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# THE LONG-MOODY CONSTRUCTION AND POLYNOMIAL FUNCTORS

ARTHUR SOULIÉ

May 17, 2017

## Abstract

In 1994, Long and Moody gave a construction on representations of braid groups which associates a representation of  $\mathbf{B}_n$  with a representation of  $\mathbf{B}_{n+1}$ . In this paper, we prove that this construction is functorial: it gives an endofunctor, called the Long-Moody functor, between the category of functors from the homogeneous category associated with the braid groupoid to a module category. Then we study the effect of the Long-Moody functor on strong polynomial functors: we prove that it increases by one the degree of strong polynomiality.

## Introduction

Linear representations of Artin braid group on  $n$  strands  $\mathbf{B}_n$  is a rich subject which appears in diverse contexts in mathematics (see for example [6] or [22] for an overview). Even if, at the present time, a complete classification of these representations is probably out of reach, any new result which would allow us to gain a better understanding of them would be a useful contribution.

In 1994, in a joint work with Moody (see [20]), Long gave a method to construct from a linear representation  $\rho : \mathbf{B}_{n+1} \rightarrow GL(V)$  a new linear representation  $lm(\rho) : \mathbf{B}_n \rightarrow GL(V^{\oplus n})$  of  $\mathbf{B}_n$ . Moreover, the construction complexifies in a sense the initial representation. For example, applying it to a one dimensional representation of  $\mathbf{B}_{n+1}$ , the construction gives a mild variation of the unreduced Burau representation of  $\mathbf{B}_n$ . This method was in fact already implicitly present in two previous articles of Long dated 1989 (see [18, 19]). In the article [3] dated 2008, Bigelow and Tian are interested in the Long-Moody construction from a purely matricial point of view. They give alternative and purely algebraic proofs of some results of [20], and they slightly extend some of them. In addition, in a survey on braid groups (see the Open Problem 7 in [6]), Birman and Brendle underline the fact that the Long-Moody construction should be studied in greater detail.

Our work focuses on the study of the Long-Moody construction from a functorial point of view. More precisely, we consider the homogenous category  $\mathcal{UB}$  associated with braid groups. This category is an example of a general construction introduced by Randal-Williams and Wahl in [23] on the braid groupoid. The category  $\mathcal{UB}$  has  $\mathbb{N}$  as objects and for each natural integer  $n$ , its automorphism group  $Aut_{\mathcal{UB}}(n)$  is the braid group  $\mathbf{B}_n$ . Let  $\mathbb{K}\text{-Mod}$  be the category of  $\mathbb{K}$ -modules, with  $\mathbb{K}$  a commutative ring. A functor  $F : \mathcal{UB} \rightarrow \mathbb{K}\text{-Mod}$  gives by evaluation a family of representations of braid groups  $\{F_n : \mathbf{B}_n \rightarrow GL(F(n))\}_{n \in \mathbb{N}}$ , which satisfies some compatibility properties. For instance, Randal-Williams and Wahl define in [23, Example 4.3] a functor  $\mathfrak{Bur} : \mathcal{UB} \rightarrow \mathbb{K}\text{-Mod}$  such that the representation  $\mathfrak{Bur}_n : \mathbf{B}_n \rightarrow GL(\mathfrak{Bur}(n))$  is the unreduced Burau representation. In the same way, we define for instance in Example 2.25 a functor  $\mathfrak{TMM} : \mathcal{UB} \rightarrow \mathbb{K}\text{-Mod}$  such that the representation  $\mathfrak{TMM}_n : \mathbf{B}_n \rightarrow GL(\mathfrak{TMM}(n))$  is the representation considered by Tong, Yang and Ma in [24].

In Proposition 3.6, we prove that the Long-Moody construction is functorial. More precisely, we show:

**Theorem A (Proposition 3.6)**. *There is a functor  $\mathbf{LM} : \mathcal{UB}\text{-Mod} \rightarrow \mathcal{UB}\text{-Mod}$ , called the Long-Moody functor, which satisfies for  $\sigma \in \mathbf{B}_n$*

$$\mathbf{LM}(F)(\sigma) = lm(F_{n+1})(\sigma).$$

Among the functors in  $\mathcal{U}\beta\text{-Mod}$ , the strong polynomial functors play a key role. This notion extends the classical one of polynomial functors, which were first defined by Eilenberg and Mac Lane in [11]. Their definition uses cross effects and concerns module categories. This initial definition can be extended to monoidal categories where the monoidal unit is also a null object. Djament and Vespa introduce in [9] the definition of strong polynomial functors for symmetric monoidal categories with the monoidal unit being an initial object. Here, we will see that the category  $\mathcal{U}\beta$  is not symmetric, nor braided, but pre-braided in the sense of [23]. However, the notion of strong polynomial functor may be extended to the wider context of pre-braided monoidal categories (see Definition 2.6). Therefore, we investigate the effects of the Long-Moody functor on strong polynomial functors. We establish the following theorem.

**Theorem B (Corollary 4.17)** . *Let  $F$  be a strong polynomial functor of  $\mathcal{U}\beta\text{-Mod}$  of degree  $n$ , such that  $\text{colim}_{n \in \mathbb{N}} F(n) \neq 0$ . Then, the functor  $\mathbf{LM}(F)$  is a strong polynomial functor of  $\mathcal{U}\beta\text{-Mod}$  of degree  $n + 1$ .*

Thus, iterating the Long-Moody functor on a strong polynomial functor of  $\mathcal{U}\beta\text{-Mod}$  of degree  $n$ , we generate polynomial functors of  $\mathcal{U}\beta\text{-Mod}$ , of any degree bigger than  $n$ . For instance, the functors  $\mathbf{Bur}$  and  $\mathfrak{T}\mathfrak{M}$  happen to be strong polynomial functors of degree one, and we prove that the functor  $\mathbf{Bur}$  is equivalent to a functor obtained by applying the Long-Moody construction. Strong polynomial functors turn out inter alia to be very useful for homological stability problems. For example, in [23], Randal-Williams and Wahl construct a general framework to prove homological stability for different families of groups. They obtain the stability for coefficients given by a specific kind of strong polynomial functors (namely coefficient systems of finite degree). Thus, the Long-Moody functor will provide new examples of twisted coefficients corresponding to the framework of Randal-Williams and Wahl.

The paper is organized as follows. In section 1, we first recall definitions, folklore facts and properties about braid groups and free groups, especially focusing on the link between them. Then, following [23], we aim at explaining the construction of a homogeneous category from a braided monoidal groupoid, using Quillen's construction. In Section 2, we review the notion of strong polynomial functors, and slightly extend the framework of [9] for pre-braided monoidal categories. We will especially dwell on the interesting case of the coefficient system of finite degree, which  $\mathbf{Bur}$  and  $\mathfrak{T}\mathfrak{M}$  happen to be examples. In Section 3, we prove Theorem A and give some properties of the Long-Moody functor. Section 4 is devoted to the proof of Theorem B.

*Notation.* We will consider a commutative ring  $\mathbb{K}$  throughout this work (the classical example is  $\mathbb{K} = \mathbb{C}$ ). For all natural integers  $n$ , considering a canonical basis  $\{e_1, \dots, e_n\}$  (respectively  $\{f_1, \dots, f_{n+1}\}$ ) of  $\mathbb{K}^{\oplus n}$  (respectively of  $\mathbb{K}^{\oplus n+1}$ ), we define  $\gamma_n^{\mathbb{K}} : \mathbb{K}^{\oplus n} \hookrightarrow \mathbb{K}^{\oplus n+1}$  the canonical inclusion morphism by:

$$\forall i \in \{1, \dots, n\} \gamma_n^{\mathbb{K}}(e_i) = f_i.$$

We define inductively  $\gamma_{n,n'}^{\mathbb{K}} = \gamma_{n'-1}^{\mathbb{K}} \circ \dots \circ \gamma_n^{\mathbb{K}} : \mathbb{K}^{\oplus n} \hookrightarrow \mathbb{K}^{\oplus n'}$  for all natural integers  $n$  and  $n'$  such that  $n' \geq n$ .

We denote by  $\mathbb{K}\text{-Mod}$  the category of  $\mathbb{K}$ -modules. We will also consider another commutative ring  $R$ .

Let  $\mathbf{Cat}$  denote the category of small categories. For  $\mathcal{C} \in \text{Obj}(\mathbf{Cat})$ , the core  $\mathcal{G}\mathbf{r}(\mathcal{C})$  is the subcategory of  $\mathcal{C}$  which has the same objects as  $\mathcal{C}$  and of which the morphisms are the isomorphisms of  $\mathcal{C}$ . We denote by  $\mathcal{G}\mathbf{r} : \mathbf{Cat} \rightarrow \mathbf{Cat}$  the functor which associates to a category its core.

Let  $\mathcal{C}$  be a category. For two objects  $A$  and  $B$  of the category  $\mathcal{C}$ , if  $A$  embeds in  $B$  and if no explicit notation is given, then we denote by  $I_A^B$  the associated embedding. If  $0$  is an initial object in the category  $\mathcal{C}$ , then we denote by  $\iota_A : 0 \rightarrow A$  the unique morphism from  $0$  to  $A$ .

For  $\mathcal{C}$  an object of  $\mathbf{Cat}$  and  $\mathcal{D}$  a category, we denote by  $\mathbf{Fct}(\mathcal{C}, \mathcal{D})$  the category of functors from  $\mathcal{C}$  to  $\mathcal{D}$ . For the particular case where  $\mathcal{D} = \mathbb{K}\text{-Mod}$ , we denote by  $\mathcal{C}\text{-Mod}$  the functor category  $\mathbf{Fct}(\mathcal{C}, \mathbb{K}\text{-Mod})$ . A monoidal category is denoted by  $(\mathcal{C}, \natural, 0, \alpha, \lambda, \rho)$  with  $\natural$  the monoidal product,  $0$  the unit,  $\alpha$  the associator,  $\lambda$  is the left unitor and  $\rho$  is the right unitor. If the category is braided, we denote by  $b^{\mathcal{C}}$  its braiding.

We denote by  $\mathcal{G}\mathbf{r}$  the category of groups. Let  $\mathbf{gr}$  be the full subcategory of  $\mathcal{G}\mathbf{r}$  of finitely generated free groups.

Let  $\mathbb{N}$  denote the category of natural integers considered as a poset. For all natural integers  $n$ , we denote by  $\gamma_n$  the unique element of  $\text{Hom}_{\mathbb{N}}(n, n+1)$ . for all natural integers  $n$  and  $n'$  such that  $n' \geq n$ , we denote by  $\gamma_{n,n'} : n \rightarrow n'$  the unique element of  $\text{Hom}_{\mathbb{N}}(n, n')$ , composition of the morphisms  $\gamma_{n'-1} \circ \gamma_{n'-2} \circ \dots \circ \gamma_{n+1} \circ \gamma_n$ . The addition defines a strict monoidal structure on  $\mathbb{N}$ , denoted by  $(\mathbb{N}, +, 0)$ .

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## 1 Recollections on braid and free groups and homogeneous categories

### 1.1 Braid groups and free groups

#### 1.1.1 Generalities

The aim of this section is to describe the necessary tools for our study. First, we recall some classical facts about braid groups and their links with free groups. Then, we give some notions and properties about Quillen's construction from a monoidal groupoid, pre-braided categories and homogeneous categories introduced by Randal-Williams and Wahl in [23].

**Basic notions** We recall that the braid group on  $n$  strands denoted by  $\mathbf{B}_n$  is the free group generated by  $\sigma_1, \dots, \sigma_{n-1}$  satisfying the relations:

- $\forall i \in \{1, \dots, n-2\}, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ ;
- $\forall i, j \in \{1, \dots, n-1\}$  such that  $|i-j| \geq 2, \sigma_i \sigma_j = \sigma_j \sigma_i$ .

**Definition 1.1.** Let  $\mathbf{B}_\infty$  denote the colimit of the family of groups  $\{\mathbf{B}_n\}_{n \in \mathbb{N}}$  and  $\mathbf{B}_\infty^{ab}$  its abelianisation.

Let us introduce the groupoid  $\beta$  associated with the family of braid groups.

**Definition 1.2.** The braid groupoid  $\beta$  is the groupoid with objects the natural integers  $n \in \mathbb{N}$  and morphisms (for  $n, m \in \mathbb{N}$ ):

$$\text{Hom}_\beta(n, m) = \begin{cases} \mathbf{B}_n & \text{if } n = m \\ \emptyset & \text{if } n \neq m \end{cases}$$

For each natural integer  $n$ , one can identify  $\mathbf{B}_n$  with a subgroup of  $\mathbf{B}_{n+1}$ , inducing an inclusion morphism  $\mathbf{B}_n \hookrightarrow \mathbf{B}_{n+1}$ . In the sequel, we will fix such a family of inclusion morphisms and denote them by  $\gamma_n^{b, \bullet} : \mathbf{B}_n \hookrightarrow \mathbf{B}_{n+1}$  for all natural integers  $n$ . One then defines the inclusion morphisms  $\gamma_{n, n'}^{b, \bullet} = \gamma_{n'-1}^{b, \bullet} \circ \dots \circ \gamma_n^{b, \bullet} : \mathbf{B}_n \hookrightarrow \mathbf{B}_{n'}$ , when  $n \leq n'$ .

**Example 1.3.** We may identify  $\mathbf{B}_n$  with the subgroup generated by the first generators  $\sigma_1, \dots, \sigma_{n-1}$  or by the last generators  $\sigma_2, \dots, \sigma_n$  of  $\mathbf{B}_{n+1}$ . This inclusion will be denoted by  $\gamma_n^{b, f} : \mathbf{B}_n \hookrightarrow \mathbf{B}_{n+1}$  respectively  $\gamma_n^{b, l} : \mathbf{B}_n \hookrightarrow \mathbf{B}_{n+1}$ , explicitly defined for all  $i \in \{1, \dots, n\}$  by  $\gamma_n^{g, f}(\sigma_i) = \sigma_i$  and  $\gamma_n^{f, l}(\sigma_i) = \sigma_{i+1}$ .

We also consider the free group on  $n$  generators, which we denote by:

$$\mathbf{F}_n = \langle g_1, \dots, g_n \rangle.$$

For each natural integer  $n$ , one can see  $\mathbf{F}_n$  as a subgroup of  $\mathbf{F}_{n+1}$ .

**Definition 1.4.** For all natural integers  $n$  and  $n'$  such that  $n' \geq n$ , we consider morphisms  $\gamma_{n, n'}^{f, \bullet} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n'}$ , defined by:

$$\forall i \in \{1, \dots, n\}, \gamma_{n, n'}^{f, \bullet}(g_i) = g_{k_i}$$

with  $g_{k_i} = g_{k_j}$  if and only if  $i = j$ .

*Notation 1.5.* For simplicity, the map  $\gamma_{n, n+1}^{f, \bullet}$  will be denoted by  $\gamma_n^{f, \bullet}$ .

**Example 1.6.** Classical morphisms from  $\mathbf{F}_n$  to  $\mathbf{F}_{n+1}$  are the identifications of  $\mathbf{F}_n$  as the subgroup of  $\mathbf{F}_{n+1}$  generated by the  $n$  first respectively last copies of  $\mathbf{F}_1$  in

$$\mathbf{F}_{n+1} = \begin{cases} \underbrace{\mathbf{F}_1 * \mathbf{F}_1 * \dots * \mathbf{F}_1}_{n \text{ times}} * \mathbf{F}_1 \\ \mathbf{F}_1 * \underbrace{\mathbf{F}_1 * \mathbf{F}_1 * \dots * \mathbf{F}_1}_{n \text{ times}} \end{cases}$$

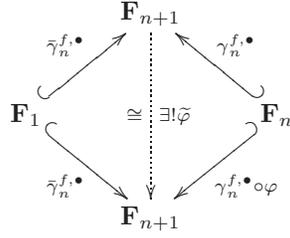
They will be denoted by  $\gamma_n^{f, f} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n+1}$  respectively  $\gamma_n^{f, l} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n+1}$ . Explicitly, these morphisms are defined for all  $i \in \{1, \dots, n\}$  by  $\gamma_n^{f, f}(g_i) = g_i$  and  $\gamma_n^{f, l}(g_i) = g_{i+1}$ .

**Definition 1.7.** Let  $n$  and  $n'$  be natural integers such that  $n' \geq n$ . To a morphism  $\gamma_{n, n'}^{f, \bullet} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n'}$  defined in Definition 1.4, we associate the morphism  $\bar{\gamma}_{n, n'}^{f, \bullet} : \mathbf{F}_{n'-n} \hookrightarrow \mathbf{F}_{n'}$  defined for all  $i \in \{1, \dots, n' - n\}$  by  $\bar{\gamma}_{n, n'}^{f, \bullet}(g_i) = g_{l_i}$  with  $l_i \notin \{k_1, \dots, k_n\}$  and  $l_i < l_j$  if and only if  $i < j$ .

*Notation 1.8.* For simplicity, the map  $\bar{\gamma}_{n, n+1}^{f, \bullet}$  will be denoted by  $\bar{\gamma}_n^{f, \bullet}$ .

*Remark 1.9.* In particular, since the free product is the coproduct in the category  $\mathfrak{Gr}$ , we have a group isomorphism  $\mathbf{F}_{n'} \cong \text{Im}(\bar{\gamma}_{n, n'}^{f, \bullet}) * \text{Im}(\gamma_{n, n'}^{f, \bullet})$ .

*Remark 1.10.* In addition, for all natural integers  $n$ , we can extend elements of  $\text{Aut}(\mathbf{F}_n)$  to elements of  $\text{Aut}(\mathbf{F}_{n+1})$  using  $\gamma_n^{f, \bullet} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n+1}$ . Namely, let  $\varphi$  be an element of  $\text{Aut}(\mathbf{F}_{n+1})$ . Thanks to the universal property of the coproduct, one defines uniquely an element  $\tilde{\varphi}$  of  $\text{Aut}(\mathbf{F}_n)$  such that the following diagram is commutative.



Explicitly, the automorphism  $\tilde{\varphi}$  is defined by:

$$\tilde{\varphi}(g_i) = \begin{cases} \gamma_n^{f,\bullet} \circ \varphi(g_i) & \text{if } i \in \{1, \dots, n\} \\ g_{n+1} = \tilde{\gamma}_n^{f,\bullet}(g_1) & \text{if } i = n+1. \end{cases}$$

Hence, for all natural integers  $n$ , we define  $\tilde{\gamma}_n^{f,\bullet} : Aut(\mathbf{F}_n) \hookrightarrow Aut(\mathbf{F}_{n+1})$  assigning  $\tilde{\gamma}_n^{f,\bullet}(\varphi) = \tilde{\varphi}$  for all  $\varphi \in Aut(\mathbf{F}_n)$ .

Furthermore, the group  $\mathbf{F}_n$  can be viewed as a subgroup of  $\mathbf{B}_{n+1}$ . For example, let us consider the pure braid group on  $n$  strands  $\mathbf{PB}_n$  (see [1] for a presentation by generators and relations). Then, the group  $\mathbf{F}_n$  is isomorphic to the kernel of the homomorphism  $\mathbf{p}_n : \mathbf{PB}_{n+1} \rightarrow \mathbf{PB}_n$  which consists in forgetting a string (classically the first or last one) of a pure braid on  $n+1$  strands. Then, we have the following commutative diagram, where the line is a short exact sequence:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbf{F}_n & \longrightarrow & \mathbf{PB}_{n+1} & \xrightarrow{\mathbf{p}_n} & \mathbf{PB}_n \longrightarrow 0. \\
& & & \searrow & \downarrow \mathbf{b}_{n+1} & & \\
& & & & \mathbf{B}_{n+1} & & 
\end{array}$$

We denote the associated embedding by  $\varsigma_{n,\bullet} : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}$ .

**Example 1.11.** A classical identification, called the pure braid local system in the literature (see [20, Remark p.223]) and denoted by  $\varsigma_{n,1}$ , is defined by the following assignment.

$$\begin{aligned}
\varsigma_{n,1} : \mathbf{F}_n & \hookrightarrow \mathbf{B}_{n+1} \\
g_i & \longmapsto \begin{cases} \sigma_1^2 & \text{if } i = 1 \\ \sigma_i \varsigma_{n,1}(g_{i-1}) \sigma_i^{-1} & \text{if } i \in \{2, \dots, n\} \end{cases}
\end{aligned}$$

Another identification  $\varsigma_{n,2}$  is the one called inner automorphism local system (see [10]), and is defined by assigning:

$$\begin{aligned}
\varsigma_{n,2} : \mathbf{F}_n & \hookrightarrow \mathbf{B}_{n+1} \\
g_i & \longmapsto \begin{cases} (\sigma_2 \cdots \sigma_n)^n & \text{if } i = 1 \\ \sigma_{i-1} \varsigma_{n,2}(g_{i-1}) \sigma_{i-1}^{-1} & \text{if } i \in \{2, \dots, n\} \end{cases}
\end{aligned}$$

### 1.1.2 Action of braid groups on automorphism groups of free groups

From a historical perspective (see for example [4] or [14]), the geometric point of view of topology gives us different actions of  $\mathbf{B}_n$  on the free group  $\mathbf{F}_n$ . As a consequence, there are several ways to consider the group  $\mathbf{B}_n$  as a subgroup of  $Aut(\mathbf{F}_n)$ . Geometrically speaking, it comes from the identification of  $\mathbf{B}_n$  with the mapping class group of a  $n$ -punctured disk  $\Sigma_{0,1}^n$ : fixing a point  $y$  on the boundary of the disk  $\Sigma_{0,1}^n$ , each free generators  $g_i$  can be taken as a loop of the disk based on  $y$  turning around certain fixed points. Each element  $\sigma$  of  $\mathbf{B}_n$ , as an automorphism up to isotopy of the disk  $\Sigma_{0,1}^n$ , induces a well-defined action on the fundamental group  $\pi_1(\Sigma_{0,1}^n) \cong \mathbf{F}_n$ .

In the sequel, we will fix such a family of group actions of  $\mathbf{B}_n$  on the free group  $\mathbf{F}_n$  and denote by  $a_{n,\bullet} : \mathbf{B}_n \rightarrow Aut(\mathbf{F}_n)$  the induced group morphism for all natural integers  $n$ .

**Example 1.12.** A first classical action is called the Artin representation (see for example [5, Section 1]) . It is defined for all elementary braids  $\sigma_i$  where  $i \in \{1, \dots, n-1\}$  by:

$$a_{n,1}(\sigma_i) : \mathbf{F}_n \longrightarrow \mathbf{F}_n$$

$$g_j \longmapsto \begin{cases} g_i^{-1} g_{i+1} g_i & \text{if } j = i \\ g_i & \text{if } j = i + 1 \\ g_j & \text{if } j \notin \{i, i + 1\} \end{cases}$$

**Wada representations** In 1992, Wada introduced in [26] a certain type of representations of braid groups. The Artin representation falls in fact into the framework of this study. Explicitly, Wada investigated on representations  $\rho : \mathbf{B}_n \longrightarrow \text{Aut}(\mathbf{F}_n)$  such that for all generators  $\sigma_i$  of  $\mathbf{B}_n$  and  $g_j$  of  $\mathbf{F}_n$ :

$$[\rho(\sigma_i)](g_j) = \begin{cases} W(g_i, g_{i+1}) & \text{if } j = i \\ V(g_i, g_{i+1}) & \text{if } j = i + 1 \\ g_j & \text{if } j \notin \{i, i + 1\} \end{cases}$$

where  $W(g_i, g_{i+1})$  and  $V(g_i, g_{i+1})$  are reduced words on  $\{g_i^{\pm 1}, g_{i+1}^{\pm 1}\}$ . He conjectured a classification of these Wada-type representations. This conjecture was proved by Ito in [13]. We give here a functorial approach of their work. Let us formalize some of our tools before, so as to introduce this work.

**Definition 1.13.** Let  $\mathbf{B}_\bullet : \mathbb{N} \longrightarrow \mathfrak{Gr}$ ,  $\mathbf{F}_\bullet : \mathbb{N} \longrightarrow \mathfrak{Gr}$ ,  $GL_\bullet : \mathbb{N} \longrightarrow \mathfrak{Gr}$  and  $Aut_\bullet : \mathbb{N} \longrightarrow \mathfrak{Gr}$  be the functors defined by:

- Objects: for all natural integers  $n$ ,  $\mathbf{B}_\bullet(n) = \mathbf{B}_n$  the braid group on  $n$  strands,  $\mathbf{F}_\bullet(n) = \mathbf{F}_n$  the free group on  $n$  generators,  $GL_\bullet(n) = GL_n(\mathbb{K})$  general linear group of degree  $n$  on  $\mathbb{K}$  and  $Aut_\bullet(n) = \text{Aut}(\mathbf{F}_n)$  the automorphism group of the free group on  $n$  generators;
- Morphisms: for all natural integers  $n$ , we define  $\mathbf{B}_\bullet(\gamma_n) = \gamma_n^{b, \bullet}$ ,  $\mathbf{F}_\bullet(\gamma_n) = \gamma_n^{f, \bullet}$ ,  $GL_\bullet(\gamma_n) = \gamma_n^{\mathbb{K}}$  and  $Aut_\bullet(\gamma_n) = \tilde{\gamma}_n^{f, \bullet}$  (recall that  $\tilde{\gamma}_n^{f, \bullet}$  was defined in Remark 1.10).

