



The Long-Moody construction and polynomial functors

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THE LONG-MOODY CONSTRUCTION AND POLYNOMIAL FUNCTORS

ARTHUR SOULIÉ

February 26, 2017

Abstract

In 1994, Long and Moody gave a construction on representations of braid groups which associates a representation of \mathbf{B}_n with a representation of \mathbf{B}_{n+1} . In this paper, we prove that this construction is functorial: it gives an endofunctor, called the Long-Moody functor, between the category of functors from the homogeneous category associated with the braid groupoid to a module category. Then we study the effect of the Long-Moody functor on strong polynomial functors: we prove that it increases by one the degree of strong polynomiality.

Introduction

Linear representations of Artin braid group on n strands \mathbf{B}_n is a rich subject which appears in diverse contexts in mathematics (see for example [6] or [20] for an overview). Even if, at the present time, a complete classification of these representations is probably out of reach, any new result which would allow us to gain a better understanding of them may be noteworthy.

In 1994, in a joint work with Moody (see [18]), Long gave a method to construct from a linear representation $\rho : \mathbf{B}_{n+1} \rightarrow GL(V)$ a new linear representation $lm(\rho) : \mathbf{B}_n \rightarrow GL(V^{\oplus n})$ of \mathbf{B}_n . Moreover, the construction complexifies in a sense the initial representation. For example, applying it to a one dimensional representation of \mathbf{B}_{n+1} , the construction gives a mild variation of the unreduced Burau representation of \mathbf{B}_n . This method was in fact already implicitly present in two previous articles of Long dated 1989 (see [16, 17]). In the article [3] dated 2008, Bigelow and Tian are interested in the Long-Moody construction from a purely matricial point of view. They give alternative and purely algebraic proofs of some results of [18], and they slightly extend some of them. Besides, in a survey on braid groups (see the Open Problem 7 in [6]), Birman and Brendle underline the fact that the Long-Moody construction should be studied in greater details.

Our work will focus on the study of the Long-Moody construction from a functorial point of view. More precisely, we consider the homogenous category $\mathfrak{U}\beta$ associated with braid groups. This category is an example of a general construction introduced by Randal-Williams and Wahl in [21] on the braid groupoid. The category $\mathfrak{U}\beta$ has \mathbb{N} as objects and, for each natural integer n , its automorphism group $Aut_{\mathfrak{U}\beta}(n)$ is the braid group \mathbf{B}_n . Let $\mathbb{K}\text{-Mod}$ be the category of \mathbb{K} -modules, with \mathbb{K} a commutative ring. A functor $F : \mathfrak{U}\beta \rightarrow \mathbb{K}\text{-Mod}$ gives by evaluation a family of representations of braid groups $\{F_n : \mathbf{B}_n \rightarrow GL(F(n))\}_{n \in \mathbb{N}}$, which satisfies some compatibility properties. For instance, Randal-Williams and Wahl define in [21, Example 4.3] a functor $\mathfrak{B}ur : \mathfrak{U}\beta \rightarrow \mathbb{K}\text{-Mod}$ such that the representation $\mathfrak{B}ur_n : \mathbf{B}_n \rightarrow GL(\mathfrak{B}ur(n))$ is the unreduced Burau representation. In the same way, we define for instance in Example 2.25 a functor $\mathfrak{T}\mathfrak{Y}\mathfrak{M} : \mathfrak{U}\beta \rightarrow \mathbb{K}\text{-Mod}$ such that the representation $\mathfrak{T}\mathfrak{Y}\mathfrak{M}_n : \mathbf{B}_n \rightarrow GL(\mathfrak{T}\mathfrak{Y}\mathfrak{M}(n))$ is the representation considered by Tong, Yang and Ma in [22].

In Proposition 3.14, we prove that the Long-Moody construction is functorial. More precisely, we show:

Theorem A (Proposition 3.14) . *There is a functor $\mathbf{LM} : \mathfrak{U}\beta\text{-Mod} \rightarrow \mathfrak{U}\beta\text{-Mod}$, called the Long-Moody functor, which satisfies for $\sigma \in \mathbf{B}_n$*

$$\mathbf{LM}(F)(\sigma) := lm(F_{n+1})(\sigma).$$

Among the functors in $\mathcal{U}\mathcal{B}\text{-Mod}$, the strong polynomial functors play a key role. This notion extends the classical one of polynomial functors, which were first defined by Eilenberg and Mac Lane in [11]. Their definition uses cross effects and concerns module categories. This initial definition can be extended to monoidal categories where the monoidal unit is also a null object. Djament and Vespa introduce in [9] the definition of strong polynomial functors for symmetric monoidal categories with the monoidal unit being an initial object. Here, we will see that the category $\mathcal{U}\mathcal{B}$ is not symmetric, nor braided, but pre-braided in the sense of [21]. However, the notion of strong polynomial functor may be extended to the wider context of pre-braided monoidal categories (see Definition 2.6). Therefore, we investigate the effects of the Long-Moody functor on strong polynomial functors. Indeed, they turn out inter alia to be very useful for homological stability problems. For example, in [21], Randal-Williams and Wahl construct a general framework to prove homological stability for different families of groups. They obtain the stability for coefficients given by a specific kind of strong polynomial functors (namely coefficient systems of finite degree). We establish the following theorem.

Theorem B (Corollary 4.16) . *Let F be a strong polynomial functor of $\mathcal{U}\mathcal{B}\text{-Mod}$ of degree n . Then, the functor $\mathbf{LM}(F)$ is a strong polynomial functor of $\mathcal{U}\mathcal{B}\text{-Mod}$ of degree $n + 1$.*

Thus, iterating the Long-Moody functor on a strong polynomial functor of $\mathcal{U}\mathcal{B}\text{-Mod}$ of degree n , we generate polynomial functors of $\mathcal{U}\mathcal{B}\text{-Mod}$, of any degree bigger than n . For instance, the functors \mathbf{Bur} and \mathbf{TMM} happen to be strong polynomial functors of degree one, and we prove that the functor \mathbf{Bur} is equivalent to a functor obtained by applying the Long-Moody construction.

The paper is organized as follows. In section 1, we first recall definitions, folklore facts and properties about braid groups and free groups, especially focusing on their link. Then, following [21], we aim at explaining the construction of a homogeneous category from a braided monoidal groupoid, using Quillen's construction. In Section 2, we review the notion of strong polynomial functors, and slightly extend the framework of [9] for pre-braided monoidal categories. We will especially dwell on the interesting case of coefficient system of finite degree, which \mathbf{Bur} and \mathbf{TMM} happen to be examples. In Section 3, we prove the Theorem A and give some properties of the Long-Moody functor. Section 4 is devoted to the proof of Theorem B.

Notation. We will consider a commutative ring \mathbb{K} all along this work (the classical example is $\mathbb{K} = \mathbb{C}$). We denote by $\mathbb{K}\text{-Mod}$ the category of \mathbb{K} -modules.

Let \mathcal{Cat} denote the category of small categories. For $\mathcal{C} \in \mathbf{Obj}(\mathcal{Cat})$, the underlying groupoid $\mathcal{Gr}(\mathcal{C})$ is the subcategory of \mathcal{C} which has the same objects as \mathcal{C} and whose morphisms are the isomorphisms of \mathcal{C} . We denote by $\mathcal{Gr} : \mathcal{Cat} \rightarrow \mathcal{Cat}$ the functor which associates to a category its underlying groupoid.

Let \mathcal{C} be a category. For two objects A and B of the category \mathcal{C} , if A embeds in B and if no explicit notation is given, then we denote by I_A^B the associated embedding. If 0 is an initial object in the category \mathcal{C} , then we denote by $\iota_A : 0 \rightarrow A$ the unique morphism from 0 to A .

For \mathcal{C} an object of \mathcal{Cat} and \mathcal{D} a category, we denote by $\mathbf{Fct}(\mathcal{C}, \mathcal{D})$ the category of functors from \mathcal{C} to \mathcal{D} . For the particular case where $\mathcal{D} = \mathbb{K}\text{-Mod}$, we denote by $\mathcal{C}\text{-Mod}$ the functor category $\mathbf{Fct}(\mathcal{C}, \mathbb{K}\text{-Mod})$. A monoidal category is denoted by $(\mathcal{C}, \mathfrak{h}, 0, \alpha, \lambda, \rho)$ where \mathfrak{h} is the monoidal product, 0 is the unit, α is the associator, λ is the left unitor and ρ is the right unitor. If the category is braided, we denote by $b^{\mathcal{C}}$ its braiding.

We denote by \mathcal{Gr} the category of groups.

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1 Recollections on braid and free groups and homogenous categories

The aim of this section is to remind or present the necessary tools of our study. First, we recall some classical facts about braid groups and their links with free groups. Then, we give some notions and properties about Quillen's construction from a monoidal groupoid, pre-braided categories and homogenous categories introduced by Randal-Williams and Wahl in [21].

1.1 Braid groups and free groups

1.1.1 Generalities

We recall that the braid group on n strands denoted by \mathbf{B}_n is the free group generated by $\sigma_1, \dots, \sigma_{n-1}$ satisfying the relations:

- $Rel(1) : \forall i \in \{1, \dots, n-2\}, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1};$
- $Rel(2) : \forall i, j \in \{1, \dots, n-1\} \text{ such that } |i - j| \geq 2, \sigma_i \sigma_j = \sigma_j \sigma_i.$

Notation 1.1. For each natural integer n , one can identify \mathbf{B}_n with a subgroup of \mathbf{B}_{n+1} , denoting by $\gamma_n^b : \mathbf{B}_n \hookrightarrow \mathbf{B}_{n+1}$ the induced natural inclusion. Inductively, one defines the monomorphisms $\gamma_{n,n'}^b : \mathbf{B}_n \hookrightarrow \mathbf{B}_{n'}$, when $n \leq n'$.

Let us introduce the groupoid β associated with the family of braid groups.

Definition 1.2. The braid groupoid β is the groupoid with objects the natural integers $n \in \mathbb{N}$ and morphisms (for $n, m \in \mathbb{N}$):

$$Hom_{\beta}(n, m) := \begin{cases} \mathbf{B}_n & \text{if } n = m \\ \emptyset & \text{if } n \neq m \end{cases}$$

Let \mathfrak{S}_n denote the symmetric group on n elements:

$$\mathfrak{S}_n = \langle \tau_1, \dots, \tau_{n-1} \mid Rel(1), Rel(2) \text{ and } \forall i \in \{1, \dots, n-1\}, \tau_i^2 = 1 \rangle.$$

Definition 1.3. Let n be a fixed natural integer. Let $\mathfrak{s}_n : \mathbf{B}_n \longrightarrow \mathfrak{S}_n$ be the surjective group homomorphism defined by:

$$i \in \{1, \dots, n\}, \mathfrak{s}_n(\sigma_i) = \tau_i.$$

The kernel of this homomorphism \mathfrak{s}_n is called the pure braid group on n strands and denoted by \mathbf{PB}_n . We denote by $\mathbf{b}_n : \mathbf{PB}_n \hookrightarrow \mathbf{B}_n$ the associated monomorphism. We have the following short exact sequence:

$$1 \longrightarrow \mathbf{PB}_n \xrightarrow{\mathbf{b}_n} \mathbf{B}_n \xrightarrow{\mathfrak{s}_n} \mathfrak{S}_n \longrightarrow 1.$$

Proposition 1.4. [1] Let n be a natural integer. The pure braid group \mathbf{PB}_n has a presentation with generators $\{a_{ij}\}_{1 \leq i < j \leq n}$, defined by:

$$i, j \in \{1, \dots, n\}, i < j, a_{ij} := \sigma_{j-1}^{-1} \cdots \sigma_{i+1}^{-1} \sigma_i^2 \sigma_{i+1} \cdots \sigma_{j-1},$$

and relations

$$\begin{cases} a_{rs} a_{ij} a_{rs}^{-1} = a_{ij} & \text{if } r < s < i < j \text{ or } i < r < s < j, \\ a_{rs} a_{ij} a_{rs}^{-1} = a_{rj}^{-1} a_{ij} a_{rj} & \text{if } r < s = i < j, \\ a_{rs} a_{ij} a_{rs}^{-1} = [a_{sj}, a_{rj}]^{-1} a_{ij} [a_{sj}, a_{rj}] & \text{if } r < i < s < j, \\ a_{rs} a_{ij} a_{rs}^{-1} = (a_{sj} a_{rj})^{-1} a_{ij} (a_{sj} a_{rj}) & \text{if } r = i < s < j. \end{cases}$$

We also consider the free group on n generators, which we denote by:

$$\mathbf{F}_n = \langle g_1, \dots, g_n \rangle.$$

Notation 1.5. For each natural integer n , one can identify \mathbf{F}_n with a subgroup of \mathbf{F}_{n+1} , denoting by $\gamma_n^f : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n+1}$ the associated monomorphism. Inductively, one defines the monomorphisms $\gamma_{n,n'}^f : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n'}$, when $n \leq n'$.

Example 1.6. In fact, there are two classical identifications. On the one hand, we may identify \mathbf{B}_n (respectively \mathbf{F}_n) with the subgroup generated by the $n-1$ (respectively n) first generators $\sigma_1, \dots, \sigma_{n-1}$ of \mathbf{B}_{n+1} (respectively g_1, \dots, g_n of \mathbf{F}_{n+1}). The inclusions for this first case will be denoted by $\gamma_n^{b,p}$ (respectively $\gamma_n^{f,p}$). On the other hand, we can see \mathbf{B}_n (respectively \mathbf{F}_n) as the subgroups generated by the $n-1$ last generators $\sigma_2, \dots, \sigma_n$ of \mathbf{B}_{n+1} (respectively g_2, \dots, g_{n+1} of \mathbf{F}_{n+1}), denoting the inclusions by $\gamma_n^{b,l}$ (respectively $\gamma_n^{f,l}$).

The group \mathbf{F}_n can be viewed as a subgroup of \mathbf{B}_{n+1} , in different ways. For example, it can be considered as the kernel of the homomorphism $\mathbf{p}_n : \mathbf{PB}_{n+1} \rightarrow \mathbf{PB}_n$ which consists in forgetting a string (classically the first or last one) of a pure braid on $n+1$ strands. Then, we have the following commutative diagram, where the line is a short exact sequence:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{F}_n & \longrightarrow & \mathbf{PB}_{n+1} & \xrightarrow{\mathbf{p}_n} & \mathbf{PB}_n \longrightarrow 0. \\ & & & \searrow & \downarrow \mathbf{b}_{n+1} & & \\ & & & & \mathbf{B}_{n+1} & & \end{array}$$

Notation 1.7. We denote the associated embedding by $\varsigma_{n,\bullet} : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}$.

Example 1.8. A classical identification, called the pure braid local system in the literature (see [18, Remark p.223]) and denoted by $\varsigma_{n,1}$, is defined by the following assignment.

$$\begin{aligned} \varsigma_{n,1} : \mathbf{F}_n &\hookrightarrow \mathbf{B}_{n+1} \\ g_i &\longmapsto \begin{cases} \sigma_1^2 & \text{if } i = 1 \\ \sigma_{i\varsigma_{n,1}}(g_{i-1}) \sigma_i^{-1} & \text{if } i \in \{2, \dots, n\} \end{cases} \end{aligned}$$

Example 1.9. Another classical identification (implicitly used in [3]) denoted by $\varsigma_{n,2}$, is defined by assigning:

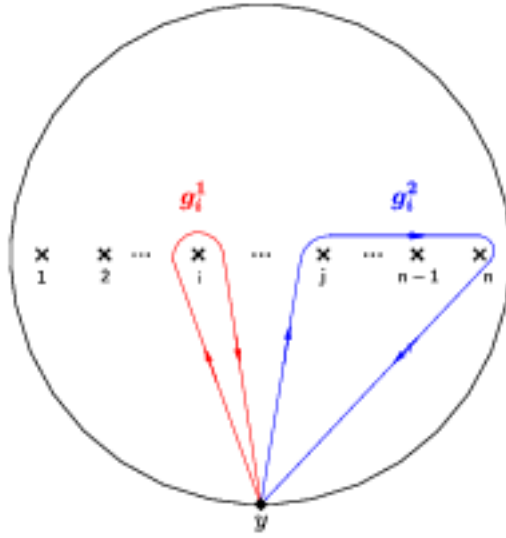
$$\begin{aligned} \varsigma_{n,2} : \mathbf{F}_n &\hookrightarrow \mathbf{B}_{n+1} \\ g_i &\longmapsto \begin{cases} \sigma_{i-1\varsigma_{n,2}}(g_{i+1}) \sigma_{i-1}^{-1} & \text{if } i \in \{1, \dots, n-1\} \\ \sigma_n^2 & \text{if } i = n \end{cases} \end{aligned}$$

Example 1.10. A last identification $\varsigma_{n,3}$ is the one called inner automorphism local system (see [10]), and is defined by assigning:

$$\begin{aligned} \varsigma_{n,3} : \mathbf{F}_n &\hookrightarrow \mathbf{B}_{n+1} \\ g_i &\longmapsto \begin{cases} (\sigma_2 \cdots \sigma_n)^n & \text{if } i = 1 \\ \sigma_{i-1\varsigma_{n,3}}(g_{i-1}) \sigma_{i-1}^{-1} & \text{if } i \in \{2, \dots, n\} \end{cases} \end{aligned}$$

Remark 1.11. In the three examples, we implicitly pass by \mathbf{PB}_{n+1} to form $\varsigma_{n,1}$ and $\varsigma_{n,2}$.

From a historical perspective (see for example [4] or [13]), the geometric point of view of topology gives us different actions of \mathbf{B}_n on the free group \mathbf{F}_n . As a consequence, there are several ways to consider the group \mathbf{B}_n as a subgroup of $\text{Aut}(\mathbf{F}_n)$. Geometrically speaking, it comes from the identification of \mathbf{B}_n with the mapping class group of a n -punctured disk $\Sigma_{0,1}^n$: fixing a point y on the boundary of the disk $\Sigma_{0,1}^n$, each free generators g_i can be taken as a loop of the disk based on y turning around certain fixed points. Two classical identifications are illustrated in the following picture: in the first case, the free generator g_i is the loop turning around the i -th fixed point, whereas in the second case, it is the one encircling the $n - j - 1$ last points. Each element σ of \mathbf{B}_n , as an automorphism up to isotopy of the disk $\Sigma_{0,1}^n$, induces a well-defined action on the fundamental group $\pi_1(\Sigma_{0,1}^n) \cong \mathbf{F}_n$.



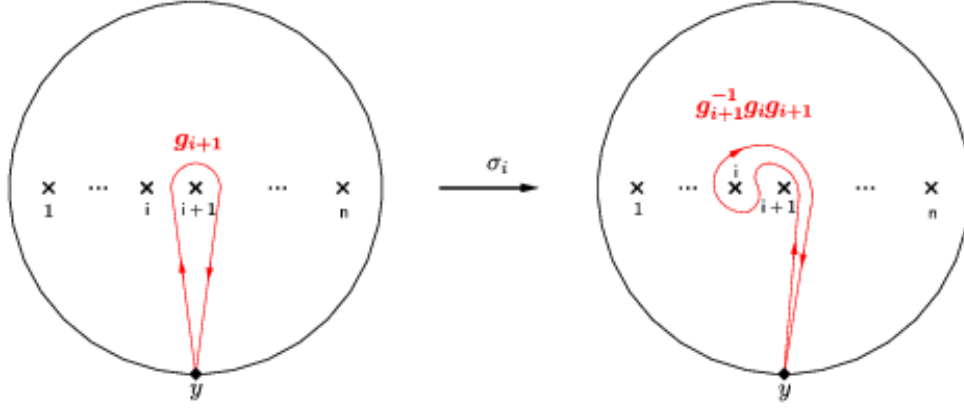
Remark 1.12. Denoting g_i^1 (resp. g_i^2) the first (resp. the second) identification, we have the following relations: for all $1 \leq i < n$, $g_i^1 = g_i^2 (g_{i+1}^2)^{-1}$ ie $g_i^2 = g_i^1 g_{i+1}^1 \cdots g_n^1$, and $g_n^1 = g_n^2$ (with the convention that we start with the left most element).

Notation 1.13. We denote the induced monomorphism by $a_{n,\bullet} : \mathbf{B}_n \hookrightarrow \text{Aut}(\mathbf{F}_n)$.

Example 1.14. A first classical action is defined (see for example [5, Section 1]) for all elementary braids σ_i where $i \in \{1, \dots, n-1\}$ by:

$$\begin{aligned} a_{n,1}(\sigma_i) : \mathbf{F}_n &\longrightarrow \mathbf{F}_n \\ g_j &\longmapsto \begin{cases} g_{i+1} & \text{if } j = i \\ g_{i+1}^{-1} g_i g_{i+1} & \text{if } j = i + 1 \\ g_j & \text{if } j \notin \{i, i + 1\} \end{cases} \end{aligned}$$

A geometrical intuition of this action is given by identifying each free generator g_i with the loop g_i^1 depicted in the previous picture. The following picture describes the action of $a_{n,1}(\sigma_i)$ on g_{i+1} .



