned} \varphi(x, t) &\leq \frac{\alpha}{1-\tau} \text{ in } Q_1, \\ \varphi(x, t) &\geq \beta \text{ in } Q_2, \\ \varphi(x, t) &\geq \gamma \text{ in } \Sigma_0. \end{aligned}$$

Noting $\frac{\beta}{\alpha} < 1 < \frac{\gamma}{\alpha}$, we can choose $0 < \theta < 1$ so that

$$\frac{1}{1-\tau} = \theta \frac{\alpha}{\beta} + (1-\theta) \frac{\gamma}{\alpha} > 1.$$

With this choice of τ , (4.3) yields

$$(4.4) \quad C \int_{Q_1} u^2 dxdt \leq e^{a\tau} \int_{Q_2} (u^2 + |\nabla u|^2) dxdt + e^{-b\tau} \int_{\Sigma_0} (u^2 + |\nabla u|^2 + (\partial_t u)^2) dxdt.$$

Here $a = 2(1-\theta)(\gamma - \beta)$ and $b = 2\theta(\gamma - \beta)$.

As we have done before, we get from (4.4)

$$C \|u\|_{L^2([B(x_0, \rho+r/2) \cap \Omega] \times (s-\tau\eta, s+\tau\eta))} \leq \|u\|_{L^2((t_0, t_1), H^1(\Omega))}^{1-\vartheta} \left(\|u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma_0))} \right)^\vartheta.$$

Here $\vartheta = \frac{a}{a+b}$.

The same argument as in Section 2 enables us to get

$$C \|u\|_{L^2([B(x_0, \rho+r/2) \cap \Omega] \times (t_0+\epsilon, \eta, t_1-\epsilon))} \leq \|u\|_{L^2((t_0, t_1), H^1(\Omega))}^{1-\vartheta} \left(\|u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma_0))} \right)^\vartheta.$$

Fix $\omega \in B(x_0, \rho + r/2) \cap \Omega$. Then the last estimates implies

$$C \|u\|_{L^2(\omega \times (t_0+\epsilon, t_1-\epsilon))} \leq \|u\|_{L^2((t_0, t_1), H^1(\Omega))}^{1-\vartheta} \left(\|u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma_0))} \right)^\vartheta,$$

which, in combination with Young's inequality, leads

$$(4.5) \quad C \|u\|_{L^2(\omega \times (t_0+\epsilon, t_1-\epsilon))} \leq \sigma^\gamma \|u\|_{L^2((t_0, t_1), H^1(\Omega))} + \sigma^{-1} \left(\|u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma_0))} \right),$$

for $\sigma > 0$, where $\gamma = \frac{\vartheta}{1-\vartheta}$. \square

Lemma 4.1. (*Caccioppoli type inequality*) Let $\omega_0 \Subset \omega_1 \Subset \Omega$. There exist a constant $C > 0$, depending only on Ω , κ , K , T_0 , ω_0 and ω_1 , so that, for $0 < \epsilon < (t_1 - t_0)/4$ and $u \in H^{2,1}(Q)$ satisfying $Lu = 0$ in Q , we have

$$(4.6) \quad C \|\nabla u\|_{L^2(\omega_0 \times (t_0+2\epsilon, t_1-2\epsilon))} \leq \|u\|_{L^2(\omega_1 \times (t_0+\epsilon, t_1-\epsilon))} + \sqrt{\epsilon} \|\partial_t u\|_{L^2(\omega_1 \times (t_0+\epsilon, t_1-\epsilon))}.$$

Proof. Let $u \in H^{2,1}(Q)$ satisfying $Lu = 0$ in Q . By Green's formula, for any $v \in L^2((t_0 + \epsilon, t_1 - \epsilon), H_0^1(\Omega))$, we have

$$(4.7) \quad \int_{t_0+\epsilon}^{t_1-\epsilon} \int_{\Omega} A \nabla u \cdot \nabla v dx dt - \int_{t_0+\epsilon}^{t_1-\epsilon} \int_{\Omega} \partial_t u v = 0.$$

Let $\phi \in C_0^\infty(\omega_1)$ satisfying $0 \leq \phi \leq 1$ and $\phi = 1$ in ω_0 . Pick $\chi \in C_0^\infty((t_0 + \epsilon, t_1 - \epsilon))$ so that $0 \leq \chi \leq 1$ and $\chi = 1$ in $(t_0 + 2\epsilon, t_1 - 2\epsilon)$. It is well known that χ can be chosen in such a way that $|\chi'| \leq c\epsilon^{-1}$, for some universal constant c .