Let us consider two words  $W(g_1, g_2)$  and  $V(g_1, g_2)$  on  $\mathbf{F}_2$ . Let  $(W, V) : \mathbf{B}_2 \longrightarrow \text{Aut}(\mathbf{F}_2)$  be the morphism defined by:

$$[(W, V)(\sigma_1)](g_j) = \begin{cases} W(g_1, g_2) & \text{if } j = 1 \\ V(g_1, g_2) & \text{if } j = 2 \end{cases}$$

Two morphisms  $(W, V) : \mathbf{B}_2 \longrightarrow \text{Aut}(\mathbf{F}_2)$  and  $(W', V') : \mathbf{B}_2 \longrightarrow \text{Aut}(\mathbf{F}_2)$  are said to be:

- swap-dual if  $W'(g_1, g_2) = V(g_2, g_1)$  and  $V'(g_1, g_2) = W(g_2, g_1)$ ;
- backward-dual if  $W'(g_1, g_2) = (W(g_1^{-1}, g_2^{-1}))^{-1}$  and  $V'(g_1, g_2) = (V(g_1^{-1}, g_2^{-1}))^{-1}$ ;
- inverse if  $(W', V') = (W, V)^{-1}$ .

For all natural integers  $n$ , for all  $i \in \{1, \dots, n-1\}$ , we denote by  $incl_i^n : \mathbf{B}_2 \hookrightarrow \mathbf{B}_n$  the inclusion morphism induced by:

$$incl_i^n(\sigma_1) = \sigma_i.$$

**Theorem 1.14.** [26, 13] Let  $W(g_1, g_2)$  and  $V(g_1, g_2)$  be two words on  $\mathbf{F}_2$ . Let  $\mathcal{W} : \mathbf{B}_\bullet \longrightarrow \text{Aut}_\bullet$  be a natural transformation. It will be said to be of Wada-type if for all natural integers  $n$ , for all  $i \in \{1, \dots, n-1\}$ , the following diagram is commutative.

$$\begin{array}{ccc} \mathbf{B}_n & \xrightarrow{\mathcal{W}_n} & \text{Aut}(\mathbf{F}_n) \\ \uparrow incl_i^n & & \uparrow id_{\mathbf{F}_{n-i}} * \dots * id_{\mathbf{F}_{n-i-1}} \\ \mathbf{B}_2 & \xrightarrow{(W, V)} & \text{Aut}(\mathbf{F}_2) \end{array}$$

Note that therefore a Wada-type natural transformation is entirely determined by the choice of  $(W, V)$ . Then, there are seven types of Wada-type natural transformation  $\mathcal{W}$  up to the swap-dual, backward-dual and inverse equivalences, listed below.

1.  $(W, V)(g_1, g_2) = (g_1^{-m}g_2g_1^m, g_1)$  where  $m \in \mathbb{Z}$ ;
2.  $(W, V)(g_1, g_2) = (g_1, g_2)$ ;
3.  $(W, V)(g_1, g_2) = (g_2^{-1}, g_1)$ ;
4.  $(W, V)(g_1, g_2) = (g_1^{-1}g_2g_1, g_1)$ ;
5.  $(W, V)(g_1, g_2) = (g_2^{-1}, g_1^{-1})$ ;
6.  $(W, V)(g_1, g_2) = (g_1g_2g_1, g_1^{-1})$ ;
7.  $(W, V)(g_1, g_2) = (g_2g_1^2, g_2g_1^{-1}g_2^{-1})$ .

*Notation 1.15.* The actions given by the  $i$ -th Wada-type natural transformation will be denoted by  $a_{n, \mathcal{W}(i)} : \mathbf{B}_n \hookrightarrow \text{Aut}(\mathbf{F}_n)$ . In particular, taking  $m = 1$ , we obtain  $a_{n, \mathcal{W}(1)} = a_{n, 1}$ .

We give a last action of  $\mathbf{B}_n$  on  $\text{Aut}(\mathbf{F}_n)$ , which will be useful throughout our work and does not fall into the framework of the Wada classification.

**Example 1.16.** An action is defined for all elementary braids  $\sigma_i$  where  $i \in \{1, \dots, n-1\}$  by:

$$a_{n, 8}(\sigma_i) : \mathbf{F}_n \longrightarrow \mathbf{F}_n$$

$$g_j \longmapsto \begin{cases} g_j & \text{if } j \neq i+1 \\ g_{j+2}g_{j+1}^{-1}g_j & \text{if } i \leq n-2 \text{ and } j = i+1 \\ g_n^{-1}g_{n-1} & \text{if } i = n-1 \text{ and } j = n \end{cases}$$

### 1.1.3 Augmentation ideal

Finally, we need to focus on the augmentation ideal of the group ring  $\mathbb{K}[\mathbf{F}_n]$ .

**Definition 1.17.** The augmentation ideal of the group ring  $\mathbb{K}[\mathbf{F}_n]$ , denoted by  $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ , is defined to be the kernel of the morphism:

$$\begin{aligned} \mathbb{K}[\mathbf{F}_n] &\xrightarrow{\mathcal{K}} \mathbb{K}. \\ \sum \lambda_g g &\longmapsto \sum \lambda_g \end{aligned}$$

**Proposition 1.18.** [27, Chapter 6, Proposition 6.2.6] *The augmentation ideal  $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$  is a free  $\mathbb{K}[\mathbf{F}_n]$ -module with basis the set  $\{(g_i - 1) \mid i \in \{1, \dots, n\}\}$ .*

*Remark 1.19.* The proof in [27, Chapter 6, Proposition 6.2.9] is done there for  $\mathbb{K} = \mathbb{Z}$ , but the general case here is exactly the same.

*Remark 1.20.* An action  $a_{n, \bullet} : \mathbf{B}_n \hookrightarrow \text{Aut}(\mathbf{F}_n)$  extends naturally to  $\mathbb{K}[\mathbf{F}_n]$ . Indeed, let  $\sum \lambda_g g \in \mathbb{K}[\mathbf{F}_n]$  and  $b \in \mathbf{B}_n$ , then, one defines  $\tilde{a}_{n, \bullet} : \mathbf{B}_n \hookrightarrow \text{Aut}(\mathbb{K}[\mathbf{F}_n])$  by:

$$\tilde{a}_{n, \bullet}(b) \left( \sum \lambda_g g \right) = \sum \lambda_g a_n(b)(g).$$

Hence, since  $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$  is a submodule of  $\mathbb{K}[\mathbf{F}_n]$ , this induces by restriction an action on  $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$  denoted by  $A_{n, \bullet} : \mathbf{B}_n \hookrightarrow \text{Aut}(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]})$ .



- Composition: let  $[X, f] : A \rightarrow B$  and  $[Y, g] : B \rightarrow C$  be two morphisms in the category  $\mathfrak{UG}$ . Then, the composition is defined by:

$$[Y, g] \circ [X, f] = [Y \natural X, g \circ (id_Y \natural f)].$$

- Identity: for all objects  $X$  of  $\mathfrak{UG}$ , the identity morphism is given by  $[0, id_X] : X \rightarrow X$ .

*Remark 1.23.* One easily checks that for all morphisms  $[X', f] : X \rightarrow A$ ,  $[X'', g] : B \rightarrow X$ ,  $[D_1, \varphi_1] : C_1 \rightarrow C_2$ ,  $[D_2, \varphi_2] : C_2 \rightarrow C_3$  and  $[D_3, \varphi_3] : C_3 \rightarrow C_4$  in the category  $\mathfrak{UG}$ :

- $[X', f] \circ [0, id_X] = [X', f]$  and  $[0, id_X] \circ [X'', g] := [X'', g]$ ;
- $([D_1, \varphi_1] \circ [D_2, \varphi_2]) \circ [D_3, \varphi_3] = [D_1, \varphi_1] \circ ([D_2, \varphi_2] \circ [D_3, \varphi_3])$ .

The Quillen's construction  $\mathfrak{UG}$  has the additional following property.

**Proposition 1.24.** [23, Proposition 1.8] *Let  $(\mathfrak{G}, \natural, 0)$  be a strict monoidal groupoid. Then, the unit  $0$  is initial in the category  $\mathfrak{UG}$ .*

## 1.2.2 Pre-braided monoidal categories

We present the notion of pre-braided category, introduced by Randal-Williams and Wahl in [23]. This is a generalization of braided category, which will prove to be necessary to understand the homogeneous category constructed from the groupoid  $\beta$ .

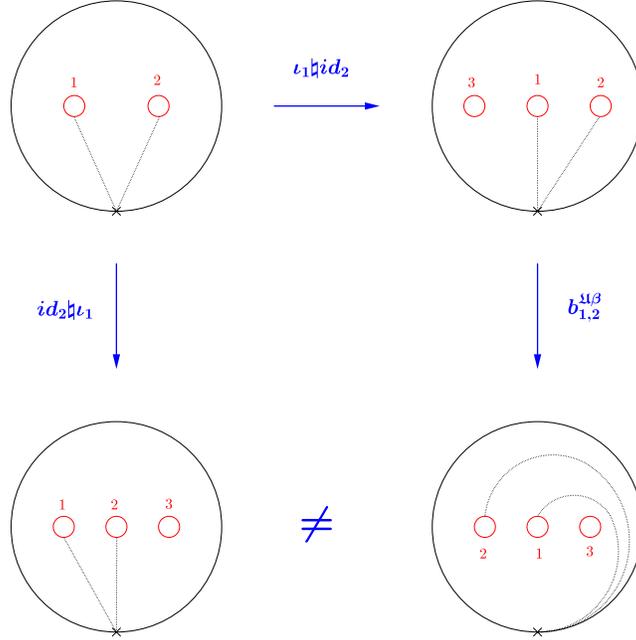
**Definition 1.25.** [23, Definition 1.5] Let  $(\mathfrak{C}, \natural, 0, \alpha, \lambda, \rho)$  be a monoidal category such that the unit  $0$  is initial. We say that the monoidal category  $(\mathfrak{C}, \natural, 0, \alpha, \lambda, \rho)$  is pre-braided if:

- the core  $\mathcal{Gt}((\mathfrak{C}, \natural, 0, \alpha, \lambda, \rho))$  is a braided monoidal category ;
- for all objects  $A$  and  $B$  of  $\mathfrak{C}$ , the groupoid braiding  $b_{A,B}^{\mathfrak{C}} : A \natural B \rightarrow B \natural A$  satisfies:

$$b_{A,B}^{\mathfrak{C}} \circ (id_A \natural \iota_B) = \iota_B \natural id_A : A \rightarrow B \natural A.$$

*Remark 1.26.* A braided monoidal category is obviously pre-braided.

A monoidal product  $\natural : \beta \times \beta \rightarrow \beta$  is defined assigning the usual addition for the objects and connecting two braids side by side for the morphisms, along with a braiding denoted  $b_{-, -}^{\beta}$ . For more details, see for example [21, Chapter XI, Part 4] for the definition of the braiding, except that we will use the reverse convention for  $b_{-, -}^{\beta}$  as the one proposed in this reference. The pre-braiding defined on  $\mathfrak{UG}$  is not a braiding. Indeed, the following figure shows that  $b_{1,2}^{\mathfrak{UG}} \circ (\iota_1 \natural id_2) \neq id_2 \natural \iota_1$  whereas this two morphisms should be equal if  $b_{-, -}^{\mathfrak{UG}}$  was a braiding. This example shows in particular that a pre-braided monoidal category is not necessarily braided.



Under some assumption, the Quillen's construction  $\mathfrak{U}\mathfrak{G}$  inherits a pre-braided property.

**Proposition 1.27.** [23, Proposition 1.8] *Let  $(\mathfrak{G}, \natural, 0)$  be a strict monoidal groupoid. If the category  $\mathfrak{G}$  is braided monoidal and has no zero divisors (i.e. for objects  $A$  and  $B$  of  $\mathfrak{G}$ ,  $A \natural B \cong 0$  if and only if  $A \cong B \cong 0$ ), then the category  $(\mathfrak{U}\mathfrak{G}, \natural, 0)$  is pre-braided monoidal. Moreover, the monoidal structure of  $\mathfrak{U}\mathfrak{G}$  is such that the map  $\mathfrak{G} \rightarrow \mathfrak{U}\mathfrak{G}$  taking an isomorphism  $f$  to  $[0, f]$  is monoidal.*

*Remark 1.28.* The monoidal structure on the category  $\mathfrak{U}\mathfrak{G}$  is defined letting for  $[X, f] \in \text{Hom}_{\mathfrak{U}\mathfrak{G}}(A, B)$  and  $[Y, g] \in \text{Hom}_{\mathfrak{U}\mathfrak{G}}(C, D)$ :

$$[X, f] \natural [Y, g] = \left[ X \natural Y, (f \natural g) \circ \left( id_X \natural b_{A,Y}^{-1} \natural id_C \right) \right].$$

### 1.2.3 Homogeneous categories

The notion of homogeneous category is introduced by Randal-Williams and Wahl in [23, Section 1], inspired by the set-up of Djament and Vespa in [8, Section 1.2]. With two additional assumptions, the Quillen's construction  $\mathfrak{U}\mathfrak{G}$  from a strict monoidal groupoid  $(\mathfrak{G}, \natural, 0)$  will be endowed with an homogeneous category structure. First, we need to give basic definitions necessary to give the one of homogeneous category.

**Definition 1.29.** Let  $(\mathfrak{C}, \natural, 0)$  be a strict monoidal category in which the unit  $0$  is also initial. For all objects  $A$  and  $B$  of  $\mathfrak{C}$ , we define:

- a preferred morphism:  $\iota_A \natural id_B : (B = 0 \natural B) \rightarrow A \natural B$ ;
- a set of morphisms characterised by this preferred morphism:

$$\text{Fix}(B) = \text{Fix}(B, A \natural B) = \{ \phi \in \text{Aut}(A \natural B) \mid \phi \circ (\iota_A \natural id_B) = \iota_A \natural id_B \}.$$

*Remark 1.30.* Since  $(\mathfrak{C}, \natural, 0)$  is assumed to be small  $\text{Hom}_{\mathfrak{C}}(A, B)$  is a set and  $\text{Aut}_{\mathfrak{C}}(B)$  defines a group (with composition of morphisms as the group product). The group  $\text{Aut}_{\mathfrak{C}}(B)$  acts by post-composition on  $\text{Hom}_{\mathfrak{C}}(A, B)$ :

$$\begin{array}{ccc} \text{Aut}_{\mathfrak{C}}(B) \times \text{Hom}_{\mathfrak{C}}(A, B) & \longrightarrow & \text{Hom}_{\mathfrak{C}}(A, B). \\ (\phi, f) & \longmapsto & \phi \circ f \end{array}$$

Now, we may introduce homogeneous categories.

**Definition 1.31.** Let  $(\mathfrak{C}, \natural, 0)$  be a strict monoidal small category. This category is homogeneous if the unit  $0$  is initial in  $\mathfrak{C}$  and if the two following assumptions are satisfied.

- **(H1)** : For all objects  $A$  and  $B$  of the category  $\mathfrak{C}$ , the action by post-composition of  $Aut(B)$  on  $Hom_{\mathfrak{C}}(A, B)$  is transitive.
- **(H2)** : For all objects  $A$  and  $B$  of the category  $\mathfrak{C}$ , the map

$$\begin{array}{ccc} Aut_{\mathfrak{C}}(A) & \longrightarrow & Aut_{\mathfrak{C}}(A \natural B) \\ f & \longmapsto & f \natural id_B \end{array}$$

is injective with image  $Fix(B) = \{\phi \in Aut_{\mathfrak{C}}(A \natural B) \mid \phi \circ (\iota_A \natural id_B) = \iota_A \natural id_B\}$ .

*Remark 1.32.* A strict monoidal category  $(\mathfrak{C}, \natural, 0)$  satisfying **(H1)** and **(H2)** is therefore determined by its core.

Let us focus on some elementary properties of homogeneous categories.

**Proposition 1.33.** [23, Remark 1.4] *Let  $(\mathfrak{C}, \natural, 0)$  be a homogeneous category. Let  $A$  and  $B$  be two objects of this category. Then:*

1.  $Hom_{\mathfrak{C}}(B, A \natural B) \cong Aut_{\mathfrak{C}}(A \natural B) / Aut_{\mathfrak{C}}(A)$ .
2.  $Hom_{\mathfrak{C}}(A, A) \cong Aut_{\mathfrak{C}}(A)$ .
3. If  $Hom_{\mathfrak{C}}(A, B) \neq \emptyset$  and  $Hom_{\mathfrak{C}}(B, A) \neq \emptyset$ , then  $A \cong B$ .

*Remark 1.34.* We will deal with objects in  $\mathbf{Fct}(\mathfrak{C}, \mathcal{A})$  for  $(\mathfrak{C}, \natural, 0)$  a homogeneous category and  $\mathcal{A}$  an abelian category. In order to prove propositions for a functor  $F$  of  $\mathbf{Fct}(\mathfrak{C}, \mathbb{K}\text{-Mod})$ , according to the first property of 1.33, it is sufficient to restrict the work on morphisms to the automorphisms. In other words, proving a result dealing with  $F$  for all the automorphisms automatically extends to check this result on  $F$  for all the morphisms.

We should now give the two additional properties that a strict monoidal groupoid  $(\mathfrak{G}, \natural, 0)$  may satisfy so as to the category  $\mathfrak{U}\mathfrak{G}$  be homogeneous.

**Definition 1.35.** Let  $(\mathfrak{G}, \natural, 0)$  be a strict monoidal groupoid. We define two assumptions.

- **(C)** : For all objects  $A, B$  and  $C$  of  $\mathfrak{G}$ , if  $A \natural C \cong B \natural C$  then  $A \cong B$ . The category  $\mathfrak{G}$  is then said to satisfy cancellation property.
- **(I)** : For all objects  $A, B$  of  $\mathfrak{G}$ , the following morphism is injective:

$$\begin{array}{ccc} Aut_{\mathfrak{G}}(A) & \longrightarrow & Aut_{\mathfrak{G}}(A \natural B). \\ f & \longmapsto & f \natural id_B \end{array}$$

**Theorem 1.36.** [23, Theorem 1.10] *Let  $(\mathfrak{G}, \natural, 0)$  be a braided monoidal groupoid with no zero divisors.*

1. *The category satisfies **(H1)** if and only if the groupoid  $\mathfrak{G}$  satisfies **(C)**.*
2. *If the groupoid  $\mathfrak{G}$  satisfies **(I)**, then  $\mathfrak{U}\mathfrak{G}$  satisfies **(H2)**.*

*In particular, if the groupoid  $\mathfrak{G}$  satisfies **(C)** and **(I)**, then  $\mathfrak{U}\mathfrak{G}$  is homogeneous.*

The relationship between the automorphisms of the groupoid  $\mathfrak{G}$  and those of its associated Quillen's construction  $\mathfrak{U}\mathfrak{G}$  is not always clear. Indeed, we intuitively expect that  $\mathfrak{G}$  is the core of  $\mathfrak{U}\mathfrak{G}$  but we need two more hypotheses in order to ensure this property.

**Proposition 1.37.** [23, Proposition 1.7] *Let  $(\mathfrak{G}, \natural, 0)$  be a monoidal groupoid and let  $\mathfrak{U}\mathfrak{G}$  denote the Quillen's construction on  $\mathfrak{G}$ . We assume that:*

- $Aut_{\mathfrak{G}}(0) = \{id_0\}$ .
- *The groupoid  $\mathfrak{G}$  has no zero divisors: if  $A \natural B \cong 0$  in  $\mathfrak{G}$ , then  $A \cong 0$  and  $B \cong 0$ .*

*Then  $\mathfrak{G}$  is the core of  $\mathfrak{U}\mathfrak{G}$ .*

**Example 1.38.** The braid groupoid  $\beta$  is braided monoidal, it has no zero divisors. Moreover its monoidal structure clearly satisfies hypotheses **(C)** and **(I)**, so its associated category  $\mathfrak{U}\beta$  is pre-braided homogeneous.

### 1.3 Coherent conditions

We consider two natural integers  $n$  and  $n'$  such that  $n' \geq n$ . Recall that we have defined four key group morphisms:  $a_{n,\bullet} : \mathbf{B}_n \hookrightarrow \text{Aut}(\mathbf{F}_n)$ ,  $\varsigma_{n,\bullet} : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}$ ,  $\gamma_{n,n'}^{b,\bullet} : \mathbf{B}_n \hookrightarrow \mathbf{B}_{n'}$  and  $\gamma_{n,n'}^{f,\bullet} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n'}$  (with abbreviations  $\gamma_n^{b,\bullet}$  and  $\gamma_n^{f,\bullet}$  when  $n' = n + 1$ ). We will need the following coherence conditions for our constructions.

**Condition 1.39.** We require  $\varsigma_{n+1,\bullet} \circ \gamma_n^{f,\bullet} = \gamma_{n+1}^{b,\bullet} \circ \varsigma_{n,\bullet}$ , ie the following diagram is commutative.

$$\begin{array}{ccc} \mathbf{B}_{n+1} & \xrightarrow{\gamma_{n+1}^{b,\bullet}} & \mathbf{B}_{n+2} \\ \varsigma_{n,\bullet} \uparrow & & \uparrow \varsigma_{n+1,\bullet} \\ \mathbf{F}_n & \xrightarrow{\gamma_n^{f,\bullet}} & \mathbf{F}_{n+1} \end{array}$$

**Condition 1.40.** We denote by  $(id_{(n'-n)\natural} -) : \mathbf{B}_n \rightarrow \mathbf{B}_{n'}$  the morphism which for an element  $\sigma$  of  $\mathbf{B}_n$  gives the element  $id_{(n'-n)\natural}\sigma$  of  $\mathbf{B}_{n'}$ . We require  $(id_{(n'-n)\natural} -) \circ \gamma_n^{b,\bullet} = \gamma_{n'}^{b,\bullet} \circ (id_{(n'-n)\natural} -)$ , ie for all element  $\sigma$  of  $\mathbf{B}_n$  the following diagram is commutative.

$$\begin{array}{ccc} \mathbf{B}_n & \xrightarrow{id_{(n'-n)\natural} -} & \mathbf{B}_{n'} \\ \gamma_n^{b,\bullet} \downarrow & & \downarrow \gamma_{n'}^{b,\bullet} \\ \mathbf{B}_{n+1} & \xrightarrow{id_{(n'-n)\natural} -} & \mathbf{B}_{n'+1} \end{array}$$

**Condition 1.41.** We require  $\gamma_{n,n'}^{f,\bullet} \circ (a_{n,\bullet}(\sigma')) = (a_{n',\bullet}(\sigma'\natural\sigma)) \circ \gamma_{n,n'}^{f,\bullet}$  for all elements  $\sigma$  of  $\mathbf{B}_n$  and  $\sigma'$  of  $\mathbf{B}_{n'-n}$ , ie the following diagram is commutative.

$$\begin{array}{ccc} \mathbf{F}_n & \xrightarrow{a_{n,\bullet}(\sigma)} & \mathbf{F}_n \\ \gamma_{n,n'}^{f,\bullet} \downarrow & & \downarrow \gamma_{n,n'}^{f,\bullet} \\ \mathbf{F}_{n'} & \xrightarrow{a_{n',\bullet}(\sigma'\natural\sigma)} & \mathbf{F}_{n'} \end{array}$$

*Remark 1.42.* Conditions 1.40 and 1.41 will give sufficient relations to define the Long-Moody functor on objects in Proposition 3.6. Moreover, Condition 1.40 will be used to prove Propositions 4.8, and Condition 1.40 will be used to prove Propositions 4.8 and 4.10.

**Condition 1.43.** Let  $m$  be a natural integer. We require  $a_{m+n',\bullet} \left( (b_{m,n'-n}^{\text{gr}})^{-1} \natural id_n \right)$  is the identity on the image of the homomorphism  $\gamma_{m+n,m+n'}^{f,\bullet} \circ \bar{\gamma}_{m,m+n'}^{f,\bullet} : \mathbf{F}_m \rightarrow \mathbf{F}_{m+n'}$ , ie for all element  $\sigma$  of  $\mathbf{B}_n$  the following diagram is commutative.

$$\begin{array}{ccc} \mathbf{F}_m & \xrightarrow{\bar{\gamma}_{m,m+n'}^{f,\bullet}} & \mathbf{F}_{m+n'} \\ \bar{\gamma}_{m,m+n}^{f,\bullet} \downarrow & & \uparrow a_{n',\bullet} \left( (b_{m,n'-n}^{\text{gr}})^{-1} \natural id_n \right) \\ \mathbf{F}_{m+n} & \xrightarrow{\gamma_{m+n,m+n'}^{f,\bullet}} & \mathbf{F}_{m+n'} \end{array}$$

**Condition 1.44.** We require  $a_{n',\bullet} (id_{n'-n}\natural -)$  is the identity on the image of the homomorphism  $\bar{\gamma}_{n'-n,n'}^{f,\bullet} : \mathbf{F}_{n'-n} \rightarrow \mathbf{F}_{n'}$ , ie for all element  $\sigma$  of  $\mathbf{B}_n$  the following diagram is commutative.

$$\begin{array}{ccc}
\mathbf{F}_{n'-n} & \xrightarrow{\bar{\gamma}_{n'-n,n'}^{f,\bullet}} & \mathbf{F}_{n'} \\
& \searrow \bar{\gamma}_{n'-n,n'}^{f,\bullet} & \nearrow a_{n',\bullet}(id_{n'-n}\natural\sigma) \\
& & \mathbf{F}_{n'}
\end{array}$$

In other words, for all element  $\sigma$  of  $\mathbf{B}_n$ , for all element  $f$  of  $\mathbf{F}_{n'-n}$ ,  $a_{n',\bullet}(id_{n'-n}\natural\sigma) \circ \bar{\gamma}_{n,n'}^{f,\bullet}(f) = \bar{\gamma}_{n,n'}^{f,\bullet}(f)$ .