Considering the groups \mathbf{F}_n and \mathbf{B}_n as subgroups of \mathbf{B}_{n+1} by $\varsigma_{n,1} : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}$ and $\gamma_n^{b,l} : \mathbf{B}_n \longrightarrow \mathbf{B}_{n+1}$, a straightforward calculation shows that the conjugation action of \mathbf{B}_n yields the map $a_{n,1} : \mathbf{B}_n \hookrightarrow \text{Aut}(\mathbf{F}_n)$. Likewise, considering \mathbf{F}_n and \mathbf{B}_n as subgroups of \mathbf{B}_{n+1} by $\varsigma_{n,3} : \mathbf{F}_n \longrightarrow \mathbf{B}_{n+1}$ and $\gamma_n^{b,p} : \mathbf{B}_n \longrightarrow \mathbf{B}_{n+1}$, the conjugation action of \mathbf{B}_n yields the map $a_{n,3} : \mathbf{B}_n \hookrightarrow \text{Aut}(\mathbf{F}_n)$. In both cases, as a subgroup of \mathbf{B}_{n+1} , the group \mathbf{F}_n is normal for the conjugation action.

Example 1.15. An alternative action (implicitly used in [3]) can be defined for all elementary braids σ_i where $i \in \{1, \dots, n-1\}$ by:

$$a_{n,2}(\sigma_i) : \mathbf{F}_n \longrightarrow \mathbf{F}_n$$

$$g_j \longmapsto \begin{cases} g_i g_{i+1} g_i^{-1} & \text{if } j = i \\ g_i & \text{if } j = i+1 \\ g_j & \text{if } j \notin \{i, i+1\} \end{cases}$$

It is in fact a “reverse” version of the previous one. If we see the groups \mathbf{F}_n and \mathbf{B}_n as subgroups of \mathbf{B}_{n+1} by $\varsigma_{n,2} : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}$ and $\gamma_{n,p}^b : \mathbf{B}_n \longrightarrow \mathbf{B}_{n+1}$, direct computations shows that the conjugation action of \mathbf{B}_n yields the map $(a_{n,2} \circ \gamma_{n,p}^b) : \mathbf{B}_n \hookrightarrow \text{Aut}(\mathbf{F}_n)$. Then, the group \mathbf{F}_n is normal as a subgroup of \mathbf{B}_{n+1} for the conjugation action.

Example 1.16. A last classical action is defined for all elementary braids σ_i where $i \in \{1, \dots, n-1\}$ by:

$$a_{n,3}(\sigma_i) : \mathbf{F}_n \longrightarrow \mathbf{F}_n$$

$$g_j \longmapsto \begin{cases} g_j & \text{if } j \neq i+1 \\ g_{j+2} g_{j+1}^{-1} g_j & \text{if } i \leq n-2 \text{ and } j = i+1 \\ g_n^{-1} g_{n-1} & \text{if } i = n-1 \text{ and } j = n \end{cases}$$

A geometrical intuition of this action is given by identifying each free generator g_i with the loop g_i^2 depicted in the first picture.

Remark 1.17. An action $a_{n,\bullet}$ gives naturally rise to a outer semidirect product structure, which will be denoted by $\mathbf{F}_n \rtimes_{a_{n,\bullet}} \mathbf{B}_n$. Moreover, the inclusions $\gamma_n^{b,\bullet} : \mathbf{B}_n \hookrightarrow \mathbf{B}_{n+1}$ and $\varsigma_{n,\bullet} : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}$ allow to consider $\mathbf{F}_n \rtimes_{a_{n,\bullet}} \mathbf{B}_n$ as a subgroup of \mathbf{B}_{n+1} , forming by universal property a unique inclusion homomorphism

$$\mathbf{F}_n \rtimes_{a_{n,\bullet}} \mathbf{B}_n \xrightarrow{\gamma_n^{\rtimes}} \mathbf{B}_{n+1} \text{ such that } \gamma_n^{\rtimes} \circ I_{\mathbf{B}_n}^{\mathbf{F}_n \rtimes \mathbf{B}_n} = \gamma_n^{b,\bullet} \text{ and } \gamma_n^{\rtimes} \circ I_{\mathbf{F}_n}^{\mathbf{F}_n \rtimes \mathbf{B}_n} = \varsigma_{n,\bullet}.$$

Finally, we need to focus on the augmentation ideal of the group ring $\mathbb{K}[\mathbf{F}_n]$.

Definition 1.18. The augmentation ideal of the group ring $\mathbb{K}[\mathbf{F}_n]$, denoted by $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$, is defined to be the kernel of the morphism:

$$\begin{aligned} \mathbb{K}[\mathbf{F}_n] &\xrightarrow{\kappa} \mathbb{K}. \\ \sum \lambda_g g &\mapsto \sum \lambda_g \end{aligned}$$

Proposition 1.19. [24, Chapter 6, Proposition 6.2.6] The augmentation ideal $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ is a free $\mathbb{K}[\mathbf{F}_n]$ -module with basis the set $\{(g_i - 1) \mid i \in \{1, \dots, n\}\}$.

Remark 1.20. The proof in [24, Chapter 6, Proposition 6.2.9] is done there for $\mathbb{K} = \mathbb{Z}$, but the general case here is exactly the same.

Remark 1.21. An action $a_{n,\bullet} : \mathbf{B}_n \hookrightarrow \text{Aut}(\mathbf{F}_n)$ extends naturally to $\mathbb{K}[\mathbf{F}_n]$. Indeed, let $\sum \lambda_g g \in \mathbb{K}[\mathbf{F}_n]$ and $b \in \mathbf{B}_n$, then, one defines $\tilde{a}_{n,\bullet} : \mathbf{B}_n \hookrightarrow \text{Aut}(\mathbb{K}[\mathbf{F}_n])$ by:

$$\tilde{a}_{n,\bullet}(b) \left(\sum \lambda_g g \right) = \sum \lambda_g g a_n(b)(g).$$

Hence, since $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ is a submodule of $\mathbb{K}[\mathbf{F}_n]$, this induces by restriction an action on $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ denoted by $A_n : \mathbf{B}_n \hookrightarrow \text{Aut}(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]})$.

Example 1.22. We may compute the three corresponding actions for the examples $a_{n,\bullet} : \mathbf{B}_n \rightarrow \text{Aut}(\mathbf{F}_n)$ in 1.14, 1.15 and 1.16. For all elementary braids σ_i where $i \in \{1, \dots, n-1\}$:

$$\begin{aligned} A_{n,1}(\sigma_i) : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} &\longrightarrow \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \\ g_j - 1 &\longmapsto \begin{cases} g_{k+1} - 1 & \text{if } j = i \\ g_{i+1}^{-1} g_i g_{i+1} - 1 = [g_i - 1] g_{i+1} + [g_{i+1} - 1] (1 - g_{i+1}^{-1} g_i g_{i+1}) & \text{if } j = i + 1 \\ g_j - 1 & \text{if } j \notin \{i, i + 1\} \end{cases} \end{aligned}$$

$$\begin{aligned} A_{n,2}(\sigma_i) : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} &\longrightarrow \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \\ g_j - 1 &\longmapsto \begin{cases} g_i g_{i+1} g_i^{-1} - 1 = g_i [g_{i+1} - 1] + (1 - g_i g_{i+1} g_i^{-1}) [g_i - 1] & \text{if } j = i \\ g_i - 1 & \text{if } j = i + 1 \\ g_j - 1 & \text{if } j \notin \{i, i + 1\} \end{cases} \end{aligned}$$

$$\begin{aligned} A_{n,3}(\sigma_i) : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} &\longrightarrow \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \\ g_j - 1 &\longmapsto \begin{cases} g_j - 1 & \text{if } j \neq i + 1 \\ g_{j+1} g_j^{-1} g_{j-1} - 1 = [g_{j-1} - 1] + [g_j - 1] g_j^{-1} g_{j-1} & \text{if } i \leq n - 2 \text{ and } j = i + 1 \\ \quad + [g_{j+1} - 1] g_j^{-1} g_{j-1} & \\ [g_{n-1} - 1] + [g_n - 1] (-g_n^{-1} g_{n-1}) & \text{if } i = n - 1 \text{ and } j = n \end{cases} \end{aligned}$$

1.2 The homogenous category associated with a groupoid

This subsection focuses on the presentation and the study of the Quillen's construction, in particular applied to the groupoid associated with the family of braid groups, and which turns out to be homogenous.

1.2.1 Quillen's construction

In [21], Randal-Williams and Wahl study a construction due to Quillen in [12, p.219], for a monoidal category S acting on a category X in the case $S = X = \mathfrak{G}$ where \mathfrak{G} is a groupoid. Our review here is based on [21].

Definition 1.23. Let $(\mathfrak{G}, \natural, 0)$ be a strict monoidal groupoid. The Quillen's construction on the groupoid \mathfrak{G} , denoted by $\mathfrak{U}\mathfrak{G}$ is defined by:

- Objects: $Obj(\mathfrak{U}\mathfrak{G}) = Obj(\mathfrak{G})$;
- Morphisms: For A and B two objects of \mathfrak{G} , a morphism $[X, f] : A \longrightarrow B$ in the category $\mathfrak{U}\mathfrak{G}$ is an equivalence class of pairs (X, f) where:
 - X is an object of \mathfrak{G} ;
 - $f : X \natural A \longrightarrow B$ is a morphism of \mathfrak{G} ;
 - the equivalence relation \sim is defined by $(X, f) \sim (X', f')$ if and only if there exists a morphism $g : X \longrightarrow X'$ in \mathfrak{G} such that the following diagram commutes.

$$\begin{array}{ccc} X \natural A & \xrightarrow{f} & B \\ g \natural id_A \downarrow & \nearrow f' & \\ X' \natural A & & \end{array}$$

- Composition: let $[X, f] : A \longrightarrow B$ and $[Y, g] : B \longrightarrow C$ be two morphisms in the category $\mathfrak{U}\mathfrak{G}$. Then, the composition is defined by:

$$[Y, g] \circ [X, f] := [Y \natural X, g \circ (id_Y \natural f)] .$$

- Identity: for all object X of $\mathfrak{U}\mathfrak{G}$, the identity morphism is given by $[0, id_X] : X \longrightarrow X$.

Remark 1.24. One easily checks that for all morphisms $[X', f] : X \longrightarrow A$, $[X'', g] : B \longrightarrow X$, $[D_1, \varphi_1] : C_1 \longrightarrow C_2$, $[D_2, \varphi_2] : C_2 \longrightarrow C_3$ and $[D_3, \varphi_3] : C_3 \longrightarrow C_4$ in the category $\mathfrak{U}\mathfrak{G}$:

- $[X', f] \circ [0, id_X] = [X', f]$ and $[0, id_X] \circ [X'', g] := [X'', g]$;
- $([D_1, \varphi_1] \circ [D_2, \varphi_2]) \circ [D_3, \varphi_3] = [D_1, \varphi_1] \circ ([D_2, \varphi_2] \circ [D_3, \varphi_3])$.

The Quillen's construction $\mathfrak{U}\mathfrak{G}$ has the additional following property.

Proposition 1.25. [21, Proposition 1.6] Let $(\mathfrak{G}, \natural, 0)$ be a strict monoidal groupoid. Then, the unit 0 is initial in the category $\mathfrak{U}\mathfrak{G}$.

1.2.2 Prebraided monoidal categories

Beforehand, we present the notion of pre-braided category, introduced by Randal-Williams and Wahl in [21]. It is a generalization of braided category, which will prove to be necessary to understand the homogenous category constructed from the groupoid β .

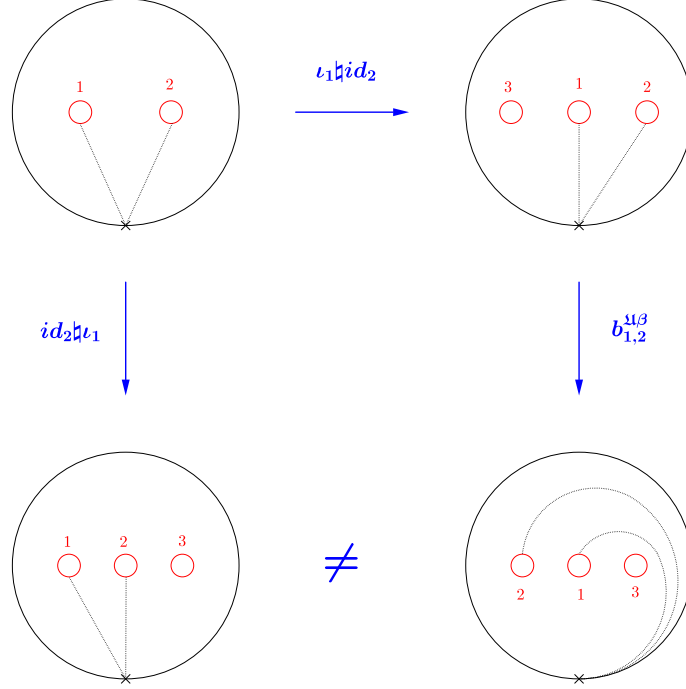
Definition 1.26. Let $(\mathfrak{C}, \natural, 0, \alpha, \lambda, \rho)$ be a monoidal category such that the unit 0 is initial. Let $(\mathfrak{G}, \natural, 0, \alpha, \lambda, \rho)$ be the groupoid $\mathcal{G}\mathfrak{r}((\mathfrak{C}, \natural, 0, \alpha, \lambda, \rho))$. We say that the monoidal category $(\mathfrak{C}, \natural, 0, \alpha, \lambda, \rho)$ is pre-braided if:

- $(\mathfrak{C}, \natural, 0, \alpha, \lambda, \rho)$ is a braided monoidal category ;
- for all objects A and B of \mathfrak{C} , the groupoid braiding $b_{A,B}^{\mathfrak{C}} : A \natural B \longrightarrow B \natural A$ satisfies:

$$b_{A,B}^{\mathfrak{C}} \circ (id_A \natural id_B) = \iota_B \natural id_A : A \longrightarrow B \natural A.$$

Remark 1.27. A braided monoidal category is obviously pre-braided.

Example 1.28. A monoidal product $\natural : \beta \times \beta \longrightarrow \beta$ is defined assigning the usual addition for the objects and connecting two braids side by side for the morphisms, along with a braiding denoted $b_{-, -}^\beta$ (for more details, see for example [19, Chapter XI, Part 4]). The pre-braiding defined on $\mathfrak{U}\beta$ is not a braiding. Indeed, the following figure shows that $b_{1,2}^{\mathfrak{U}\beta} \circ (\iota_1 \natural id_2) \neq id_2 \natural \iota_1$ whereas this two morphisms should be equal if $b_{-, -}^{\mathfrak{U}\beta}$ was a braiding. This example shows in particular that a pre-braided monoidal category is not necessarily braided.



Under some assumption, the Quillen's construction $\mathfrak{U}\mathfrak{G}$ inherits a pre-braided property.

Proposition 1.29. [21, Proposition 1.6] *Let $(\mathfrak{G}, \natural, 0)$ be a strict monoidal groupoid. If the category \mathfrak{G} is braided monoidal, then the category $(\mathfrak{U}\mathfrak{G}, \natural, 0)$ is pre-braided monoidal. Moreover, the monoidal structure of $\mathfrak{U}\mathfrak{G}$ is such that the map $\mathfrak{G} \longrightarrow \mathfrak{U}\mathfrak{G}$ taking an isomorphism f to $[0, f]$ is monoidal.*

Remark 1.30. The monoidal structure on the category $\mathfrak{U}\mathfrak{G}$ is defined letting for $[X, f] \in Hom_{\mathfrak{U}\mathfrak{G}}(A, B)$ and $[Y, g] \in Hom_{\mathfrak{U}\mathfrak{G}}(C, D)$:

$$[X, f] \natural [Y, g] := \left[X \natural Y, (f \natural g) \circ \left(id_X \natural b_{A,Y}^{-1} \natural id_C \right) \right].$$

1.2.3 Homogenous categories

The notion of homogenous category is introduced by Randal-Williams and Wahl in [21, Section 1], inspired by the set-up of Djament and Vespa in [8, Section 1.2]. With two additional assumptions, the Quillen's construction $\mathfrak{U}\mathfrak{G}$ from a strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ will be endowed with an homogenous category structure. First, we need to give basic definitions necessary to give the one of homogenous category.

Definition 1.31. Let $(\mathfrak{C}, \natural, 0)$ be a strict monoidal small category in which the unit 0 is also initial. For all objects A and B of \mathfrak{C} , we define:

- a preferred morphism: $\iota_A \natural id_B : (B = 0 \natural B) \longrightarrow A \natural B$;
- a set of morphisms characterised by this preferred morphism:

$$Fix(B) = Fix(B, A \natural B) = \{ \phi \in Aut(A \natural B) \mid \phi \circ (\iota_A \natural id_B) = \iota_A \natural id_B \}.$$

Remark 1.32. Since $(\mathfrak{C}, \natural, 0)$ is assumed to be small $Hom_{\mathfrak{C}}(A, B)$ is a set and $Aut_{\mathfrak{C}}(B)$ defines a group (with composition of morphisms as the group product). The group $Aut_{\mathfrak{C}}(B)$ acts by post-composition on $Hom_{\mathfrak{C}}(A, B)$:

$$\begin{array}{ccc} Aut_{\mathfrak{C}}(B) \times Hom_{\mathfrak{C}}(A, B) & \longrightarrow & Hom_{\mathfrak{C}}(A, B). \\ (\phi, f) & \longmapsto & \phi \circ f \end{array}$$

Now, we may introduce homogenous categories.

Definition 1.33. Let $(\mathfrak{C}, \natural, 0)$ be a strict monoidal small category. This category is homogenous if the unit 0 is initial in \mathfrak{C} and if the two following assumptions are satisfied.

- **(H1)** : For all objects A and B of the category \mathfrak{C} , the action by post-composition of $Aut(B)$ on $Hom_{\mathfrak{C}}(A, B)$ is transitive.
- **(H2)** : For all objects A and B of the category \mathfrak{C} , the group morphism

$$\begin{array}{ccc} Aut_{\mathfrak{C}}(A) & \longrightarrow & Aut_{\mathfrak{C}}(A \natural B) \\ f & \longmapsto & f \natural id_B \end{array}$$

is injective with image $Fix(B) = \{\phi \in Aut_{\mathfrak{C}}(A \natural B) \mid \phi \circ (\iota_A \natural id_B) = \iota_A \natural id_B\}$.

Remark 1.34. A strict monoidal small category $(\mathfrak{C}, \natural, 0)$ satisfying **(H1)** and **(H2)** is therefore determined by its underlying groupoid.

Let us focus on some elementary properties of homogenous categories.

Proposition 1.35. Let $(\mathfrak{C}, \natural, 0)$ be a homogenous category. Let A and B be two objects of this category. Then:

1. $Hom_{\mathfrak{C}}(B, A \natural B) \cong Aut_{\mathfrak{C}}(A \natural B) / Aut_{\mathfrak{C}}(A)$.
2. $Hom_{\mathfrak{C}}(A, A) \cong Aut_{\mathfrak{C}}(A)$.
3. If $Hom_{\mathfrak{C}}(A, B) \neq \emptyset$ and $Hom_{\mathfrak{C}}(B, A) \neq \emptyset$, then $A \cong B$.

Proof. For the first property, let us consider the map

$$\begin{array}{ccc} Aut_{\mathfrak{C}}(A \natural B) & \xrightarrow{\Psi} & Hom_{\mathfrak{C}}(B, A \natural B). \\ \psi & \longmapsto & \psi \circ (\iota_A \natural id_B) \end{array}$$

Because of the transitivity property **(H1)**, for every morphism f of $Hom_{\mathfrak{C}}(B, A \natural B)$, there exists a morphism ψ of $Aut_{\mathfrak{C}}(A \natural B)$ such that $\psi \circ (\iota_A \natural id_B) = f$: so the map Ψ is surjective. Because of the assumption **(H2)**, the map $Aut_{\mathfrak{C}}(A) \rightarrow Aut_{\mathfrak{C}}(A \natural B)$ sending f on $f \natural id_B$ is injective of image $Fix(B)$ (and therefore $Fix(B) \cong Aut_{\mathfrak{C}}(A)$). Yet:

$$\ker(\Psi) = \{\psi \in Aut_{\mathfrak{C}}(A \natural B) \mid \psi \circ (\iota_A \natural id_B) = \iota_A \natural id_B\} = Fix(B).$$

Hence $\ker(\Psi) \cong Aut_{\mathfrak{C}}(A)$ and as a consequence:

$$Hom_{\mathfrak{C}}(B, A \natural B) \cong Aut_{\mathfrak{C}}(A \natural B) / \ker(\Psi) \cong Aut_{\mathfrak{C}}(A \natural B) / Aut_{\mathfrak{C}}(A).$$

For the second property, since the unit 0 is initial in \mathfrak{C} , it has no non-trivial automorphism. Thus:

$$Hom_{\mathfrak{C}}(A, A) = Hom_{\mathfrak{C}}(A, 0 \natural A) / Aut_{\mathfrak{C}}(0) = Aut_{\mathfrak{C}}(A).$$

For the third property, let $f \in Hom_{\mathfrak{C}}(B, A)$ and $g \in Hom_{\mathfrak{C}}(A, B)$, we obtain $f \circ g = \phi \in Aut_{\mathfrak{C}}(A)$ and $g \circ f = \psi \in Aut_{\mathfrak{C}}(B)$. We deduce that $f \circ (g \circ \phi^{-1}) = id_A$ and $(\psi^{-1} \circ g) \circ f = id_B$. Moreover, we remark that $g \circ \phi^{-1} = \psi^{-1} \circ g$. Therefore, f and g are invertible. \square

Remark 1.36. We will deal with objects in $\mathbf{Fct}(\mathfrak{C}, \mathcal{A})$ for $(\mathfrak{C}, \natural, 0)$ a homogenous category and \mathcal{A} an abelian category. In order to prove propositions for a functor F of $\mathbf{Fct}(\mathfrak{C}, \mathbb{K}\text{-Mod})$, according to the first property of 1.35, it is sufficient to restrict the work on morphisms to the automorphisms. In other words, proving a result dealing with F for all the automorphisms automatically extends to check this result on F for all the morphisms.