Taking $v = \phi^2 \chi u$ in (4.7), we get in straightforward manner

$$\int_{t_0+\epsilon}^{t_1-\epsilon} \int_{\omega_1} \chi \phi^2 A \nabla u \cdot \nabla u dx dt = -2 \int_{t_0+\epsilon}^{t_1-\epsilon} \int_{\omega_1} (\sqrt{\chi} \phi \nabla u) \cdot (\sqrt{\chi} u A \nabla \phi) dx dt - \frac{1}{2} \int_{t_0+\epsilon}^{t_1-\epsilon} \int_{\omega_1} \phi u^2 \chi' dx dt.$$

But

$$\int_{t_0+\epsilon}^{t_1-\epsilon} \int_{\omega_1} A \nabla u \cdot \nabla v dx dt \geq \kappa \int_{t_0+\epsilon}^{t_1-\epsilon} \int_{\omega_1} \chi \phi^2 |\nabla u|^2 dx dt.$$

Therefore

$$(4.8) \quad \kappa \int_{t_0+\epsilon}^{t_1-\epsilon} \int_{\omega_1} \chi \phi^2 |\nabla u|^2 dx dt \leq -2 \int_{t_0+\epsilon}^{t_1-\epsilon} \int_{\omega_1} (\sqrt{\chi} \phi \nabla u) \cdot (\sqrt{\chi} u A \nabla \phi) dx dt - \frac{1}{2} \int_{t_0+\epsilon}^{t_1-\epsilon} \int_{\omega_1} \phi u^2 \chi' dx dt.$$

An elementary convexity condition yields

$$(4.9) \quad 2 \left| \int_{t_0+\epsilon}^{t_1-\epsilon} \int_{\omega_1} (\sqrt{\chi} \phi \nabla u) \cdot (\sqrt{\chi} u A \nabla \phi) dx dt \right| \leq \frac{\kappa}{2} \int_{t_0+\epsilon}^{t_1-\epsilon} \int_{\omega_1} \chi \phi^2 |\nabla u|^2 dx dt + C \int_{t_0+\epsilon}^{t_1-\epsilon} \int_{\omega_1} u^2 dx dt.$$

On the other hand, since

$$\left| \frac{1}{2} \int_{t_0+\epsilon}^{t_1-\epsilon} \int_{\omega_1} \phi u^2 \chi' dx dt \right| \leq \frac{C}{2\epsilon} \int_{t_0+\epsilon}^{t_0+2\epsilon} \int_{\omega_1} u^2 dx dt + \frac{C}{2\epsilon} \int_{t_1-2\epsilon}^{t_1-\epsilon} \int_{\omega_1} u^2 dx dt,$$

Hardy's inequality implies

$$(4.10) \quad \left| \frac{1}{2} \int_{t_0+\epsilon}^{t_1-\epsilon} \int_{\omega_1} \phi u^2 \chi' dx dt \right| \leq C \epsilon \int_{t_0+\epsilon}^{t_1-\epsilon} \int_{\omega_1} (\partial_t u)^2 dx dt.$$

The expected inequality follows by using (4.9) and (4.10) in (4.8). \square

We find as a consequence of (4.1) and (4.6) the following inequality, for some $\tilde{\omega} \Subset \Omega$,

$$C \|u\|_{L^2((t_0+2\epsilon, t_1-2\epsilon), H^1(\tilde{\omega}))} \leq \sigma^\gamma \|u\|_{L^2((t_0, t_1), H^1(\Omega))} + \sqrt{\epsilon} \|\partial_t u\|_{L^2(\omega_1 \times (t_0+\epsilon, \eta, t_1-\epsilon))} + \sigma^{-1} \left(\|u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma_0))} \right).$$