*Remark 1.45.* Condition 1.43 and 1.44 will be used to define the functor  $\Upsilon_m$  in Proposition 4.3.

For the further constructions, we will have to make relevant choices for these four morphisms: the characterization of the choice will be encoded by the quadruplet  $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$ .

**Definition 1.46.** A quadruplet  $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$  is said to be coherent if it satisfies the conditions 1.39, 1.40 and 1.41 for all natural integers  $n$  and  $n'$  such that  $n' \geq n$ . If it satisfies moreover conditions 1.43 and 1.44 for all natural integers  $n$  and  $n'$  such that  $n' \geq n$ , the coherent quadruplet  $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$  is said to be reliable.

**Example 1.47.** Considering the morphisms  $a_{n,1}$ ,  $\varsigma_{n,1}$ ,  $\gamma_{n,n'}^{b,l}$  and  $\gamma_{n,n'}^{f,l}$  for all natural integers  $n$  provide a reliable quadruplet  $(1, 1, l, l)$ .

## 2 Strong polynomial functors

We deal here with the concept of strong polynomial functor. This type of functor will be at the centre of our work in Section 4. We review (and in fact slightly extend) the definition and properties of a strong polynomial functor due to Djament and Vespa in [9] and also a particular case of coefficient systems of finite degree used by Randal-Williams and Wahl in [23].

In [9, Section 1], Djament and Vespa construct a framework to define strong polynomial functors in the category  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ , where  $\mathfrak{M}$  is a symmetric monoidal category where the unit is an initial object and  $\mathcal{A}$  is an abelian category. Here, we aim at generalizing this definition for functors from pre-braided monoidal categories having the same additional property. In particular, a notion of strong polynomial functor will be well-defined for the category  $\mathfrak{UB}\text{-Mod} = \mathbf{Fct}(\mathfrak{UB}, \mathbb{K}\text{-Mod})$ . The keypoint of this section is Proposition 2.4, in so far as it constitutes the crucial property necessary and sufficient to extend the definition of strong polynomial functor for the pre-braided case.

### 2.1 Strong polynomiality

**Definition 2.1.** We denote by  $\mathfrak{Mon}^{\text{pb}}$  the category defined by the following assignment.

- Objects : the pre-braided strict monoidal small categories  $(\mathfrak{M}, \natural, 0)$ .
- Morphisms : the pre-braided strict monoidal functors  $F : (\mathfrak{M}, \natural, 0) \rightarrow (\mathfrak{N}, \natural, 0)$ . Namely, these are strict monoidal functors  $F$  such that the functor  $\mathcal{G}\mathfrak{r}(F)$  is braided.

**Definition 2.2.** We denote by  $\mathfrak{Mon}_{\text{ini}}^{\text{pb}}$  the full subcategory of  $\mathfrak{Mon}^{\text{pb}}$  whose objects are pre-braided strict monoidal small categories  $(\mathfrak{M}, \natural, 0)$  such that the unit  $0$  is an initial object. We denote by  $\mathfrak{Mon}_{\text{null}}^{\text{pb}}$  the full subcategory of  $\mathfrak{Mon}_{\text{ini}}^{\text{pb}}$  whose objects are pre-braided strict monoidal small categories  $(\mathfrak{M}, \natural, 0)$  such that the unit  $0$  is a null object.

We introduce the translation functor, which will play a central role in the definition of strong polynomiality.

**Definition 2.3.** Let  $(\mathfrak{M}, \natural, 0)$  be an object of  $\mathfrak{Mon}^{\text{pb}}$ , let  $\mathfrak{C}$  be a category and let  $x$  be an object of  $\mathfrak{M}$ . We define the endofunctor  $x\natural_{\mathfrak{M}} : \mathfrak{M} \rightarrow \mathfrak{M}$  by:

- Objects: for all objects  $m$  of  $\mathfrak{M}$ ,  $x\mathfrak{h}id_{\mathfrak{M}}(m) = x\mathfrak{h}m$ .
- Morphisms: for all morphism  $f$  of  $Hom_{\mathfrak{M}}(m, m')$ ,  $x\mathfrak{h}id_{\mathfrak{M}}(f) = id_x\mathfrak{h}f : x\mathfrak{h}m \longrightarrow x\mathfrak{h}m'$ .

For  $x$  an object of  $\mathfrak{M}$ , we define the translation by  $x$  functor  $\tau_x : \mathbf{Fct}(\mathfrak{M}, \mathfrak{C}) \longrightarrow \mathbf{Fct}(\mathfrak{M}, \mathfrak{C})$  to be the endofunctor of  $\mathbf{Fct}(\mathfrak{M}, \mathfrak{C})$  obtained by precomposition by the functor  $x\mathfrak{h}id_{\mathfrak{M}}$ .

The following proposition establishes the commutation of two translation functors associated with two objects of  $\mathfrak{M}$ . It is the keystone property to define polynomial functors.

**Proposition 2.4.** *Let  $(\mathfrak{M}, \mathfrak{h}, 0)$  be an object of  $\mathfrak{Mon}^{\text{pb}}$  and let  $\mathfrak{C}$  be a category. Let  $x$  and  $y$  be two objects of  $\mathfrak{M}$ . Then, there exist a natural isomorphism between functors from  $\mathbf{Fct}(\mathfrak{M}, \mathfrak{C})$  to  $\mathbf{Fct}(\mathfrak{M}, \mathfrak{C})$ :*

$$\tau_x \circ \tau_y \cong \tau_y \circ \tau_x.$$

*Proof.* First, because of the associativity of the monoidal product  $\mathfrak{h}$  and of the strictness of  $\mathfrak{M}$ , we have that  $\tau_x \circ \tau_y = \tau_{x\mathfrak{h}y}$  and  $\tau_y \circ \tau_x = \tau_{y\mathfrak{h}x}$ . Let us denote by  $b_{-, -}^{\mathfrak{M}}$  the pre-braiding of  $\mathfrak{M}$ . The key point is the fact that  $b_{x,y}^{\mathfrak{M}} : x\mathfrak{h}y \xrightarrow{\cong} y\mathfrak{h}x$  is a braiding (in so far as it is a braiding defined on the core of  $\mathfrak{M}$ ). for all objects  $F$  of  $\mathbf{Fct}(\mathfrak{M}, \mathfrak{C})$ , we define a morphism between  $\tau_{x\mathfrak{h}y}(F)$  and  $\tau_{y\mathfrak{h}x}(F)$  by:

$$(b_{x,y}^{\mathfrak{M}}\mathfrak{h}id_{\mathfrak{M}})^*(F) : [\tau_{x\mathfrak{h}y}(F) = F((x\mathfrak{h}y)\mathfrak{h}-)] \longrightarrow [(F \circ (b_{x,y}^{\mathfrak{M}}\mathfrak{h}id_{\mathfrak{M}}))((x\mathfrak{h}y)\mathfrak{h}-) = F((y\mathfrak{h}x)\mathfrak{h}-) = \tau_{y\mathfrak{h}x}(F)].$$

Let  $\lambda : F \Longrightarrow G$  be a natural transformation in  $\mathbf{Fct}(\mathfrak{M}, \mathfrak{C})$ . Then:

- Let  $m$  be an object of  $\mathfrak{M}$ , since  $b_{x,y}^{\mathfrak{M}}\mathfrak{h}id_m$  is a morphism of  $\mathfrak{M}$  and  $\lambda$  is a natural transformation, we deduce that:

$$\left( (b_{x,y}^{\mathfrak{M}}\mathfrak{h}id_{\mathfrak{M}})^*(G)(m) \right) \circ \tau_{x\mathfrak{h}y}(\lambda_m) = \tau_{y\mathfrak{h}x}(\lambda_m) \circ \left( (b_{x,y}^{\mathfrak{M}}\mathfrak{h}id_{\mathfrak{M}})^*(F)(m) \right).$$

- Let us consider a morphism  $f \in Hom_{\mathfrak{M}}(m, m')$ . Since  $\lambda$  is a natural transformation, because of the functoriality of  $F$  and  $G$  and of composition rules of the monoidal product, we deduce from the result of the previous point:

$$\left( (b_{x,y}^{\mathfrak{M}}\mathfrak{h}id_{\mathfrak{M}})^*(G)(f) \right) \circ \tau_{x\mathfrak{h}y}(\lambda_m) = \tau_{y\mathfrak{h}x}(\lambda_{m'}) \circ \left( (b_{x,y}^{\mathfrak{M}}\mathfrak{h}id_{\mathfrak{M}})^*(F)(f) \right).$$

Hence we have proved that  $(b_{x,y}^{\mathfrak{M}}\mathfrak{h}id_{\mathfrak{M}})^*(G) \circ \tau_{x\mathfrak{h}y}(\lambda) = \tau_{y\mathfrak{h}x}(\lambda) \circ (b_{x,y}^{\mathfrak{M}}\mathfrak{h}id_{\mathfrak{M}})^*(F)$ . Thus, we have defined a natural transformation  $(b_{x,y}^{\mathfrak{M}}\mathfrak{h}id_{\mathfrak{M}})^* : \tau_{x\mathfrak{h}y} \Longrightarrow \tau_{y\mathfrak{h}x}$ . It happens to be an isomorphism in so far as we analogously construct a natural transformation  $\left( (b_{x,y}^{\mathfrak{M}})^{-1}\mathfrak{h}id_{\mathfrak{M}} \right)^* : \tau_{y\mathfrak{h}x} \Longrightarrow \tau_{x\mathfrak{h}y}$  and direct computations show that for all objects  $F$  of  $\mathbf{Fct}(\mathfrak{M}, \mathfrak{C})$   $\left( (b_{x,y}^{\mathfrak{M}})^{-1}\mathfrak{h}id_{\mathfrak{M}} \right)^*(F) \circ (b_{x,y}^{\mathfrak{M}}\mathfrak{h}id_{\mathfrak{M}})^*(F) = id_{\tau_{x\mathfrak{h}y}(F)}$  and  $(b_{x,y}^{\mathfrak{M}}\mathfrak{h}id_{\mathfrak{M}})^*(F) \circ \left( (b_{x,y}^{\mathfrak{M}})^{-1}\mathfrak{h}id_{\mathfrak{M}} \right)^*(F) = id_{\tau_{y\mathfrak{h}x}(F)}$ .  $\square$

*Remark 2.5.* In Proposition 2.4, the natural isomorphism is not unique: as the proof shows it, we could have used the morphism  $(b_{y,x}^{\mathfrak{M}})^{-1}\mathfrak{h}id_{\mathfrak{M}}$  instead to define an isomorphism between  $\tau_{x\mathfrak{h}y}(F)$  and  $\tau_{y\mathfrak{h}x}(F)$ . This natural isomorphism becomes unique if we additionally assume that the category  $\mathfrak{M}$  is symmetric monoidal as done by Djament and Vespa in [9, Section 1].

In the remainder of this section, we consider an abelian category  $\mathcal{A}$ . Let us move on to the introduction of the evanescence and difference functors, which will characterize the (very) strong polynomiality of a functor in  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ .

If  $\mathfrak{M}$  is a small category and  $\mathcal{A}$  is an abelian category, then the functor category  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$  is an abelian category (see [21, Chapter VIII]).

Let  $(\mathfrak{M}, \mathfrak{h}, 0)$  be an object of  $\mathfrak{Mon}_{\text{ini}}^{\text{pb}}$  and  $x$  be an object of  $\mathfrak{M}$ . for all objects  $F$  of  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ , we denote by  $i_x(F) : \tau_0(F) \longrightarrow \tau_x(F)$  the natural transformations induced by the precomposition of  $F$  by the unique morphism  $\iota_x : 0 \longrightarrow x$  of  $\mathfrak{M}$ . Since  $Id_{\mathbf{Fct}(\mathfrak{M}, \mathcal{A})}$  and  $\tau_x$  are endofunctors of  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ , they induce  $i_x : Id_{\mathbf{Fct}(\mathfrak{M}, \mathcal{A})} \longrightarrow \tau_x$  a natural transformation of  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ . Since the category  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$  is abelian, the kernel and cokernel of the natural transformation  $i_x$  exist.

**Definition 2.6.** We define  $\kappa_x = \ker(i_x)$  and  $\delta_x = \text{coker}(i_x)$ . The endofunctors  $\kappa_x$  and  $\delta_x$  of  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$  are called respectively evanescence and difference functor associated with  $x$ .

The following lemma present elementary properties of the translation, evanescence and difference functors. They are either straightforward consequences of the definitions, or direct generalizations of the framework where  $\mathfrak{M}$  is symmetric monoidal considered in [9]. Indeed, the proofs of the numbered properties are exactly the same as those of [9, Proposition 1.4]: everything works in the same way in so far as the commutation property of the translation endofunctor is still satisfied in the pre-braided case by Proposition 2.4 and if we use the convention to add objects on the right.

**Lemma 2.7.** *Let  $(\mathfrak{M}, \natural, 0)$  be an object of  $\mathfrak{Mon}_{\text{ini}}^{\text{pb}}$ , let  $x$  and  $y$  be objects of  $\mathfrak{M}$ . Then the translation functor  $\tau_x$  is exact and we have the following exact sequence in the category of endofunctors of  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ :*

$$0 \longrightarrow \kappa_x \xrightarrow{\Omega_x} Id \xrightarrow{\iota_x} \tau_x \xrightarrow{\Delta_x} \delta_x \longrightarrow 0. \quad (1)$$

Moreover, considering a short exact sequence  $0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$  in the category  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ , the snake lemma implies that we have the following exact sequence in the category  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ :

$$0 \longrightarrow \kappa_x(F) \longrightarrow \kappa_x(G) \longrightarrow \kappa_x(H) \longrightarrow \delta_x(F) \longrightarrow \delta_x(G) \longrightarrow \delta_x(H) \longrightarrow 0. \quad (2)$$

In addition :

1. The translation endofunctors  $\tau_x$  and  $\tau_y$  of  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$  commute up to natural isomorphism. They commute with limits and colimits.
2. The difference endofunctors  $\delta_x$  and  $\delta_y$  of  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$  commute up to natural isomorphism. They commute with colimits.
3. The endofunctors  $\kappa_x$  and  $\kappa_y$  of  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$  commute up to natural isomorphism. They commute with limits.
4. The natural inclusion  $\kappa_x \circ \kappa_x \hookrightarrow \kappa_x$  is an isomorphism.
5. The translation endofunctor  $\tau_x$  and the difference endofunctor  $\delta_y$  commute up to natural isomorphism.
6. The translation endofunctor  $\tau_x$  and the endofunctor  $\kappa_y$  commute up to natural isomorphism.
7. We have the following natural exact sequence in the category of endofunctors of  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ :

$$0 \longrightarrow \kappa_y \longrightarrow \kappa_{x \natural y} \longrightarrow \tau_x \kappa_y \longrightarrow \delta_y \longrightarrow \delta_{x \natural y} \longrightarrow \tau_y \delta_x \longrightarrow 0. \quad (3)$$

Thanks to lemma 2.7, we can define strong polynomial functors.

**Definition 2.8.** Let  $(\mathfrak{M}, \natural, 0)$  be an object of  $\mathfrak{Mon}_{\text{ini}}^{\text{pb}}$ . We recursively define on  $n \in \mathbb{N}$  the category  $\mathcal{Pol}_n(\mathfrak{M}, \mathcal{A})$  of strong polynomial functors of degree smaller or equal to  $n$  to be the full subcategory of  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$  as follows:

1. If  $n < 0$ ,  $\mathcal{Pol}_n(\mathfrak{M}, \mathcal{A}) = \{0\}$ ;
2. if  $n \geq 0$ , the objects of  $\mathcal{Pol}_n(\mathfrak{M}, \mathcal{A})$  are the functors  $F$  such that for all objects  $x$  of  $\mathfrak{M}$ , the functor  $\delta_x(F)$  is an object of  $\mathcal{Pol}_{n-1}(\mathfrak{M}, \mathcal{A})$ .

The following three propositions are noteworthy properties of the framework in [9] adapted in the pre-braided case. Their proofs follow directly from those of their analogues in [9, Propositions 1.7, 1.8 and 1.9].

**Proposition 2.9.** [9, Proposition 1.7] *Let  $\mathfrak{M}$  and  $\mathfrak{M}'$  be objects of  $\mathfrak{Mon}_{\text{ini}}^{\text{pb}}$  and let  $\alpha : \mathfrak{M} \longrightarrow \mathfrak{M}'$  be a strong monoidal functor. Then, the precomposition by  $\alpha$  gives rise to a functor from  $\mathcal{Pol}_n(\mathfrak{M}', \mathcal{A})$  to  $\mathcal{Pol}_n(\mathfrak{M}, \mathcal{A})$ .*

**Proposition 2.10.** [9, Proposition 1.8] Let  $(\mathfrak{M}, \natural, 0)$  be an object of  $\mathfrak{Mon}_{\text{ini}}^{\text{pb}}$  and let  $x$  be an object of  $\mathfrak{M}$ . Then, the category  $\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$  is closed under the translation endofunctor  $\tau_x$ , under quotient, under extension and under colimit. Moreover, assuming that there exists a set  $\mathfrak{E}$  of objects of  $\mathfrak{M}$  such that:

$$\forall m \in \text{Obj}(\mathfrak{M}), \exists \{e_i\}_{i \in I} \in \text{Obj}(\mathfrak{E}) \text{ (where } I \text{ is finite), } m \cong \bigsqcup_{i \in I} e_i,$$

then, an object  $F$  of  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$  belongs to  $\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$  if and only if  $\delta_e(F)$  is an object of  $\mathcal{P}ol_{n-1}(\mathfrak{M}, \mathcal{A})$  for all objects  $e$  of  $\mathfrak{E}$ .

*Remark 2.11.* The category  $\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$  is not necessarily closed under subobjects. It is the case if  $\mathfrak{M}$  is an object of  $\mathfrak{Mon}_{\text{null}}^{\text{pb}}$ , since then, for all objects  $x$  of  $\mathfrak{M}$ ,  $\kappa_x$  is the null endofunctor of  $\mathbf{Fct}(\mathfrak{Mon}_{\text{null}}^{\text{pb}}, \mathcal{A})$ . Also, we will see in the next subsection that for very strong polynomial functors, as we force  $\kappa_x$  to be null for all objects  $x$  of  $\mathfrak{M}$ , very strong polynomial functors will be closed under kernel of an epimorphism. As a consequence, in the general case where  $\mathfrak{M}$  is an object of  $\mathfrak{Mon}_{\text{ini}}^{\text{pb}}$ , a subfunctor of an object  $F$  of  $\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$  is not necessarily a strong polynomial functor.

*Remark 2.12.* If we consider  $\mathfrak{M} = \mathfrak{U}\beta$ , then all object  $n$  (ie a natural integer) is clearly the addition of  $n$  times the object 1. Hence, because of the last statement of Proposition 2.10, when we will deal with strong polynomiality of objects in  $\mathbf{Fct}(\mathfrak{U}\beta, \mathcal{A})$ , it will suffice to verify the polynomiality for  $\tau_1$ .

**Proposition 2.13.** [9, Proposition 1.9] Let  $\mathfrak{M}$  and  $\mathfrak{M}'$  be objects of  $\mathfrak{Mon}^{\text{pb}}$ . Let  $F$  be an object of  $\mathbf{Fct}(\mathfrak{Mon}_{\text{ini}}^{\text{pb}}, \mathcal{A})$ . Then, the functor  $F$  is an object of  $\mathcal{P}ol_0(\mathfrak{M}, \mathcal{A})$  if and only if it the quotient of a constant functor of  $\mathbf{Fct}(\mathfrak{Mon}_{\text{ini}}^{\text{pb}}, \mathcal{A})$ .

**Example 2.14.** By Proposition 1.27, the category  $\mathfrak{U}\beta$  is a pre-braided monoidal category. This example is the first one which led us to extend the definition of [9]. Thus, we have a well-defined notion of strong polynomial functor on the category  $\mathfrak{U}\beta$ .

## 2.2 Very strong polynomial functors

A certain type of functors, called coefficient systems of finite degree, closely related to the strong polynomial one, is used by Randal-Williams and Wahl in [23, Definition 4.10] for their homological stability theorems, generalizing the concept introduced by van der Kallen for general linear groups [25]. Let us define a new type of strong polynomial functor, related to coefficient systems of finite degree.

**Definition 2.15.** Let  $(\mathfrak{M}, \natural, 0)$  be an object of  $\mathfrak{Mon}_{\text{ini}}^{\text{pb}}$ . We define the category  $\mathcal{V}\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$  of very strong polynomial functors of degree less or equal to  $n$  to be the full subcategory of  $\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$  such that a strong polynomial functor  $F$  is very strong polynomial if for all objects  $x$  of  $\mathfrak{M}$ ,  $\kappa_x(F) = 0$ . We define the category  $\mathcal{V}\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$  of very strong polynomial functors of degree less or equal to  $n$  to be the full subcategory of  $\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$  as follows:

1. If  $n < 0$ ,  $\mathcal{V}\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A}) = \{0\}$ ;
2. if  $n \geq 0$ , a functor  $F \in \mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$  is an object of  $\mathcal{V}\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$  if for all objects  $x$  of  $\mathfrak{M}$ ,  $\kappa_x(F) = 0$  and the functor  $\delta_x(F)$  is an object of  $\mathcal{V}\mathcal{P}ol_{n-1}(\mathfrak{M}, \mathcal{A})$ .

*Remark 2.16.* Using the framework introduced by Randal-Williams and Wahl in [23, Definition 4.10], a coefficient system in every object  $x$  of  $\mathfrak{M}$  of degree  $n$  at  $N = 0$  is a very strong polynomial functor.

**Proposition 2.17.** Let  $\mathfrak{M}$  be an object of  $\mathfrak{Mon}_{\text{ini}}^{\text{pb}}$  and let  $x$  be an object of  $\mathfrak{M}$ . Then, the category  $\mathcal{V}\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$  is closed under the translation endofunctor  $\tau_x$ , under kernel of epimorphism and under extension. Moreover, assuming that there exists a set  $\mathfrak{E}$  of objects of  $\mathfrak{M}$  such that:

$$\forall m \in \text{Obj}(\mathfrak{M}), \exists \{e_i\}_{i \in I} \in \text{Obj}(\mathfrak{E}) \text{ (where } I \text{ is finite), } m \cong \bigsqcup_{i \in I} e_i,$$

then, an object  $F$  of  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$  belongs to  $\mathcal{V}\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$  if and only if  $\kappa_e(F) = 0$  and  $\delta_e(F)$  is an object of  $\mathcal{P}ol_{n-1}(\mathfrak{M}, \mathcal{A})$  for all objects  $e$  of  $\mathfrak{E}$ .

*Proof.* The first point follows from the fact that for all objects  $x$  of  $\mathfrak{M}$ , the endofunctor  $\tau_x$  commutes with the endofunctors  $\delta_x$  and  $\kappa_x$  (see Lemma 2.7). For the second and third points, let us consider two short exact sequences of  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ :  $0 \rightarrow G \rightarrow F_1 \rightarrow F_2 \rightarrow 0$  and  $0 \rightarrow F_3 \rightarrow H \rightarrow F_4 \rightarrow 0$  with  $F_i$  a very strong polynomial functor of degree  $n$  for all  $i$ . Let  $x$  be an object of  $\mathfrak{M}$ . We use the exact sequence (2) of Lemma 2.7 to obtain the two following exact sequences in the category  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ :

$$\begin{aligned} 0 &\rightarrow \kappa_x(G) \rightarrow 0 \rightarrow 0 \rightarrow \delta_x(G) \rightarrow \delta_x(F_1) \rightarrow \delta_x(F_2) \rightarrow 0; \\ 0 &\rightarrow 0 \rightarrow \kappa_x(H) \rightarrow 0 \rightarrow \delta_x(F_3) \rightarrow \delta_x(H) \rightarrow \delta_x(F_4) \rightarrow 0. \end{aligned}$$

Therefore,  $\kappa_x(F) = \kappa_x(H) = 0$  and the result follows directly by induction on the degree of polynomiality. For the last point, we consider the long exact sequence (3) of Lemma 2.7 applied to an object  $F$  of  $\mathcal{VPol}_n(\mathfrak{M}, \mathcal{A})$  to obtain the following exact sequence in the category  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ :

$$0 \rightarrow \kappa_y(F) \rightarrow \kappa_{x \natural y}(F) \rightarrow \tau_x \kappa_y(F) \rightarrow \delta_y(F) \rightarrow \delta_{x \natural y}(F) \rightarrow \tau_y \delta_x(F) \rightarrow 0.$$

Hence, by induction on the length of objects as monoidal product of  $\{e_i\}_{i \in I}$ , we deduce that  $\kappa_m(F) = 0$  for all objects  $m$  of  $\mathfrak{M}$  if and only if  $\kappa_e(F) = 0$  for all objects  $e$  of  $\mathfrak{E}$ . Moreover, since  $\mathcal{VPol}_n(\mathfrak{M}, \mathcal{A})$  is closed under extension and by the translation endofunctor  $\tau_y$ , the result follows directly by induction on the degree of polynomiality  $n$ .  $\square$

**Proposition 2.18.** *Let  $\mathfrak{M}$  be an object of  $\mathfrak{Mon}_{\text{ini}}^{\text{pb}}$ . Let  $F$  be an object of  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ . The functor  $F$  is an object of  $\mathcal{VPol}_0(\mathfrak{M}, \mathcal{A})$  if and only if it is equivalent to a constant functor.*

*Proof.* Using the long exact sequence 1 of Lemma 2.7 applied to  $F$ , we deduce that  $F$  is an object of  $\mathcal{VPol}_0(\mathfrak{M}, \mathcal{A})$  if and only if  $F \cong \tau_x F$  for all objects  $x$  of  $\mathfrak{M}$ . It is equivalent to say that for all objects  $x$  of  $\mathfrak{M}$ , the morphism  $F(t_x)$  is an isomorphism. Hence,  $F$  is an object of  $\mathcal{VPol}_0(\mathfrak{M}, \mathcal{A})$  if and only if it is equivalent to the constant functor equals to  $F(0)$ .  $\square$

**Lemma 2.19.** *Let  $n$  be a natural integer. Let  $F$  be a strong polynomial functor of degree  $n$  in the category  $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-Mod})$ . Then a direct summand of  $F$  is necessarily an object of the category  $\mathcal{Pol}_n(\mathfrak{U}\beta, \mathbb{K}\text{-Mod})$ .*

*Proof.* Let  $G$  be a direct summand of  $F$ , in other words we can write  $F = G \oplus G'$ . for all natural integers  $m$ , since the translation functor  $\tau_m$  is an exact functor by Lemma 2.7 and the difference functor  $\delta_m$  is a colimit in the category  $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-Mod})$ , we deduce that the difference functor commutes with the direct sum:

$$\delta_m F \cong \delta_m G \oplus \delta_m G'.$$

Let  $d_0, \dots, d_k$  be natural integers. Hence:

$$\delta_{d_0} \cdots \delta_{d_k} F \cong (\delta_{d_0} \cdots \delta_{d_k} G) \oplus (\delta_{d_0} \cdots \delta_{d_k} G').$$

A fortiori, if  $\delta_{d_0} \cdots \delta_{d_n} F = 0$  then  $\delta_{d_0} \cdots \delta_{d_n} G = 0$ .  $\square$

There exist strong polynomial functors which are not very strong polynomial in any degree.