Proposition 1.37. *Let $(\mathfrak{C}, \natural, 0)$ be a pre-braided homogenous category. Let A and B be two objects of the category \mathfrak{C} . Let us consider the morphism:*

$$\begin{array}{ccc} \text{Aut}_{\mathfrak{C}}(B) & \longrightarrow & \text{Aut}_{\mathfrak{C}}(A \natural B) \\ f & \longmapsto & id_A \natural f \end{array}$$

This map gives the following isomorphism:

$$\text{Aut}_{\mathfrak{C}}(B) \xrightarrow{\cong} \text{Fix}(A) = \text{Fix}(A, A \natural B) := \{\phi \in \text{Aut}_{\mathfrak{C}}(A \natural B) \mid \phi \circ (id_A \natural \iota_B) = id_A \natural \iota_B\}.$$

Proof. By definition of the braiding $b_{A,B} : A \natural B \rightarrow B \natural A$, conjugation by $b_{A,B}$ sends the morphism $\varphi \natural id_A$ on $id_A \natural \varphi$ for $\varphi \in \text{Aut}_{\mathfrak{C}}(B)$. Moreover, we consider the group morphism:

$$\begin{array}{ccc} \text{Fix}(A, B \natural A) & \xrightarrow{\Phi} & \text{Fix}(A, A \natural B) \\ \phi & \longmapsto & b_{A,B}^{-1} \circ \phi \circ b_{A,B} \end{array}$$

Let $\phi' \in \text{Fix}(A, A \natural B)$, then $b_{A,B} \circ \phi' \circ b_{A,B}^{-1} \in \text{Fix}(A, B \natural A)$ and $b_{A,B}^{-1} \circ (b_{A,B} \circ \phi' \circ b_{A,B}^{-1}) \circ b_{A,B} = \phi'$. So Φ is surjective. Otherwise, let $\phi \in \text{Fix}(A, B \natural A)$ such that $\phi \in \ker(\Phi)$. Composing on the left by $b_{A,B}$ and on the right by $b_{A,B}^{-1}$, we obtain that $\phi = id_{A \natural B}$. Hence Φ is injective and so an isomorphism. Because of the assumption **(H2)**, we deduce that $\text{Fix}(A, B \natural A) \cong \text{Aut}_{\mathfrak{C}}(B)$. \square

We should now give the two additional properties that a strict monoidal groupoid $(\mathfrak{G}, \natural, 0)$ may satisfy so as to the category $\mathcal{U}\mathfrak{G}$ be homogenous.

Definition 1.38. Let $(\mathfrak{G}, \natural, 0)$ be a strict monoidal groupoid. We define two assumptions.

- **(C)** : For all objects A, B and C of \mathfrak{G} , if $A \natural C \cong B \natural C$ then $A \cong B$. The category \mathfrak{G} is then said to satisfy cancellation property.
- **(I)** : For all objects A, B of \mathfrak{G} , the following morphism is injective:

$$\begin{array}{ccc} \text{Aut}_{\mathfrak{G}}(A) & \longrightarrow & \text{Aut}_{\mathfrak{G}}(A \natural B) \\ f & \longmapsto & f \natural id_B \end{array}$$

Theorem 1.39. [21, Theorem 1.8] *Let $(\mathfrak{G}, \natural, 0)$ be a braided monoidal groupoid and $\mathcal{U}\mathfrak{G}$ its associated pre-braided category.*

1. *The category satisfies **(H1)** if and only if the groupoid \mathfrak{G} satisfies **(C)**.*
2. *If the groupoid \mathfrak{G} , satisfies **(I)**, then $\mathcal{U}\mathfrak{G}$ satisfies **(H2)**.*

*In particular, if the groupoid \mathfrak{G} satisfies **(C)** and **(I)**, then $\mathcal{U}\mathfrak{G}$ is homogenous.*

The relationship between the automorphisms of the groupoid \mathfrak{G} and those of its associated Quillen's construction $\mathcal{U}\mathfrak{G}$ is not always clear. Indeed, we intuitively expect that \mathfrak{G} is the underlying groupoid of $\mathcal{U}\mathfrak{G}$ but we need two more hypotheses in order to ensure this property.

Proposition 1.40. [21, Proposition 1.10] *Let $(\mathfrak{G}, \natural, 0)$ be a monoidal groupoid and let $\mathcal{U}\mathfrak{G}$ denote the Quillen's construction on \mathfrak{G} . We assume that:*

- $\text{Aut}_{\mathfrak{G}}(0) = \{id_0\}$.
- *The groupoid \mathfrak{G} has no zero divisors: if $A \natural B \cong 0$ in \mathfrak{G} , then $A \cong 0$ and $B \cong 0$.*

Then \mathfrak{G} is the underlying groupoid of $\mathcal{U}\mathfrak{G}$.

Now, we will have all along our work the braid groupoid β , its associated homogenous category $\mathcal{U}\beta$, the category of \mathbb{K} -modules $\mathbb{K}\text{-Mod}$ and the category $\mathcal{U}\beta\text{-Mod}$ of functors from $\mathcal{U}\beta$ to $\mathbb{K}\text{-Mod}$ with natural transformations for morphisms.

2 Strong polynomial functors

We deal here with the concept of strong polynomial functor. This type of functor will be in the centre of our work in Section 4. We review (and in fact slightly extend) the definition and properties of a strong polynomial functor due to Djament and Vespa in [9] and also a particular case of coefficient systems of finite degree used by Randal-Williams and Wahl in [21].

In [9, Section 1], Djament and Vespa construct a framework to define strong polynomial functors in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$, where \mathfrak{M} is a symmetric monoidal category where the unit is an initial object and \mathcal{A} is an abelian category. Here, we aim at generalizing this definition for functors from pre-braided monoidal categories having the same additional property. In particular, a notion of strong polynomial functor will be well-defined for the category $\mathcal{UB}\text{-}\mathbf{Mod} = \mathbf{Fct}(\mathcal{UB}, \mathbb{K}\text{-Mod})$. The keypoint of this section is the Proposition 2.4, in so far as it constitutes the crucial property necessary and sufficient to extend the definition of strong polynomial functor for the prebraided case.

2.1 Strong polynomiality

Definition 2.1. We denote by \mathbf{Mon}^{pb} the category defined by the following assignment.

- Objects : the pre-braided strict monoidal small categories $(\mathfrak{M}, \natural, 0)$.
- Morphisms : the pre-braided strict monoidal functors $F : (\mathfrak{M}, \natural, 0) \longrightarrow (\mathfrak{N}, \natural, 0)$. Namely, these are strict monoidal functors F such that the functor $\mathcal{Gr}(F)$ is braided.

Definition 2.2. We denote by $\mathbf{Mon}_{\text{ini}}^{\text{pb}}$ the full subcategory of \mathbf{Mon}^{pb} whose objects are pre-braided strict monoidal small categories $(\mathfrak{M}, \natural, 0)$ such that the unit 0 is an initial object. We denote by $\mathbf{Mon}_{\text{null}}^{\text{pb}}$ the full subcategory of $\mathbf{Mon}_{\text{ini}}^{\text{pb}}$ whose objects are pre-braided strict monoidal small categories $(\mathfrak{M}, \natural, 0)$ such that the unit 0 is a null object.

We introduce the translation functor, which will play a central role in the definition of strong polynomiality.

Definition 2.3. Let $(\mathfrak{M}, \natural, 0)$ be an object of \mathbf{Mon}^{pb} , let \mathfrak{C} be a category and let x be an object of \mathfrak{M} . We define the endofunctor $x \natural id_{\mathfrak{M}} : \mathfrak{M} \longrightarrow \mathfrak{M}$ by:

- Objects : for all object m of \mathfrak{M} , $x \natural id_{\mathfrak{M}}(m) := x \natural m$.
- Morphisms : for all morphism f of $Hom_{\mathfrak{M}}(m, m')$, $x \natural id_{\mathfrak{M}}(f) := id_x \natural f : x \natural m \longrightarrow x \natural m'$.

For x an object of \mathfrak{M} , we define the translation by x functor $\tau_x : \mathbf{Fct}(\mathfrak{M}, \mathfrak{C}) \longrightarrow \mathbf{Fct}(\mathfrak{M}, \mathfrak{C})$ to be the endofunctor of $\mathbf{Fct}(\mathfrak{M}, \mathfrak{C})$ obtained by precomposition by the functor $x \natural id_{\mathfrak{M}}$.

The following proposition establishes the commutation of two translation functors associated with two objects of \mathfrak{M} . It is the keystone property to define polynomial functors.

Proposition 2.4. *Let $(\mathfrak{M}, \natural, 0)$ be an object of \mathbf{Mon}^{pb} and let \mathfrak{C} be a category. Let x and y be two objects of \mathfrak{M} . Then, we have a natural isomorphism of $\mathbf{Fct}(\mathbf{Fct}(\mathfrak{M}, \mathfrak{C}), \mathbf{Fct}(\mathfrak{M}, \mathfrak{C}))$:*

$$\tau_x \circ \tau_y \cong \tau_y \circ \tau_x.$$

Proof. First, it is worth noting that because of the associativity of the monoidal product \natural and of the strictness of \mathfrak{M} , we have that $\tau_x \circ \tau_y = \tau_{x \natural y}$ and $\tau_y \circ \tau_x = \tau_{y \natural x}$. We denote by $b_{-, -}^{\mathfrak{M}}$ the pre-braiding of \mathfrak{M} . The key point is the fact that $b_{x, y}^{\mathfrak{M}} : x \natural y \xrightarrow{\cong} y \natural x$ is a braiding (in so far as it is the braiding defined on the underlying groupoid of \mathfrak{M}). For all object F of $\mathbf{Fct}(\mathfrak{M}, \mathfrak{C})$, we define a morphism between $\tau_{x \natural y}(F)$ and $\tau_{y \natural x}(F)$ by:

$$(b_{x, y}^{\mathfrak{M}} \natural id_{\mathfrak{M}})^*(F) : [\tau_{x \natural y}(F) = F((x \natural y) \natural -)] \longrightarrow [(F \circ (b_{x, y}^{\mathfrak{M}} \natural id_{\mathfrak{M}}))((x \natural y) \natural -) = F((y \natural x) \natural -) = \tau_{y \natural x}(F)].$$

Let $\lambda : F \Longrightarrow G$ be a natural transformation in $\mathbf{Fct}(\mathfrak{M}, \mathfrak{C})$. Then:

- Let m be an object of \mathfrak{M} , we consider on the one hand

$$\left((b_{x,y}^{\mathfrak{M}} \natural id_{\mathfrak{M}})^* (G) (m) \right) \circ \tau_{x \natural y} (\lambda_m) = (G (b_{x,y}^{\mathfrak{M}} \natural id_m)) \circ \lambda_{(x \natural y) \natural m}$$

and on the other hand

$$\tau_{y \natural x} (\lambda_m) \circ (b_{x,y}^{\mathfrak{M}} \natural id_{\mathfrak{M}})^* (F) (m) = \lambda_{(y \natural x) \natural m} \circ F (b_{x,y}^{\mathfrak{M}} \natural id_m).$$

Since $b_{x,y}^{\mathfrak{M}} \natural id_m$ is a morphism of \mathfrak{M} and λ is a natural transformation, the following diagram commutes:

$$\begin{array}{ccc} F((x \natural y) \natural m) & \xrightarrow{F(b_{x,y}^{\mathfrak{M}} \natural id_m)} & F((y \natural x) \natural m) \\ \lambda_{(x \natural y) \natural m} \downarrow & & \downarrow \lambda_{(y \natural x) \natural m} \\ G((x \natural y) \natural m) & \xrightarrow{G(b_{x,y}^{\mathfrak{M}} \natural id_m)} & G((y \natural x) \natural m) \end{array}$$

We deduce that:

$$\left((b_{x,y}^{\mathfrak{M}} \natural id_{\mathfrak{M}})^* (G) (m) \right) \circ \tau_{x \natural y} (\lambda_m) = \tau_{y \natural x} (\lambda_m) \circ (b_{x,y}^{\mathfrak{M}} \natural id_{\mathfrak{M}})^* (F) (m).$$

- Let us consider a morphism $f \in Hom_{\mathfrak{M}}(m, m')$. In the following diagram, the left and right squares commute because λ is a natural transformation, the top and bottom squares commute because of the functoriality of F and composition rules of the monoidal product, the front and back squares commute because of the result of previous point.

$$\begin{array}{ccccc} F((x \natural y) \natural m) & \xrightarrow{F(b_{x,y}^{\mathfrak{M}} \natural id_m)} & & \xrightarrow{\quad} & F((y \natural x) \natural m) \\ & \searrow F((x \natural y) \natural f) & & & \searrow F((y \natural x) \natural f) \\ & & F((x \natural y) \natural m') & \xrightarrow{F(b_{x,y}^{\mathfrak{M}} \natural id_{m'})} & F((y \natural x) \natural m') \\ & & \downarrow \lambda_{(x \natural y) \natural m'} & & \downarrow \lambda_{(y \natural x) \natural m'} \\ \lambda_{(x \natural y) \natural m} \downarrow & & & & \downarrow \lambda_{(y \natural x) \natural m} \\ G((x \natural y) \natural m) & \xrightarrow{\quad} & G((y \natural x) \natural m) & \xrightarrow{\quad} & \\ & \searrow G((x \natural y) \natural f) & & & \searrow G((y \natural x) \natural f) \\ & & G((x \natural y) \natural m') & \xrightarrow{G(b_{x,y}^{\mathfrak{M}} \natural id_{m'})} & G((y \natural x) \natural m') \end{array}$$

Hence this diagram is commutative.

We deduce that:

$$(b_{x,y}^{\mathfrak{M}} \natural id_{\mathfrak{M}})^* (G) \circ \tau_{x \natural y} (\lambda) = \tau_{y \natural x} (\lambda) \circ (b_{x,y}^{\mathfrak{M}} \natural id_{\mathfrak{M}})^* (F).$$

Thus, we have defined a natural transformation:

$$(b_{x,y}^{\mathfrak{M}} \natural id_{\mathfrak{M}})^* : \tau_{x \natural y} \Longrightarrow \tau_{y \natural x}.$$

It happens to be an isomorphism in so far as we analogously construct a natural transformation

$$\left((b_{x,y}^{\mathfrak{M}})^{-1} \natural id_{\mathfrak{M}} \right)^* : \tau_{y \natural x} \Longrightarrow \tau_{x \natural y}$$

and direct computations show that for all object F of $\mathbf{Fct}(\mathfrak{M}, \mathcal{C})$

$$\left((b_{x,y}^{\mathfrak{M}})^{-1} \natural id_{\mathfrak{M}} \right)^* (F) \circ (b_{x,y}^{\mathfrak{M}} \natural id_{\mathfrak{M}})^* (F) = id_{\tau_{x \natural y}(F)}$$

and

$$(b_{x,y}^{\mathfrak{M}} \natural id_{\mathfrak{M}})^* (F) \circ \left((b_{x,y}^{\mathfrak{M}})^{-1} \natural id_{\mathfrak{M}} \right)^* (F) = id_{\tau_{y \natural x}(F)}.$$

□

Let us move on to the introduction of the evanescence and difference functors, which will characterize the (very) strong polynomiality of a functor in $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$, with \mathcal{A} an abelian category.

Remark 2.5. If \mathfrak{M} is a small category and \mathcal{A} is an abelian category, then the functor category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ is an abelian category (see [19, Chapter VIII] or [24, Chapter 1]).

Let $(\mathfrak{M}, \natural, 0)$ be an object of $\mathfrak{Mon}_{\text{ini}}^{\text{pb}}$, let \mathcal{A} be an abelian category, let x be an object of \mathfrak{M} . For all object F of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$, we denote by $i_x(F) : \tau_0(F) \rightarrow \tau_x(F)$ the natural transformations induced by the precomposition of F by the unique morphism $\iota_x : 0 \rightarrow x$ of \mathfrak{M} . Since $Id_{\mathbf{Fct}(\mathfrak{M}, \mathcal{A})}$ and τ_x are endofunctors of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$, they induce $i_x : Id_{\mathbf{Fct}(\mathfrak{M}, \mathcal{A})} \rightarrow \tau_x$ a natural transformation of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$. Since the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ is abelian, the kernel and cokernel of the natural transformation i_x exist.

Definition 2.6. We define $\kappa_x := \ker(i_x)$ and $\delta_x := \text{coker}(i_x)$. The endofunctors κ_x and δ_x of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ are called respectively called evanescence and difference functor associated with x .

The following lemma present elementary properties of the translation, evanescence and difference functors. They are either straightforward consequences of the definitions (properties from 1 to 3), or direct generalizations of the framework where \mathfrak{M} is symmetric monoidal considered in [9] (properties from 4 to 10). Indeed, the proofs of properties from 4 to 10 are exactly the same as those of [9, Proposition 1.4]: everything works in the same way in so far as the commutation property of the translation endofunctor is still satisfied in the pre-braided case by Proposition 2.4.

Lemma 2.7. *Let $(\mathfrak{M}, \natural, 0)$ be an object of $\mathfrak{Mon}_{\text{ini}}^{\text{pb}}$, let \mathcal{A} be an abelian category, let x and y be objects of \mathfrak{M} .*

1. The translation functor τ_x is exact.
2. We have the following exact sequence in the category of endofunctors of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$:

$$0 \longrightarrow \kappa_x \xrightarrow{\Omega_x} Id \xrightarrow{\iota_x} \tau_x \xrightarrow{\Delta_x} \delta_x \longrightarrow 0. \quad (1)$$

3. Consider a short exact sequence in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$:

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0.$$

The snake lemma implies that we have the following exact sequence in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$:

$$0 \longrightarrow \kappa_x(F) \longrightarrow \kappa_x(G) \longrightarrow \kappa_x(H) \longrightarrow \delta_x(F) \longrightarrow \delta_x(G) \longrightarrow \delta_x(H) \longrightarrow 0. \quad (2)$$

4. The translation endofunctors τ_x and τ_y of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ commute up to natural isomorphism. They commute with limits and colimits.
5. The difference endofunctors δ_x and δ_y of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ commute up to natural isomorphism. They commute with colimits.
6. The endofunctors κ_x and κ_y of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ commute up to natural isomorphism. They commute with limits.
7. The natural inclusion $\kappa_x \circ \kappa_x \hookrightarrow \kappa_x$ is an isomorphism.
8. The translation endofunctor τ_x and the difference endofunctor δ_y commute up to natural isomorphism.

9. The translation endofunctor τ_x and the endofunctor κ_y commute up to natural isomorphism.
10. We have the following natural exact sequence in the category of endofunctors of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$:

$$0 \longrightarrow \kappa_y \longrightarrow \kappa_{x \natural y} \longrightarrow \tau_x \kappa_y \longrightarrow \delta_y \longrightarrow \delta_{x \natural y} \longrightarrow \tau_y \delta_x \longrightarrow 0. \quad (3)$$

Thanks to lemma 2.7, we may define strong polynomial functors.

Definition 2.8. Let $(\mathfrak{M}, \natural, 0)$ be an object of $\mathfrak{Mon}_{\text{ini}}^{\text{pb}}$, let \mathcal{A} be an abelian category. We recursively define on $n \in \mathbb{N}$ the category $\mathcal{Pol}_n(\mathfrak{M}, \mathcal{A})$ of strong polynomial functors of degree smaller or equal n to be the full subcategory of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ as follows:

1. If $n < 0$, $\mathcal{Pol}_n(\mathfrak{M}, \mathcal{A}) = \{0\}$;
2. if $n \geq 0$, the objects of $\mathcal{Pol}_n(\mathfrak{M}, \mathcal{A})$ are the functors F such that for all object x of \mathfrak{M} , the functor $\delta_x(F)$ is an object of $\mathcal{Pol}_{n-1}(\mathfrak{M}, \mathcal{A})$.

The following three propositions are noteworthy properties of the framework in [9] adapted in the pre-braided case. Their proofs follow directly from those of their analogues in [9, Propositions 1.7, 1.8 and 1.9].