Using one more time Hardy's inequality, we get

$$C \|u\|_{L^2((t_0, t_1), H^1(\tilde{\omega}))} \leq \sigma^\gamma \|u\|_{L^2((t_0, t_1), H^1(\Omega))} + \sqrt{\epsilon} \|\partial_t u\|_{L^2(\omega_1 \times (t_0+\epsilon, \eta, t_1-\epsilon))} + \sigma^{-1} \left(\|u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma_0))} \right) + \epsilon \|u\|_{H^1((t_0, t_1), H^1(\Omega))}.$$

Making $\epsilon \rightarrow 0$, we conclude that the following corollary is a consequence Proposition 4.1 and Lemma 4.1.

Corollary 4.1. *Let Γ_0 be a nonempty open subset of Γ . There exist $\omega \Subset \Omega$ and two constants $C > 0$ and $\gamma > 0$, that can depend on Ω , κ , K , T_0 and Γ_0 so that, for any $u \in H^{2,1}(Q) \cap H^1((t_0, t_1), H^1(\Omega))$ satisfying $Lu = 0$ and $\sigma > 0$, we have*

$$(4.11) \quad C \|u\|_{L^2((t_0, t_1), H^1(\omega))} \leq \sigma^\gamma \|u\|_{L^2((t_0, t_1), H^1(\Omega))} + \sigma^{-1} \left(\|u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma_0))} \right).$$

We are now in position to complete the proof of Theorem 1.1. We recall that

$$\mathcal{C}(u, \Gamma_0) = \|u\|_{H^1((t_0, t_1), L^2(\Gamma_0))} + \|\nabla u\|_{L^2((t_0, t_1), L^2(\Gamma_0))}.$$

From (3.32), we have

$$C \|u\|_{L^2((t_0, t_1), H^1(\Omega))} \leq \sigma^\gamma \|u\|_{\mathcal{X}^\alpha(Q)} + e^{\frac{c}{\sigma}} \|u\|_{L^2((t_0, t_1), H^1(\omega))}, \quad 0 < \sigma < 1,$$

and by (4.11)

$$C\|u\|_{L^2((t_0,t_1),H^1(\omega))} \leq \sigma_1^{\gamma_1}\|u\|_{L^2((t_0,t_1),H^1(\Omega))} + \sigma_1^{-1}\mathcal{C}(u,\Gamma_0) \quad \sigma_1 > 0.$$

Whence

$$C\|u\|_{L^2((t_0,t_1),H^1(\Omega))} \leq (\sigma^\gamma + \sigma_1^{\gamma_1}e^{\frac{c}{\sigma}})\|u\|_{\mathcal{X}^\alpha(Q)} + \sigma_1^{-1}e^{\frac{c}{\sigma}}\mathcal{C}(u,\Gamma_0), \quad 0 < \sigma < 1, \quad \sigma_1 > 0.$$

We choose σ_1 in such a way that $\sigma_1^{\gamma_1}e^{\frac{c}{\sigma}} = \sigma^\gamma$. In that case, changing the constants c and C if necessary, we get in a straightforward manner

$$C\|u\|_{L^2((t_0,t_1),H^1(\Omega))} \leq \sigma^\gamma\|u\|_{\mathcal{X}^\alpha(Q)} + e^{\frac{c}{\sigma}}\mathcal{C}(u,\Gamma_0), \quad 0 < \sigma < 1.$$

This estimate yields in a classical way the expected inequality.

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IECL, UMR CNRS 7502, UNIVERSITÉ DE LORRAINE, BOULEVARD DES AIGUILLETES BP 70239 54506 VANDOEUVRE LES NANCY CEDEX- ILE DU SAULCY - 57 045 METZ CEDEX 01 FRANCE

E-mail address: mourad.choulli@univ-lorraine.fr

DEPARTMENT OF MATHEMATICAL SCIENCES, THE UNIVERSITY OF TOKYO 3-8-1, KOMABA, MEGURO, TOKYO 153, JAPAN

E-mail address: myama@ms.u-tokyo.ac.jp