**Example 2.20.** Let us consider the category  $\mathfrak{U}\beta$  and  $n$  a natural integer. Let  $A$  be an object of  $\mathcal{A}$  and let  $0_{\mathcal{A}}$  be the null object of  $\mathcal{A}$ . We denote by  $t_A : A \rightarrow 0_{\mathcal{A}}$  the unique morphism in  $\text{Hom}_{\mathcal{A}}(A, 0_{\mathcal{A}})$ . Let  $\mathfrak{A}_n$  be an object of  $\mathbf{Fct}(\mathfrak{U}\beta, \mathcal{A})$ , defined by:

- Objects:  $\forall m \in \mathbb{N}, \mathfrak{A}_n(m) = \begin{cases} A & \text{if } n = m \\ 0_{\mathcal{A}} & \text{otherwise} \end{cases}$ .

- Morphisms: let  $[j - i, f]$  with  $f \in \mathbf{B}_n$  be a morphism from  $i$  to  $j$  in the category  $\mathfrak{U}\beta$ . Then:

$$\mathfrak{A}_n(f) = \begin{cases} id_A & \text{if } i = j = n \\ t_A & \text{if } i = n \leq j \\ \iota_A & \text{if } i \leq j = n \\ id_{0_{\mathcal{A}}} & \text{otherwise.} \end{cases}$$

For coherence, we fix  $\mathfrak{A}_{-1}$  to be the null functor of  $\mathbf{Fct}(\mathfrak{U}\beta, \mathcal{A})$ . Then, it is clear that  $i_p(\mathfrak{A}_n)$  is the zero natural transformation. On the one hand, we deduce the following natural equivalence  $\kappa_1(\mathfrak{A}_n) \cong \mathfrak{A}_n$  and a fortiori  $\mathfrak{A}_n$  is not a very strong polynomial functor. On the other hand, it is worth noting the natural equivalence  $\delta_1(\mathfrak{A}_n) \cong \tau_1(\mathfrak{A}_n)$  and the fact that  $\tau_1(\mathfrak{A}_n) \cong \mathfrak{A}_{n-1}$ . Therefore, we recursively prove that  $\mathfrak{A}_n$  is a strong polynomial functor of degree  $n$ .

*Remark 2.21.* On the contrary of  $\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$ , a quotient of an object  $F$  of  $\mathcal{V}Pol_n(\mathfrak{M}, \mathcal{A})$  is not necessarily a very strong polynomial functor. For example, for  $\mathfrak{M} = \mathfrak{U}\beta$  and  $A$  be an object of  $\mathcal{A}$ , let us consider the functor  $\mathfrak{A}_0$  defined in Example 2.20, which we proved to be a strong polynomial functor of degree 0. Let  $\mathfrak{A}$  be the constant object of  $\mathbf{Fct}(\mathfrak{U}\beta, \mathcal{A})$  equals to  $A$ . Then, we define a natural transformation  $\alpha : \mathfrak{A} \implies \mathfrak{A}_0$  assigning:

$$\forall n \in \mathbb{N}, \alpha_n = \begin{cases} id_A & \text{if } n = 0 \\ t_A & \text{otherwise.} \end{cases}$$

Moreover, it is an epimorphism in the category  $\mathbf{Fct}(\mathfrak{U}\beta, \mathcal{A})$  since for all natural integers  $n$ ,  $coker(\alpha_n) = 0_{\mathcal{A}}$ . We proved in Example 2.20 that  $\mathfrak{A}_0$  is not a very strong polynomial functor of degree 0 whereas  $\mathfrak{A}$  is a very strong polynomial functor of degree 0 by Proposition 2.18.

### 2.3 Examples of polynomial functors associated with braid representations

Different families of representations of braid group can be interpreted as very strong polynomial functors. First, we recall a result due to Tong, Yang and Ma.

**Tong-Yang-Ma results** In 1996, in the article [24], Tong, Yang and Ma interested in the representations of  $B_n$  where the  $i$ th generator is sent to a matrix of the form  $Id_{i-1} \oplus T \oplus Id_{n-i-1}$ , with  $T$  a  $m \times m$  non-singular matrix and  $m \geq 2$ . In particular, for  $m = 2$ , they prove that there exist up to equivalence only two non trivial representations of this type. We give here the result and an interpretation of their work from a functorial point of view.

**Theorem 2.22.** [24, Part II] Let  $\eta : \mathbf{B}_\bullet \longrightarrow GL_\bullet$  be a natural transformation. Assume that for all natural integers  $n$ , for all  $i \in \{1, \dots, n-1\}$ , the following diagram is commutative.

$$\begin{array}{ccc} \mathbf{B}_n & \xrightarrow{\eta_n} & GL_n(\mathbb{K}) \\ \uparrow \text{incl}_i^n & & \uparrow id_{i-1} * - * id_{n-i-1} \\ \mathbf{B}_2 & \xrightarrow{\eta_2} & GL_2(\mathbb{K}) \end{array}$$

Two natural transformations  $\eta$  and  $\eta'$  will be said here to be equivalent if there exists a natural equivalence  $\mu : GL_\bullet \longrightarrow GL_\bullet$  such that  $\mu \circ \eta = \eta'$ . Then, the natural transformation  $\eta$  is equivalent to the one of the following natural transformations.

1. The trivial natural transformation, denoted by  $\mathbf{id}$ : for every generator  $\sigma_i$  of  $\mathbf{B}_n$ ,  $\mathbf{id}_n(\sigma_i) = Id_{GL_n(\mathbb{K})}$
2. The unreduced Burau natural transformation, denoted by  $\mathbf{bur}$ : for every generator  $\sigma_i$  of  $\mathbf{B}_n$ ,

$$\mathbf{bur}_{n,t}(\sigma_i) = Id_{i-1} \oplus B(t) \oplus Id_{n-i-1},$$

with

$$B(t) = \begin{bmatrix} 1-t & t \\ 1 & 0 \end{bmatrix}.$$

3. A natural transformation based on a representation, which we call the Tong-Yang-Ma representation, denoted by  $\mathbf{tjm}$ : for every generator  $\sigma_i$  of  $\mathbf{B}_n$  if  $n \geq 2$ ,

$$\mathbf{tjm}_{n,t}(\sigma_i) = Id_{i-1} \oplus TYM(t) \oplus Id_{n-i-1},$$

with

$$TYM(t) = \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix}.$$

*Remark 2.23.* The unreduced Burau representation (see [14, Section 3.1] or [6, Section 4.2] for more details about this family of representations) is reducible but indecomposable, whereas the Tong-Yang-Ma representation is irreducible [24, Part II]. We may also define a reduced Burau natural transformation (see [14, Section 3.3] for more details about this family of representations), denoted by  $\overline{\text{bur}}$ . For  $n = 2$ , one assigns  $\overline{\text{bur}}(\sigma_1) := -t$ . for all natural integers  $n \geq 3$ , we define for every Artin generator  $\sigma_i$  of  $\mathbf{B}_n$  with  $i \in \{2, \dots, n-2\}$ :

$$\overline{\text{bur}}_{n,t}(\sigma_i) = Id_{i-2} \oplus \overline{B}(t) \oplus Id_{n-i-2}$$

with

$$\overline{B}(t) = \begin{bmatrix} 1 & t & 0 \\ 0 & -t & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and

$$\overline{\text{bur}}_{n,t}(\sigma_1) = \begin{bmatrix} -t & 0 \\ 1 & 1 \end{bmatrix} \oplus Id_{n-3} \quad ; \quad \overline{\text{bur}}_{n,t}(\sigma_{n-1}) = Id_{n-3} \oplus \begin{bmatrix} 1 & t \\ 0 & -t \end{bmatrix}.$$

Let us move on to a presentation of examples of polynomial functors associated with these families of braid representations. A first example is based on the family introduced by Tong, Yang and Ma.

*Remark 2.24.* Since,  $\mathbf{B}_\infty^{ab}$  is isomorphic to  $\mathbb{Z}$  as abelian groups, the group rings  $R[\mathbf{B}_\infty^{ab}]$  and  $R[\{t_i^{\pm 1} \mid i \in \{1, \dots, n\}\}]$  are isomorphic as  $R$ -modules.

**Example 2.25.** Let  $\mathfrak{T}\mathfrak{Y}\mathfrak{M} : \mathfrak{A}\mathfrak{B} \longrightarrow R[\mathbf{B}_\infty^{ab}]\text{-Mod}$  be the functor defined by:

- Objects:  $\forall n \in \mathbb{N}$ ,  $\mathfrak{T}\mathfrak{Y}\mathfrak{M}(n) = R[\mathbf{B}_\infty^{ab}]^{\oplus n}$ .
- Morphisms:

– Automorphisms: for all interger  $n \geq 2$ , for every Artin generator  $\sigma_i$  of  $\mathbf{B}_n$ :

$$\mathfrak{T}\mathfrak{Y}\mathfrak{M}(\sigma_i) = \text{tym}_{n,t_1}(\sigma_i).$$

– General morphisms: let  $n, n' \in \mathbb{N}$ , such that  $n' \geq n$ , and  $[n' - n, \sigma] \in \text{Hom}_{\mathfrak{A}\mathfrak{B}}(n, n')$ . We define:

$$\mathfrak{T}\mathfrak{Y}\mathfrak{M}([n' - n, \sigma]) = \mathfrak{T}\mathfrak{Y}\mathfrak{M}(\sigma) \circ \gamma_{n,n'}^{R[\mathbf{B}_\infty^{ab}]}.$$

We call this functor the Tong-Yang-Ma functor. The assignment of  $\mathfrak{T}\mathfrak{Y}\mathfrak{M}$  defines a functor since  $\mathfrak{T}\mathfrak{Y}\mathfrak{M}(id_n) = id_{R[\mathbf{B}_\infty^{ab}]^{\oplus n}}$  and for  $i, j \in \{1, \dots, n-1\}$ , such that  $j \geq i$ , by our definition  $\mathfrak{T}\mathfrak{Y}\mathfrak{M}(\sigma_j \circ \sigma_i) = \mathfrak{T}\mathfrak{Y}\mathfrak{M}(\sigma_j) \circ \mathfrak{T}\mathfrak{Y}\mathfrak{M}(\sigma_i)$  (Remark (1.34) ensuring that assignment of  $\mathfrak{T}\mathfrak{Y}\mathfrak{M}$  defines properly a functor on  $\mathfrak{A}\mathfrak{B}$ ).

Another example naturally arises from the unreduced Burau representations.

**Example 2.26.** Let  $\mathfrak{B}\text{ur} : \mathfrak{A}\mathfrak{B} \longrightarrow R[\mathbf{B}_\infty^{ab}]\text{-Mod}$  be the functor defined by:

- Objects:  $\forall n \in \mathbb{N}$ ,  $\mathfrak{B}\text{ur}(n) = R[\mathbf{B}_\infty^{ab}]^{\oplus n}$ .
- Morphisms:

– Automorphisms: for all natural integers  $n \geq 2$ , for every Artin generator  $\sigma_i$  of  $\mathbf{B}_n$ :

$$\mathfrak{B}\text{ur}(\sigma_i) = \text{bur}_{n,t_1}(\sigma_i).$$

– General morphisms: Let  $n, n' \in \mathbb{N}$ , such that  $n' \geq n$ , and  $[n' - n, \sigma_i] \in \text{Hom}_{\mathfrak{A}\mathfrak{B}}(n, n')$ . We define:

$$\mathfrak{B}\text{ur}([n' - n, \sigma_i]) = \mathfrak{B}\text{ur}(\sigma_i) \circ \gamma_{n,n'}^{R[\mathbf{B}_\infty^{ab}]}.$$

The functor  $\mathfrak{B}\text{ur}$  already appears in [23, Example 4.3 and 4.15]. We call this functor the unreduced Burau functor.

The following example corresponds to the family of the reduced Burau representations.

**Example 2.27.** Let  $\overline{\mathfrak{B}ur} : \mathcal{U}\beta \rightarrow R[\mathbf{B}_\infty^{ab}]\text{-}\mathcal{M}od$  be the functor defined by:

- Objects:  $\forall n \in \mathbb{N}^*$ ,  $\overline{\mathfrak{B}ur}(n) = R[\mathbf{B}_\infty^{ab}]^{\oplus n-1}$  and  $\overline{\mathfrak{B}ur}(0) = 0$ .
- Morphisms:

– Automorphisms: for all interger  $n \geq 2$ , for every Artin generator  $\sigma_i$  of  $\mathbf{B}_n$ :

$$\overline{\mathfrak{B}ur}(\sigma_i) = \overline{\mathfrak{bur}}_{n,t_1}(\sigma_1).$$

– General morphisms: Let  $n, n' \in \mathbb{N}$ , such that  $n' \geq n$ , and  $[n' - n, \sigma_i] \in Hom_{\mathcal{U}\beta}(n, n')$ . We define:

$$\overline{\mathfrak{B}ur}([n' - n, \sigma_i]) = \overline{\mathfrak{B}ur}(\sigma_i) \circ \gamma_{n-1, n'-1}^{R[\mathbf{B}_\infty^{ab}]}$$

We call this functor the reduced Burau functor. The assignment of  $\overline{\mathfrak{B}ur}$  defines a functor since  $\overline{\mathfrak{B}ur}(id_n) = id_{R[\mathbf{B}_\infty^{ab}]^{\oplus n-1}}$  and for  $i, j \in \{1, \dots, n-1\}$ , such that  $j \geq i$ , by our definition  $\overline{\mathfrak{B}ur}(\sigma_j \circ \sigma_i) = \overline{\mathfrak{B}ur}(\sigma_j) \circ \overline{\mathfrak{B}ur}(\sigma_i)$  (Remark (1.34) ensures that assignment of  $\overline{\mathfrak{B}ur}$  defines properly a functor on  $\mathcal{U}\beta$ ).

**Proposition 2.28.** *The functors  $\mathfrak{B}ur$  and  $\mathfrak{T}\mathfrak{Y}\mathfrak{M}$  are very strong polynomial functors of degree 1.*

*Proof.* For the functor  $\mathfrak{B}ur$ , it is a consequence of [23, Example 4.15]. So we will focus on the case of the functor  $\mathfrak{T}\mathfrak{Y}\mathfrak{M}$ . Let  $n$  be a natural integer. By the statement of Remark 2.12, it is sufficient to consider the application  $i_1 \mathfrak{T}\mathfrak{Y}\mathfrak{M}([0, id_n]) = \gamma_{n, 1+n}^{R[\mathbf{B}_\infty^{ab}]}$ . This map is a monomorphism and its cokernel is  $R[\mathbf{B}_\infty^{ab}]$ . So  $\kappa_1 \mathfrak{T}\mathfrak{Y}\mathfrak{M} = 0$  is the null functor of  $\mathbf{Fct}(\mathcal{U}\beta, R[\mathbf{B}_\infty^{ab}]\text{-}\mathcal{M}od)$ . Let  $n'$  be a natural integer such that  $n' \geq n$  and let  $\varphi_n^{n'} = [n' - n, \phi] \in Hom_{\mathcal{U}\beta}(n, n')$ . By naturality and the universal property of the cokernel, there exists a unique endomorphism of  $R[\mathbf{B}_\infty^{ab}]$  such that the following diagram commutes, where the lines are exact. It is exactly the definition of  $\delta_1 \mathfrak{T}\mathfrak{Y}\mathfrak{M}([n' - n, \phi])$ .

$$\begin{array}{ccccccc} 0 & \longrightarrow & R[\mathbf{B}_\infty^{ab}]^n & \xrightarrow{\gamma_n^{R[\mathbf{B}_\infty^{ab}]}} & R[\mathbf{B}_\infty^{ab}]^{\oplus n+1} & \xrightarrow{\pi_{n+1}} & R[\mathbf{B}_\infty^{ab}] \longrightarrow 0 \\ & & \mathfrak{T}\mathfrak{Y}\mathfrak{M}(\varphi_n^{n'}) \downarrow & & \downarrow \tau_1(\mathfrak{T}\mathfrak{Y}\mathfrak{M})(\varphi_n^{n'}) & & \downarrow \exists! \\ 0 & \longrightarrow & R[\mathbf{B}_\infty^{ab}]^{\oplus n'} & \xrightarrow{\gamma_{n'}^{R[\mathbf{B}_\infty^{ab}]}} & R[\mathbf{B}_\infty^{ab}]^{\oplus n'+1} & \xrightarrow{\pi_{n'+1}} & R[\mathbf{B}_\infty^{ab}] \longrightarrow 0 \end{array}$$

For all  $(b, a) \in R[\mathbf{B}_\infty^{ab}]^{\oplus n} \oplus R[\mathbf{B}_\infty^{ab}] = R[\mathbf{B}_\infty^{ab}]^{\oplus n+1}$ :

$$\tau_1(\mathfrak{T}\mathfrak{Y}\mathfrak{M})(\varphi_n^{n'})(b, a) = (\mathfrak{T}\mathfrak{Y}\mathfrak{M}([n' - n, \phi])(b, a).$$

Therefore:

$$\left( \pi_{n'+1} \circ \tau_1(\mathfrak{T}\mathfrak{Y}\mathfrak{M})(\varphi_n^{n'}) \right)(b, a) = a = \pi_{n+1}(b, a).$$

Hence,  $id_{R[\mathbf{B}_\infty^{ab}]}$  also makes the diagram commutative and thus  $\delta_1 \mathfrak{T}\mathfrak{Y}\mathfrak{M}([n' - n, \phi]) = id_{R[\mathbf{B}_\infty^{ab}]}$ . Hence,  $\delta_1 \mathfrak{T}\mathfrak{Y}\mathfrak{M}$  is the constant functor equals to  $R[\mathbf{B}_\infty^{ab}]$ . A fortiori, because of Proposition 2.13,  $\delta_1 \mathfrak{T}\mathfrak{Y}\mathfrak{M}$  is a degree 0 very strong polynomial functor.  $\square$

*Notation 2.29.* We could have defined the same functors with another parameter as  $t_1$ . In this case, we put this other parameter  $t' \in R[\mathbf{B}_\infty^{ab}]$  in index of the notation (e.g.  $\mathfrak{B}ur_{t'}$ ). For  $y \in \mathbb{K}$  and  $F \in \mathbf{Fct}(\mathcal{U}\beta, \mathbb{K}\text{-}\mathcal{M}od)$ , let  $yF : \mathcal{U}\beta \rightarrow \mathbb{K}\text{-}\mathcal{M}od$  be the functor defined in the same way as  $F$  for objects, such that for all natural integers  $n$ , for every Artin generator  $\sigma_i$  of  $\mathbf{B}_n$  (with  $i \in \{1, \dots, n-1\}$ ):  $(yF)(\sigma_i) = yF(\sigma_i)$ . Also, we could have defined the Burau functor with the assignement  $(R[\mathbf{B}_\infty^{ab}])[\mathbf{B}_\infty^{ab}]^{\oplus n}$  on each object  $n \in \mathbb{N}$  and the same assignement for morphisms. We will denote this version by  $\mathfrak{B}ur_{t_1} : \mathcal{U}\beta \rightarrow (R[\mathbf{B}_\infty^{ab}])[\mathbf{B}_\infty^{ab}]\text{-}\mathcal{M}od$ . We prove that  $\mathfrak{B}ur_{t_1}$  is also very strong polynomial of degree one exact exactly the same way as for  $\mathfrak{B}ur$ .

A link between the unreduced Burau and the reduced Burau functors can be established: the functor  $\overline{\mathfrak{B}ur}$  is in fact a subfunctor of the functor  $\mathfrak{B}ur$ .

**Definition 2.30.** Let  $\mathcal{T}_1 : \mathfrak{U}\mathfrak{B} \rightarrow R[\mathbf{B}_\infty^{ab}]\text{-Mod}$  be the functor defined by:

- Objects:  $\forall n \in \mathbb{N}^*$ ,  $\mathcal{T}_1(n) = R[\mathbf{B}_\infty^{ab}]$  and  $\mathcal{T}_1(0) = 0$ .
- Morphisms: for all elements  $b$  of  $\mathbf{B}_n$ ,  $\mathcal{T}_1(b) = id_{R[\mathbf{B}_\infty^{ab}]}$ .

*Remark 2.31.* The functor  $\mathcal{T}_1$  is obviously a strong polynomial functor of degree 1. It is not very strong polynomial of degree 1 since  $\delta_1 \mathcal{T}_1 \cong \mathfrak{A}_0$ . Nevertheless, it is worth noting that  $\kappa_1 \mathcal{T}_1 = 0$ .

**Lemma 2.32.** *There exists a natural equivalence  $\mathfrak{B}ur \xrightarrow{r} \tau_1 \overline{\mathfrak{B}ur}$ , defined for all natural integers  $n$  by:*

$$r_n = \begin{matrix} & \overbrace{\hspace{10em}}^n & \\ \left[ \begin{array}{cccccc} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & -1 \\ 0 & \cdots & \cdots & 0 & 1 \end{array} \right] \end{matrix}$$

*Proof.* Considering the objects, we know that  $\mathfrak{B}ur(n) = R[\mathbf{B}_\infty^{ab}]^{\oplus n} = \tau_1 \overline{\mathfrak{B}ur}(n)$  for all natural integers  $n$ . Moreover, it follows from straightforward matrix computations that  $r_n \circ \mathfrak{B}ur(\sigma_i) = \tau_1 \overline{\mathfrak{B}ur}(\sigma_i) \circ r_n$  for all  $i \in \{1, \dots, n-1\}$ .  $\square$

**Proposition 2.33.** *There exists a short exact sequence of  $\mathbf{Fct}(\mathfrak{U}\mathfrak{B}, R[\mathbf{B}_\infty^{ab}]\text{-Mod})$  with the following form:*

$$0 \longrightarrow \overline{\mathfrak{B}ur} \longrightarrow \mathfrak{B}ur \longrightarrow \mathcal{T}_1 \longrightarrow 0 .$$

*It does not split.*

*Proof.* One defines a natural transformation  $r' : \overline{\mathfrak{B}ur} \Rightarrow \tau_1 \overline{\mathfrak{B}ur}$  by assigning for all natural integers  $n$ :

$$r'_n = \left[ \begin{array}{c} 0 \\ Id_{n-1} \end{array} \right] : R[\mathbf{B}_\infty^{ab}]^{\oplus n-1} \hookrightarrow R[\mathbf{B}_\infty^{ab}]^{\oplus n} .$$

We thus define the natural transformation:

$$0 \longrightarrow \overline{\mathfrak{B}ur} \xrightarrow{r^{-1} \circ r'} \mathfrak{B}ur .$$

Repeating mutatis mutandis the work done in the proof of Proposition 2.28, we conclude that for all natural integers  $n$  we have a short exact sequence, natural in  $n$ :

$$0 \longrightarrow \overline{\mathfrak{B}ur}(n) \longrightarrow \mathfrak{B}ur(n) \longrightarrow \mathcal{T}_1(n) \longrightarrow 0 .$$

$\square$

*Remark 2.34.* Furthermore, it follows from the proof of Proposition 2.33, we have the equivalence of  $\mathfrak{U}\mathfrak{B}$ -modules  $\tau_1 \mathfrak{B}ur \cong \tau_1 \overline{\mathfrak{B}ur} \oplus \mathcal{T}_1$ . Indeed, it follows directly from the definition of the Burau functor that for all generator  $\sigma_i$  of  $\mathbf{B}_n$ ,  $\tau_1 \mathfrak{B}ur(\sigma_i) = \mathfrak{B}ur(\sigma_i) \oplus \mathcal{T}_1(\sigma_i)$ . The verification for the objects being trivial and because of Remark 1.34, we directly deduce that we have an equality  $\tau_1 \mathfrak{B}ur = \mathfrak{B}ur \oplus \mathcal{T}_1$ . Therefore, we obtain the desired equivalence.