Proposition 2.9. [9, Proposition 1.7] Let \mathfrak{M} and \mathfrak{M}' be objects of $\mathfrak{Mon}_{\text{ini}}^{\text{pb}}$, let \mathcal{A} be an abelian category and let $\alpha : \mathfrak{M} \longrightarrow \mathfrak{M}'$ be a strong monoidal functor. Then, the endofunctor of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ obtained by precomposition by the functor α restricts to a functor:

$$\begin{aligned} \alpha_{\mathcal{Pol}}^* : \mathcal{Pol}_n(\mathfrak{M}', \mathcal{A}) &\longrightarrow \mathcal{Pol}_n(\mathfrak{M}, \mathcal{A}) . \\ F &\longmapsto F \circ \alpha \end{aligned}$$

Proposition 2.10. [9, Proposition 1.8] Let \mathfrak{M} be an object of $\mathfrak{Mon}_{\text{ini}}^{\text{pb}}$ and let \mathcal{A} be an abelian category. Let $\{F_j\}_{j \in \text{obj}(J)}$ be a family of objects of $\mathcal{Pol}_n(\mathfrak{M}, \mathcal{A})$ (where the category J is an index small category), and G be an object of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$. Then:

- the functor $\tau_x F_j$ is an object of $\mathcal{Pol}_n(\mathfrak{M}, \mathcal{A})$, ie $\mathcal{Pol}_n(\mathfrak{M}, \mathcal{A})$ is closed under the translation endofunctor τ_x ;
- $0 \longrightarrow F_j \longrightarrow F_{j'} \longrightarrow G \longrightarrow 0$ is a short exact sequence, then G object of $\mathcal{Pol}_n(\mathfrak{M}, \mathcal{A})$, ie $\mathcal{Pol}_n(\mathfrak{M}, \mathcal{A})$ is closed under quotient;
- if $0 \longrightarrow F_j \longrightarrow G \longrightarrow F_{j'} \longrightarrow 0$ is a short exact sequence, then G object of $\mathcal{Pol}_n(\mathfrak{M}, \mathcal{A})$, ie $\mathcal{Pol}_n(\mathfrak{M}, \mathcal{A})$ is closed under extension;
- the colimit $\text{Colim}_{j \in J}(F_j)$ is an object of $\mathcal{Pol}_n(\mathfrak{M}, \mathcal{A})$, ie $\mathcal{Pol}_n(\mathfrak{M}, \mathcal{A})$ is closed under colimit.

Moreover, assuming that there exists a set \mathfrak{E} of objects of \mathfrak{M} such that:

$$\forall m \in \text{Obj}(\mathfrak{M}), \exists \{e_i\}_{i \in I} \in \text{Obj}(\mathfrak{E}) \text{ (where } I \text{ is finite), } m \cong \natural_{i \in I} e_i,$$

then, an object F of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ belongs to $\mathcal{Pol}_n(\mathfrak{M}, \mathcal{A})$ if and only if $\delta_e(F)$ is an object of $\mathcal{Pol}_{n-1}(\mathfrak{M}, \mathcal{A})$ for all object e of \mathfrak{E} .

Remark 2.11. The category $\mathcal{Pol}_n(\mathfrak{M}, \mathcal{A})$ is not necessarily closed under the subobject. It is the case if \mathfrak{M} is an object of $\mathfrak{Mon}_{\text{nul}}^{\text{pb}}$, since then, for all object x of \mathfrak{M} , κ_x is the null endofunctor of $\mathbf{Fct}(\mathfrak{Mon}_{\text{nul}}^{\text{pb}}, \mathcal{A})$. Also, we will see in the next subsection that for very strong polynomial functors, as we force κ_x to be null for all object x of \mathfrak{M} , coefficient systems of finite degree will be closed under kernel of an epimorphism. As a consequence, in the general case where \mathfrak{M} is an object of $\mathfrak{Mon}_{\text{ini}}^{\text{pb}}$, a subfunctor of an object F of $\mathcal{Pol}_n(\mathfrak{M}, \mathcal{A})$ is not necessarily a strong polynomial functor.

Example 2.12. If we consider $\mathfrak{M} = \mathcal{U}\beta$, then all object n (ie a natural integer) is clearly the addition of n times the object 1. Hence, because of the last statement of Proposition 2.10, when we will deal with strong polynomiality of objects in $\mathbf{Fct}(\mathcal{U}\beta, \mathcal{A})$, it will suffice to verify the polynomiality for τ_1 .

Proposition 2.13. [9, Proposition 1.9] Let \mathfrak{M} and \mathfrak{M}' be objects of $\mathfrak{Mon}^{\text{pb}}$ and let \mathcal{A} be an abelian category. Let F be an object of $\mathbf{Fct}(\mathfrak{Mon}_{\text{ini}}^{\text{pb}}, \mathcal{A})$. The three following statements are equivalent.

1. The functor F is an object of $\mathcal{P}ol_0(\mathfrak{M}, \mathcal{A})$.
2. There exists a constant functor C of $\mathbf{Fct}(\mathfrak{Mon}_{\text{ini}}^{\text{pb}}, \mathcal{A})$ and a morphism c of $\mathbf{Fct}(\mathfrak{Mon}_{\text{ini}}^{\text{pb}}, \mathcal{A})(C, F)$ (ie a natural transformation) such that the following sequence is exact:

$$C \xrightarrow{c} F \longrightarrow 0.$$

3. For all object x of \mathfrak{M} , the morphism $F(\iota_x) : F(0) \longrightarrow F(x)$ is an epimorphism.

Example 2.14. By Proposition 1.29, the category $\mathcal{U}\beta$ is a prebraided monoidal category. This example is the first one which led us to extend the definition of [9]. Thus, we have a well-defined notion of strong polynomial functor on the category $\mathcal{U}\beta$.

2.2 Very strong polynomial functors

A certain type of functors, called coefficient systems of finite degree, closely related to the strong polynomial one, is used by Randal-Williams and Wahl in [21] for their homological stability theorems, generalizing the concept introduced by van der Kallen for general linear groups [23]. We introduce here the framework of this notion, adapted to the case of the category $\mathcal{U}\beta$, using the tools of [9]. Hence, with the notations of [21], this reduces the context to the situation of $A = 0$.

Definition 2.15. Let $(\mathfrak{M}, \natural, 0)$ be an object of \mathfrak{Mon} and let x be an object of \mathfrak{M} . We denote by \mathfrak{M}_x the full subcategory of \mathfrak{M} whose objects are the objects $x^{\natural n}$ for all natural integer n .

Definition 2.16. Let $(\mathfrak{M}, \natural, 0)$ be an object of $\mathfrak{Mon}_{\text{ini}}^{\text{pb}}$, let x be an object of \mathfrak{M} and let \mathcal{A} be an abelian category. We consider an object F of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$. We recursively define on $n \in \mathbb{N}$ the category $\mathcal{CS}_n^x(\mathfrak{M}, \mathcal{A})$ of coefficient systems in x of degree n at $N \in \mathbb{N}^*$ to be the full subcategory of $\mathbf{Fct}(\mathfrak{M}_x, \mathcal{A})$ as follows:

1. The functor F is a coefficient system of degree $n < 0$ at $N \geq 0$ if:

$$\forall k \geq N, F(x^{\natural k}) = 0.$$

2. The functor F is a coefficient system of degree $n \geq 0$ at $N \geq 0$ if:

- the functor $\kappa_x(F)$ has degree -1 at N ;
- the functor $\delta_x(F)$ has degree $n - 1$ at N .

Let us define a new type of strong polynomial functor, related to coefficient systems of finite degree.

Definition 2.17. Let $(\mathfrak{M}, \natural, 0)$ be an object of $\mathfrak{Mon}_{\text{ini}}^{\text{pb}}$, let \mathcal{A} be an abelian category. We define the category $\mathcal{V}Pol_n(\mathfrak{M}, \mathcal{A})$ of very strong polynomial functors of degree smaller or equal n to be the full subcategory of $\mathcal{P}ol_n(\mathfrak{M}, \mathcal{A})$ such that: a strong polynomial functor F is very strong polynomial if for all object x of \mathfrak{M} , $\kappa_x(F) = 0$.

Remark 2.18. A coefficient system in every object x of \mathfrak{M} , of degree n at $N = 0$ is this way a very strong polynomial functor.

Proposition 2.19. Let \mathfrak{M} be an object of $\mathfrak{Mon}_{\text{ini}}^{\text{pb}}$ and let \mathcal{A} be an abelian category. Then:

- the category $\mathcal{VPol}_n(\mathfrak{M}, \mathcal{A})$ is closed under the translation endofunctor τ_x ;
- the category $\mathcal{VPol}_n(\mathfrak{M}, \mathcal{A})$ is closed under kernels of epimorphisms;
- the category $\mathcal{VPol}_n(\mathfrak{M}, \mathcal{A})$ is closed under extension;
- assuming that there exists a set \mathfrak{E} of objects of \mathfrak{M} such that:

$$\forall m \in \text{Obj}(\mathfrak{M}), \exists \{e_i\}_{i \in I} \in \text{Obj}(\mathfrak{E}) \text{ (where } I \text{ is finite), } m \cong \bigsqcup_{i \in I} e_i,$$

then, an object F of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$ belongs to $\mathcal{VPol}_n(\mathfrak{M}, \mathcal{A})$ if and only if $\kappa_e(F) = 0$ and $\delta_e(F)$ is an object of $\mathcal{Pol}_{n-1}(\mathfrak{M}, \mathcal{A})$ for all object e of \mathfrak{E} .

Proof. The first point follows from the fact that for all object x of \mathfrak{M} , the endofunctor τ_x commutes with the endofunctors δ_x and κ_x (see Lemma 2.7). For the second and third point, let us consider two short exact sequences of $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$: $0 \rightarrow G \rightarrow F_1 \rightarrow F_2 \rightarrow 0$ and $0 \rightarrow F_3 \rightarrow H \rightarrow F_4 \rightarrow 0$ where F_i is a very strong polynomial functor of degree n for all i . Let x be an object of \mathfrak{M} . We use the long exact sequence 2 of Lemma 2.7 to obtain the two following exact sequences in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$:

$$0 \rightarrow \kappa_x(G) \rightarrow 0 \rightarrow 0 \rightarrow \delta_x(G) \rightarrow \delta_x(F_1) \rightarrow \delta_x(F_2) \rightarrow 0;$$

$$0 \rightarrow 0 \rightarrow \kappa_x(H) \rightarrow 0 \rightarrow \delta_x(F_3) \rightarrow \delta_x(H) \rightarrow \delta_x(F_4) \rightarrow 0.$$

Therefore, $\kappa_x(F) = \kappa_x(H) = 0$ and the result follows directly by induction on the degree of polynomiality n . For the last point, we consider the long exact sequence 3 of Lemma 2.7 applied to an object F of $\mathcal{VPol}_n(\mathfrak{M}, \mathcal{A})$ to obtain the following exact sequence in the category $\mathbf{Fct}(\mathfrak{M}, \mathcal{A})$:

$$0 \rightarrow \kappa_y(F) \rightarrow \kappa_{x \sqcup y}(F) \rightarrow \tau_x \kappa_y(F) \rightarrow \delta_y(F) \rightarrow \delta_{x \sqcup y}(F) \rightarrow \tau_y \delta_x(F) \rightarrow 0.$$

Hence, by induction on the length of objects as monoidal product of $\{e_i\}_{i \in I}$, we deduce that $\kappa_x(F) = 0$ for all object e of \mathfrak{M} if and only if $\kappa_e(F) = 0$ for all object e of \mathfrak{E} . Moreover, since $\mathcal{VPol}_n(\mathfrak{M}, \mathcal{A})$ is closed under extension and by the translation endofunctor τ_y , the result follows directly by induction on the degree of polynomiality n . \square

Remark 2.20. On the contrary of $\mathcal{Pol}_n(\mathfrak{M}, \mathcal{A})$, the category $\mathcal{VPol}_n(\mathfrak{M}, \mathcal{A})$ is not necessarily closed under the quotient, ie a quotient of an object F of $\mathcal{VPol}_n(\mathfrak{M}, \mathcal{A})$ is not necessarily a very strong polynomial functor.

Proposition 2.21. *Let \mathfrak{M} and \mathfrak{M}' be objects of $\mathfrak{Mon}^{\text{pb}}$ and let \mathcal{A} be an abelian category. Let F be an object of $\mathbf{Fct}(\mathfrak{Mon}_{\text{ini}}^{\text{pb}}, \mathcal{A})$. The functor F is an object of $\mathcal{VPol}_0(\mathfrak{M}, \mathcal{A})$ if and only if it is a constant functor.*

Proof. Using the long exact sequence 1 of Lemma 2.7 applied to F , we deduce that F is an object of $\mathcal{VPol}_0(\mathfrak{M}, \mathcal{A})$ if and only if $F \cong \tau_x F$ for all object x of \mathfrak{M} . \square

2.3 Examples of polynomial functors associated with braid representations

Different families of representations of braid group can be interpreted as very strong polynomial functors. First of all, let us focus on the unreduced Burau functor: it has been shaped from the family of the unreduced Burau representations (see [13, Section 3.1] or [6, Section 4.2] for more details about this family).

Example 2.22. Let $\mathfrak{Bur} : \mathcal{UB} \rightarrow \mathbb{K}\text{-Mod}$ be the functor defined by:

- Objects: $\forall n \in \mathbb{N}, \mathfrak{Bur}(n) := \mathbb{K}[t, t^{-1}]^{\oplus n}$.
- Morphisms:

- Automorphisms: for all natural integer $n \geq 2$, for every Artin generator σ_i of \mathbf{B}_n (with $i \in \{1, \dots, n-1\}$):

$$\mathfrak{Bur}(\sigma_i) := \begin{bmatrix} \overbrace{1 & 0}^{i-1} & & & \overbrace{0 & \dots & 0}^{n-i-1} \\ 0 & \ddots & 0 & \dots & \vdots \\ & \ddots & 1 & & \\ \vdots & & 0 & B(t) & 0 & \vdots \\ & & \vdots & & 1 & \ddots \\ & & & \dots & 0 & \ddots & 0 \\ 0 & \dots & 0 & & 0 & 1 \end{bmatrix}$$

where

$$B(t) := \begin{bmatrix} 1-t & t \\ 1 & 0 \end{bmatrix}.$$

- General morphisms: Let $n, n' \in \mathbb{N}$, such that $n' \geq n$, and $[n' - n, \sigma_i] \in \text{Hom}_{\mathfrak{U}\mathfrak{B}}(n, n')$. We define:

$$\mathfrak{Bur}([n' - n, \sigma_i]) := \mathfrak{Bur}(\sigma_i) \circ I_{\mathbb{K}[t, t^{-1}]^{\oplus n}}^{\mathbb{K}[t, t^{-1}]^{\oplus n'}}.$$

The functor \mathfrak{Bur} already appears in [21, Example 4.3 and 4.15]. We call this functor the unreduced Burau functor.

Remark 2.23. The canonical embedding $I_{\mathbb{K}[t, t^{-1}]^{\oplus n}}^{\mathbb{K}[t, t^{-1}]^{\oplus n'}}$ can be matricially described by:

$$I_{\mathbb{K}[t, t^{-1}]^{\oplus n}}^{\mathbb{K}[t, t^{-1}]^{\oplus n'}} := \begin{bmatrix} \overbrace{1 & 0}^n \\ 0 & \ddots & 0 \\ & \ddots & 1 \\ \vdots & & 0 \\ & & \vdots \\ 0 & \dots & 0 \end{bmatrix}.$$

The following example corresponds to the family of the reduced Burau representations.

Example 2.24. Let $\overline{\mathfrak{Bur}} : \mathfrak{U}\mathfrak{B} \rightarrow \mathbb{K}\text{-Mod}$ be the functor defined by:

- Objects: $\forall n \in \mathbb{N}^*$, $\overline{\mathfrak{Bur}}(n) := \mathbb{K}[t, t^{-1}]^{\oplus n-1}$ and $\overline{\mathfrak{Bur}}(0) := 0$.
- Morphisms:
 - Automorphisms: one assigns

$$\overline{\mathfrak{Bur}}(\sigma_1) := \begin{bmatrix} -t & 0 & \overbrace{0 & \dots & 0}^{n-3} \\ 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & & \\ \vdots & \vdots & Id_{n-3} & & \\ 0 & 0 & & & \end{bmatrix},$$

for all natural integer $n \geq 3$, for every Artin generator σ_i of \mathbf{B}_n with $i \in \{2, \dots, n-2\}$:

$$\overline{\mathfrak{Bur}}(\sigma_i) := \begin{bmatrix} \overbrace{1 & 0}^{i-2} & & & \overbrace{0 & \dots & 0}^{n-i-2} \\ 0 & \ddots & 0 & \dots & \vdots \\ & \ddots & 1 & & \\ \vdots & & 0 & \overline{B}(t) & 0 & \vdots \\ & & \vdots & & 1 & \ddots \\ 0 & \dots & 0 & \dots & 0 & \ddots & 0 \\ & & & & 0 & 1 \end{bmatrix}$$

where

$$\overline{B}(t) := \begin{bmatrix} 1 & t & 0 \\ 0 & -t & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

and

$$\overline{\mathfrak{Bur}}(\sigma_{n-1}) := \begin{bmatrix} \overbrace{\quad\quad\quad}^{n-3} & 0 & 0 \\ & \vdots & \vdots \\ & Id_{n-3} & \vdots & \vdots \\ & & \vdots & \vdots \\ & & 0 & 0 \\ 0 & \dots & 0 & 1 & t \\ 0 & \dots & 0 & 0 & -t \end{bmatrix}.$$

For $n = 2$, one assigns $\overline{\mathfrak{Bur}}(\sigma_1) := -t$.

– General morphisms: Let $n, n' \in \mathbb{N}$, such that $n' \geq n$, and $[n' - n, \sigma_i] \in Hom_{\mathfrak{U}\beta}(n, n')$. We define:

$$\overline{\mathfrak{Bur}}([n' - n, \sigma_i]) := \overline{\mathfrak{Bur}}(\sigma_i) \circ I_{\mathbb{K}[t, t^{-1}]^{\oplus n-1}}^{\mathbb{K}[t, t^{-1}]^{\oplus n'-1}}.$$

We call this functor the reduced Burau functor. The assignment of $\overline{\mathfrak{Bur}}$ defines a functor since $\overline{\mathfrak{Bur}}(id_n) = id_{\mathbb{K}[t, t^{-1}]^{\oplus n-1}}$ and for $i, j \in \{1, \dots, n-1\}$, such that $j \geq i$, by our definition $\overline{\mathfrak{Bur}}(\sigma_j \circ \sigma_i) = \overline{\mathfrak{Bur}}(\sigma_j) \circ \overline{\mathfrak{Bur}}(\sigma_i)$ (Remark (1.36) ensures that assignment of $\mathfrak{T}\mathfrak{Y}\mathfrak{M}$ defines properly a functor on $\mathfrak{U}\beta$).

Another example is based on the family introduced by Tong, Yang and Ma in [22].

Example 2.25. Let $\mathfrak{T}\mathfrak{Y}\mathfrak{M} : \mathfrak{U}\beta \longrightarrow \mathbb{K}\text{-Mod}$ be the functor defined by:

- Objects: $\forall n \in \mathbb{N}$, $\mathfrak{T}\mathfrak{Y}\mathfrak{M}(n) := \mathbb{K}[t, t^{-1}]^{\oplus n}$.
- Morphisms:

– Automorphisms: for all interger $n \geq 2$, for every Artin generator σ_i of \mathbf{B}_n (with $i \in \{1, \dots, n-1\}$):

$$\mathfrak{T}\mathfrak{Y}\mathfrak{M}(\sigma_i) := \begin{bmatrix} \overbrace{1 & 0}^{i-1} & & & \overbrace{0 & \dots & 0}^{n-i-1} \\ 0 & \ddots & 0 & \dots & \vdots \\ & \ddots & 1 & & \\ \vdots & & 0 & TYM(t) & 0 & \vdots \\ & & \vdots & & 1 & \ddots \\ 0 & \dots & 0 & \dots & 0 & \ddots & 0 \\ & & & & 0 & 1 \end{bmatrix}$$

where

$$TYM(t) := \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix}.$$

– General morphisms: let $n, n' \in \mathbb{N}$, such that $n' \geq n$, and $[n' - n, \sigma] \in Hom_{\mathcal{U}\beta}(n, n')$. We define:

$$\mathfrak{TYM}([n' - n, \sigma]) := \mathfrak{TYM}(\sigma) \circ I_{\mathbb{K}[t, t^{-1}]^{\oplus n}}^{\mathbb{K}[t, t^{-1}]^{\oplus n'}}.$$

We call this functor the Tong-Yang-Ma functor. The assignment of \mathfrak{TYM} defines a functor since $\mathfrak{TYM}(id_n) = id_{\mathbb{K}[t, t^{-1}]^{\oplus n}}$ and for $i, j \in \{1, \dots, n-1\}$, such that $j \geq i$, by our definition $\mathfrak{TYM}(\sigma_j \circ \sigma_i) = \mathfrak{TYM}(\sigma_j) \circ \mathfrak{TYM}(\sigma_i)$ (Remark (1.36) ensuring that assignment of \mathfrak{TYM} defines properly a functor on $\mathcal{U}\beta$).