**Corollary 2.35.** *The functor  $\overline{\mathfrak{B}ur}$  is a strong polynomial functor of degree 2.*

*Proof.* According to Proposition 2.33 and Remark 2.31, the functor  $\overline{\mathfrak{Bur}}$  is the kernel of the epimorphism  $\mathfrak{Bur} \longrightarrow \mathcal{T}_1 \longrightarrow 0$ . Thanks to the exact sequence (2) of Lemma 2.7, we obtain the following exact sequence in the category  $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ :

$$0 \longrightarrow \delta_1(\overline{\mathfrak{Bur}}) \longrightarrow \delta_1(\mathfrak{Bur}) \longrightarrow \delta_1(\mathcal{T}_1) \cong \mathfrak{A}_0 \longrightarrow 0.$$

The following exact sequence is given applying again the exact sequence (2) of Lemma 2.7, Proposition 2.28 and Example 2.20:

$$0 \longrightarrow \kappa_1 \delta_1(\overline{\mathfrak{Bur}}) \longrightarrow 0 \longrightarrow \mathfrak{A}_0 \longrightarrow \delta_1 \delta_1(\overline{\mathfrak{Bur}}) \longrightarrow 0.$$

Hence,  $\mathfrak{A}_0 \cong \delta_1 \delta_1(\overline{\mathfrak{Bur}})$  and the result follows from the fact that  $\mathfrak{A}_0$  is a strong polynomial functor of degree 0 according to Example 2.20.  $\square$

A last example is given by the family of the Lawrence-Krammer representations (see [2, 16, 15]).

**Example 2.36.** Let  $\mathfrak{L}\mathfrak{K} : \mathfrak{M}\beta \longrightarrow (R[\mathbf{B}_\infty^{ab}])[\mathbf{B}_\infty^{ab}]\text{-Mod}$  be the functor defined by:

- Objects: for all natural integers  $n \geq 2$ ,  $\mathfrak{L}\mathfrak{K}(n) = \bigoplus_{1 \leq j < k \leq n} V_{j,k}$ , with for all  $1 \leq j < k \leq n$ ,  $V_{j,k}$  is a free

$(R[\mathbf{B}_\infty^{ab}])[\mathbf{B}_\infty^{ab}]$ -module of rank one. Hence,  $\mathfrak{L}\mathfrak{K}(n) \cong ((R[\mathbf{B}_\infty^{ab}])[\mathbf{B}_\infty^{ab}])^{\oplus n(n-1)/2}$  as  $(R[\mathbf{B}_\infty^{ab}])[\mathbf{B}_\infty^{ab}]$ -modules. Moreover, one assigns  $\mathfrak{L}\mathfrak{K}(1) = 0$  and  $\mathfrak{L}\mathfrak{K}(0) = 0$ .

- Morphisms:

- Automorphisms: for all natural integers  $n$ , for every Artin generator  $\sigma_i$  of  $\mathbf{B}_n$  (with  $i \in \{1, \dots, n-1\}$ ), for all  $v_{j,k} \in V_{j,k}$  (with  $1 \leq j < k \leq n$ ),

$$\mathfrak{L}\mathfrak{K}(\sigma_i)v_{j,k} = \begin{cases} v_{j,k} & \text{if } i \notin \{j-1, j, k-1, k\}, \\ t_1 v_{i,k} + (t_1^2 - t_1)v_{i,i+1} + (1-t_1)v_{i+1,k} & \text{if } i = j-1, \\ v_{i+1,k} & \text{if } i = j \neq k-1, \\ t_1 v_{j,i} + (1-t_1)v_{j,i+1} + (t_1^2 - t_1)t_2 v_{i,i+1} & \text{if } i = k-1 \neq j, \\ v_{j,i+1} & \text{if } i = k, \\ -t_2 t_1^2 v_{i,i+1} & \text{if } i = j = k-1. \end{cases}$$

- General morphisms: let  $n, n' \in \mathbb{N}$ , such that  $n' \geq n$ , and  $[n' - n, \sigma] \in \text{Hom}_{\mathfrak{M}\beta}(n, n')$ . We define:

$$\mathfrak{L}\mathfrak{K}([n' - n, \sigma]) = \mathfrak{L}\mathfrak{K}(\sigma) \circ \gamma_{n(n-1)/2, n'(n'-1)/2}^{(R[\mathbf{B}_\infty^{ab}])[\mathbf{B}_\infty^{ab}]}$$

*Remark 2.37.* The Lawrence-Krammer representations are usually defined as modules over a group ring of type  $R[t^{\pm 1}, q^{\pm 1}]$ . However, since we have lead all our work for commutative rings, it is the reason why we have chosen to consider  $R[\mathbb{Z} \oplus \mathbb{Z}]$  which is restrictive, but does not change the frame of the Lawrence-Krammer representations.

**Lemma 2.38.** *There is a natural equivalence  $\delta_1 \mathfrak{L}\mathfrak{K} \xrightarrow{r''} \check{\mathfrak{Bur}}_{t_1}$ , defined for all natural integers  $n$  by:*

$$r''_n = \begin{bmatrix} \overbrace{0 \quad \cdots \quad 0 \quad 1}^n \\ \vdots \quad \ddots \quad \ddots \quad 0 \\ 0 \quad \ddots \quad \ddots \quad \vdots \\ 1 \quad 0 \quad \cdots \quad 0 \end{bmatrix}.$$

*Proof.* Let  $n$  and  $n'$  be two natural integers such that  $n' \geq n$ . Let  $\varphi_n^{n'} = [n' - n, \phi] \in \text{Hom}_{\mathfrak{M}\beta}(n, n')$ . By naturality and because of the universal property of the cokernel, there exists a unique endomorphism of

$(R[\mathbf{B}_\infty^{ab}])[\mathbf{B}_\infty^{ab}]$ -modules such that the following diagram commutes, where the lines are exact. It is exactly the definition of  $\delta_1 \mathfrak{L}\mathfrak{K}([n' - n, \phi])$ .

$$\begin{array}{ccccccc}
0 & \longrightarrow & \bigoplus_{1 \leq j < k \leq n} V_{j,k} & \xrightarrow{i_1 \mathfrak{L}\mathfrak{K}([0, id_n])} & \bigoplus_{1 \leq i < l \leq n+1} V_{i,l} & \xrightarrow{\pi_n} & \bigoplus_{1 \leq i \leq n} V_{i,n+1} \longrightarrow 0 \\
& & \mathfrak{L}\mathfrak{K}(\varphi_n^{n'}) \downarrow & & \downarrow \tau_1(\mathfrak{L}\mathfrak{K})(\varphi_n^{n'}) & & \downarrow \exists! \\
0 & \longrightarrow & \bigoplus_{1 \leq j' < k' \leq n'} V_{j',k'} & \xrightarrow{i_1 \mathfrak{L}\mathfrak{K}([0, id_{n'}])} & \bigoplus_{1 \leq i' < l' \leq n'+1} V_{i',l'} & \xrightarrow{\pi_{n'}} & \bigoplus_{1 \leq i' \leq n'} V_{i',n'+1} \longrightarrow 0
\end{array}$$

Let  $i \in \{1, \dots, n-1\}$ , let  $j \in \{1, \dots, n-1\}$  and let  $v_{j,n+1}$  be an element of  $V_{j,n+1}$ . Then we compute:

$$\mathfrak{L}\mathfrak{K}(\sigma_i) v_{j,n+1} = \begin{cases} v_{j,n+1} & \text{if } i \notin \{j-1, j\}, \\ t_1 v_{j-1,n+1} + (t_1^2 - t_1) v_{j-1,j} + (1-t_1) v_{j,n+1} & \text{if } i = j-1, \\ v_{j+1,n+1} & \text{if } i = j \neq n. \end{cases}$$

We deduce that in the canonical basis  $\{\mathbf{e}_{1,n+1}, \mathbf{e}_{2,n+1}, \dots, \mathbf{e}_{n,n+1}\}$  of  $\bigoplus_{1 \leq i \leq n} V_{i,n+1}$ .

$$\delta_1 \mathfrak{L}\mathfrak{K}(\sigma_i) = Id_{i-1} \oplus \tilde{B}(t_1) \oplus Id_{n-i-1},$$

with:

$$\tilde{B}(t_1) = \begin{bmatrix} 0 & 1 \\ t_1 & 1-t_1 \end{bmatrix}.$$

□

**Proposition 2.39.** *The functor  $\mathfrak{L}\mathfrak{K}$  is a very strong polynomial functor of degree 2.*

*Proof.* Let  $n$  be a natural integer. By Remark 2.12, we only have to consider the application  $i_1 \mathfrak{L}\mathfrak{K}([0, id_n])$ . This map is clearly a monomorphism and its cokernel is  $\bigoplus_{1 \leq i \leq n} V_{i,n+1}$ . Hence,  $\kappa_1 \mathfrak{L}\mathfrak{K}$  is the null constant

functor of  $\mathbf{Fct}(\mathfrak{U}\beta, (R[\mathbf{B}_\infty^{ab}])[\mathbf{B}_\infty^{ab}]\text{-Mod})$ . Since the functor  $\mathfrak{B}ut_{t_1}$  is very strong polynomial of degree one by Proposition 2.28, we deduce from Lemma 2.38 that  $\mathfrak{L}\mathfrak{K}$  is very strong polynomial of degree two. □

### 3 Functoriality of the Long-Moody construction

#### 3.1 Review of the Long-Moody construction

We present here the Long-Moody construction for representations of braid groups, which appears in the following theorem, corresponding to Theorem 2.1 of [20]. It is worth noting that for  $\rho : \mathbf{B}_{n+1} \rightarrow GL(V)$  a representation of  $\mathbf{B}_{n+1}$ , with  $V$  a  $\mathbb{K}$ -module,  $\rho$  induces representations  $\rho \circ \gamma_n^{b,\bullet} : \mathbf{B}_n \rightarrow GL(V)$  and  $\rho \circ \varsigma_{n,\bullet} : \mathbf{F}_n \rightarrow GL(V)$ . As a consequence,  $V$  is naturally equipped with a structure of  $\mathbb{K}[\mathbf{B}_n]$ -module and of  $\mathbb{K}[\mathbf{F}_n]$ -module.

**Theorem 3.1.** [20] *Let  $\rho : \mathbf{B}_{n+1} \rightarrow GL(V)$  be a representation of  $\mathbf{B}_{n+1}$ , with  $V$  a  $\mathbb{K}$ -module, and let  $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$  be a coherent quadruplet. The Long-Moody construction associated with  $\rho$ , which is defined as follows, is a representation of  $\mathbf{B}_n$ :*

$$lm_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(\rho) : \mathbf{B}_n \longrightarrow GL\left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} V\right)$$

where for all  $\sigma \in \mathbf{B}_n$ ,  $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$  and  $v \in V$ :

$$lm_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(\rho)(\sigma) \left( i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) = a_{n,\bullet}(\sigma)(i) \otimes_{\mathbb{K}[\mathbf{F}_n]} \rho(\gamma_n^{b,\bullet}(\sigma))(v).$$

*Notation 3.2.* When there is no ambiguity, once the quadruplet  $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$  is clearly given, we forget it in the notation  $lm_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$  for convenience (especially for proofs).

*Proof.* Let  $\sigma, \sigma' \in \mathbf{B}_n$ . For all elements  $v$  of  $V$  and  $i$  of  $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ , since  $\rho$  is a representation of  $\mathbf{B}_{n+1}$  and the maps  $a_{n, \bullet} : \mathbf{B}_n \hookrightarrow \text{Aut}(\mathbf{F}_n)$  and  $\gamma_n^{b, \bullet} : \mathbf{B}_n \hookrightarrow \mathbf{B}_{n+1}$  are group morphisms, we directly prove that:

$$lm(\rho)(\sigma\sigma') \left( i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) = lm(\rho)(\sigma) \left( lm(\rho)(\sigma') \left( i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) \right).$$

Hence  $lm(\rho)(\sigma\sigma') = lm(\rho)(\sigma) lm(\rho)(\sigma')$ .  $\square$

*Remark 3.3.* For all  $k \in \{1, \dots, n\}$ , we denote by  $V_k := \mathbb{K}[(g_k - 1)] \otimes_{\mathbb{K}[\mathbf{F}_n]} V$  with  $g_k$  a generator of  $\mathbf{F}_n$ . The action of  $\mathbf{B}_n$  on  $\mathbf{F}_n$  extends naturally on  $\mathbb{K}[\mathbf{F}_n]$ . Then  $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} V$  is isomorphic to  $V^{\oplus n}$ . Indeed, we have an isomorphism  $\Lambda : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} V \longrightarrow \bigoplus_{k=1}^n V_k \cong V^{\oplus n}$  defined by:

$$\forall v \in V, \forall k \in \{1, \dots, n\}, \Lambda \left( (g_k - 1) \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) := \left( 0, \dots, 0, \overbrace{v}^{k-th}, 0, \dots, 0 \right).$$

In fact, we may have a matricial point of view on this construction (see [20, Theorem 2.2]). In the same way, the study of Bigelow and Tian in [3] is performed from a purely matricial point of view.

*Remark 3.4.* In the original Long-Moody construction for representations, one could modify the construction adding an extra parameter  $s \in \mathbb{K}^*$  (see [20, Corollary 2.6]). Indeed, considering  $\rho : \mathbf{B}_{n+1} \rightarrow GL(V)$  a representation, we can define:

$$lm_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(s\rho) : \mathbf{B}_n \longrightarrow GL \left( \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} V \right)$$

assigning for all  $b \in \mathbf{B}_n$ ,  $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$  and  $v \in V$ :

$$lm_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(s\rho)(b) \left( i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) = \left( a_{n, \bullet}(b)(i) \otimes_{\mathbb{K}[\mathbf{F}_n]} s\rho(\gamma_n^{b, \bullet}(b))(v) \right).$$

For  $s = 1$ , the construction can be functorialized as we will see in the next section. It is not the case if  $s \neq 1$  because of the composition axiom which must be satisfied for a functor.

### 3.2 The Long-Moody functor

In this subsection, we prove that the Long-Moody construction defined in [20] provides a functor  $\mathbf{LM} : \mathcal{U}\beta\text{-Mod} \longrightarrow \mathcal{U}\beta\text{-Mod}$ . Let  $F$  be a  $\mathcal{U}\beta$ -module and  $n$  be a natural integer. For all  $\sigma \in \mathbf{B}_n$ , since  $F(\sigma) \in GL(F(n))$ ,  $F$  defines in fact a representation of  $\mathbf{B}_n$ :

$$F|_{\text{Hom}_{\mathcal{U}\beta}(n, n)} : \mathbf{B}_n \longrightarrow GL(F(n)).$$

A fortiori, the  $\mathbb{K}$ -module  $F(n+1)$  is endowed with a  $\mathbb{K}[\mathbf{B}_{n+1}]$ -module structure with the action  $F|_{\text{Hom}_{\mathcal{U}\beta}(n+1, n+1)}$ . Thanks to the monomorphism  $\varsigma_{n, \bullet} : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}$ , it is also a  $\mathbb{K}[\mathbf{F}_n]$ -module. Hence, one can form the tensor product  $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} F(n+1)$ , which is isomorphic to  $F(n+1)^{\oplus n}$  by the isomorphism  $\Lambda$  (see Remark 3.3).

*Remark 3.5.* Consider two natural integers  $n$  and  $n'$  such that  $n' \geq n$ , and  $[n' - n, \sigma] \in \text{Hom}_{\mathcal{U}\beta}(n, n')$ . Let  $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$  be a coherent quadruplet. As  $[n' - n, \gamma_{n'}^{b, \bullet}(\sigma)]$  belongs to  $\text{Hom}_{\mathcal{U}\beta}(n+1, n'+1)$ , we remark that  $F \left( [n' - n, \gamma_{n'}^{b, \bullet}(\sigma)] \right)$  belongs to  $\text{Hom}_{\mathbb{K}\text{-Mod}}(F(n+1), F(n'+1))$ . One can generalize the definition of the Long-Moody construction, defining the  $\mathbb{K}$ -module homomorphism:

$$lm(F)_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}([n' - n, \sigma]) : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} F(n+1) \longrightarrow \mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'}]} \otimes_{\mathbb{K}[\mathbf{F}_{n'}]} F(n'+1)$$

by  $\forall i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ ,  $\forall v \in F(n+1)$ :

$$lm_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)\left(\left[[n' - n, \gamma_{n'}^{b, \bullet}(\sigma)]\right]\right)\left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v\right) = a_{n', \bullet}(\sigma)\left(\gamma_{n', n'}^{f, \bullet}(i)\right) \otimes_{\mathbb{K}[\mathbf{F}_{n'}]} F\left(\left[[n' - n, \gamma_{n'}^{b, \bullet}(\sigma)]\right]\right)(v).$$

**Proposition 3.6.** For  $F \in \text{Obj}(\mathfrak{LB}\text{-Mod})$  and  $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$  a coherent quadruplet, the following assignment defines a functor  $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F) : \mathfrak{LB} \rightarrow \mathfrak{KB}\text{-Mod}$ .

- *Objects:*  $\forall n \in \mathbb{N}$ ,  $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)(n) = \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} F(n+1)$ .
- *Morphisms:* Let  $n, n' \in \mathbb{N}$ , such that  $n' \geq n$ , and  $[n' - n, \sigma] \in \text{Hom}_{\mathfrak{LB}}(n, n')$ . One assigns:

$$\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)([n' - n, \sigma]) = lm_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)\left(\left[[n' - n, \gamma_{n'}^{b, \bullet}(\sigma)]\right]\right).$$

*Notation 3.7.* When there is no ambiguity, once the quadruplet  $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$  is clearly given, we forget it in the notation  $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$  for convenience (especially for proofs).

*Proof.* Let  $n$  be a fixed natural integer. Let  $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$  and  $v \in F(n+1)$ . On the one hand, according to our assignment and since  $a_{n, \bullet}$ ,  $\gamma_n^{b, \bullet}$  and  $\gamma_n^{f, \bullet}$  are group morphisms, we easily prove that  $\mathbf{LM}(F)(id_{\mathbf{B}_n}) = id_{\mathbf{LM}(F)(n)}$ . On the other hand, let  $n, n'$  and  $n''$  be natural integers such that  $n'' \geq n' \geq n$ , let  $[n' - n, \sigma]$  and  $[n'' - n', \sigma']$  be morphisms respectively in  $\text{Hom}_{\mathfrak{LB}}(n, n')$  and in  $\text{Hom}_{\mathfrak{LB}}(n', n'')$ . Since  $a_{n, \bullet}$  and  $\gamma_n^{b, \bullet}$  are group morphisms, we deduce from the definition that:

$$\begin{aligned} & \mathbf{LM}(F)\left(\left([n'' - n', \sigma']\right) \circ \left([n' - n, \sigma]\right)\right)\left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v\right) \\ &= \left(a_{n'', \bullet}(\sigma') \circ a_{n', \bullet}(id_{n'' - n'} \natural \sigma)\right)\left(\gamma_{n', n''}^{f, \bullet} \circ \gamma_{n, n'}^{f, \bullet}(i)\right) \otimes_{\mathbb{K}[\mathbf{F}_{n''}]} F\left(\left[[n'' - n', \gamma_{n''}^{b, \bullet}(\sigma')] \circ \gamma_{n'}^{b, \bullet}((id_{n'' - n'} \natural \sigma))\right]\right)(v). \end{aligned}$$

As  $\gamma_{n''}^{b, \bullet}((id_{n'' - n'} \natural \sigma)) = id_{(n''+1) - (n'+1)} \natural \gamma_{n'}^{b, \bullet}(\sigma)$  by Condition 1.40, because of the definition of morphisms in  $\mathfrak{LB}$ :

$$\left[[n'' - n', \gamma_{n''}^{b, \bullet}(\sigma')] \circ \gamma_{n'}^{b, \bullet}((id_{n'' - n'} \natural \sigma))\right] = \left[[n'' - n', \gamma_{n''}^{b, \bullet}(\sigma')\right] \circ \left[[n' - n, \gamma_{n'}^{b, \bullet}(\sigma)]\right].$$

Moreover, Condition 1.41 asserts that for all elements  $i'$  of  $\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'}]}$ :

$$\gamma_{n', n''}^{f, \bullet} \circ a_{n', \bullet}(\sigma)(i') = a_{n'', \bullet}(id_{n'' - n'} \natural \sigma)\left(\gamma_{n', n''}^{f, \bullet}(i')\right).$$

So, for all elements  $i$  of  $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ :  $\gamma_{n', n''}^{f, \bullet}\left(a_{n, \bullet}(\sigma)\left(\gamma_{n, n'}^{f, \bullet}(i)\right)\right) = a_{n'', \bullet}(id_{n'' - n'} \natural \sigma)\left(\gamma_{n', n''}^{f, \bullet} \circ \gamma_{n, n'}^{f, \bullet}(i)\right)$ . Therefore, the associativity axiom is satisfied:

$$\begin{aligned} & \mathbf{LM}(F)\left(\left([n'' - n', \sigma']\right) \circ \left([n' - n, \sigma]\right)\right)\left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v\right) \\ &= \left(\mathbf{LM}(F)([n'' - n', \sigma']) \circ \mathbf{LM}(F)([n' - n, \sigma])\right)\left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v\right). \end{aligned}$$

□

In the following proposition, we define an endofunctor of  $\mathfrak{LB}\text{-Mod}$  corresponding to the Long-Moody construction. It will be called the Long-Moody functor.

**Proposition 3.8.** For a coherent quadruplet  $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$ , the following assignment defines a functor  $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} : \mathfrak{LB}\text{-Mod} \rightarrow \mathfrak{LB}\text{-Mod}$ .

- *Objects* : Let  $F \in \text{Obj}(\mathfrak{U}\beta\text{-Mod})$ ,  $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)$  is defined as in Proposition 3.6.
- *Morphisms* : Let  $F$  and  $G$  be two  $\mathfrak{U}\beta$ -modules, and  $\eta : F \Rightarrow G$  be a natural transformation. We define  $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(\eta) : \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F) \Rightarrow \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(G)$  for all natural integers  $n$  by:

$$\forall i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}, \forall v \in F(n+1), \left( \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(\eta) \right)_n \left( i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) = i \otimes_{\mathbb{K}[\mathbf{F}_n]} \eta_{n+1}(v).$$

In other words:

$$\left( \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(\eta) \right)_n = id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}} \otimes_{\mathbb{K}[\mathbf{F}_n]} \eta_{n+1}.$$

*Proof.* The remaining point to check for  $\mathbf{LM}$  to be well-defined is the consistency of our definition on morphisms. Let  $F$  and  $G$  be two  $\mathfrak{U}\beta$ -modules, and let  $\eta : F \Rightarrow G$  be a natural transformation. Let  $n, n' \in \mathbb{N}$ , such that  $n' \geq n$ , and  $[n' - n, \sigma] \in \text{Hom}_{\mathfrak{U}\beta}(n, n')$ . Since  $\eta$  is a natural transformation:

$$\forall v \in F(n), G \left( \left[ n' - n, \gamma_{n'}^{b, \bullet}(\sigma) \right] \right) (\eta_{n+1}(v)) = \eta_{n'+1} \left( F \left( \left[ n' - n, \gamma_{n'}^{b, \bullet}(\sigma) \right] \right) (v) \right).$$

Hence, it follows from our definitions that the following diagram commutes, and therefore  $\mathbf{LM}$  is a morphism of  $\mathfrak{U}\beta$ -modules.

$$\begin{array}{ccc} \mathbf{LM}(F)(n) & \xrightarrow{\mathbf{LM}(\eta)_n} & \mathbf{LM}(G)(n) \\ \downarrow \mathbf{LM}(F)([n'-n, \sigma]) & & \downarrow \mathbf{LM}(G)([n'-n, \sigma]) \\ \mathbf{LM}(F)(n') & \xrightarrow{\mathbf{LM}(\eta)_{n'}} & \mathbf{LM}(G)(n') \end{array}$$

For  $F$  a  $\mathfrak{U}\beta$ -module and  $id_F : F \Rightarrow F$  the identity natural transformation (ie for all natural integers  $n$  and for all elements  $v$  of  $F(n+1)$ ,  $id_{F(n+1)}(v) = v$ ), it comes directly that  $\mathbf{LM}(id_F) = id_{\mathbf{LM}(F)}$ . Finally, let us check the associativity axiom. Let  $\eta : F \Rightarrow G$  and  $\mu : G \Rightarrow H$  be natural transformations. Let  $n$  be a natural integer,  $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$  and  $v \in F(n+1)$ . Yet, because  $\mu$  and  $\eta$  are morphisms in the category of functors:

$$\mathbf{LM}(\mu \circ \eta)_n \left( i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) = i \otimes_{\mathbb{K}[\mathbf{F}_n]} (\mu_{n+1} \circ \eta_{n+1})(v) = \mathbf{LM}(\mu)_n \circ \mathbf{LM}(\eta)_n \left( i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right).$$

□

*Remark 3.9.* For  $\eta : F \Rightarrow G$  a natural transformation, with  $\Lambda$  the isomorphism of Remark 3.3:

$$\forall n \in \mathbb{N}, \Lambda \left( \left( \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(\eta) \right)_n \right) = \eta_{n+1}^{\oplus n}.$$

**Proposition 3.10.** For a coherent quadruplet  $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$ , the functor  $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} : \mathfrak{U}\beta\text{-Mod} \rightarrow \mathfrak{U}\beta\text{-Mod}$  is reduced and exact.