A link between the unreduced Burau and the reduced Burau functors can be established: the functor $\overline{\mathfrak{Bur}}$ is in fact a subfunctor of the functor \mathfrak{Bur} .

Definition 2.26. Let $\mathcal{T}_1 : \mathcal{U}\beta \rightarrow \mathbb{K}\text{-Mod}$ be the constant functor defined by:

- Objects: $\forall n \in \mathbb{N}^*$, $\mathcal{T}_1(n) := \mathbb{K}[t, t^{-1}]$ and $\mathcal{T}_1(0) := 0$.
- Morphisms: for all element b of \mathbf{B}_n , $\mathcal{T}_1(b) := id_{\mathbb{K}}$.

we define the translation on the right by $x \in \mathbb{N}$ functor $\tau_x^r : \mathbf{Fct}(\mathcal{U}\beta, \mathbb{K}\text{-Mod}) \rightarrow \mathbf{Fct}(\mathcal{U}\beta, \mathbb{K}\text{-Mod})$ to be the endofunctor of $\mathbf{Fct}(\mathcal{U}\beta, \mathbb{K}\text{-Mod})$ obtained by precomposition by the functor $id_{\mathfrak{M}} \natural x$.

Remark 2.27. For all element σ of \mathbf{B}_n , $\tau_1^r(\sigma) = \gamma_n^{b,p}(\sigma)$.

Proposition 2.28. The following short sequence of $\mathbf{Fct}(\mathcal{U}\beta, \mathbb{K}\text{-Mod})$ is exact.

$$0 \longrightarrow \overline{\mathfrak{Bur}} \xrightarrow{r} \mathfrak{Bur} \longrightarrow \mathcal{T}_1 \longrightarrow 0$$

It does not split. Furthermore, we have the equivalence of $\mathcal{U}\beta$ -modules:

$$\tau_1^r \mathfrak{Bur} \cong \tau_1^r \overline{\mathfrak{Bur}} \oplus \mathcal{T}_1.$$

Proof. Since $\gamma_{n,1}^b$ is a group morphism, it respects composition, and a fortiori it is clear that $\tau_1^r : \mathcal{U}\beta \rightarrow \mathcal{U}\beta$ is an endofunctor. Considering the objects, we know that $\mathfrak{Bur}(n) = \mathbb{K}[t, t^{-1}]^{\oplus n} = \tau_1^r \overline{\mathfrak{Bur}}(n)$ for all natural integer n . The assingment for all natural integer n :

$$r_n := \begin{bmatrix} \overbrace{1 & -1 & 0 & \dots & 0}^n \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & -1 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix} \in GL\left(\mathbb{K}[t, t^{-1}]^{\oplus n}\right)$$

defines the natural transformation $r : \mathfrak{Bur} \Rightarrow \tau_1^r \overline{\mathfrak{Bur}}$. Indeed, for all natural integer n , it suffices (by Remark (1.36)) to verify that for all generator σ_i of \mathbf{B}_n :

$$r_n \mathfrak{Bur}(\sigma_i) r_n^{-1} = \begin{bmatrix} \overbrace{1 & 0}^{i-2} & & \overbrace{0 & \dots & 0}^{n-i-1} \\ 0 & \ddots & 0 & \dots & \vdots \\ & \ddots & 1 & & \\ \vdots & & 0 & \overline{B}(t) & 0 & \vdots \\ & & \vdots & 1 & \ddots & \\ 0 & \dots & 0 & \dots & 0 & \ddots & 0 \\ & & & & 0 & 1 \end{bmatrix} = \overline{\mathfrak{Bur}}(\gamma_{n,1}^b(\sigma_i)) = \tau_1^r \overline{\mathfrak{Bur}}(\sigma_i).$$

It is a natural equivalence with inverse given for all natural integer n by:

$$r_n^{-1} := \overbrace{\begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 1 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}}^n \in GL\left(\mathbb{K}[t, t^{-1}]^{\oplus n}\right).$$

Hence, we have an equivalence $\mathbf{Bur} \xrightarrow{r} \tau_1^r \overline{\mathbf{Bur}}$. Furthermore, one defines a natural transformation $r' : \overline{\mathbf{Bur}} \Rightarrow \tau_1^r \overline{\mathbf{Bur}}$ by assigning for all natural integer n , $r'_n = I_{\mathbb{K}[t, t^{-1}]^{\oplus n-1}}^{\mathbb{K}[t, t^{-1}]^{\oplus n}} : \mathbb{K}[t, t^{-1}]^{\oplus n-1} \hookrightarrow \mathbb{K}[t, t^{-1}]^{\oplus n}$ which is a monomorphism. Indeed, for all natural integer n , it suffices (by Remark (1.36)) to verify that for all generator σ_i of \mathbf{B}_n :

$$\begin{aligned} (\tau_1^r \overline{\mathbf{Bur}}(\sigma_i)) \circ r'_n &= \begin{bmatrix} \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & \ddots & 0 \\ & \ddots & 1 \\ \vdots & & 0 \\ & \vdots & \\ 0 & \cdots & 0 \end{bmatrix}}^{i-2} & & \overbrace{\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & & \\ 1 & \ddots & \\ 0 & \ddots & 0 \end{bmatrix}}^{n-i-1} \\ & \overline{B}(t) & \end{bmatrix} \\ &= \begin{bmatrix} \overbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}}^{n-1} & \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & \ddots & 0 \\ & \ddots & 1 \\ \vdots & & 0 \\ & \vdots & \\ 0 & \cdots & 0 \end{bmatrix}}^{i-2} & \overbrace{\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & & \\ 1 & \ddots & \\ 0 & \ddots & 0 \\ 0 & 1 \end{bmatrix}}^{n-i-2} \\ & \overline{B}(t) & \end{bmatrix} \\ &= r'_n \circ \overline{\mathbf{Bur}}(\sigma_i). \end{aligned}$$

Repeating mutatis mutandis the work done in the proof of Proposition 2.29, we conclude that for all natural integer n we have a short exact sequence, natural in n :

$$0 \longrightarrow \overline{\mathbf{Bur}}(n) \longrightarrow \tau_1^r \overline{\mathbf{Bur}}(n) \longrightarrow \mathcal{T}_1(n) \longrightarrow 0.$$

Hence, because of the naturality in n and the first equivalence proved, we deduce the first result of the

proposition. Moreover, for all generator σ_i of \mathbf{B}_n :

$$\begin{aligned} \tau_1^r \mathfrak{Bur}(\sigma_i) &= \begin{bmatrix} \overbrace{\begin{matrix} 1 & 0 \\ 0 & \ddots & 0 \end{matrix}}^{i-1} & & \overbrace{\begin{matrix} 0 & \cdots & 0 \end{matrix}}^{n-i} \\ & \ddots & \\ & & B(t) & 0 & \vdots \\ & & & 1 & \ddots \\ & & & 0 & \ddots & 0 \\ 0 & \cdots & 0 & & 0 & 1 \end{bmatrix} = \begin{bmatrix} \overbrace{\quad\quad\quad}^n & 0 \\ & \vdots \\ \mathfrak{Bur}(\sigma_i) & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix} \\ &= \mathfrak{Bur}(\sigma_i) \oplus \mathcal{T}_1(\sigma_i). \end{aligned}$$

The verification for the objects being trivial and because of Remark (1.36), we directly deduce that we have an equality $\tau_1^r \mathfrak{Bur} = \mathfrak{Bur} \oplus \mathcal{T}_1$. Therefore, we obtain the desired equivalence:

$$\tau_1^r \mathfrak{Bur} = \mathfrak{Bur} \oplus \mathcal{T}_1 \cong \tau_1^r \overline{\mathfrak{Bur}} \oplus \mathcal{T}_1.$$

□

Proposition 2.29. *The functors \mathfrak{Bur} , $\overline{\mathfrak{Bur}}$ and \mathfrak{IYM} are very strong polynomial functors of degree 1.*

Proof. For the functor \mathfrak{Bur} , it is a consequence of [21, Example 4.15]. According to Proposition 2.28, the functor $\overline{\mathfrak{Bur}}$ is a subfunctor of \mathfrak{Bur} . The functor $\overline{\mathfrak{Bur}}$ is obviously not constant, and a fortiori not a very strong polynomial functor of degree 0 by Proposition 2.21. Hence, the result follows from the fact that the category $\mathcal{VPol}_1(\mathcal{U}\mathcal{B}, \mathbb{K}\text{-Mod})$ is closed under kernels of epimorphisms (see Proposition 2.19). So we will focus on the case of the functor \mathfrak{IYM} . Let n be a natural integer. By the statement of Example 2.12, it is sufficient to consider the application $i_1 \mathfrak{IYM}([0, id_n]) = I_{\mathbb{K}[t, t^{-1}]^{\oplus n}}^{\mathbb{K}[t, t^{-1}]^{\oplus 1+n}}$. This map is a monomorphism and its cokernel is $\mathbb{K}[t, t^{-1}]$. So $\kappa_1 \mathfrak{IYM} = 0$ is the null functor of $\mathbf{Fct}(\mathcal{U}\mathcal{B}, \mathbb{K}\text{-Mod})$. Let n' be a natural integer such that $n' \geq n$ and let $\varphi_n^{n'} = [n' - n, \phi] \in Hom_{\mathcal{U}\mathcal{B}}(n, n')$. By naturality and the universal property of the cokernel, there exists a unique endomorphism of $\mathbb{K}[t, t^{-1}]$ such that the following diagram commutes, where the lines are exact. It is exactly the definition of $\delta_1 \mathfrak{IYM}([n' - n, \phi])$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{K}[t, t^{-1}]^n & \xrightarrow{I_{\mathbb{K}[t, t^{-1}]^{\oplus n}}^{\mathbb{K}[t, t^{-1}]^{\oplus 1+n}}} & \mathbb{K}[t, t^{-1}]^{\oplus 1+n} & \xrightarrow{\pi_{1+n}} & \mathbb{K}[t, t^{-1}] \longrightarrow 0 \\ & & \downarrow \mathfrak{IYM}(\varphi_n^{n'}) & & \downarrow \tau_1(\mathfrak{IYM})(\varphi_n^{n'}) & & \downarrow \exists! \\ 0 & \longrightarrow & \mathbb{K}[t, t^{-1}]^{\oplus n'} & \xrightarrow{I_{\mathbb{K}[t, t^{-1}]^{\oplus n'}}^{\mathbb{K}[t, t^{-1}]^{\oplus 1+n}}} & \mathbb{K}[t, t^{-1}]^{\oplus 1+n'} & \xrightarrow{\pi_{1+n'}} & \mathbb{K}[t, t^{-1}] \longrightarrow 0 \end{array}$$

For all $(a, b) \in \mathbb{K}[t, t^{-1}] \oplus \mathbb{K}[t, t^{-1}]^{\oplus n} = \mathbb{K}[t, t^{-1}]^{\oplus 1+n}$:

$$\begin{aligned} \tau_1(\mathfrak{IYM})(\varphi_n^{n'})(a, b) &= \left(\mathfrak{IYM}(id_1 \natural \phi) \circ I_{\mathbb{K}[t, t^{-1}]^{\oplus 1+n}}^{\mathbb{K}[t, t^{-1}]^{\oplus 1+n'}} \right)(a, b) \\ &= (id_{\mathbb{K}[t, t^{-1}]} \oplus \mathfrak{IYM}(\phi)) \circ I_{\mathbb{K}[t, t^{-1}]^{\oplus 1+n}}^{\mathbb{K}[t, t^{-1}]^{\oplus 1+n'}}(a, b) \\ &= (a, \mathfrak{IYM}([n' - n, \phi])(b)). \end{aligned}$$

Therefore:

$$(\pi_{1+n'} \circ \tau_1(\mathfrak{IYM})(\varphi_n^{n'}))(a, b) = a = \pi_{1+n}(a, b).$$

Hence, $id_{\mathbb{K}[t,t^{-1}]}$ also makes the diagram commutative and thus $\delta_1 \mathfrak{T}\mathfrak{Y}\mathfrak{M}([n' - n, \phi]) = id_{\mathbb{K}[t,t^{-1}]}$. Hence, $\delta_1 \mathfrak{T}\mathfrak{Y}\mathfrak{M}$ is the constant functor $\mathcal{T}_1 : \mathfrak{U}\beta \rightarrow \mathbb{K}\text{-Mod}$. A fortiori, because of Proposition 2.13, $\delta_1 \mathfrak{T}\mathfrak{Y}\mathfrak{M}$ is a degree 0 very strong polynomial functor. \square

A last example is given by the family of the Lawrence-Krammer representations (see [2, 15, 14]).

Example 2.30. Let $\mathfrak{L}\mathfrak{K} : \mathfrak{U}\beta \rightarrow \mathbb{K}\text{-Mod}$ be the functor defined by:

- Objects: for all natural integer $n \geq 2$, $\mathfrak{L}\mathfrak{K}(n) := \bigoplus_{1 \leq j < k \leq n} V_{j,k}$, where for all $1 \leq j < k \leq n$, $V_{j,k}$ is a free $\mathbb{K}[t^{\pm 1}, q^{\pm 1}]$ -module of rank one. Hence, $\mathfrak{L}\mathfrak{K}(n) \cong (\mathbb{K}[t^{\pm 1}, q^{\pm 1}])^{\oplus n(n-1)/2}$ as $\mathbb{K}[t^{\pm 1}, q^{\pm 1}]$ -modules. Moreover, oen assigns $\mathfrak{L}\mathfrak{K}(1) := 0$ and $\mathfrak{L}\mathfrak{K}(0) := 0$.
- Morphisms:
 - Automorphisms: for all natural integer n , for every Artin generator σ_i of \mathbf{B}_n (with $i \in \{1, \dots, n-1\}$), for all $v_{j,k} \in V_{j,k}$ (with $1 \leq j < k \leq n$),

$$\mathfrak{L}\mathfrak{K}(\sigma_i) v_{j,k} := \begin{cases} v_{j,k} & \text{if } i \notin \{j-1, j, k-1, k\}, \\ qv_{i,k} + (q^2 - q)v_{i,j} + (1-q)v_{j,k} & \text{if } i = j-1, \\ v_{j+1,k} & \text{if } i = j \neq k-1, \\ qv_{j,i} + (1-q)v_{j,k} + (q^2 - q)tv_{i,k} & \text{if } i = k-1 \neq j, \\ v_{j,k+1} & \text{if } i = k, \\ -tq^2v_{j,k} & \text{if } i = j = k-1. \end{cases}$$

- General morphisms: let $n, n' \in \mathbb{N}$, such that $n' \geq n$, and $[n' - n, \sigma] \in Hom_{\mathfrak{U}\beta}(n, n')$. We define:

$$\mathfrak{L}\mathfrak{K}([n' - n, \sigma]) := \mathfrak{L}\mathfrak{K}(\sigma) \circ I_{\mathbb{K}[t,t^{-1}]^{\oplus n(n-1)/2} \rightarrow \mathbb{K}[t,t^{-1}]^{\oplus n'(n'-1)/2}}.$$

Proposition 2.31. *The functor $\mathfrak{L}\mathfrak{K}$ is a very strong polynomial functor of degree 2.*

Proof. Let n be a natural integer. By the statement of Example 2.12, we only have to consider the application $i_1 \mathfrak{L}\mathfrak{K}([0, id_n])$. This map is clearly a monomorphism and its cokernel is $\bigoplus_{1 \leq i \leq n} V_{i,n+1}$. It is easy to conclude that $\kappa_1 \mathfrak{L}\mathfrak{K}$ is the null constant functor of $\mathbf{Fct}(\mathfrak{U}\beta, \mathbb{K}\text{-Mod})$. Let n' be another natural integer (we can suppose that $n' \geq n$). Let $\varphi_n^{n'} = [n' - n, \phi] \in Hom_{\mathfrak{U}\beta}(n, n')$. By naturality and because of the universal property of the cokernel, there exists a unique endomorphism of $\mathbb{K}[t^{\pm 1}, q^{\pm 1}]$ such that the following diagram commutes, where the lines are exact. It is exactly the definition of $\delta_1 \mathfrak{L}\mathfrak{K}([n' - n, \phi])$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{1 \leq j < k \leq n} V_{j,k} & \xrightarrow{i_1 \mathfrak{L}\mathfrak{K}([0, id_n])} & \bigoplus_{1 \leq i < l \leq n+1} V_{i,l} & \xrightarrow{\pi_n} & \bigoplus_{1 \leq i \leq n} V_{i,n+1} \longrightarrow 0 \\ & & \downarrow \mathfrak{L}\mathfrak{K}(\varphi_n^{n'}) & & \downarrow \tau_1(\mathfrak{L}\mathfrak{K})(\varphi_n^{n'}) & & \downarrow \exists! \\ 0 & \longrightarrow & \bigoplus_{1 \leq j' < k' \leq n'} V_{j',k'} & \xrightarrow{i_1 \mathfrak{L}\mathfrak{K}([0, id_{n'}])} & \bigoplus_{1 \leq i' < l' \leq n'+1} V_{i',l'} & \xrightarrow{\pi_{n'}} & \bigoplus_{1 \leq i' \leq n'} V_{i',n'+1} \longrightarrow 0 \end{array}$$

Let $i \in \{1, \dots, n-1\}$, let $j \in \{1, \dots, n-1\}$ and let $v_{j,n+1}$ be an element of $V_{j,n+1}$. Then we compute:

$$\mathfrak{L}\mathfrak{K}(\sigma_i) v_{j,n+1} = \begin{cases} v_{j,n+1} & \text{if } i \notin \{j-1, j\}, \\ qv_{j-1,n+1} + (q^2 - q)v_{j-1,j} + (1-q)v_{j,n+1} & \text{if } i = j-1, \\ v_{j+1,n+1} & \text{if } i = j \neq n. \end{cases}$$

We deduce that in the canonical basis $\{\mathbf{e}_{1,n+1}, \mathbf{e}_{2,n+1}, \dots, \mathbf{e}_{n,n+1}\}$ of $\bigoplus_{1 \leq i \leq n} V_{i,n+1}$.

$$\delta_1 \mathfrak{L}\mathfrak{K}(\sigma_i) = \begin{bmatrix} \overbrace{1 & 0}^{i-1} & & & \overbrace{0 & \dots & 0}^{n-i-1} \\ 0 & \ddots & 0 & \dots & \vdots \\ & \ddots & 1 & & \\ \vdots & & 0 & \tilde{B}(t) & 0 & \vdots \\ & & \vdots & & 1 & \ddots \\ & & & \dots & 0 & \ddots & 0 \\ 0 & \dots & 0 & & & 0 & 1 \end{bmatrix}$$

where:

$$\tilde{B}(t) = \begin{bmatrix} 0 & 1 \\ q & 1-q \end{bmatrix}.$$

Then, we define the natural transformation $\delta_1 \mathfrak{L}\mathfrak{K} \xrightarrow{r''} \widetilde{\mathfrak{B}\mathfrak{u}\mathfrak{r}}$ (where $\widetilde{\mathfrak{B}\mathfrak{u}\mathfrak{r}}$ is defined as $\mathfrak{B}\mathfrak{u}\mathfrak{r}$ for morphisms and for objects $\widetilde{\mathfrak{B}\mathfrak{u}\mathfrak{r}}(n) = (\mathbb{K}[t^{\pm 1}, q^{\pm 1}])^{\oplus n}$) with for all integer n , $\bigoplus_{1 \leq i \leq n} V_{i,n+1} \cong (\mathbb{K}[t^{\pm 1}, q^{\pm 1}])^{\oplus n} \xrightarrow{r''_n} (\mathbb{K}[t^{\pm 1}])^{\oplus n}$

being the matrix $\begin{bmatrix} \overbrace{0 & \dots & 0 & 1}^n \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix}$, which is invertible. Hence, we have the equivalence: $\delta_1 \mathfrak{L}\mathfrak{K} \cong \widetilde{\mathfrak{B}\mathfrak{u}\mathfrak{r}}$.

Since the functor $\mathfrak{B}\mathfrak{u}\mathfrak{r}$ is strong polynomial of degree one (see Proposition (2.29)), we deduce that $\mathfrak{L}\mathfrak{K}$ is strong polynomial of degree two. \square

3 Functoriality of the Long-Moody construction

We recall that for all natural integer n , we are given four key group morphisms: $a_{n,\bullet} : \mathbf{B}_n \hookrightarrow \text{Aut}(\mathbf{F}_n)$, $\varsigma_{n,\bullet} : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}$, $\gamma_{n,n'}^{b,\bullet} : \mathbf{B}_n \hookrightarrow \mathbf{B}_{n'}$ and $\gamma_{n,n'}^{f,\bullet} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n'}$ (when $n \leq n'$ and with abbreviations $\gamma_n^{b,\bullet}$ and $\gamma_n^{f,\bullet}$ when $n' = n+1$).

3.1 Coherent morphisms

For all natural integer n , we have defined four key group morphisms: $a_{n,\bullet} : \mathbf{B}_n \hookrightarrow \text{Aut}(\mathbf{F}_n)$, $\varsigma_{n,\bullet} : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}$, $\gamma_{n,n'}^{b,\bullet} : \mathbf{B}_n \hookrightarrow \mathbf{B}_{n'}$ and $\gamma_{n,n'}^{f,\bullet} : \mathbf{F}_n \hookrightarrow \mathbf{F}_{n'}$ (when $n \leq n'$) (with abbreviations $\gamma_n^{b,\bullet}$ and $\gamma_n^{f,\bullet}$ when $n' = n+1$). We will need the following coherence conditions for our constructions.

Remark 3.1. For all natural integers n and n' such that $n' \geq n$, the morphism $\gamma_{n,n'}^{f,\bullet} : \mathbf{F}_n \rightarrow \mathbf{F}_{n'}$ admits a left inverse $\overline{\gamma}_{n,n'}^{f,\bullet} : \mathbf{F}_{n'} \rightarrow \mathbf{F}_n$, ie there exists a family of morphisms $\{\overline{\gamma}_{n,n'}^{f,\bullet} : \mathbf{F}_{n'} \rightarrow \mathbf{F}_n\}_{n \in \mathbb{N}}$ such that for all element f of \mathbf{F}_n , $\overline{\gamma}_{n,n'}^{f,\bullet} \circ \gamma_{n,n'}^{f,\bullet}(f) = f$.

Condition 3.2. For all natural integer n , we require $\varsigma_{n+1,\bullet} \circ \gamma_n^{f,\bullet} = \gamma_{n+1}^{b,\bullet} \circ \varsigma_{n,\bullet}$, ie the following diagram is

commutative.

$$\begin{array}{ccc} \mathbf{B}_{n+1} & \xrightarrow{\gamma_{n+1}^{b,\bullet}} & \mathbf{B}_{n+2} \\ \uparrow \varsigma_{n,\bullet} & & \uparrow \varsigma_{n+1,\bullet} \\ \mathbf{F}_n & \xrightarrow{\gamma_n^{f,\bullet}} & \mathbf{F}_{n+1} \end{array}$$

Condition 3.3. For all natural integers n and n' such that $n' \geq n$, we require $(id_{(n'-n)} \lrcorner -) \circ \gamma_n^{b,\bullet} = \gamma_{n'}^{b,\bullet} \circ (id_{(n'-n)} \lrcorner -)$, ie the following diagram is commutative. We denote by $(id_{(n'-n)} \lrcorner -) : \mathbf{B}_n \rightarrow \mathbf{B}_{n'}$ the morphism which for an element σ of \mathbf{B}_n gives the element $id_{(n'-n)} \lrcorner \sigma$ of $\mathbf{B}_{n'}$.

$$\begin{array}{ccc} \mathbf{B}_n & \xrightarrow{\gamma_n^{b,\bullet}} & \mathbf{B}_{n+1} \\ id_{(n'-n)} \lrcorner - \downarrow & & \downarrow id_{(n'-n)} \lrcorner - \\ \mathbf{B}_{n'} & \xrightarrow{\gamma_{n'}^{b,\bullet}} & \mathbf{B}_{n'+1} \end{array}$$

Condition 3.4. For all natural integers n and n' such that $n' \geq n$, we require $\gamma_{n,n'}^{f,\bullet} \circ (a_{n,\bullet}(-)) = (a_{n',\bullet}(id_{n'-n} \lrcorner -)) \circ \gamma_{n,n'}^{f,\bullet}$ as elements of $Hom_{Grp}(\mathbf{B}_n, Hom_{\mathfrak{Gr}}(\mathbf{F}_n, \mathbf{F}_{n'}))$, ie for all element σ of \mathbf{B}_n the following diagram is commutative.

$$\begin{array}{ccccc} & & \mathbf{F}_n & & \\ & \nearrow a_{n,\bullet}(\sigma) & & \searrow \gamma_{n,n'}^{f,\bullet} & \\ \mathbf{F}_n & & & & \mathbf{F}_{n'} \\ & \searrow \gamma_{n,n'}^{f,\bullet} & & \nearrow a_{n',\bullet}(id_{n'-n} \lrcorner \sigma) & \\ & & \mathbf{F}_{n'} & & \end{array}$$

Condition 3.5. For all natural integers n and n' such that $n' \geq n$, we require $a_{n',\bullet}(id_{n'-n} \lrcorner -)$ is the identity element of $Hom_{Grp}(\mathbf{B}_n, Hom_{Grp}(\mathbf{F}_{n'-n}, \mathbf{F}_{n'}))$, ie the following diagram is commutative. We denote by $id_{Aut(\mathbf{F}_{n'})} : \mathbf{B}_n \rightarrow Aut(\mathbf{F}_{n'})$ the trivial action of \mathbf{B}_n on $Aut(\mathbf{F}_{n'})$ and by $(\gamma_{n'-n,n'}^{f,\bullet})^* : Aut(\mathbf{F}_{n'}) \rightarrow Hom_{\mathfrak{Gr}}(\mathbf{F}_{n'-n}, \mathbf{F}_{n'})$ the morphism which acts by pre-composition by $\gamma_{n'-n,n'}^{f,\bullet}$.

$$\begin{array}{ccccc} \mathbf{B}_n & \xrightarrow{id_{Aut(\mathbf{F}_{n'})}} & Aut(\mathbf{F}_{n'}) & \xrightarrow{(\gamma_{n'-n,n'}^{f,\bullet})^*} & Hom_{\mathfrak{Gr}}(\mathbf{F}_{n'-n}, \mathbf{F}_{n'}) \\ id_{(n'-n)} \lrcorner - \downarrow & & & & \uparrow \\ \mathbf{B}_{n'} & \xrightarrow{a_{n',\bullet}} & Aut(\mathbf{F}_{n'}) & \xrightarrow{(\gamma_{n'-n,n'}^{f,\bullet})^*} & Hom_{\mathfrak{Gr}}(\mathbf{F}_{n'-n}, \mathbf{F}_{n'}) \end{array}$$

In other words, for all element σ of \mathbf{B}_n , $a_{n',\bullet}(id_{n'-n} \lrcorner \sigma) \circ \gamma_{n'-n,n'}^{f,\bullet} = \gamma_{n'-n,n'}^{f,\bullet}$.

Condition 3.6. For all natural integers m, n and n' such that $n' \geq n$, we require $a_{m+n',\bullet}(id_m \lrcorner -) \circ \gamma_{n,m+n'}^{f,\bullet} = \gamma_{n',m+n'}^{f,\bullet} \circ a_{n',\bullet}(-) \circ \gamma_{n,n'}^{f,\bullet}$ as elements of $Hom_{\mathfrak{Gr}}(\mathbf{B}_{n'}, Hom_{\mathfrak{Gr}}(\mathbf{F}_n, \mathbf{F}_{m+n'}))$, ie the following diagram is commutative. We denote by $(\gamma_{m,m+n'}^{f,\bullet})_* : Aut(\mathbf{F}_{n'}) \rightarrow Hom_{\mathfrak{Gr}}(\mathbf{F}_{n'}, \mathbf{F}_{m+n'})$ the morphism which acts by post-composition by $\gamma_{m,m+n'}^{f,\bullet}$.

$$\begin{array}{ccccc}
\mathbf{B}_{n'} & \xrightarrow{a_{n', \bullet}} & \text{Aut}(\mathbf{F}_{n'}) & \xrightarrow{(\gamma_{n', m+n'}^{f, \bullet})^*} & \text{Hom}_{\mathfrak{S}_{\mathbf{r}}}(\mathbf{F}_{n'}, \mathbf{F}_{m+n'}) \\
\downarrow \text{id}_m \wr - & & & & \searrow (\gamma_{n, n'}^{f, \bullet})^* \\
\mathbf{B}_{m+n'} & \xrightarrow{a_{m+n', \bullet}} & \text{Aut}(\mathbf{F}_{m+n'}) & \xrightarrow{(\gamma_{n, m+n'}^{f, \bullet})^*} & \text{Hom}_{\mathfrak{S}_{\mathbf{r}}}(\mathbf{F}_n, \mathbf{F}_{m+n'})
\end{array}$$

In other words, for all element σ of $\mathbf{B}_{n'}$, $a_{m+n', \bullet}(\text{id}_m \wr \sigma) \circ \gamma_{n, m+n'}^{f, \bullet} = \gamma_{n', m+n'}^{f, \bullet} \circ a_{n', \bullet}(\sigma) \circ \gamma_{n, n'}^{f, \bullet}$.

For the further constructions, we will have to make relevant choices for these four morphisms: the characterization of the choice will be encoded by the quadruplet $(\bullet_a, \bullet_{\varsigma}, \bullet_{\gamma^b}, \bullet_{\gamma^f})$.

Definition 3.7. A quadruplet $(\bullet_a, \bullet_{\varsigma}, \bullet_{\gamma^b}, \bullet_{\gamma^f})$ is said to be coherent if it satisfies the conditions 3.2, 3.3, 3.4, 3.5 and 3.6.

3.2 Review of the Long-Moody construction

We present here the Long-Moody construction for representations of braid groups, which appears in the following theorem, corresponding to Theorem 2.1 of [18]. It is worth noting that for $\rho : \mathbf{B}_{n+1} \rightarrow GL(V)$ a representation of \mathbf{B}_{n+1} , with V a \mathbb{K} -module, ρ induces representations $\rho \circ \gamma_n^{b, \bullet} : \mathbf{B}_n \rightarrow GL(V)$ and $\rho \circ \varsigma_n, \bullet : \mathbf{F}_n \rightarrow GL(V)$. As a consequence, V is naturally equipped with a structure of $\mathbb{K}[\mathbf{B}_n]$ -module and of $\mathbb{K}[\mathbf{F}_n]$ -module.

Theorem 3.8. [18] Let $\rho : \mathbf{B}_{n+1} \rightarrow GL(V)$ be a representation of \mathbf{B}_{n+1} , where V is a \mathbb{K} -module, and let $(\bullet_a, \bullet_{\varsigma}, \bullet_{\gamma^b}, \bullet_{\gamma^f})$ be a coherent quadruplet. The Long-Moody construction associated with ρ , which is defined as follows, is a representation of \mathbf{B}_n :

$$lm_{(\bullet_a, \bullet_{\varsigma}, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(\rho) : \mathbf{B}_n \longrightarrow GL\left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} V\right)$$

where for all $\sigma \in \mathbf{B}_n$, $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v \in V$:

$$lm_{(\bullet_a, \bullet_{\varsigma}, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(\rho)(\sigma) \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) := a_{n, \bullet}(\sigma)(i) \otimes_{\mathbb{K}[\mathbf{F}_n]} \rho(\gamma_n^{b, \bullet}(\sigma))(v).$$

Notation 3.9. When there is no ambiguity, once the quadruplet $(\bullet_a, \bullet_{\varsigma}, \bullet_{\gamma^b}, \bullet_{\gamma^f})$ is clearly given, we forget it in the notation $lm_{(\bullet_a, \bullet_{\varsigma}, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$ for convenience (especially for proofs).

Proof. Let $\sigma, \sigma' \in \mathbf{B}_n$. For all elements v of V and i of $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$, since ρ is a representation of \mathbf{B}_{n+1} and the maps $a_{n, \bullet} : \mathbf{B}_n \hookrightarrow \text{Aut}(\mathbf{F}_n)$ and $\gamma_n^{b, \bullet} : \mathbf{B}_n \hookrightarrow \mathbf{B}_{n+1}$ are group morphisms, we obtain that:

$$\begin{aligned}
lm(\rho)(\sigma\sigma') \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) &= a_{n, \bullet}(\sigma\sigma')(i) \otimes_{\mathbb{K}[\mathbf{F}_n]} \rho(\gamma_n^{b, \bullet}(\sigma\sigma')) \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) \\
&= (a_{n, \bullet}(\sigma) a_{n, \bullet}(\sigma'))(i) \otimes_{\mathbb{K}[\mathbf{F}_n]} (\rho(\gamma_n^{b, \bullet}(\sigma)) \rho(\gamma_n^{b, \bullet}(\sigma'))) \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) \\
&= lm(\rho)(\sigma) \left(lm(\rho)(\sigma') \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) \right).
\end{aligned}$$

Hence $lm(\rho)(\sigma\sigma') = lm(\rho)(\sigma) lm(\rho)(\sigma')$. □

Remark 3.10. For all $k \in \{1, \dots, n\}$, we denote $V_k := \mathbb{K}[(g_k - 1)] \otimes_{\mathbb{K}[\mathbf{F}_n]} V$. The action of \mathbf{B}_n on \mathbf{F}_n extends naturally on $\mathbb{K}[\mathbf{F}_n]$. Then $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} V$ is isomorphic to $V^{\oplus n}$. Indeed, we have an identification isomorphism

$\Lambda : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} V \longrightarrow \bigoplus_{k=1}^n V_k \cong V^{\oplus n}$ defined by:

$$\forall v \in V, \forall k \in \{1, \dots, n\}, \Lambda \left((g_k - 1) \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) := \left(0, \dots, 0, \overbrace{v}^{k-th}, 0, \dots, 0 \right)^T$$

where $-^T$ denotes the transpose matrix.

In fact, we may have a matricial point of view on this construction. In the original article [18], the result given by Long is written with a line-matrix basis. In the same way, the study of Bigelow and Tian in [3] is performed from a purely matricial point of view.

Theorem 3.11. [18, Theorem 2.2] *Let $\rho : \mathbf{B}_{n+1} \rightarrow GL(V)$ be a representation of \mathbf{B}_{n+1} , where V is a \mathbb{K} -vector space. Let $(1, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$ be a coherent quadruplet where $a_{n,\bullet} = a_{n,1}$. The Long-Moody construction associated with ρ , can be matricially written in the following way, in the basis $\{V_k\}_{k=1,\dots,n}$ of $V^{\oplus n}$:*

$$\forall i \in \{1, \dots, n-1\}, lm_{(1, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(\rho)(\sigma_i) = \begin{bmatrix} \overbrace{\begin{matrix} \rho(\sigma_i) & 0 \\ 0 & \ddots \\ & \rho(\sigma_i) \end{matrix}}^{i-1} & & & \overbrace{\begin{matrix} & & & 0 \\ & & & \vdots \\ & & \rho(\sigma_i) & \\ & \ddots & & 0 \end{matrix}}^{n-i-1} \\ & \dots & & \\ & & R_1(\rho) & \\ & \vdots & & \\ & 0 & \dots & \begin{matrix} \ddots & 0 \\ 0 & \rho(\sigma_i) \end{matrix} \end{bmatrix}$$

where:

$$R_1(\rho) := \begin{bmatrix} 0 & \rho(g_{i+1})\rho(\sigma_i) \\ \rho(\sigma_i) & (1_V - \rho(g_{i+1}^{-1}g_i g_{i+1}))\rho(\sigma_i) \end{bmatrix}.$$

Proof. Let $i \in \{1, \dots, n-1\}$ fixed. We compute how $lm(\rho)$ acts on each generator of the basis of $\bigoplus_{i=1}^n V_i$. There are three different cases which we treat following the computations of Example 1.22.

1. $lm(\rho)(\sigma_i) \left((g_i - 1) \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) = (g_{i+1} - 1) \otimes_{\mathbb{K}[\mathbf{F}_n]} \rho(\sigma_i)(v)$. Then:

$$\Lambda \left(lm_{(1, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(\rho)(\sigma_i)_{|V_i} \right) = \rho(\sigma_i) id_{V_{i+1}}.$$

2. $lm(\rho)(\sigma_i) \left((g_{i+1} - 1) \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) = [(g_{i+1} - 1)(1 - g_{i+1}^{-1}g_i g_{i+1}) + (g_i - 1)g_{i+1}] \otimes_{\mathbb{K}[\mathbf{F}_n]} \rho(\sigma_i)(v)$.

We conclude that:

$$\Lambda \left(lm(\rho)(\sigma_i)_{|V_{i+1}} \right) = (1 - \rho(g_{i+1}^{-1}g_i g_{i+1}))\rho(\sigma_i) id_{V_{i+1}} + \rho(g_{i+1}\sigma_i)(\sigma_i) id_{V_i}.$$

3. Let $k \in \{1, \dots, n-1\}$ such that $k \neq i, i+1$. So:

$$lm(\rho)(\sigma_i) \left((g_k - 1) \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) = (g_k - 1) \otimes_{\mathbb{K}[\mathbf{F}_n]} \rho(\sigma_i)(v).$$

Therefore:

$$\Lambda \left(lm(\rho)(\sigma_i)_{|V_k} \right) = \rho(\sigma_i) id_{V_k}.$$

□

Remark 3.12. One can add an extra parameter $q \in \mathbb{K}^*$ to the construction. Indeed, considering $\rho : \mathbf{B}_{n+1} \rightarrow GL(V)$ a representation, we can define:

$$lm_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f}), q}(\rho) : \mathbf{B}_n \longrightarrow GL\left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} V\right)$$

assigning for all $b \in \mathbf{B}_n$, $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v \in V$:

$$\begin{aligned} lm_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f}), q}(\rho)(b) \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) &= \left[lm_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(\rho)(b) \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) \right] \otimes_{\mathbb{K}} q \\ &= \left(a_{n, \bullet}(b)(i) \otimes_{\mathbb{K}[\mathbf{F}_n]} \rho(\gamma_n^{b, \bullet}(b))(v) \right) \otimes_{\mathbb{K}} q \\ &= a_{n, \bullet}(b)(i) \otimes_{\mathbb{K}[\mathbf{F}_n]} \left(\rho(\gamma_n^b(b))(v) \otimes_{\mathbb{K}} q \right). \end{aligned}$$

3.3 The Long-Moody functor

In this subsection, we prove that the Long-Moody construction defined in [18] provides a functor $\mathbf{LM} : \mathcal{U}\beta\text{-}\mathcal{M}\mathfrak{od} \longrightarrow \mathcal{U}\beta\text{-}\mathcal{M}\mathfrak{od}$. Let F be a $\mathcal{U}\beta$ -module and n be a natural integer. For all $\sigma \in \mathbf{B}_n$, since $F(\sigma) \in GL(F(n))$, F defines in fact a representation of \mathbf{B}_n :

$$F|_{Hom_{\mathcal{U}\beta}(n, n)} : \mathbf{B}_n \longrightarrow GL(F(n)).$$

A fortiori, the \mathbb{K} -module $F(n+1)$ is in fact endowed with a $\mathbb{K}[\mathbf{B}_{n+1}]$ -module structure with this action $F|_{Hom_{\mathcal{U}\beta}(n+1, n+1)}$. Thanks to the monomorphism $\varsigma_{n, \bullet} : \mathbf{F}_n \hookrightarrow \mathbf{B}_{n+1}$, it is also a $\mathbb{K}[\mathbf{F}_n]$ -module. Hence, one can form the tensor product $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} F(n+1)$, which is isomorphic to $F(n+1)^{\oplus n}$ by the isomorphism Λ (see Remark 3.10).

Remark 3.13. Consider two natural integers n and n' such that $n' \geq n$, and $[n' - n, \sigma] \in Hom_{\mathcal{U}\beta}(n, n')$. Let $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$ be a coherent quadruplet. As $[n' - n, \gamma_n^{b, \bullet}(\sigma)]$ belongs to $Hom_{\mathcal{U}\beta}(n+1, n'+1)$, we remark that $F\left([n' - n, \gamma_n^{b, \bullet}(\sigma)]\right)$ belongs to $Hom_{\mathbb{K}\text{-}Mod}(F(n+1), F(n'+1))$. One can generalize the definition of the Long-Moody construction, defining the \mathbb{K} -module homomorphism:

$$lm(F)_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}([n' - n, \sigma]) : \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} F(n+1) \longrightarrow \mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'}]} \otimes_{\mathbb{K}[\mathbf{F}_{n'}]} F(n'+1)$$

by $\forall i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}, \forall v \in F(n+1)$:

$$lm_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)([n' - n, \sigma]) \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) = a_{n', \bullet}(\sigma) \left(\gamma_{n, n'}^{f, \bullet}(i) \right) \otimes_{\mathbb{K}[\mathbf{F}_{n'}]} F\left([n' - n, \gamma_n^{b, \bullet}(\sigma)]\right)(v).$$

Proposition 3.14. *For $F \in Obj(\mathcal{U}\beta\text{-}\mathcal{M}\mathfrak{od})$ and $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$ a coherent quadruplet, the following assignment defines a functor $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F) : \mathcal{U}\beta \longrightarrow \mathbb{K}\text{-}Mod$.*

- *Objects:* $\forall n \in \mathbb{N}$, $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)(n) = \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} F(n+1)$.
- *Morphisms:* Let $n, n' \in \mathbb{N}$, such that $n' \geq n$, and $[n' - n, \sigma] \in Hom_{\mathcal{U}\beta}(n, n')$. One assigns:

$$\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)([n' - n, \sigma]) = lm_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)\left([n' - n, \gamma_n^{b, \bullet}(\sigma)]\right).$$

Notation 3.15. When there is no ambiguity, once the quadruplet $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$ is clearly given, we forget it in the notation $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$ for convenience (especially for proofs).