*Proof.* Let  $0_{\mathfrak{U}\beta\text{-Mod}} : \mathfrak{U}\beta \rightarrow \mathfrak{K}\text{-Mod}$  denotes the null functor. It comes straightforward from the definition of the Long-Moody functor that  $\mathbf{LM}(0_{\mathfrak{U}\beta\text{-Mod}}) = 0_{\mathfrak{U}\beta\text{-Mod}}$ .

To prove the exactness property, let us consider the short exact sequence  $0 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 0$  in the category  $\mathfrak{U}\beta\text{-Mod}$ . Let  $n$  be a natural integer. Since the augmentation ideal  $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$  is a free  $\mathbb{K}[\mathbf{F}_n]$ -module (see for example [27, Chapter 2, Proposition 6.2.6]), it is therefore a flat  $\mathbb{K}[\mathbf{F}_n]$ -module. Then, by definition, it ensures that the functor  $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} - : \mathfrak{K}\text{-Mod} \rightarrow \mathfrak{K}\text{-Mod}$  is an exact functor for all natural

integers  $n$ . Hence, applying it to the morphisms  $\alpha_{n+1}$  and  $\beta_{n+1}$ , we obtain that the following short sequence of  $\mathbb{K}$ -modules is exact. We denote  $(\mathbf{LM}(\alpha))_n$  resp.  $(\mathbf{LM}(\beta))_n$  by  $\tilde{\alpha}_{n+1}$  resp.  $\tilde{\beta}_{n+1}$ .

$$0 \longrightarrow \mathbf{LM}(F)(n) \xrightarrow{\tilde{\alpha}_{n+1}} \mathbf{LM}(G)(n) \xrightarrow{\tilde{\beta}_{n+1}} \mathbf{LM}(H)(n) \longrightarrow 0$$

It remains to check the naturality. Let  $n'$  be another natural integer such that  $n' \geq n$  and let  $[n' - n, \sigma] \in \text{Hom}_{\mathcal{U}\beta}(n, n')$ . Let  $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ ,  $v \in F(n+1)$  and  $w \in G(n+1)$ . On the one hand, since  $\alpha_{n+1}$  is a natural transformation, then  $G\left([n' - n, \gamma_{n'}^{b, \bullet}(\sigma)]\right)(\alpha_{n+1}(v)) = \alpha_{n'+1} \circ F\left([n' - n, \gamma_{n'}^{b, \bullet}(\sigma)]\right)(v)$ . Hence:

$$\mathbf{LM}(G)([n' - n, \sigma]) \circ \tilde{\alpha}_{n+1} \left( i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) = (\tilde{\alpha}_{n'+1} \circ \mathbf{LM}(F)([n' - n, \sigma])) \left( i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right).$$

On the other hand, since  $\beta_{n+1}$  is a natural transformation, then  $H\left([n' - n, \gamma_{n'}^{b, \bullet}(\sigma)]\right)(\beta_{n+1}(v)) = \beta_{n'+1} \circ G\left([n' - n, \gamma_{n'}^{b, \bullet}(\sigma)]\right)(v)$ . Hence:

$$\mathbf{LM}(H)([n' - n, \sigma]) \circ \tilde{\beta}_{n+1} \left( i \otimes_{\mathbb{K}[\mathbf{F}_n]} w \right) = (\tilde{\beta}_{n'+1} \circ \mathbf{LM}(G)([n' - n, \sigma])) \left( i \otimes_{\mathbb{K}[\mathbf{F}_n]} w \right).$$

□

*Remark 3.11.* Assume that, for a coherent quadruplet  $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$  and an object  $F$  of the category  $\mathcal{U}\beta\text{-}\mathcal{M}\text{od}$ ,  $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F) = 0$ . Then it follows directly from the definitions that  $F$  is the null functor.

**Corollary 3.12.** *For a coherent quadruplet  $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$ , the functor  $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$  commutes with all finite limits and finite colimits.*

*Proof.* The functor  $\mathbf{LM}$  is exact according to Proposition 3.10. The result is a property of exact functors (see for example [21, Chapter 8, section 3]). □

### 3.3 Examples of Long-Moody functors

Let us give the examples of Long-Moody functors which arise using the actions that we have recorded in Theorem 1.14 and Example 1.16. We will always use the identification  $\varsigma_{n,1} : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}$  of Example 1.11.

**Definition 3.13.** Let  $a_{n, \mathcal{W}(i)} : \mathbf{B}_n \hookrightarrow \text{Aut}(\mathbf{F}_n)$  be the action given by the  $i$ -th Wada-type natural transformation and let  $a_{n,8} : \mathbf{B}_n \hookrightarrow \text{Aut}(\mathbf{F}_n)$  be the group morphism described in Example 1.16. For  $(\mathcal{W}(i), 1, \bullet_{\gamma^b}, \bullet_{\gamma^f})$  respectively  $(8, 1, \bullet_{\gamma^b}, \bullet_{\gamma^f})$  a coherent quadruplet associated with  $a_{n, \mathcal{W}(i)}$  respectively  $a_{n,8}$ , we denote by  $\mathbf{LM}_i : \mathcal{U}\beta\text{-}\mathcal{M}\text{od} \rightarrow \mathcal{U}\beta\text{-}\mathcal{M}\text{od}$  the corresponding Long-Moody functor defined in Proposition 3.8 for  $i \in \{1, \dots, 8\}$ .

A first step to understand the effect of these endofunctors is to investigate on their effect on a functor equivalent to a constant one.

**Example 3.14.** We denote by  $\mathfrak{X} : \mathcal{U}\beta \rightarrow R[\mathbf{B}_\infty^{ab}] \text{-}\mathcal{M}\text{od}$  the functor defined by:

- Objects:  $\forall n \in \mathbb{N}$ ,  $\mathfrak{X}(n) = R[\mathbf{B}_\infty^{ab}]$ .
- Morphisms:
  - Automorphisms: for all natural integers  $n$ , for every Artin generator  $\sigma_i$  of  $\mathbf{B}_n$  (with  $i \in \{1, \dots, n-1\}$ ):
$$\mathfrak{X}(\sigma_i) = 1.$$
  - General morphisms: Let  $n, n' \in \mathbb{N}$ , such that  $n' \geq n$ , and  $[n' - n, \sigma_i] \in \text{Hom}_{\mathcal{U}\beta}(n, n')$ . We define:

$$\mathfrak{X}([n' - n, \sigma_i]) = \mathfrak{X}(\sigma_i) \circ i_{R[\mathbf{B}_\infty^{ab}] \hookrightarrow R[\mathbf{B}_\infty^{ab}]} = 1.$$

As a consequence, we already know that for all  $i \in \{1, \dots, 8\}$ ,  $\mathbf{LM}_i(\mathfrak{X})(n) \cong R[\mathbf{B}_\infty^{ab}]^{\oplus n}$ . Moreover, once defined on automorphisms,  $\mathbf{LM}_i(\mathfrak{X})$  is defined on general morphisms precomposition by  $i_{R[\mathbf{B}_\infty^{ab}]^{\oplus n} \hookrightarrow R[\mathbf{B}_\infty^{ab}]^{\oplus n'}}$  for all natural integers  $n$  and  $n'$ .

### 3.3.1 The case $i = 1$

**The case  $m = 0$ :** We obtain the functor  $t_1^{-1}\mathbf{LM}_{1,0}(t_1\mathfrak{X}) : \mathfrak{U}\beta \rightarrow R[\mathbf{B}_\infty^{ab}]$ - $\mathfrak{M}\mathfrak{o}\mathfrak{d}$ , defined on automorphisms for all natural integers  $n$ , for every Artin generator  $\sigma_i$  of  $\mathbf{B}_n$ , by:

$$t_1^{-1}\mathbf{LM}_{1,0}(t_1\mathfrak{X})(\sigma_i) = (Id_{i-1} \oplus AD(t_1^2) \oplus Id_{n-i-1}),$$

with:

$$AD(t_1^2) = \begin{bmatrix} 0 & t_1^2 \\ t_1^2 & 0 \end{bmatrix}.$$

**The case  $m \neq 0$ :** Let us first assume that  $m \neq 0$ . We obtain the functor  $t_1^{-1}\mathbf{LM}_{1,m \neq 0}(t_1\mathfrak{X}) : \mathfrak{U}\beta \rightarrow (R[\mathbf{B}_\infty^{ab}])[\mathbf{B}_\infty^{ab}]$ - $\mathfrak{M}\mathfrak{o}\mathfrak{d}$ , defined on automorphisms for all natural integers  $n$ , for every Artin generator  $\sigma_i$  of  $\mathbf{B}_n$ , by:

$$t_1^{-1}\mathbf{LM}_{1,m \neq 0}(t_1\mathfrak{X})(\sigma_i) = Id_{i-1} \oplus \tilde{B}_m(t_1) \oplus Id_{n-i-1},$$

with:

$$\tilde{B}_m(t_1) = \begin{bmatrix} (1 + t_1^2 + \dots + t_1^{2m-2})(1 - t_1^2) & 1 \\ t_1^{2m} & 0 \end{bmatrix}.$$

If we assume that  $m = 1$ , then the functor  $t_1^{-1}\mathbf{LM}_{(1,1,\bullet_{\gamma b},\bullet_{\gamma f})}(t_1\mathfrak{X})$  is very similar to  $\mathfrak{B}\mathfrak{u}\mathfrak{r}$ : in fact,  $t_1^{-1}\mathbf{LM}_{(1,\bullet_s,\bullet_{\gamma b},\bullet_{\gamma f})}(t_1\mathfrak{X}) = \mathfrak{B}\mathfrak{u}\mathfrak{r}_{t_1^2}$ .

**Recovering of the Lawrence-Krammer functor:** Let us first introduce the following result due to Long in [20]. In this paragraph, we assume that  $\mathfrak{X}(n) = (R[\mathbf{B}_\infty^{ab}])[\mathbf{B}_\infty^{ab}]$ .

**Proposition 3.15.** [20, special case of Corollary 2.10] *Let  $n$  be a natural integer such that  $n \geq 4$ . Then, the Lawrence-Krammer representation  $\mathfrak{L}\mathfrak{K}_{|\mathbf{B}_n}$  is a subrepresentation of  $t_2^{-1}\mathbf{LM}_{(1,1,f,f)}(t_2(t_1^{-1}\mathbf{LM}_{(1,1,f,f)}(t_1\mathfrak{X})))_{|\mathbf{B}_n}$ .*

The two following results due to Long will be crucial to prove this proposition.

**Proposition 3.16.** [20, Corollary 2.7] *Let  $n$  be a natural integer and  $\rho : \mathbf{B}_{n+1} \rightarrow GL(V)$  be a representation of  $\mathbf{B}_n$  with  $V$  a  $R[\mathbf{B}_\infty^{ab}]$ -module. Then, the representation induced by the Long-Moody functor is defined as a group morphism:*

$$t_2^{-1}\mathbf{LM}_{(1,1,f,f)}(t_2\rho)_{|\mathbf{B}_n} : \mathbf{B}_n \rightarrow H^1(Y_{\underline{w}_n,1}, E_\rho)$$

for  $E_\rho$  a flat vector bundle associated with  $\rho$  (see [20, p. 225-226]).

**Lemma 3.17.** [20, Lemma 2.9] *For all natural integers  $m$ , there is an isomorphism of abelian groups:*

$$H^{m+1}(Y_{\underline{w}_n,m+1}, E_{t_1\mathfrak{X}_{|\mathbf{B}_n}}) \cong H^1\left(Y_{\underline{w}_n,1}, H^m\left(Y_{\underline{w}_{n+1},m}, E_{t_1\mathfrak{X}_{|\mathbf{B}_n}}\right)\right).$$

In particular, for  $m = 1$ :

$$H^2(Y_{\underline{w}_n,2}, E_{t_1\mathfrak{X}_{|\mathbf{B}_n}}) \cong H^1\left(Y_{\underline{w}_n,1}, H^1\left(Y_{\underline{w}_{n+1},2}, E_{t_1\mathfrak{X}_{|\mathbf{B}_n}}\right)\right).$$

*Proof of Proposition 3.15.* We first need to introduce new tools. Let  $\underline{w}_n = (w_1, \dots, w_n) \in R^n$  such that  $w_i \neq w_j$  if  $i \neq j$ . For  $n$  and  $m$  two natural integers, we define the configuration space:

$$Y_{\underline{w}_n,1} = \{(z_1, \dots, z_m) \mid z_i \in R \setminus \underline{w}_n, z_i \neq z_j \text{ if } i \neq j\}.$$

By Proposition 3.16, we can write as a representation:

$$t_2^{-1}\mathbf{LM}_{(1,1,f,f)}(t_2(t_1^{-1}\mathbf{LM}_{(1,1,f,f)}(t_1\mathfrak{X})))_{|\mathbf{B}_n} : \mathbf{B}_n \rightarrow H^1\left(Y_{\underline{w}_n,1}, E_{t_1^{-1}\mathbf{LM}_{(1,1,f,f)}(t_1\mathfrak{X})}\right).$$

A fortiori by Lemma 3.17,  $t_2^{-1}\mathbf{LM}_{(1,1,f,f)}(t_2(t_1^{-1}\mathbf{LM}_{(1,1,f,f)}(t_1\mathfrak{X})))_{|\mathbf{B}_n}$  is an action of  $\mathbf{B}_n$  on  $H^2(Y_{\underline{w}_n,2}, E_{t_1\mathfrak{X}_{|\mathbf{B}_n}})$ . In particular, for  $m = 2$  and  $n \geq 4$ , according to [20, 17, Theorem 5.1], the representation of  $\mathbf{B}_n$  factorizing through the Hecke algebra  $H_n(t_1)$  corresponding to the Young diagram  $(n-2, 2)$  (which is in fact the Lawrence-Krammer representation) is a subrepresentation of  $t_2^{-1}\mathbf{LM}_{(1,1,f,f)}(t_2(t_1^{-1}\mathbf{LM}_{(1,1,f,f)}(t_1\mathfrak{X})))_{|\mathbf{B}_n}$ .  $\square$

We denote  $\mathfrak{L}\mathfrak{K}^{\geq 4} : \mathfrak{U}\mathfrak{B} \rightarrow \mathbb{K}\text{-}\mathfrak{M}\mathfrak{o}\mathfrak{d}$  the Lawrence-Krammer defined in Example 2.36 restricted on natural integers such that  $n \geq 4$  on objects. The result of Proposition 3.15 induces the existence of an intertwining  $l_n : (R[\mathbf{B}_\infty^{ab}])[\mathbf{B}_\infty^{ab}]^{\oplus n(n-1)/2} \hookrightarrow (R[\mathbf{B}_\infty^{ab}])[\mathbf{B}_\infty^{ab}]^{\oplus n(n+1)}$  for all natural integers  $n \geq 4$ , ie a  $R[\mathbf{B}_\infty^{ab}]$ -module morphism such that for all  $\sigma \in \mathbf{B}_n$  the following diagram is commutative.

$$\begin{array}{ccc} (R[\mathbf{B}_\infty^{ab}])[\mathbf{B}_\infty^{ab}]^{\oplus n(n-1)/2} & \xrightarrow{l_n} & (R[\mathbf{B}_\infty^{ab}])[\mathbf{B}_\infty^{ab}]^{\oplus n(n+1)} \\ \mathfrak{L}\mathfrak{K}(\sigma) \downarrow & & \downarrow \mathfrak{L}\mathfrak{K}(\sigma) \\ (R[\mathbf{B}_\infty^{ab}])[\mathbf{B}_\infty^{ab}]^{\oplus n(n-1)/2} & \xrightarrow{l_n} & (R[\mathbf{B}_\infty^{ab}])[\mathbf{B}_\infty^{ab}]^{\oplus n(n+1)} \end{array}$$

According to Remark 1.34, this intertwining defines in fact a natural transformation  $l : \mathfrak{L}\mathfrak{K}^{\geq 4} \Longrightarrow t_2^{-1}\mathbf{LM}_{(1,1,f,f)}(t_2(t_1^{-1}\mathbf{LM}_{(1,1,f,f)}(t_1\mathfrak{X}))) \Big|_{\mathbf{B}_n}^{\geq 4}$ . We deduce that  $\mathfrak{L}\mathfrak{K}^{\geq 4}$  is a subfunctor of  $t_2^{-1}\mathbf{LM}_{(1,1,f,f)}(t_2(t_1^{-1}\mathbf{LM}_{(1,1,f,f)}(t_1\mathfrak{X}))) \Big|_{\mathbf{B}_n}^{\geq 4}$ .

### 3.3.2 The case $i = 2$

We obtain the functor  $t_1^{-1}\mathbf{LM}_2(t_1\mathfrak{X}) : \mathfrak{U}\mathfrak{B} \rightarrow R[\mathbf{B}_\infty^{ab}]\text{-}\mathfrak{M}\mathfrak{o}\mathfrak{d}$ , defined on automorphisms for all natural integers  $n$ , for every Artin generator  $\sigma_i$  of  $\mathbf{B}_n$ , by:

$$t_1^{-1}\mathbf{LM}_2(t_1\mathfrak{X})(\sigma_i) = Id_n.$$

In fact, for all  $F \in \text{Obj}(\mathfrak{U}\mathfrak{B}\text{-}\mathfrak{M}\mathfrak{o}\mathfrak{d})$ ,  $\mathbf{LM}_2(F) : \mathfrak{U}\mathfrak{B} \rightarrow R[\mathbf{B}_\infty^{ab}]\text{-}\mathfrak{M}\mathfrak{o}\mathfrak{d}$  is defined on automorphisms for all natural integers  $n$ , for every Artin generator  $\sigma_i$  of  $\mathbf{B}_n$ , by:

$$\mathbf{LM}_2(F)(\sigma_i) = (F(\sigma_i))^{\oplus n}.$$

### 3.3.3 The case $i = 3$

We obtain the functor  $t_1^{-1}\mathbf{LM}_3(t_1\mathfrak{X}) : \mathfrak{U}\mathfrak{B} \rightarrow R[\mathbf{B}_\infty^{ab}]\text{-}\mathfrak{M}\mathfrak{o}\mathfrak{d}$ , defined on automorphisms for all natural integers  $n$ , for every Artin generator  $\sigma_i$  of  $\mathbf{B}_n$ , by:

$$t_1^{-1}\mathbf{LM}_3(t_1\mathfrak{X})(\sigma_i) = Id_{i-1} \oplus TYM(-t_1^2) \oplus Id_{n-i-1}.$$

Hence, the functor  $t_1^{-1}\mathbf{LM}_3(t_1\mathfrak{X})$  is very similar to  $\mathfrak{T}\mathfrak{Y}\mathfrak{M}$ : we directly obtain that  $t_1^{-1}\mathbf{LM}_3(t_1\mathfrak{X}) = \mathfrak{T}\mathfrak{Y}\mathfrak{M}_{-t_1^2}$ .

### 3.3.4 The case $i = 8$

Here we assume moreover that we use the morphisms  $\gamma_n^{b,l}$  and  $\gamma_n^{f,l}$ . Therefore we use the coherent quadruplet  $(8, 1, l, l)$ . We obtain the functor  $t_1^{-1}\mathbf{LM}_8(t_1\mathfrak{X}) : \mathfrak{U}\mathfrak{B} \rightarrow R[\mathbf{B}_\infty^{ab}]\text{-}\mathfrak{M}\mathfrak{o}\mathfrak{d}$ , defined on automorphisms for all natural integers  $n$ , for every Artin generator  $\sigma_i$  of  $\mathbf{B}_n$ , by:

$$t_1^{-1}\mathbf{LM}_8(t_1\mathfrak{X})(\sigma_i) = Id_{i-2} \oplus \overline{B}(1) \oplus Id_{n-i-2}$$

if  $i \leq n - 2$ , and:

$$t_1^{-1}\mathbf{LM}_8(t_1\mathfrak{X})(\sigma_{n-1}) = Id_{n-3} \oplus \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

Hence, the functor  $\mathbf{LM}_3(\mathfrak{X})$  is very similar to  $\overline{\mathfrak{B}\mathfrak{u}\mathfrak{t}}_1$ : indeed, direct computation shows that  $\mathbf{LM}_3(\mathfrak{X}) = \tau_1(\overline{\mathfrak{B}\mathfrak{u}\mathfrak{t}}_1)$ .

### 3.3.5 The other cases

For the actions given by the Wada-type natural transformation 4, 5, 6 and 7 in Theorem 1.14, the produced functors  $\mathbf{LM}_i(\mathfrak{X}) : \mathfrak{U}\mathfrak{B} \rightarrow R[\mathbf{B}_\infty^{ab}]\text{-}\mathfrak{M}\mathfrak{o}\mathfrak{d}$  are mild variations of what is given by the case  $i = 1$ .

## 4 The Long-Moody functor applied to polynomial functors

Let us move on the effect of the Long-Moody construction on polynomial functors. Let  $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma b}, \bullet_{\gamma f})$  be a reliable quadruplet, which we fix for all the work of this Section (in particular, we forget it in many notations).

### 4.1 The intermediary functors

We have to introduce two new functors which will play a key role in the main result of the study. First, let us recall the following crucial property of the augmentation ideal of a free product of groups. The proof of this proposition is a consequence of combining [7, Section 4, Lemma 4.3] and [7, Section 4, Theorem 4.7].

**Proposition 4.1.** *Let  $G$  and  $H$  be groups. Then, there is a natural  $\mathbb{K}[G * H]$ -module isomorphism:*

$$\mathcal{I}_{\mathbb{K}[G * H]} \cong \left( \mathcal{I}_{\mathbb{K}[G]} \otimes_{\mathbb{K}G} \mathbb{K}[G * H] \right) \oplus \left( \mathcal{I}_{\mathbb{K}[H]} \otimes_{\mathbb{K}H} \mathbb{K}[G * H] \right).$$

For a fixed natural integer  $m$  and  $F$  a  $\mathfrak{U}\beta$ -module, let us consider the  $\mathfrak{U}\beta$ -module  $\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma b}, \bullet_{\gamma f})}(F)$  for all natural integers  $n$ , by Proposition 4.1, we have a  $\mathbb{K}[\mathbf{F}_{m+n}]$ -module isomorphism:

$$\begin{aligned} & \mathcal{I}_{\mathbb{K}[\mathbf{F}_{m+n}]} \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} F(m+n+1) \\ & \cong \left( \left( \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]} \otimes_{\mathbb{K}[\mathbf{F}_m]} \mathbb{K}[\mathbf{F}_{m+n}] \right) \oplus \left( \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} \mathbb{K}[\mathbf{F}_{m+n}] \right) \right) \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} F(m+n+1). \end{aligned}$$

Therefore, because of the distributivity of tensor product with respect to the direct sum, we prove the following proposition.

**Proposition 4.2.** *Let  $F \in \text{Obj}(\mathfrak{U}\beta\text{-Mod})$  and  $m, n$  be natural integers. Then, we have the following  $\mathbb{K}$ -module isomorphism:*

$$\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma b}, \bullet_{\gamma f})}(F)(n) \cong \left( \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]} \otimes_{\mathbb{K}[\mathbf{F}_m]} F(m+n+1) \right) \oplus \left( \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} F(m+n+1) \right). \quad (4)$$

The aim of this section is in fact to show that this  $\mathbb{K}$ -module decomposition will lead to a  $\mathfrak{U}\beta$ -module decomposition of  $\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma b}, \bullet_{\gamma f})}(F)$  (see Theorem (4.13)).

**Proposition 4.3.** *Let  $m$  be a fixed natural integer. for all natural integers  $n$  and  $F \in \text{Obj}(\mathfrak{U}\beta\text{-Mod})$ , the submodules  $\mathcal{I}_{\mathbb{K}[\mathbf{F}_m]} \otimes_{\mathbb{K}[\mathbf{F}_m]} F(m+n+1)$  of  $\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma b}, \bullet_{\gamma f})}(F)(n)$  naturally define a subfunctor of*

$\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma b}, \bullet_{\gamma f})}(F)$ , which will be denoted by  $\Upsilon_m^{(\bullet_\varsigma, \bullet_{\gamma b}, \bullet_{\gamma f})}(F)$ .

*This way, for all  $F \in \text{Obj}(\mathfrak{U}\beta\text{-Mod})$ , the subfunctors  $\Upsilon_m^{(\bullet_\varsigma, \bullet_{\gamma b}, \bullet_{\gamma f})}(F)$  of  $\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma b}, \bullet_{\gamma f})}(F)$  define a subfunctor of  $\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma b}, \bullet_{\gamma f})}$ , which will be denoted by  $\Upsilon_m^{(\bullet_\varsigma, \bullet_{\gamma b}, \bullet_{\gamma f})}$ .*

*Remark 4.4.* As the notation suggests it, the functor  $\Upsilon_m^{(\bullet_\varsigma, \bullet_{\gamma b}, \bullet_{\gamma f})}$  does not depend on the morphism  $a_{n, \bullet}$ . We will see the reason why in the proof of Proposition 4.3.

*Notation 4.5.* When there is no ambiguity, once a triplet  $(\bullet_\varsigma, \bullet_{\gamma b}, \bullet_{\gamma f})$  is clearly given, we forget it in the notation  $\Upsilon_m^{(\bullet_\varsigma, \bullet_{\gamma b}, \bullet_{\gamma f})}$  for convenience (especially for the proofs).

*Proof.* First, let us show that  $\mathcal{Y}_m(F)$  is a subfunctor of  $\tau_m \mathbf{LM}(F)$  for every  $F \in \text{Obj}(\mathfrak{U}\beta\text{-}\mathfrak{Mod})$ . for all natural integers  $n$ :

$$\mathcal{Y}_m(F)(n) = \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]} \otimes_{\mathbb{K}[\mathbf{F}_m]} F(m+n+1).$$

Because of Proposition 4.2, the  $\mathbb{K}$ -module  $\mathcal{Y}_m(F)(n)$  is a submodule of the  $\mathbb{K}$ -module  $\tau_m \mathbf{LM}(F)(n)$ . Then, we define the monomorphism associated with the direct sum

$$v_{m,n}(F) = \bar{\gamma}_{m,m+n}^{f,\bullet} \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} id_{F(m+n+1)} : \mathcal{Y}_m(F)(n) \hookrightarrow \tau_m \mathbf{LM}(F)(n)$$

by the assignment:

$$\forall i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}, \forall v \in F(m+n+1), v_{m,n}(F) \left( i \otimes_{\mathbb{K}[\mathbf{F}_m]} v \right) = \bar{\gamma}_{m,m+n}^{f,\bullet}(i) \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} v.$$

Let  $n$  and  $n'$  be natural integers such that  $n' \geq n$ , and  $[n' - n, \sigma] \in \text{Hom}_{\mathfrak{U}\beta}(n, n')$  with  $\sigma \in \mathbf{B}_{n'}$ . Let  $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}$  and  $v \in F(m+n+1)$ . We naturally define the functor  $\mathcal{Y}_m^{(\bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)$  on morphisms by:

$$\mathcal{Y}_m(F)([n' - n, \sigma]) \left( i \otimes_{\mathbb{K}[\mathbf{F}_m]} v \right) = i \otimes_{\mathbb{K}[\mathbf{F}_m]} F \left( \left[ n' - n, \gamma_{m+n'}^{b,\bullet} \left( (id_m \natural \sigma) \circ \left( (b_{m,n'-n}^{\mathfrak{M}})^{-1} \natural id_n \right) \right) \right] \right) (v).$$

It remains to show that we define a natural transformation  $v_m(F) : \mathcal{Y}_m(F) \implies \tau_m \mathbf{LM}(F)$ . It follows from Condition 1.43 that  $a_{m+n',\bullet} \left( (b_{m,n'-n}^{\mathfrak{M}})^{-1} \natural id_n \right) \circ \left( \gamma_{m+n,m+n'}^{f,\bullet} \circ \bar{\gamma}_{m,m+n}^{f,\bullet} \right) = \bar{\gamma}_{m,m+n'}^{f,\bullet}$ . Since by Condition 1.44,  $a_{m+n',\bullet} \left( (id_m \natural \sigma) \right) \circ \bar{\gamma}_{m,m+n'}^{f,\bullet} = \bar{\gamma}_{m,m+n'}^{f,\bullet}$  for all elements  $\sigma$  of  $\mathbf{B}_{n'}$ , naturality then follows from the following equality:

$$(\tau_m \mathbf{LM}(F)([n' - n, \sigma]) \circ v_{m,n}(F)) \left( i \otimes_{\mathbb{K}[\mathbf{F}_m]} v \right) = v_{m,n'}(F) \circ \mathcal{Y}_m(F)([n' - n, \sigma]) \left( i \otimes_{\mathbb{K}[\mathbf{F}_m]} v \right).$$

Therefore, for every  $F \in \text{Obj}(\mathfrak{U}\beta\text{-}\mathfrak{Mod})$ ,  $\mathcal{Y}_m(F)$  is a subfunctor of  $\tau_m \mathbf{LM}(F)$ . We naturally define the functor  $\mathcal{Y}_m$  on natural transformations by:

$$\forall n \in \mathbb{N}, (\mathcal{Y}_m(\eta))_n = id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}} \otimes_{\mathbb{K}[\mathbf{F}_m]} \eta_{m+n+1}.$$

Let us show that we may define a natural transformation  $v_m : \mathcal{Y}_m \implies \tau_m \mathbf{LM}$  in the category  $\mathfrak{U}\beta\text{-}\mathfrak{Mod}$ . Let  $F$  and  $G$  be two  $\mathfrak{U}\beta$ -modules, and  $\eta : F \implies G$  a natural transformation. Then, it follows from our definitions that for all natural integers  $n$ , for all  $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}$  and  $v \in F(m+n+1)$ :

$$(v_{m,n}(G) \circ (\mathcal{Y}_m(\eta))_n) \left( i \otimes_{\mathbb{K}[\mathbf{F}_m]} v \right) = ((\tau_m \mathbf{LM}(\eta))_n \circ v_{m,n}(F)) \left( i \otimes_{\mathbb{K}[\mathbf{F}_m]} v \right).$$

□

The functor  $\mathcal{Y}_m^{(\bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$  satisfies some convenient properties.

**Lemma 4.6.** *for all natural integers  $m$ , the functor  $\mathcal{Y}_m^{(\bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$  is exact.*

*Proof.* Let us consider the short exact sequence  $0 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 0$  in the category  $\mathfrak{U}\beta\text{-}\mathfrak{Mod}$ . Let  $m$  and  $n$  be natural integers. Since the augmentation ideal  $\mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}$  is a free  $\mathbb{K}[\mathbf{F}_m]$ -module (see for example [27, Chapter 2, Proposition 6.2.6]), it is therefore a flat  $\mathbb{K}[\mathbf{F}_m]$ -module. Then, by definition, it ensures that the functor  $\mathcal{I}_{\mathbb{K}[\mathbf{F}_m]} \otimes_{\mathbb{K}[\mathbf{F}_m]} - : \mathfrak{K}\text{-}\mathfrak{Mod} \longrightarrow \mathbb{K}\text{-}\mathfrak{Mod}$  is an exact functor for all natural integers  $m$ . Hence,

applying it to the morphisms  $\alpha_{m+n+1}$  and  $\beta_{m+n+1}$ , we obtain that the following short sequence of  $\mathbb{K}$ -modules is exact. We denote  $(\mathcal{Y}_m(\alpha))_n$  resp.  $(\mathcal{Y}_m(\beta))_n$  by  $\tilde{\alpha}_{n+1}$  resp.  $\tilde{\beta}_{n+1}$ .

$$0 \longrightarrow \mathcal{Y}_m(F)(n) \xrightarrow{\tilde{\alpha}_{n+1}} \mathcal{Y}_m(G)(n) \xrightarrow{\tilde{\beta}_{n+1}} \mathcal{Y}_m(H)(n) \longrightarrow 0$$

The naturality in  $n$  follows directly from the definitions and our constructions. □

**Proposition 4.7.** For all natural integers  $m$  and  $l$ , the functor  $\Upsilon_m^{(\bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$  commutes with the difference functor  $\delta_l$  and the evanescence functor  $\kappa_l$  for all natural integers  $l$ .

*Proof.* Let  $n$  be a natural integer and  $F$  be a  $\mathfrak{U}\beta$ -module. For all natural integers  $l$ , it follows directly from the definition of  $i_l$  that

$$i_l(\Upsilon_m(F))_n = \Upsilon_m(F)([l, id_{l+n}]) = id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}} \otimes_{\mathbb{K}[\mathbf{F}_m]} F \left( \left[ l, \gamma_{m+n+l}^{b, \bullet} \left( (b_{m,l}^{\mathfrak{M}})^{-1} \natural id_n \right) \right] \right).$$

and

$$(\Upsilon_m(i_l(F)))_n = id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}} \otimes_{\mathbb{K}[\mathbf{F}_m]} F([l, id_{l+m+n+1}]).$$

Since we know that the two following sequences are exact (because of Lemma 4.6 for the bottom one), the following diagram is commutative.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \kappa_l(\Upsilon_m F)(n) & & \tau_l(\Upsilon_m F)(n) & \longrightarrow & \delta_l(\Upsilon_m F)(n) \longrightarrow 0 \\ & & \searrow & & \nearrow & & \downarrow \cong \\ & & & \Upsilon_m F(n) & & & id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}} \otimes_{\mathbb{K}[\mathbf{F}_m]} F \left( \left[ 0, \gamma_{m+n+l}^{b, \bullet} \left( (b_{m,l}^{\mathfrak{M}})^{-1} \natural id_n \right) \right] \right) \\ & & \nearrow & & \searrow & & \downarrow \\ 0 & \longrightarrow & \Upsilon_m(\kappa_l F)(n) & & \Upsilon_m(\tau_l F)(n) & \longrightarrow & \Upsilon_m(\delta_l F)(n) \longrightarrow 0 \end{array}$$

Therefore, by the universal properties of the kernel and the cokernel, we deduce that  $\kappa_l(\Upsilon_m F)(n) \cong \Upsilon_m(\kappa_l F)(n)$  and  $\delta_l(\Upsilon_m F)(n) \cong \Upsilon_m(\delta_l F)(n)$ . The naturality in  $n$  follows directly from the definitions and our constructions.  $\square$

**Proposition 4.8.** Let  $m$  be a natural integer and  $F$  be a  $\mathfrak{U}\beta$ -module. Then the functor  $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(\tau_m F)$  is a subfunctor of  $\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)$ .

This way, for all  $F \in \text{Obj}(\mathfrak{U}\beta\text{-}\mathfrak{M}\text{od})$ , the subfunctors  $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(\tau_m F)$  of  $\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)$  define a subfunctor of  $\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$ , which will be denoted by  $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \circ \tau_m$ .

*Proof.* For all natural integers  $n$ ,  $\mathbf{LM}(\tau_m F)(n) = \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} F(m+n+1)$ . Because of Proposition 4.2, the  $\mathbb{K}$ -module  $\mathbf{LM}(\tau_m F)(n)$  is a submodule of the  $\mathbb{K}$ -module  $\tau_m \mathbf{LM}(F)(n)$ . Then, we define the monomorphism associated with the direct sum  $\varepsilon_{m,n}(F) = \gamma_{n,m+n}^{f, \bullet} \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} id_{F(m+n+1)} : \mathbf{LM}(\tau_m F)(n) \hookrightarrow \tau_m \mathbf{LM}(F)(n)$  by the assignment:

$$\forall i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}, \forall v \in F(m+n+1), \varepsilon_{m,n}(F) \left( i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) = \gamma_{n,m+n}^{f, \bullet}(i) \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} v.$$

It remains to show that we define a natural transformation  $\varepsilon_m(F) : \mathbf{LM}(\tau_m F) \implies \tau_m \mathbf{LM}(F)$ . Let  $n$  and  $n'$  be natural integers such that  $n' \geq n$ , and  $[n' - n, \sigma] \in \text{Hom}_{\mathfrak{U}\beta}(n, n')$  with  $\sigma \in \mathbf{B}_{n'}$ . Let  $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$  and  $v \in F(m+n+1)$ . By Condition 1.40, we deduce that the functor  $\mathbf{LM}(\tau_m F)$  is defined on morphisms by:

$$\begin{aligned} & \mathbf{LM}(\tau_m F)([n' - n, \sigma]) \left( i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) \\ &= a_{n', \bullet}(\sigma) \left( \gamma_{n,n'}^{f, \bullet}(i) \otimes_{\mathbb{K}[\mathbf{F}_{n'}]} F \left( \left[ n' - n, \gamma_{m+n'}^{b, \bullet} \left( (id_m \natural \sigma) \circ \left( (b_{m,n'-n}^{\mathfrak{M}})^{-1} \natural id_n \right) \right] \right) \right) (v). \end{aligned}$$

We remark that by Condition 1.41, we know that  $a_{m+n', \bullet} \left( (b_{m,n'-n}^{\mathfrak{M}})^{-1} \natural id_n \right) \circ \gamma_{n,m+n'}^{f, \bullet} = \gamma_{n,m+n'}^{f, \bullet}$ . Since, again by Condition 1.41,  $a_{m+n', \bullet} (id_m \natural \sigma) \circ \gamma_{n',m+n'}^{f, \bullet} = \gamma_{n',m+n'}^{f, \bullet} \circ a_{n', \bullet}(\sigma)$  for all elements  $\sigma$  of  $\mathbf{B}_{n'}$ , naturality then follows from the following equality:

$$(\tau_m \mathbf{LM}(F)([n' - n, \sigma]) \circ \varepsilon_{m,n}(F)) \left( i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) = (\varepsilon_{m,n'}(F) \circ \mathbf{LM}(\tau_m F))([n' - n, \sigma]) \left( i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right).$$

Therefore, for every  $F \in \text{Obj}(\mathfrak{U}\beta\text{-Mod})$ ,  $\mathbf{LM}(\tau_m F)$  is a subfunctor of  $\tau_m \mathbf{LM}(F)$ . Let us show that we may define a natural transformation  $\varepsilon_m : \mathbf{LM}\tau_m \Rightarrow \tau_m \mathbf{LM}$  in the category  $\mathfrak{U}\beta\text{-Mod}$ . Let  $F$  and  $G$  be two  $\mathfrak{U}\beta$ -modules, and  $\eta : F \Rightarrow G$  a natural transformation. Then, it follows from our definitions that for all natural integers  $n$ , for all  $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}$  and  $v \in F(m+n+1)$ :

$$(\varepsilon_{m,n}(G) \circ (\mathbf{LM} \circ \tau_m(\eta))_n) \left( i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) = ((\tau_m \mathbf{LM}(\eta))_n \circ \varepsilon_{m,n}(F)) \left( i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right).$$

□

**Proposition 4.9.** *Let  $m$  be a natural integer. Then, there is an isomorphism in the category  $\mathbf{Fct}(\mathfrak{U}\beta\text{-Mod}, \mathfrak{U}\beta\text{-Mod})$ :*

$$\tau_m \mathbf{LM}_{(\bullet_a, \bullet_s, \bullet_{\gamma b}, \bullet_{\gamma f})} \cong \Upsilon_m^{(\bullet_s, \bullet_{\gamma b}, \bullet_{\gamma f})} \bigoplus \left( \mathbf{LM}_{(\bullet_a, \bullet_s, \bullet_{\gamma b}, \bullet_{\gamma f})} \circ \tau_m \right).$$

*Proof.* Let  $F$  be a  $\mathfrak{U}\beta$ -module. By Propositions 4.3 and 4.8, by the universal property of the direct sum and the naturality which comes directly, we deduce that  $\Upsilon_m F \bigoplus \mathbf{LM}(\tau_m F)$  is a subfunctor of  $\tau_m \mathbf{LM}(F)$ . for all natural integers  $n$ , we have an isomorphism of  $\mathbb{K}$ -modules according to Propostion 4.2:

$$\Upsilon_m F(n) \bigoplus \mathbf{LM}(\tau_m F)(n) \cong \tau_m \mathbf{LM}(F)(n).$$

Hence by the universal property of the direct sum, for all natural integers  $n$ , there exists a unique  $\mathbb{K}$ -module isomorphism  $\Xi_{\oplus, n, F} = v_{m,n}(F) \oplus \varepsilon_{m,n}(F) : \Upsilon_m F(n) \bigoplus \mathbf{LM}(\tau_m F)(n) \xrightarrow{\cong} \tau_m \mathbf{LM}(F)(n)$  such that the following diagram is commutative.

$$\begin{array}{ccc} & \mathbf{LM}(\tau_m F)(n) & \\ \swarrow 0 \oplus id_{\mathbf{LM}(\tau_m F)} & & \searrow \varepsilon_{m,n}(F) \\ \Upsilon_m F(n) \bigoplus \mathbf{LM}(\tau_m F)(n) & \xrightarrow[\Xi_{\oplus, n, F}]{\cong} & \tau_m \mathbf{LM}(F)(n) \\ \swarrow id_{\Upsilon_m F} \oplus 0 & & \searrow v_{m,n}(F) \\ & \Upsilon_m F(n) & \end{array}$$

Let  $n$  and  $n'$  be natural integers such that  $n' \geq n$ , and  $[n' - n, \sigma] \in \text{Hom}_{\mathfrak{U}\beta}(n, n')$ . In each of the following two diagrams, since the top and bottom triangles and the lateral trapezoids are commutative by our definitions, the exterior rectangle is commutative.

$$\begin{array}{ccc} \Upsilon_m F(n) \bigoplus \mathbf{LM}(\tau_m F)(n) & \xrightarrow[\Xi_{\oplus, n, F}]{\cong} & \tau_m \mathbf{LM}(F)(n) \\ \downarrow & \swarrow 0 \oplus id_{\mathbf{LM}(\tau_m F)(n)} & \searrow \varepsilon_{m,n}(F) \\ & \mathbf{LM}(\tau_m F)(n) & \\ & \downarrow \mathbf{LM}(\tau_m F)([n' - n, \sigma]) & \\ \Upsilon_m F(n) \bigoplus \mathbf{LM}(\tau_m F)([n' - n, \sigma]) & & \tau_m \mathbf{LM}(F)([n' - n, \sigma]) \\ & \swarrow 0 \oplus id_{\mathbf{LM}(\tau_m F)(n')} & \searrow \varepsilon_{m,n'}(F) \\ & \mathbf{LM}(\tau_m F)(n') & \\ \Upsilon_m F(n') \bigoplus \mathbf{LM}(\tau_m F)(n') & \xrightarrow[\Xi_{\oplus, n', F}]{\cong} & \tau_m \mathbf{LM}(F)(n') \\ \downarrow & & \downarrow \\ \Upsilon_m F(n') \bigoplus \mathbf{LM}(\tau_m F)([n' - n, \sigma]) & & \tau_m \mathbf{LM}(F)([n' - n, \sigma]) \end{array}$$

$$\begin{array}{ccc}
\Upsilon_m F(n) \oplus \mathbf{LM}(\tau_m F)(n) & \xrightarrow[\cong]{\Xi_{\oplus, n, F}} & \tau_m \mathbf{LM}(F)(n) \\
\downarrow & \swarrow \text{id}_{\Upsilon_m F(n)} \oplus 0 & \searrow v_{m, n}(F) \\
& \Upsilon_m F(n) & \\
& \downarrow \Upsilon_m F([n'-n, \sigma]) & \\
& \Upsilon_m F(n') & \\
& \swarrow \text{id}_{\Upsilon_m F(n')} \oplus 0 & \searrow v_{m, n'}(F) \\
\Upsilon_m F(n') \oplus \mathbf{LM}(\tau_m F)(n') & \xrightarrow[\cong]{\Xi_{\oplus, n', F}} & \tau_m \mathbf{LM}(F)(n') \\
\downarrow & & \downarrow \\
\Upsilon_m F(n) \oplus \mathbf{LM}(\tau_m F)([n'-n, \sigma]) & & \tau_m \mathbf{LM}(F)([n'-n, \sigma])
\end{array}$$

This shows that we have a natural equivalence:

$$\Xi_{\oplus, F} = v_m(F) \oplus \varepsilon_m(F) : \Upsilon_m(F) \bigoplus \mathbf{LM}_{\tau_m}(F) \xrightarrow{\cong} \tau_m \mathbf{LM}(F).$$

It remains to show the naturality. Let  $F$  and  $G$  be two  $\mathfrak{U}\beta$ -modules, and  $\eta : F \Rightarrow G$  be a natural transformation. Then, it follows from our definitions and constructions that for all natural integers  $n$ , for all  $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}$ ,  $j \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$  and  $v, w \in F(m+n+1)$ :

$$\begin{aligned}
& \left( \Upsilon_m(\eta) \bigoplus \mathbf{LM}_{\tau_m}(\eta) \right)_n \left( \left( i \otimes_{\mathbb{K}[\mathbf{F}_m]} v \right) \oplus \left( j \otimes_{\mathbb{K}[\mathbf{F}_n]} w \right) \right) \\
&= \left( \Xi_{\oplus, n, G}^{-1} \circ (\tau_m \mathbf{LM}(\eta))_n \circ \Xi_{\oplus, n, F} \right) \left( \left( i \otimes_{\mathbb{K}[\mathbf{F}_m]} v \right) \oplus \left( j \otimes_{\mathbb{K}[\mathbf{F}_n]} w \right) \right).
\end{aligned}$$

Hence, we have a natural equivalence  $\Xi_{\oplus} = v_m \oplus \varepsilon_m$  in the category  $\mathfrak{U}\beta\text{-Mod}$ .  $\square$

## 4.2 Splitting of the difference functor

Our aim is to study the effect of the Long-Moody construction on strong polynomial functors. To that purpose, we focus on the behaviour of the difference functor on the Long-Moody construction. Thus, we will be interested in the study of the cokernel of the map  $i_m \mathbf{LM}_{(\bullet_a, \bullet_{\varsigma}, \bullet_{\gamma b}, \bullet_{\gamma f})}(F)$ . Let us recall some properties of this map.

- for all natural integers  $n$ ,  $i_m(\mathbf{LM}(F))_n = \mathbf{LM}(F)([m, id_{m+n}])$ . Explicitly for all elements  $k$  of  $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ , for all elements  $v$  of  $F(n+1)$ :

$$\mathbf{LM}(F)([m, id_{m+n}]) \left( k \otimes_{\mathbb{K}[\mathbf{F}_n]} w \right) = \gamma_{n, m+n}^{f, \bullet}(k) \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} F([m, id_{m+n+1}]) (w).$$

- we have the natural exact sequence:

$$0 \longrightarrow \kappa_m \mathbf{LM}(F)(n) \longrightarrow \mathbf{LM}(F)(n) \xrightarrow{i_m \mathbf{LM}(F)(n)} \tau_m \mathbf{LM}(F)(n) \longrightarrow \delta_m \mathbf{LM}(F)(n) \longrightarrow 0.$$

**Lemma 4.10.** *Let  $m$  be a natural integer and  $F$  be a  $\mathfrak{U}\beta$ -module. The submodules  $Im \left[ \mathbf{LM}_{(\bullet_a, \bullet_{\varsigma}, \bullet_{\gamma b}, \bullet_{\gamma f})}(F)([m, id_{m+n}]) \right]$  of  $\tau_m \mathbf{LM}_{(\bullet_a, \bullet_{\varsigma}, \bullet_{\gamma b}, \bullet_{\gamma f})}(F)(n)$  define a subfunctor of  $\mathbf{LM}_{(\bullet_a, \bullet_{\varsigma}, \bullet_{\gamma b}, \bullet_{\gamma f})} \tau_m(F)$ , which will be denoted by  $Im_m(F)$ .*

*This way, for all  $F \in Obj(\mathfrak{U}\beta\text{-Mod})$ , the subfunctors  $Im_m(F)$  of  $\mathbf{LM}_{(\bullet_a, \bullet_{\varsigma}, \bullet_{\gamma b}, \bullet_{\gamma f})} \tau_m(F)$  define a subfunctor of  $\mathbf{LM}_{(\bullet_a, \bullet_{\varsigma}, \bullet_{\gamma b}, \bullet_{\gamma f})} \tau_m$ , which will be denoted by  $Im_m$ .*

*Proof.* Let  $n$  be natural integer. Let  $k \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$  and  $w \in F(n+1)$ . The module  $Im_m(F)(n) = Im \left[ \mathbf{LM}_{(\bullet_a, \bullet_{\varsigma}, \bullet_{\gamma b}, \bullet_{\gamma f})}(F)([m, id_{m+n}]) \right]$  is the submodule of  $\tau_m \mathbf{LM}_{(\bullet_a, \bullet_{\varsigma}, \bullet_{\gamma b}, \bullet_{\gamma f})}(F)(n)$  with elements of the form  $\gamma_{n, m+n}^{f, \bullet}(k) \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} F([m, id_{m+n+1}]) (w)$ .