Proof. Let n be a fixed natural integer. Let $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v \in F(n+1)$. On the one hand, according to our assignment and since $a_{n,\bullet}$, $\gamma_n^{b,\bullet}$ and $\gamma_n^{f,\bullet}$ are group morphisms:

$$\begin{aligned} \mathbf{LM}(F)(id_{\mathbf{B}_n}) \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) &= a_{n,\bullet}(id_{\mathbf{B}_n})(i) \otimes_{\mathbb{K}[\mathbf{F}_n]} F([0, \gamma_n^{b,\bullet}(id_{\mathbf{B}_n})])(v) \\ &= i \otimes_{\mathbb{K}[\mathbf{F}_n]} v. \end{aligned}$$

So, $\mathbf{LM}(F)(id_{\mathbf{B}_n}) = id_{\mathbf{LM}(F)(n)}$. On the other hand, let n , n' and n'' be natural integers such that $n'' \geq n' \geq n$, let $([n' - n, \sigma])$ and $([n'' - n', \sigma'])$ be morphisms respectively in $Hom_{\mathcal{UB}}(n, n')$ and in $Hom_{\mathcal{UB}}(n', n'')$. Since $a_{n,\bullet}$ and $\gamma_n^{b,\bullet}$ are group morphisms, we deduce from the definition that:

$$\begin{aligned} &\mathbf{LM}(F)(([n'' - n', \sigma']) \circ ([n' - n, \sigma])) \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) \\ &= (a_{n'',\bullet}(\sigma') \circ a_{n',\bullet}(id_{n''-n'} \natural \sigma)) \left(\gamma_{n',n''}^{f,\bullet} \circ \gamma_{n,n'}^{f,\bullet}(i) \right) \otimes_{\mathbb{K}[\mathbf{F}_{n''}]} F \left([n'' - n', \gamma_{n',n''}^{b,\bullet}(\sigma') \circ \gamma_{n,n'}^{b,\bullet}((id_{n''-n'} \natural \sigma))] \right) (v). \end{aligned}$$

As $\gamma_{n',n''}^{b,\bullet}((id_{n''-n'} \natural \sigma)) = id_{(n''+1)-(n'+1)} \natural \gamma_{n'}^{b,\bullet}(\sigma)$ by Condition 3.3, because of the definition of morphisms in \mathcal{UB} :

$$\begin{aligned} [n'' - n', \gamma_{n',n''}^{b,\bullet}(\sigma') \circ \gamma_{n,n'}^{b,\bullet}((id_{n''-n'} \natural \sigma))] &= [n'' - n', \gamma_{n',n''}^{b,\bullet}(\sigma') \circ (id_{(n''+1)-(n'+1)} \natural \gamma_{n'}^{b,\bullet}(\sigma))] \\ &= [n'' - n', \gamma_{n',n''}^{b,\bullet}(\sigma')] \circ [n' - n, \gamma_{n,n'}^{b,\bullet}(\sigma)]. \end{aligned}$$

Moreover, Condition 3.4 asserts that for all element i' of $\mathcal{I}_{\mathbb{K}[\mathbf{F}_{n'}]}$:

$$\gamma_{n',n''}^{f,\bullet} \circ a_{n',\bullet}(\sigma)(i') = a_{n'',\bullet}(id_{n''-n'} \natural \sigma) \left(\gamma_{n',n''}^{f,\bullet}(i') \right).$$

So, for all element i of $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$: $\gamma_{n',n''}^{f,\bullet} \left(a_{n',\bullet}(\sigma) \left(\gamma_{n,n'}^{f,\bullet}(i) \right) \right) = a_{n'',\bullet}(id_{n''-n'} \natural \sigma) \left(\gamma_{n',n''}^{f,\bullet} \circ \gamma_{n,n'}^{f,\bullet}(i) \right)$. Therefore:

$$\begin{aligned} &\mathbf{LM}(F)(([n'' - n', \sigma']) \circ ([n' - n, \sigma])) \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) \\ &= \left(a_{n'',\bullet}(\sigma') \circ \gamma_{n',n''}^{f,\bullet} \right) \left(a_{n',\bullet}(\sigma) \left(\gamma_{n,n'}^{f,\bullet}(i) \right) \right) \otimes_{\mathbb{K}[\mathbf{F}_{n''}]} F \left([n'' - n', \gamma_{n',n''}^{b,\bullet}(\sigma')] \circ [n' - n, \gamma_{n,n'}^{b,\bullet}(\sigma)] \right) (v) \\ &= (\mathbf{LM}(F)([n'' - n', \sigma']) \circ \mathbf{LM}(F)([n' - n, \sigma])) \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right). \end{aligned}$$

Hence, the axiom of associativity is satisfied. \square

In the following proposition, we define an endofunctor of $\mathcal{UB}\text{-}\mathcal{Mod}$ corresponding to the Long-Moody construction. It will be called the Long-Moody functor.

Proposition 3.16. *For a coherent quadruplet $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$, the following assignment defines a functor $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} : \mathcal{UB}\text{-}\mathcal{Mod} \rightarrow \mathcal{UB}\text{-}\mathcal{Mod}$.*

- *Objects :* Let $F \in \text{Obj}(\mathcal{UB}\text{-}\mathcal{Mod})$, $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)$ is defined as in Proposition 3.14.
- *Morphisms :* Let F and G be two \mathcal{UB} -modules, and $\eta : F \Rightarrow G$ be a natural transformation. We define $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(\eta) : \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F) \Rightarrow \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(G)$ for all natural integer n by:

$$\forall i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}, \forall v \in F(n+1), \left(\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(\eta) \right)_n \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) = i \otimes_{\mathbb{K}[\mathbf{F}_n]} \eta_{n+1}(v).$$

In other words:

$$\left(\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(\eta) \right)_n = id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}} \otimes_{\mathbb{K}[\mathbf{F}_n]} \eta_{n+1}.$$

Proof. The remaining point to check for \mathbf{LM} to be well-defined is the consistency of our definition on morphisms. Let F and G be two $\mathfrak{U}\beta$ -modules, and let $\eta : F \Rightarrow G$ be a natural transformation. Let $n, n' \in \mathbb{N}$, such that $n' \geq n$, and $[n' - n, \sigma] \in \text{Hom}_{\mathfrak{U}\beta}(n, n')$. Since η is a natural transformation:

$$\forall v \in F(n), G\left([n' - n, \gamma_{n'}^{b, \bullet}(\sigma)]\right)(\eta_{n+1}(v)) = \eta_{n'+1}\left(F\left([n' - n, \gamma_{n'}^{b, \bullet}(\sigma)]\right)(v)\right).$$

Hence, it follows from our definitions that the following diagram commutes, and therefore \mathbf{LM} is a morphism of $\mathfrak{U}\beta$ -modules.

$$\begin{array}{ccc} \mathbf{LM}(F)(n) & \xrightarrow{\mathbf{LM}(\eta)_n} & \mathbf{LM}(G)(n) \\ \downarrow \mathbf{LM}(F)([n' - n, \sigma]) & & \downarrow \mathbf{LM}(G)([n' - n, \sigma]) \\ \mathbf{LM}(F)(n') & \xrightarrow{\mathbf{LM}(\eta)_{n'}} & \mathbf{LM}(G)(n') \end{array}$$

For F a $\mathfrak{U}\beta$ -module and $id_F : F \Rightarrow F$ the identity natural transformation (ie for all natural integer n and for all element v of $F(n+1)$, $id_{F(n+1)}(v) = v$), it comes directly that $\mathbf{LM}(id_F) = id_{\mathbf{LM}(F)}$. Finally, let us check the associativity axiom. Let $\eta : F \Rightarrow G$ and $\mu : G \Rightarrow H$ be natural transformations. Let n be a natural integer, $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v \in F(n+1)$. Yet, because μ and η are morphisms in the category of functors:

$$\mathbf{LM}(\mu \circ \eta)_n \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) = i \otimes_{\mathbb{K}[\mathbf{F}_n]} (\mu_{n+1} \circ \eta_{n+1})(v) = \mathbf{LM}(\mu)_n \circ \mathbf{LM}(\eta)_n \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right).$$

□

Remark 3.17. For $\eta : F \Rightarrow G$ a natural transformation, with Λ the isomorphism of Remark 3.10:

$$\forall n \in \mathbb{N}, \Lambda \left(\left(\mathbf{LM}_{(\bullet_a, \bullet_s, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(\eta) \right)_n \right) = \eta_{n+1}^{\oplus n}.$$

Proposition 3.18. *For a coherent quadruplet $(\bullet_a, \bullet_s, \bullet_{\gamma^b}, \bullet_{\gamma^f})$, the functor $\mathbf{LM}_{(\bullet_a, \bullet_s, \bullet_{\gamma^b}, \bullet_{\gamma^f})} : \mathfrak{U}\beta\text{-Mod} \rightarrow \mathfrak{U}\beta\text{-Mod}$ is reduced and exact.*

Proof. Let $0_{\mathfrak{U}\beta\text{-Mod}} : \mathfrak{U}\beta \rightarrow \mathbb{K}\text{-Mod}$ denotes the null functor. It comes straightforward from the definition of the Long-Moody functor that $\mathbf{LM}(0_{\mathfrak{U}\beta\text{-Mod}}) = 0_{\mathfrak{U}\beta\text{-Mod}}$.

To prove the exactness property, let us consider the short exact sequence $0 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 0$ in the category $\mathfrak{U}\beta\text{-Mod}$. Let n be a natural integer. Since the augmentation ideal $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ is a free $\mathbb{K}[\mathbf{F}_n]$ -module (see for example [24, Chapter 2, Proposition 6.2.6]), it is therefore a flat $\mathbb{K}[\mathbf{F}_n]$ -module. Then, by definition, it ensures that the functor $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} - : \mathbb{K}\text{-Mod} \rightarrow \mathbb{K}\text{-Mod}$ is an exact functor for all natural integer n . Hence, applying it to the morphisms α_{n+1} and β_{n+1} , we obtain that the following short sequence of \mathbb{K} -modules is exact. We denote $(\mathbf{LM}(\alpha))_n$ resp. $(\mathbf{LM}(\beta))_n$ by $\tilde{\alpha}_{n+1}$ resp. $\tilde{\beta}_{n+1}$.

$$0 \longrightarrow \mathbf{LM}(F)(n) \xrightarrow{\tilde{\alpha}_{n+1}} \mathbf{LM}(G)(n) \xrightarrow{\tilde{\beta}_{n+1}} \mathbf{LM}(H)(n) \longrightarrow 0$$

It remains to check the naturality. Let n' be another natural integer such that $n' \geq n$ and let $[n' - n, \sigma] \in \text{Hom}_{\mathfrak{U}\beta}(n, n')$. Let $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$, $v \in F(n+1)$ and $w \in G(n+1)$. On the one hand, since α_{n+1} is a natural transformation, then $G\left([n' - n, \gamma_{n'}^{b, \bullet}(\sigma)]\right)(\alpha_{n+1}(v)) = \alpha_{n'+1} \circ F\left([n' - n, \gamma_{n'}^{b, \bullet}(\sigma)]\right)(v)$. Hence:

$$\mathbf{LM}(G)([n' - n, \sigma]) \circ \tilde{\alpha}_{n+1} \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) = (\tilde{\alpha}_{n'+1} \circ \mathbf{LM}(F)([n' - n, \sigma])) \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right).$$

On the other hand, since β_{n+1} is a natural transformation, then $H\left([n' - n, \gamma_{n'}^{b, \bullet}(\sigma)]\right)(\beta_{n+1}(v)) = \beta_{n'+1} \circ G\left([n' - n, \gamma_{n'}^{b, \bullet}(\sigma)]\right)(v)$. Hence:

$$\mathbf{LM}(H)([n' - n, \sigma]) \circ \tilde{\beta}_{n+1} \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} w \right) = \left(\tilde{\beta}_{n'+1} \circ \mathbf{LM}(G)([n' - n, \sigma]) \right) \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} w \right).$$

□

Corollary 3.19. *For a coherent quadruplet $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$, the functor $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$ commutes with all finite limits and finite colimits.*

Proof. The functor \mathbf{LM} is exact according to Proposition 3.18. The result is a property of exact functors (see for example [19, Chapter 8, section 3]). □

4 The Long-Moody functor applied to polynomial functors

Let us move on the effect of the Long-Moody construction on polynomial functors.

4.1 The intermediary functors

We have to introduce two new functors which will play a key role in the main result of the study. First, let us recall the following crucial property of the augmentation ideal of a free product of groups. The proof of this proposition is a consequence of combining [7, Section 4, Lemma 4.3] and [7, Section 4, Theorem 4.7].

Proposition 4.1. *Let G and H be groups. Then, there is a natural $\mathbb{K}[G * H]$ -module isomorphism:*

$$\mathcal{I}_{\mathbb{K}[G * H]} \cong \left(\mathcal{I}_{\mathbb{K}[G]} \otimes_{\mathbb{K}G} \mathbb{K}[G * H] \right) \oplus \left(\mathcal{I}_{\mathbb{K}[H]} \otimes_{\mathbb{K}H} \mathbb{K}[G * H] \right).$$

For a fixed natural integer m and F a $\mathfrak{U}\beta$ -module, let us consider the $\mathfrak{U}\beta$ -module $\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)$. For all natural integer n , by Proposition 4.1, we have a $\mathbb{K}[\mathbf{F}_{m+n}]$ -module isomorphism:

$$\begin{aligned} & \tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)(n) \\ &= \mathcal{I}_{\mathbb{K}[\mathbf{F}_{m+n}]} \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} F(m+n+1) \\ &\cong \left(\left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_m]} \otimes_{\mathbb{K}[\mathbf{F}_m]} \mathbb{K}[\mathbf{F}_{m+n}] \right) \oplus \left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} \mathbb{K}[\mathbf{F}_{m+n}] \right) \right) \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} F(m+n+1). \end{aligned}$$

Therefore, because of the distributivity of tensor product with respect to the direct sum, we prove the following proposition.

Proposition 4.2. *Let $F \in \text{Obj}(\mathfrak{U}\beta\text{-Mod})$ and m, n be natural integers. Then, we have the following \mathbb{K} -module isomorphism:*

$$\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)(n) \cong \left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_m]} \otimes_{\mathbb{K}[\mathbf{F}_m]} F(m+n+1) \right) \oplus \left(\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} F(m+n+1) \right). \quad (4)$$

The aim of this section is in fact to show that this \mathbb{K} -module decomposition will lead to a $\mathfrak{U}\beta$ -module decomposition of $\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$ (see Theorem (4.13)).

Proposition 4.3. Let m be a fixed natural integer. For all natural integer n and $F \in \text{Obj}(\mathfrak{U}\beta\text{-}\mathfrak{M}\mathfrak{o}\mathfrak{d})$, the submodules $\mathcal{I}_{\mathbb{K}[\mathbf{F}_m]} \otimes_{\mathbb{K}[\mathbf{F}_m]} F(m+n+1)$ of $\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)(n)$ naturally define a subfunctor of

$\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)$, which will be denoted by $\Upsilon_m^{(\bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)$.

This way, for all $F \in \text{Obj}(\mathfrak{U}\beta\text{-}\mathfrak{M}\mathfrak{o}\mathfrak{d})$, the subfunctors $\Upsilon_m^{(\bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)$ of $\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)$ define a subfunctor of $\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$, which will be denoted by $\Upsilon_m^{(\bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$.

Remark 4.4. As the notation suggests it, the functor $\Upsilon_m^{(\bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$ does not depend on the morphism $a_{n, \bullet}$. We will see the reason why in the proof of Proposition 4.3.

Notation 4.5. When there is no ambiguity, once a triplet $(\bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$ is clearly given, we forget it in the notation $\Upsilon_m^{(\bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$ for convenience (especially for the proofs).

Proof. First, let us show that $\Upsilon_m(F)$ is a subfunctor of $\tau_m \mathbf{LM}(F)$ for every $F \in \text{Obj}(\mathfrak{U}\beta\text{-}\mathfrak{M}\mathfrak{o}\mathfrak{d})$. For all natural integer n :

$$\Upsilon_m(F)(n) = \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]} \otimes_{\mathbb{K}[\mathbf{F}_m]} F(m+n+1).$$

Because of Proposition 4.2, the \mathbb{K} -module $\Upsilon_m(F)(n)$ is a submodule of the \mathbb{K} -module $\tau_m \mathbf{LM}(F)(n)$. Then, we define the monomorphism associated with the direct sum

$$v_{m,n}(F) = \gamma_{m,m+n}^{f, \bullet} \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} id_{F(m+n+1)} : \Upsilon_m(F)(n) \hookrightarrow \tau_m \mathbf{LM}(F)(n)$$

by the assignment:

$$\forall i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}, \forall v \in F(m+n+1), v_{m,n}(F) \left(i \otimes_{\mathbb{K}[\mathbf{F}_m]} v \right) = \gamma_{m,m+n}^{f, \bullet}(i) \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} v.$$

Let n and n' be natural integers such that $n' \geq n$, and $[n' - n, \sigma] \in \text{Hom}_{\mathfrak{U}\beta}(n, n')$ with $\sigma \in \mathbf{B}_{n'}$. Let $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}$ and $v \in F(m+n+1)$. We naturally define the functor $\Upsilon_m^{(\bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)$ on morphisms by:

$$\Upsilon_m(F)([n' - n, \sigma]) \left(i \otimes_{\mathbb{K}[\mathbf{F}_m]} v \right) = i \otimes_{\mathbb{K}[\mathbf{F}_m]} F \left([n' - n, \gamma_{m+n'}^{b, \bullet}(id_m \natural \sigma)] \right) (v).$$

It remains to show that we define a natural transformation $v_m(F) : \Upsilon_m(F) \Rightarrow \tau_m \mathbf{LM}(F)$. Since by Condition 3.5, $a_{m+n', \bullet}(id_m \natural \sigma) \circ \gamma_{m,m+n'}^{f, \bullet} = \gamma_{m,m+n'}^{f, \bullet}$ for all element σ of $\mathbf{B}_{n'}$, naturality then follows from the following equality:

$$(\tau_m \mathbf{LM}(F)([n' - n, \sigma]) \circ v_{m,n}(F)) \left(i \otimes_{\mathbb{K}[\mathbf{F}_m]} v \right) = v_{m,n'}(F) \circ \Upsilon_m(F)([n' - n, \sigma]) \left(i \otimes_{\mathbb{K}[\mathbf{F}_m]} v \right).$$

Therefore, for every $F \in \text{Obj}(\mathfrak{U}\beta\text{-}\mathfrak{M}\mathfrak{o}\mathfrak{d})$, $\Upsilon_m(F)$ is a subfunctor of $\tau_m \mathbf{LM}(F)$. Let us show that we may define a natural transformation $v_m : \Upsilon_m \Rightarrow \tau_m \mathbf{LM}$ in the category $\mathfrak{U}\beta\text{-}\mathfrak{M}\mathfrak{o}\mathfrak{d}$. Let F and G be two $\mathfrak{U}\beta$ -modules, and $\eta : F \Rightarrow G$ a natural transformation. We naturally define the functor Υ_m on natural transformations by:

$$\forall n \in \mathbb{N}, (\Upsilon_m(\eta))_n = id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}} \otimes_{\mathbb{K}[\mathbf{F}_m]} \eta_{m+n+1}.$$

Then, it follows from our definitions that for all natural integer n , for all $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}$ and $v \in F(m+n+1)$:

$$\begin{aligned} (v_{m,n}(G) \circ (\Upsilon_m(\eta))_n) \left(i \otimes_{\mathbb{K}[\mathbf{F}_m]} v \right) &= \gamma_{m,m+n}^{f, \bullet}(i) \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} \eta_{m+n+1}(v) \\ &= ((\tau_m \mathbf{LM}(\eta))_n \circ v_{m,n}(F)) \left(i \otimes_{\mathbb{K}[\mathbf{F}_m]} v \right). \end{aligned}$$

□

The functor $\Upsilon_m^{(\bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$ satisfies some convenient properties.