We denote by

$$im_m(F)_n : Im_m(F)(n) \hookrightarrow \tau_m \mathbf{LM}_{(\bullet_a, \bullet_{\varsigma}, \bullet_{\gamma b}, \bullet_{\gamma f})}(F)(n)$$

the associated inclusion morphism. Recalling  $\varepsilon_m : \mathbf{LM}\tau_m \Longrightarrow \tau_m \mathbf{LM}$  the natural transformation defined in Proposition 4.8, we have:

$$\mathbf{LM}(F)([m, id_{n+m}]) \left( k \otimes_{\mathbb{K}[\mathbf{F}_n]} w \right) = \varepsilon_{m,n}(F) \circ (\mathbf{LM}(\iota_m F)([0, id_{n+1}])) \left( k \otimes_{\mathbb{K}[\mathbf{F}_n]} w \right).$$

Hence, by the definition of  $\varepsilon_{m,n}(F)$ , using the decomposition of Proposition 4.2, the projection on the summand  $\mathcal{I}_m F(n)$  is zero. By the universal property of the kernel, we deduce that there exists a unique morphism of  $\mathbb{K}$ -modules:

$$\vartheta_{m,n}(F) : Im_m(F)(n) \longrightarrow \mathbf{LM}(\tau_m F)(n)$$

such that the following diagram is commutative.

$$\begin{array}{ccc} Im_m(F)(n) & \xrightarrow{im_m(F)_n} & \tau_m \mathbf{LM}(\bullet_{a,\bullet_s,\bullet_{\gamma b},\bullet_{\gamma f}})(F)(n) \\ & \searrow \vartheta_{m,n}(F) & \swarrow \varepsilon_{m,n}(F) \\ & \mathbf{LM}(\tau_m F)(n) & \end{array}$$

A fortiori,  $\vartheta_{m,n}(F)$  is a monomorphism and for all  $k \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ , and for all  $w \in F(n+1)$ :

$$\vartheta_{m,n}(F) \left( \gamma_{n,m+n}^{f,\bullet}(k) \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} \iota_m F([0, id_{n+1}]) (w) \right) = k \otimes_{\mathbb{K}[\mathbf{F}_n]} F([m, id_{m+n+1}]) (w).$$

Hence, the  $\mathbb{K}$ -module  $Im_m(F)(n)$  is a submodule of the  $\mathbb{K}$ -module  $\mathbf{LM}(\tau_m F)(n)$  for all natural integers  $n$ . Let  $n$  and  $n'$  be natural integers such that  $n' \geq n$ , and  $[n' - n, \sigma] \in Hom_{\mathcal{U}\beta}(n, n')$ . Let  $k \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$  and  $w \in F(n+1)$ . Let  $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}$  and  $v \in F(m+n+1)$ . We naturally define the functor  $Im_m(F)$  on morphisms by:

$$\begin{aligned} & Im_m(F)([n' - n, \sigma]) \left( \gamma_{n,m+n}^{f,\bullet}(k) \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} F([m, id_{m+n+1}]) (w) \right) \\ &= a_{m+n',\bullet} \left( (id_m \natural \sigma) \circ \left( (b_{m,n'-n}^{\mathfrak{M}})^{-1} \natural id_n \right) \right) \left( \gamma_{n,m+n'}^{f,\bullet}(k) \otimes_{\mathbb{K}[\mathbf{F}_{m+n'}]} F \left( [m+n'-n, \gamma_{m+n'}^{b,\bullet} \left( (id_m \natural \sigma) \circ \left( (b_{m,n'-n}^{\mathfrak{M}})^{-1} \natural id_n \right) \right) \right) \right) \end{aligned}$$

It is worth noting that Let us check that  $Im_m(F)$  is a subfunctor of  $\mathbf{LM}(\tau_m F)$  for every  $F \in Obj(\mathcal{U}\beta\text{-Mod})$ . First, we remark that by Condition 1.41, we know that  $a_{m+n',\bullet} \left( (b_{m,n'-n}^{\mathfrak{M}})^{-1} \natural id_n \right) \circ \gamma_{n,m+n'}^{f,\bullet} = \gamma_{n,m+n'}^{f,\bullet}$ . By Condition 1.41,  $a_{m+n',\bullet} (id_m \natural \sigma) \circ \gamma_{n,m+n'}^{f,\bullet} = \gamma_{n',m+n'}^{f,\bullet} \circ a_{n',\bullet}(\sigma) \circ \gamma_{n,n'}^{f,\bullet}$  for all elements  $\sigma$  of  $\mathbf{B}_{n'}$ , so we deduce the following equality:

$$\begin{aligned} & (\vartheta_{m,n'}(F) \circ Im_m(F)([n' - n, \sigma])) \left( \gamma_{n,m+n}^{f,\bullet}(k) \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} F([m, id_{m+n+1}]) (w) \right) \\ &= (\mathbf{LM}(\tau_m F)([n' - n, \sigma]) \circ \vartheta_{m,n}(F)) \left( \gamma_{n,m+n}^{f,\bullet}(k) \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} F([m, id_{m+n+1}]) (w) \right). \end{aligned}$$

Let us show that we may define a natural transformation  $\vartheta_m : Im_m \Longrightarrow \mathbf{LM}\tau_m$  in the category  $\mathcal{U}\beta\text{-Mod}$ . Let  $F$  and  $G$  be two  $\mathcal{U}\beta$ -modules, and  $\eta : F \Longrightarrow G$  a natural transformation. We naturally define the functor  $Im_m$  on natural transformations by:

$$\forall n \in \mathbb{N}, (Im_m(\eta))_n = id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{m+n}]}} \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} \eta_{m+n+1}.$$

Then, it follows from our definitions that for all natural integers  $n$ , for all  $k \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$  and  $w \in F(m+n+1)$ :

$$\begin{aligned} & (\vartheta_{m,n}(G) \circ (Im_m(\eta))_n) \left( \gamma_{n,m+n}^{f,\bullet}(k) \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} F([m, id_{m+n+1}]) (w) \right) \\ &= ((\mathbf{LM}\tau_m(\eta))_n \circ \vartheta_{m,n}(F)) \left( \gamma_{n,m+n}^{f,\bullet}(k) \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} F([m, id_{m+n+1}]) (w) \right). \end{aligned}$$

□

The following proposition is thus a direct consequence of this lemma and Proposition 4.9.

**Proposition 4.11.** *Let  $m$  be a natural integer. For a reliable quadruplet  $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$ , there is an isomorphism in the category  $\mathbf{Fct}(\mathfrak{U}\beta\text{-}\mathfrak{M}\text{od}, \mathfrak{U}\beta\text{-}\mathfrak{M}\text{od})$ :*

$$\delta_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \cong \Upsilon_m^{(\bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \bigoplus \frac{\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \circ \tau_m}{\mathit{Im}_m}.$$

We may refine this proposition using the following lemma.

**Lemma 4.12.** *Let  $m$  be a natural integer. There is an isomorphism in the category  $\mathbf{Fct}(\mathfrak{U}\beta\text{-}\mathfrak{M}\text{od}, \mathfrak{U}\beta\text{-}\mathfrak{M}\text{od})$ :*

$$\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \circ \delta_m \cong \frac{\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \circ \tau_m}{\mathit{Im}_m}.$$

*Proof.* Let  $F$  be a  $\mathfrak{U}\beta$ -module. Let us consider the exact sequence 1 of Lemma 2.7 applied to  $F$ :

$$0 \longrightarrow \kappa_m(F) \xrightarrow{\Omega_m(F)} F \xrightarrow{i_m(F)} \tau_m(F) \xrightarrow{\Delta_m(F)} \delta_m(F) \longrightarrow 0.$$

Since  $\mathbf{LM}$  is an exact functor (see Proposition 3.10), we deduce that the following sequence is exact:

$$0 \longrightarrow \mathbf{LM}(\kappa_m(F)) \xrightarrow{\mathbf{LM}(\Omega_m(F))} \mathbf{LM}(F) \xrightarrow{\mathbf{LM}(i_m(F))} \mathbf{LM}(\tau_m(F)) \xrightarrow{\mathbf{LM}(\Delta_m(F))} \mathbf{LM}(\delta_m(F)) \longrightarrow 0.$$

Therefore, because of the exactness of this sequence and Lemma 4.10, for all natural integers  $n$ :

$$\ker(\mathbf{LM}(\Delta_m(F))_n) = \mathit{Im}(\mathbf{LM}(i_m(F))_n) = \mathit{Im}_m(F)(n).$$

The first row of the following diagram is a short exact sequence. The universal property of the cokernel ensures that there exists a unique  $\mathbb{K}$ -module morphism  $\Psi_{n,F} : \frac{\mathbf{LM}(\tau_m F)(n)}{\mathit{Im}_m(F)(n)} \longrightarrow \mathbf{LM}(\delta_m F)(n)$  such that the following diagram commutes.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathit{Im}_m(F)(n) & \xrightarrow{\vartheta_{m,n}(F)} & \mathbf{LM}(\tau_m F)(n) & \xrightarrow{\pi_n(F)} & \frac{\mathbf{LM}(\tau_m F)(n)}{\mathit{Im}_m(F)(n)} \longrightarrow 0 \\ & & \searrow 0 & & \downarrow \mathbf{LM}(\Delta_m(F))_n & \swarrow \exists! \Psi_{n,F} & \\ & & & & \mathbf{LM}(\delta_m(F))(n) & & \end{array}$$

Moreover, since  $\ker(\mathbf{LM}(\Delta_m(F))_n) = \mathit{Im}_m(F)(n)$ , we know that  $\Psi_{n,F}$  is an isomorphism of  $\mathbb{K}$ -modules.

Let us check the naturality on  $n$ . Let  $n, n' \in \mathbb{N}$ , such that  $n' \geq n$ , and  $[n' - n, \sigma] \in \mathit{Hom}_{\mathfrak{U}\beta}(n, n')$ . Since the functor  $\mathit{Im}_m(F)$  is a subfunctor of  $\mathbf{LM}(\tau_m F)$  by Lemma 4.10, we define the quotient functor  $\frac{\mathbf{LM}(\tau_m F)}{\mathit{Im}_m(F)}$  on morphisms by the universal property of cokernel:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathit{Im}_m(F)(n) & \xrightarrow{\vartheta_{m,n}(F)} & \mathbf{LM}(\tau_m(F))(n) & \xrightarrow{\pi_n(F)} & \frac{\mathbf{LM}(\tau_m F)(n)}{\mathit{Im}_m(F)(n)} \longrightarrow 0 \\ \mathit{Im}_m(F)([n' - n, \sigma]) \downarrow & & \downarrow & & \downarrow \mathbf{LM}(\tau_m(F))[n' - n, \sigma] & & \downarrow \exists! \frac{\mathbf{LM}(\tau_m F)}{\mathit{Im}_m(F)}([n' - n, \sigma]) \\ 0 & \longrightarrow & \mathit{Im}_m(F)(n') & \xrightarrow{\vartheta_{m,n'}(F)} & \mathbf{LM}(\tau_m(F))(n') & \xrightarrow{\pi_{n'}(F)} & \frac{\mathbf{LM}(\tau_m F)(n')}{\mathit{Im}_m(F)(n')} \longrightarrow 0 \end{array}$$

Thus, the following diagram commutes.

$$\begin{array}{ccc} \frac{\mathbf{LM}(\tau_m F)(n)}{\mathit{Im}_m(F)(n)} & \xrightarrow{\Psi_{n,F}} & \mathbf{LM}(\delta_m(F))(n) \\ \downarrow \frac{\mathbf{LM}(\tau_m F)}{\mathit{Im}_m(F)}([n' - n, \sigma]) & & \downarrow \mathbf{LM}(\delta_m(F))([n' - n, \sigma]) \\ \frac{\mathbf{LM}(\tau_m F)(n')}{\mathit{Im}_m(F)(n')} & \xrightarrow{\Psi_{n',F}} & \mathbf{LM}(\delta_m(F))(n') \end{array}$$

But for all natural integers  $n$ ,  $\pi_n(F)$  is an epimorphism. So, we obtain the commuting proving that  $\Psi_F : \frac{\mathbf{LM}(\tau_m F)}{Im_m(F)} \xrightarrow{\cong} \mathbf{LM}(\delta_m F)$  is an isomorphism of  $\mathbf{Fct}(\mathcal{U}\beta\text{-}\mathcal{M}\text{od}, \mathcal{U}\beta\text{-}\mathcal{M}\text{od})$ .

It remains to show that we can define a natural isomorphism  $\Psi : \frac{\mathbf{LM} \circ \tau_m}{Im_m} \xrightarrow{\cong} \mathbf{LM} \circ \delta_m$  in the category  $\mathcal{U}\beta\text{-}\mathcal{M}\text{od}$ . Let  $F$  and  $G$  be two  $\mathcal{U}\beta$ -modules, and  $\eta : F \Rightarrow G$  a natural transformation. We naturally define the quotient functor  $\frac{\mathbf{LM} \circ \tau_m}{Im_m}$  on natural transformations to be the quotient:

$$\begin{array}{ccc} \mathbf{LM}(\tau_m(F))(n) & \xrightarrow{\pi_n(F)} & \frac{\mathbf{LM}(\tau_m F)(n)}{Im_m(F)(n)} \\ (\mathbf{LM}\tau_m(\eta))_n \downarrow & & \downarrow \left(\frac{\mathbf{LM} \circ \tau_m}{Im_m}(\eta)\right)_n \\ \mathbf{LM}(\tau_m(G))(n) & \xrightarrow{\pi_n(G)} & \frac{\mathbf{LM}(\tau_m G)(n)}{Im_m(G)(n)} \end{array}$$

Moreover, by naturality in objects of  $\mathcal{U}\beta\text{-}\mathcal{M}\text{od}$  of the exact sequence 1 of Lemma 2.7, because of the exactness of the Long-Moody functor (see Proposition 3.10) and by definition of  $\Psi_{n,F}$ , the natural transformation  $\mathbf{LM} \circ \delta_m(\eta)$  is defined to make the following square commutative for all natural integers  $n$ .

$$\begin{array}{ccc} \frac{\mathbf{LM}(\tau_m F)(n)}{Im_m(F)(n)} & \xrightarrow{\Psi_{n,F}} & \mathbf{LM}(\delta_m(F))(n) \\ \left(\frac{\mathbf{LM} \circ \tau_m}{Im_m}(\eta)\right)_n \downarrow & & \downarrow (\mathbf{LM} \circ \delta_m(\eta))_n \\ \frac{\mathbf{LM}(\tau_m G)(n)}{Im_m(G)(n)} & \xrightarrow{\Psi_{n,G}} & \mathbf{LM}(\delta_m(G)) \end{array}$$

□

Lemma 4.12 leads therefore to the following result.

**Theorem 4.13.** *Let  $m$  be a natural integer. There is a natural isomorphism in the category  $\mathcal{U}\beta\text{-}\mathcal{M}\text{od}$ :*

$$\delta_m \mathbf{LM}_{(\bullet_a, \bullet_c, \bullet_{\gamma b}, \bullet_{\gamma f})} \cong \Upsilon_m \bigoplus \left( \mathbf{LM}_{(\bullet_a, \bullet_c, \bullet_{\gamma b}, \bullet_{\gamma f})} \circ \delta_m \right).$$

Furthermore, we may determine the behaviour of the evanescence functor.

**Theorem 4.14.** *Let  $m$  be a natural integer. Then, the endofunctor  $\kappa_m$  commutes with the endofunctor  $\mathbf{LM}_{(\bullet_a, \bullet_c, \bullet_{\gamma b}, \bullet_{\gamma f})}$  of the category  $\mathbf{Fct}(\mathcal{U}\beta\text{-}\mathcal{M}\text{od}, \mathcal{U}\beta\text{-}\mathcal{M}\text{od})$ . In other words, there is a natural isomorphism in the category  $\mathbf{Fct}(\mathcal{U}\beta\text{-}\mathcal{M}\text{od}, \mathcal{U}\beta\text{-}\mathcal{M}\text{od})$ :*

$$\kappa_m \circ \mathbf{LM}_{(\bullet_a, \bullet_c, \bullet_{\gamma b}, \bullet_{\gamma f})} \cong \mathbf{LM}_{(\bullet_a, \bullet_c, \bullet_{\gamma b}, \bullet_{\gamma f})} \circ \kappa_m.$$

*Proof.* Since the Long-Moody functor is exact (see Proposition 3.10), we have the following exact sequence.

$$0 \longrightarrow \mathbf{LM} \circ \kappa_m \xrightarrow{\mathbf{LM} \circ \Omega_m} \mathbf{LM} \xrightarrow{\mathbf{LM} \circ i_m} \mathbf{LM} \circ \tau_m \xrightarrow{\mathbf{LM} \circ \Delta_m} \mathbf{LM} \circ \delta_m \longrightarrow 0$$

Therefore, the image functor  $Im(\mathbf{LM} \circ i_m)$  is isomorphic to the kernel functor  $Ker(\mathbf{LM} \circ \Delta_m)$ . Applying the exact sequence 1 of Lemma 2.7 to  $\mathbf{LM}(F)$ , we obtain the following diagram, where the first line is an exact sequence and the square is commutative.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \kappa_m \circ \mathbf{LM} & \xrightarrow{\Omega_m \circ \mathbf{LM}} & \mathbf{LM} & \xrightarrow{i_m \circ \mathbf{LM}} & \tau_m \mathbf{LM} & \xrightarrow{\Delta_m \circ \mathbf{LM}} & \delta_m \mathbf{LM} & \longrightarrow & 0 \\ & & & & & & \downarrow \cong & & \downarrow \cong & & \\ & & & & & & \text{by Proposition 4.9} & & \text{by Theorem 4.13} & & \\ & & & & & & \Upsilon_m \bigoplus (\mathbf{LM} \circ \tau_m) & \xrightarrow[\text{by Lemma 4.10}]{id \bigoplus (\mathbf{LM} \circ \Delta_m)} & \Upsilon_m \bigoplus (\mathbf{LM} \circ \delta_m) & & \end{array}$$

By the exactness property of the first row, we deduce that:

$$Im(\mathbf{LM} \circ i_m) \cong Ker(\mathbf{LM} \circ \Delta_m) \cong Im(i_m \circ \mathbf{LM}).$$

As the functor  $\kappa_m \circ \mathbf{LM}$  (resp.  $\mathbf{LM} \circ \kappa_m$ ) is defined to be the kernel of  $\mathbf{LM} \xrightarrow{i_m \circ \mathbf{LM}} \tau_m \mathbf{LM}$  (resp. of  $\mathbf{LM} \xrightarrow{\mathbf{LM} \circ i_m} \mathbf{LM} \circ \tau_m$ ), by the unicity up to isomorphism of the kernel, we conclude that  $\kappa_m \circ \mathbf{LM} \cong \mathbf{LM} \circ \kappa_m$ .  $\square$

### 4.3 Increasing of the polynomial degree

**Proposition 4.15.** *Let  $m$  be a natural integer and  $F$  be a non-null  $\mathfrak{U}\beta$ -module. If the functor  $F$  is strong (resp. very strong) polynomial of degree  $n$ , then:*

- the functor  $\Upsilon_m^{(\bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)$  belongs to  $\mathcal{P}ol_n(\mathfrak{U}\beta, \mathbb{K}\text{-Mod})$ ;
- the functor  $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)$  belongs to  $\mathcal{P}ol_{n+1}(\mathfrak{U}\beta, \mathbb{K}\text{-Mod})$ .

*Proof.* The result for a strong (resp. very strong) polynomial functor for  $\Upsilon_m$  follows from Corollary 4.7 by induction on the degree of polynomiality. We proceed by induction for  $\mathbf{LM}$ . For a strong polynomial functor of degree 0, by Theorem 4.13, we obtain that:

$$\delta_m \mathbf{LM}(F) \cong \Upsilon_m(F).$$

Therefore  $\mathbf{LM}(F)$  is a degree 1 strong polynomial functor. Now, assume that  $F$  is a strong polynomial functor of degree  $n$ . By Theorem 4.13:

$$\delta_m \mathbf{LM}(F) \cong \mathbf{LM}(\delta_m F) \oplus \Upsilon_m(F).$$

By the induction hypothesis and the result on  $\Upsilon_m$ , we deduce that  $\mathbf{LM}(F)$  is a strong polynomial functor of degree  $n+1$ . Moreover, if we assume that  $F$  is a very strong polynomial functor of degree  $n$ , the result follows from Corollary 4.7 and Theorem 4.14.  $\square$

**Corollary 4.16.** *For all natural integers  $n$ , the endofunctor  $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$  of  $\mathfrak{U}\beta\text{-Mod}$  restricts to a functor:*

$$\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}^{\mathcal{P}ol, n} : \mathcal{P}ol_n(\mathfrak{U}\beta, \mathbb{K}\text{-Mod}) \longrightarrow \mathcal{P}ol_{n+1}(\mathfrak{U}\beta, \mathbb{K}\text{-Mod})$$

which sends every degree  $n$  polynomial functor to a degree  $n+1$  polynomial functor.

**Corollary 4.17.** *Let  $n$  be a natural integer and  $F$  be an object  $\mathcal{P}ol_n(\mathfrak{U}\beta, \mathbb{K}\text{-Mod})$  such that  $\text{colim}_{n \in \mathbb{N}} F(n) \neq 0$ . Then:*

- the functor  $\Upsilon_m^{(\bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)$  is a strong polynomial (resp. very strong) functor of degree  $n$ ;
- the functor  $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)$  is a strong (resp. very strong) polynomial functor of equal to  $n+1$ .

**Example 4.18.** By Proposition 2.18,  $\mathfrak{X}$  is a degree 0 strong polynomial functor. Now applying the Long-Moody functor, we proved (see 3.3.1) that  $\mathbf{LM}_{1,1}(\mathfrak{X})$  is a isomorphic of  $\mathfrak{B}ur_{t_1, t_1^2}$  (which is clearly very strong polynomial by Proposition (2.28)), and therefore  $\mathbf{LM}(\mathfrak{X})$  is a very strong polynomial functor of degree 1.

### 4.4 Non essential surjectivity

**Proposition 4.19.** *Let  $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$  be a reliable quadruplet and  $n$  be a natural integer. Let  $F$  a strong polynomial functor of degree  $n+1$  in the category  $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-Mod})$ . Assume that there exists an object  $G$  of the category  $\mathfrak{U}\beta\text{-Mod}$  such that  $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(G) = F$ . Then, the functor  $G$  is a strong polynomial functor of degree  $n$  in the category  $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-Mod})$ .*

*Proof.* It follows from Theorem 4.13 that:

$$\delta_1 F \cong \mathcal{T}_1 G \oplus (\mathbf{LM}(\delta_1 G) v).$$

According to Lemma 2.19, the functor  $G$  an object of the category  $\mathcal{P}ol_n(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{M}od)$ , and because of Proposition 4.15 the functor  $\mathcal{T}_1 G$  is an object of the category  $\mathcal{P}ol_n(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{M}od)$ . A fortiori, thanks to Corollary 4.16,  $\mathbf{LM}(\delta_1 G)$  is also an object of the category  $\mathcal{P}ol_n(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{M}od)$ . Since the functor  $\delta_1 F$  is a strong polynomial functor of degree exactly  $n$ , it follows that the functor  $G$  is a strong polynomial functor of degree exactly equals to  $n$ .  $\square$

**Corollary 4.20.** *Let  $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$  be a reliable quadruplet. Let  $F$  be an object of the category  $\mathcal{V}\mathcal{P}ol_{n+1}(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{M}od)$ , of degree exactly equals to  $n + 1$ . Assume that there exists an object  $G$  of the category  $\mathfrak{U}\beta\text{-}\mathfrak{M}od$  such that  $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(G) = F$ . Then, the functor  $G$  is an object of  $\mathcal{V}\mathcal{P}ol_n(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{M}od)$ , of degree exactly equals to  $n$ .*

*Proof.* According to Lemma 2.19, the functor  $G$  is a strong polynomial functor of degree  $n$ . Moreover, for all natural integers  $m$ , thanks to Theorem 4.14 we have that:

$$\mathbf{LM}(\kappa_m G) \cong \kappa_m \mathbf{LM}(G) = 0$$

and a fortiori  $\kappa_m G = 0$  (see Remark 3.11).  $\square$

**Proposition 4.21.** *Let  $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$  be a reliable quadruplet. The functor  $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} : \mathfrak{U}\beta\text{-}\mathfrak{M}od \rightarrow \mathfrak{U}\beta\text{-}\mathfrak{M}od$  is not essentially surjective.*

*Proof.* Assume that there exist two objects  $T_1$  and  $T_2$  of the category  $\mathfrak{U}\beta\text{-}\mathfrak{M}od$  such that  $\mathbf{LM}(T_1) \cong \mathfrak{B}ur$  and  $\mathbf{LM}(T_2) \cong \mathfrak{T}\mathfrak{Y}\mathfrak{M}$ . According to Corollary 4.20, the functors  $T_1$  and  $T_2$  are very strong polynomial functors of degree 0 since the Burau functor and the Tong-Yang-Ma functors are very strong polynomial of degree one (see Proposition 2.28). Thanks to Proposition 2.18, there exist two constant functors  $C_1$  and  $C_2$  that  $T_1 \cong C_1$  and  $T_2 \cong C_2$ .

Furthermore, for all natural integers  $n$ ,  $\mathfrak{T}\mathfrak{Y}\mathfrak{M}(n) = \mathfrak{B}ur(n)$ . We deduce that:

$$\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} C_1(n) = \mathbf{LM}(C_1)(n) \cong \mathbf{LM}(C_2)(n) = \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} C_2(n).$$

It follows directly that the constant functors  $C_1$  and  $C_2$  are isomorphic. Therefore, the functors  $\mathfrak{B}ur \cong \mathbf{LM}(C_1)$  and  $\mathfrak{T}\mathfrak{Y}\mathfrak{M} \cong \mathbf{LM}(C_2)$  are isomorphic. There is a contradiction since the unreduced Burau representation and the Tong-Yang-Ma representation are not equivalent according to [24, Part II].  $\square$

*Remark 4.22.* In [6, Section 4.7, Open Problem 7], Birman and Brendle ask “wether all finite dimensional unitary matrix representations of  $\mathbf{B}_n$  arise in a manner which is related to the construction” recalled in Theorem 3.1. Since the Tong-Yang-Ma and unreduced Burau representations recalled in Theorem 2.22 are unitary representations, the proof of Proposition 4.21 shows that any Long-Moody functor (and especially the one based on the first construction of in Theorem 3.1) cannot provide all the functors encoding unitary representations. Therefore, we refine the problem asking wether all functors encoding families of unitary representations of braid groups lies in the image of a Long-Moody functor.

Another question is to ask wether we can obtain the reduced Burau functor  $\overline{\mathfrak{B}ur}$  by a modified Long-Moody functor: the modification would be to take the tensor product with  $\mathcal{I}_{\mathbf{F}_{n-1}}$  on  $\mathbf{F}_{n-1}$ , since for  $F$  an element of  $\mathbf{F}ct(\mathfrak{U}\beta, \mathbb{K}\text{-}\mathfrak{M}od)$ , the  $\mathbb{K}$ -module  $F(n+1)$  is in fact a  $\mathbb{K}[\mathbf{F}_{n-1}]$ -module thanks to the morphism  $\varsigma_n \circ \varsigma_{n-1} : \mathbf{F}_{n-1} \hookrightarrow \mathbf{B}_{n+1}$  for all natural integers  $n$ .

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