Lemma 4.6. *For all natural integer m , the functor $\Upsilon_m^{(\bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$ is exact.*

Proof. Let us consider the short exact sequence $0 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 0$ in the category $\mathcal{UB}\text{-}\mathcal{Mod}$. Let m be a natural integer. Since the augmentation ideal $\mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}$ is a free $\mathbb{K}[\mathbf{F}_m]$ -module (see for example [24, Chapter 2, Proposition 6.2.6]), it is therefore a flat $\mathbb{K}[\mathbf{F}_m]$ -module. Then, by definition, it ensures that the functor $\mathcal{I}_{\mathbb{K}[\mathbf{F}_m]} \otimes_{\mathbb{K}[\mathbf{F}_m]} -: \mathbb{K}\text{-}\mathcal{Mod} \longrightarrow \mathbb{K}\text{-}\mathcal{Mod}$ is an exact functor for all natural integer m . Hence, applying it to the morphisms α_{m+n+1} and β_{m+n+1} , we obtain that the following short sequence of \mathbb{K} -modules is exact. We denote $(\Upsilon_m(\alpha))_n$ resp. $(\Upsilon_m(\beta))_n$ by $\tilde{\alpha}_{n+1}$ resp. $\tilde{\beta}_{n+1}$.

$$0 \longrightarrow \Upsilon_m(F) \xrightarrow{\tilde{\alpha}_{n+1}} \Upsilon_m(G) \xrightarrow{\tilde{\beta}_{n+1}} \Upsilon_m(H) \longrightarrow 0$$

We have to check the naturality. Let n' be another natural integer such that $n' \geq n$ and let $[n' - n, \sigma] \in \text{Hom}_{\mathcal{UB}}(n, n')$. Let $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}$, $v \in F(m + n + 1)$ and $w \in G(m + n + 1)$. On the one hand, since α_{n+1} is a natural transformation, then:

$$G\left(\left[n' - n, \gamma_{m+n'}^{b, \bullet}(id_m \natural \sigma)\right]\right)(\alpha_{m+n+1}(v)) = \alpha_{n'+1} \circ F\left(\left[n' - n, \gamma_{m+n'}^{b, \bullet}(id_m \natural \sigma)\right]\right)(v).$$

Hence:

$$\Upsilon_m(G)([n' - n, \sigma]) \circ \tilde{\alpha}_{n+1}\left(i \otimes_{\mathbb{K}[\mathbf{F}_m]} v\right) = (\tilde{\alpha}_{n'+1} \circ \Upsilon_m(F)([n' - n, \sigma]))\left(i \otimes_{\mathbb{K}[\mathbf{F}_m]} v\right).$$

On the other hand, since β_{n+1} is a natural transformation, then:

$$H\left(\left[n' - n, \gamma_{m+n'}^{b, \bullet}(id_m \natural \sigma)\right]\right)(\beta_{m+n+1}(v)) = \beta_{n'+1} \circ G\left(\left[n' - n, \gamma_{m+n'}^{b, \bullet}(id_m \natural \sigma)\right]\right)(v).$$

Hence:

$$\Upsilon_m(H)([n' - n, \sigma]) \circ \tilde{\beta}_{n+1}\left(i \otimes_{\mathbb{K}[\mathbf{F}_m]} w\right) = (\tilde{\beta}_{n'+1} \circ \Upsilon_m(G)([n' - n, \sigma]))\left(i \otimes_{\mathbb{K}[\mathbf{F}_m]} w\right).$$

□

Proposition 4.7. *For all natural integers m and l , the functor $\Upsilon_m^{(\bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$ commutes with the difference functor δ_l .*

Proof. Let n be a natural integer and F be a \mathcal{UB} -module. For all natural integer l , it follows directly from the definition of i_l that $i_l(\Upsilon_m(F))(n) = \Upsilon_m(i_l(F))(n)$. Since we know that the two following sequences are exact (because of Lemma 4.6 for the second one), the following diagram is commutative.

$$\begin{array}{ccccccc} \Upsilon_m F(n) & \xrightarrow{i_l(\Upsilon_m(F))(n)} & \Upsilon_l(\Upsilon_m F)(n) & \longrightarrow & \delta_l(\Upsilon_m F)(n) & \longrightarrow & 0 \\ \parallel & & \parallel & & & & \\ \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]} \otimes_{\mathbb{K}[\mathbf{F}_m]} F(m + n + 1) & & \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]} \otimes_{\mathbb{K}[\mathbf{F}_m]} F(m + l + n + 1) & & & & \\ \parallel & & \parallel & & & & \\ \Upsilon_m F(n) & \xrightarrow{\Upsilon_m(i_l(F))(n)} & \Upsilon_m(\Upsilon_l F)(n) & \longrightarrow & \Upsilon_m(\delta_l F)(n) & \longrightarrow & 0 \end{array}$$

Therefore, by the universal property of the cokernel, we deduce:

$$\delta_l(\Upsilon_m F)(n) \cong \Upsilon_m(\delta_l F)(n).$$

The naturality in n follows directly because of our constructions.

□

Proposition 4.8. *Let m be a natural integer and F be a $\mathfrak{U}\beta$ -module. Then the functor $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(\tau_m F)$ is a subfunctor of $\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)$.*

This way, for all $F \in \text{Obj}(\mathfrak{U}\beta\text{-}\mathfrak{Mod})$, the subfunctors $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(\tau_m F)$ of $\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)$ define a subfunctor of $\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$, which will be denoted by $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \circ \tau_m$.

Proof. For all natural integer n , $\mathbf{LM}(\tau_m F)(n) = \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \otimes_{\mathbb{K}[\mathbf{F}_n]} F(m+n+1)$. Because of Proposition 4.2, the \mathbb{K} -module $\mathbf{LM}(\tau_m F)(n)$ is a submodule of the \mathbb{K} -module $\tau_m \mathbf{LM}(F)(n)$. Then, we define the monomorphism associated with the direct sum $\varepsilon_{m,n}(F) = \gamma_{n,m+n}^{f,\bullet} \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} id_{F(m+n+1)} : \mathbf{LM}(\tau_m F)(n) \hookrightarrow \tau_m \mathbf{LM}(F)(n)$ by the assignment:

$$\forall i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}, \forall v \in F(m+n+1), \varepsilon_{m,n}(F) \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) = \gamma_{n,m+n}^{f,\bullet}(i) \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} v.$$

It remains to show that we define a natural transformation $\varepsilon_m(F) : \mathbf{LM}(\tau_m F) \Rightarrow \tau_m \mathbf{LM}(F)$. Let n and n' be natural integers such that $n' \geq n$, and $[n' - n, \sigma] \in \text{Hom}_{\mathfrak{U}\beta}(n, n')$ with $\sigma \in \mathbf{B}_{n'}$. Let $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v \in F(m+n+1)$. By Condition 3.3, we deduce that the functor $\mathbf{LM}(\tau_m F)$ is defined on morphisms by:

$$\mathbf{LM}(\tau_m F)([n' - n, \sigma]) \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) = a_{n',\bullet}(\sigma) \left(\gamma_{n,n'}^{f,\bullet}(i) \right) \otimes_{\mathbb{K}[\mathbf{F}_{n'}]} F \left([n' - n, \gamma_{m+n'}^{b,\bullet}(id_m \natural \sigma)] \right) (v).$$

Since by Condition 3.6, $a_{m+n',\bullet}(id_m \natural \sigma) \circ \gamma_{n,m+n'}^{f,\bullet} = \gamma_{n',m+n'}^{f,\bullet} \circ a_{n',\bullet}(\sigma) \circ \gamma_{n,n'}^{f,\bullet}$ for all element σ of $\mathbf{B}_{n'}$, naturality then follows from the following equality:

$$(\tau_m \mathbf{LM}(F)([n' - n, \sigma]) \circ \varepsilon_{m,n}(F)) \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) = (\varepsilon_{m,n'}(F) \circ \mathbf{LM}(\tau_m F))([n' - n, \sigma]) \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right).$$

Therefore, for every $F \in \text{Obj}(\mathfrak{U}\beta\text{-}\mathfrak{Mod})$, $\mathbf{LM}(\tau_m F)$ is a subfunctor of $\tau_m \mathbf{LM}(F)$. Let us show that we may define a natural transformation $\varepsilon_m : \mathbf{LM}\tau_m \Rightarrow \tau_m \mathbf{LM}$ in the category $\mathfrak{U}\beta\text{-}\mathfrak{Mod}$. Let F and G be two $\mathfrak{U}\beta$ -modules, and $\eta : F \Rightarrow G$ a natural transformation. Then, it follows from our definitions that for all natural integer n , for all $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}$ and $v \in F(m+n+1)$:

$$\begin{aligned} (\varepsilon_{m,n}(G) \circ (\mathbf{LM}\tau_m(\eta))_n) \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right) &= \gamma_{n,m+n}^{f,\bullet}(i) \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} \eta_{m+n+1}(v) \\ &= ((\tau_m \mathbf{LM}(\eta))_n \circ \varepsilon_{m,n}(F)) \left(i \otimes_{\mathbb{K}[\mathbf{F}_n]} v \right). \end{aligned}$$

□

Proposition 4.9. *Let m be a natural integer. Then, there is an isomorphism in the category $\mathbf{Fct}(\mathfrak{U}\beta\text{-}\mathfrak{Mod}, \mathfrak{U}\beta\text{-}\mathfrak{Mod})$:*

$$\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \cong \gamma_m^{(\bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \bigoplus \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \circ \tau_m.$$

Proof. Let F be a $\mathfrak{U}\beta$ -module. By Propositions 4.3 and 4.8, by the universal property of the direct sum and the naturality which comes directly, we deduce that $\gamma_m F \bigoplus \mathbf{LM}(\tau_m F)$ is a subfunctor of $\tau_m \mathbf{LM}(F)$. For all natural integer n , we have an isomorphism of \mathbb{K} -modules according to Proposition 4.2:

$$\gamma_m F(n) \bigoplus \mathbf{LM}(\tau_m F)(n) \cong \tau_m \mathbf{LM}(F)(n).$$

Hence by the universal property of the direct sum, for all natural integer n , there exists a unique \mathbb{K} -module isomorphism $\Xi_{\oplus,n,F} = \nu_{m,n}(F) \oplus \varepsilon_{m,n}(F) : \gamma_m F(n) \bigoplus \mathbf{LM}(\tau_m F)(n) \xrightarrow{\cong} \tau_m \mathbf{LM}(F)(n)$ such that the

following diagram is commutative.

$$\begin{array}{ccc}
& \mathbf{LM}(\tau_m F)(n) & \\
I_{\mathbf{LM}(\tau_m F)}^\oplus \swarrow & & \searrow \varepsilon_{m,n}(F) \\
\Upsilon_m F(n) \oplus \mathbf{LM}(\tau_m F)(n) & \xrightarrow[\Xi_{\oplus,n,F}]{\cong} & \tau_m \mathbf{LM}(F)(n) \\
I_{\Upsilon_m F}^\oplus \swarrow & & \searrow v_{m,n}(F) \\
& \Upsilon_m F(n) &
\end{array}$$

Let n and n' be natural integers such that $n' \geq n$, and $[n' - n, \sigma] \in \text{Hom}_{\mathcal{UB}}(n, n')$. In each of the following two diagrams, since the top and bottom triangles and the lateral trapezoids are commutative by our definitions, the exterior rectangle is commutative.

$$\begin{array}{ccc}
\Upsilon_m F(n) \oplus \mathbf{LM}(\tau_m F)(n) & \xrightarrow[\cong]{\Xi_{\oplus,n,F}} & \tau_m \mathbf{LM}(F)(n) \\
\downarrow \Upsilon_m F(n) \oplus \mathbf{LM}(\tau_m F)([n' - n, \sigma]) & & \downarrow \tau_m \mathbf{LM}(F)([n' - n, \sigma]) \\
& \begin{array}{ccc}
& \mathbf{LM}(\tau_m F)(n) & \\
I_{\mathbf{LM}(\tau_m F)(n)}^\oplus \swarrow & & \searrow \varepsilon_{m,n}(F) \\
& \mathbf{LM}(\tau_m F)(n') & \\
I_{\mathbf{LM}(\tau_m F)(n')}^\oplus \swarrow & & \searrow \varepsilon_{m,n'}(F) \\
& \mathbf{LM}(\tau_m F)(n') &
\end{array} & \\
\Upsilon_m F(n') \oplus \mathbf{LM}(\tau_m F)(n') & \xrightarrow[\cong]{\Xi_{\oplus,n',F}} & \tau_m \mathbf{LM}(F)(n')
\end{array}$$

$$\begin{array}{ccc}
\Upsilon_m F(n) \oplus \mathbf{LM}(\tau_m F)(n) & \xrightarrow[\cong]{\Xi_{\oplus,n,F}} & \tau_m \mathbf{LM}(F)(n) \\
\downarrow \Upsilon_m F(n) \oplus \mathbf{LM}(\tau_m F)([n' - n, \sigma]) & & \downarrow \tau_m \mathbf{LM}(F)([n' - n, \sigma]) \\
& \begin{array}{ccc}
& \Upsilon_m F(n) & \\
I_{\Upsilon_m F(n)}^\oplus \swarrow & & \searrow v_{m,n}(F) \\
& \Upsilon_m F(n') & \\
I_{\Upsilon_m F(n')}^\oplus \swarrow & & \searrow v_{m,n'}(F) \\
& \Upsilon_m F(n') &
\end{array} & \\
\Upsilon_m F(n') \oplus \mathbf{LM}(\tau_m F)(n') & \xrightarrow[\cong]{\Xi_{\oplus,n',F}} & \tau_m \mathbf{LM}(F)(n')
\end{array}$$

This shows that we have a natural equivalence:

$$\Xi_{\oplus,F} = v_m(F) \oplus \varepsilon_m(F) : \Upsilon_m(F) \bigoplus \mathbf{LM}\tau_m(F) \xrightarrow{\cong} \tau_m \mathbf{LM}(F).$$

It remains to show the naturality. Let F and G be two \mathcal{UB} -modules, and $\eta : F \Rightarrow G$ be a natural transformation. Then, for all natural integer n , for all $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}$, $j \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $v, w \in F(m + n + 1)$:

$$\begin{aligned}
& \left(\Upsilon_m \bigoplus \mathbf{LM}\tau_m(\eta) \right)_n \left(\left(i \underset{\mathbb{K}[\mathbf{F}_m]}{\otimes} v \right) \oplus \left(j \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} w \right) \right) \\
&= \left(\left(\gamma_{m,m+n}^{f,\bullet}(i) \underset{\mathbb{K}[\mathbf{F}_m]}{\otimes} \eta_{m+n+1}(v) \right) \oplus \left(\gamma_{m,m+n}^{f,\bullet}(j) \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} \eta_{m+n+1}(w) \right) \right) \\
&= \left(\Xi_{\oplus,n,G}^{-1} \circ (\tau_m \mathbf{LM}(\eta))_n \circ \Xi_{\oplus,n,F} \right) \left(\left(i \underset{\mathbb{K}[\mathbf{F}_m]}{\otimes} v \right) \oplus \left(j \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} w \right) \right).
\end{aligned}$$

Hence, we have a natural equivalence $\Xi_{\oplus} = v_m \oplus \varepsilon_m$ in the category $\mathcal{UB}\text{-Mod}$. \square

4.2 Splitting of the difference functor

Our aim is to study the effect of the Long-Moody construction on strong polynomial functors. To that purpose, we focus on the behaviour of the difference functor on the Long-Moody construction. Let $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$ be a coherent quadruplet, which we fix for all the work of this subsection (in particular, we forget it in many notations). Thus, we will be interested in the study of the cokernel of the map $i_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)$. Let us recall some properties of this map.

- for all natural integer n , $i_m(\mathbf{LM}(F))(n) = \mathbf{LM}(F)([m, id_{m+n}])$. Explicitly for all element k of $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$, for all element v of $F(n+1)$:

$$\mathbf{LM}(F)([m, id_{m+n}]) \left(k \otimes_{\mathbb{K}[\mathbf{F}_n]} w \right) = \gamma_{n, m+n}^{f, \bullet}(k) \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} F([m, id_{m+n+1}]) (w).$$

- we have the natural exact sequence:

$$0 \longrightarrow \kappa_m \mathbf{LM}(F)(n) \longrightarrow \mathbf{LM}(F)(n) \xrightarrow{i_m \mathbf{LM}(F)(n)} \tau_m \mathbf{LM}(F)(n) \longrightarrow \delta_m \mathbf{LM}(F)(n) \longrightarrow 0.$$

Lemma 4.10. *Let m be a natural integer and F be a $\mathfrak{U}\beta$ -module. The submodules $Im \left[\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)([m, id_{m+n}]) \right]$ of $\tau_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(F)(n)$ define a subfunctor of $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \tau_m(F)$, which will be denoted by $Im_m(F)$.*

This way, for all $F \in \text{Obj}(\mathfrak{U}\beta\text{-Mod})$, the subfunctors $Im_m(F)$ of $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \tau_m(F)$ define a subfunctor of $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \tau_m$, which will be denoted by Im_m .

Proof. Let n be natural integer. Let $k \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $w \in F(n+1)$. Recalling $\varepsilon_m : \mathbf{LM} \tau_m \implies \tau_m \mathbf{LM}$ the natural transformation defined in Proposition 4.8, we have:

$$\begin{aligned} \mathbf{LM}(F)([m, id_{m+n}]) \left(k \otimes_{\mathbb{K}[\mathbf{F}_n]} w \right) &= \gamma_{n, m+n}^{f, \bullet}(k) \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} F([m, id_{m+n+1}]) (w) \\ &= \varepsilon_{m, n}(F) \left(k \otimes_{\mathbb{K}[\mathbf{F}_n]} \tau_m F([0, id_{n+1}]) (w) \right). \end{aligned}$$

Hence, by the definition of $\varepsilon_{m, n}(F)$, using the decomposition of Proposition 4.2, the projection on the summand $\mathcal{I}_{\mathbb{K}[\mathbf{F}_m]} \otimes_{\mathbb{K}[\mathbf{F}_m]} F(m+n+1)$ is zero. By the universal property of the kernel, we deduce that there exists a unique monomorphism of \mathbb{K} -modules:

$$\vartheta_{m, n}(F) : Im_m(F)(n) \hookrightarrow \mathbf{LM}(\tau_m F)(n)$$

such that for all $k \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$, and for all $w \in F(n+1)$:

$$\vartheta_{m, n}(F) \left(\mathbf{LM}(F)([m, id_{m+n}]) \left(k \otimes_{\mathbb{K}[\mathbf{F}_n]} w \right) \right) = k \otimes_{\mathbb{K}[\mathbf{F}_n]} F([m, id_{m+n+1}]) (w).$$

Hence, the \mathbb{K} -module $Im_m(F)(n)$ is a submodule of the \mathbb{K} -module $\mathbf{LM}(\tau_m F)(n)$ for all natural integer n . Let n and n' be natural integers such that $n' \geq n$, and $[n' - n, \sigma] \in \text{Hom}_{\mathfrak{U}\beta}(n, n')$. Let $k \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $w \in F(n+1)$. Let $i \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_m]}$ and $v \in F(m+n+1)$. We naturally define the functor $Im_m(F)$ on morphisms by:

$$\begin{aligned} &Im_m(F)([n' - n, \sigma]) \left(\gamma_{n, m+n}^{f, \bullet}(k) \otimes_{\mathbb{K}[\mathbf{F}_{m+n}]} F([m, id_{m+n+1}]) (w) \right) \\ &= a_{m+n', \bullet}(id_m \natural \sigma) \left(\gamma_{n, m+n'}^{f, \bullet}(k) \right) \otimes_{\mathbb{K}[\mathbf{F}_{m+n'}]} F \left(\left[m + n' - n, id_m \natural (\gamma_{n'}^{f, \bullet}(\sigma)) \right] (w) \right). \end{aligned}$$

Let us check that $Im_m(F)$ is a subfunctor of $\mathbf{LM}(\tau_m F)$. By Condition 3.6, $a_{m+n', \bullet}(id_m \natural \sigma) \circ \gamma_{n, m+n'}^{f, \bullet} = \gamma_{n', m+n'}^{f, \bullet} \circ a_{n', \bullet}(\sigma) \circ \gamma_{n, n'}^{f, \bullet}$ for all element σ of $\mathbf{B}_{n'}$. Hence:

$$\begin{aligned} & (\vartheta_{m, n'}(F) \circ Im_m(F)([n' - n, \sigma])) \left(\gamma_{n, m+n}^{f, \bullet}(k) \underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes} F([m, id_{m+n+1}](w)) \right) \\ &= a_{n', \bullet}(\sigma) \left(\gamma_{n, n'}^{f, \bullet}(k) \underset{\mathbb{K}[\mathbf{F}_{n'}]}{\otimes} F \left([m + n' - n, id_m \natural (\gamma_{n', \bullet}^{f, \bullet}(\sigma))] \right) (v) \right) \\ &= (\mathbf{LM}(\tau_m F)([n' - n, \sigma]) \circ \vartheta_{m, n}(F)) \left(\gamma_{n, m+n}^{f, \bullet}(k) \underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes} F([m, id_{m+n+1}](w)) \right). \end{aligned}$$

Hence, $Im_m(F)$ is a subfunctor of $\mathbf{LM}(\tau_m F)$ for every $F \in \text{Obj}(\mathcal{U}\beta\text{-}\mathfrak{Mod})$. Let us show that we may define a natural transformation $\vartheta_m : Im_m \Rightarrow \mathbf{LM}\tau_m$ in the category $\mathcal{U}\beta\text{-}\mathfrak{Mod}$. Let F and G be two $\mathcal{U}\beta$ -modules, and $\eta : F \Rightarrow G$ a natural transformation. We naturally define the functor Im_m on natural transformations by:

$$\forall n \in \mathbb{N}, (Im_m(\eta))_n = id_{\mathcal{I}_{\mathbb{K}[\mathbf{F}_{m+n}]}} \underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes} \eta_{m+n+1}.$$

Then, it follows from our definitions that for all natural integer n , for all $k \in \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ and $w \in F(m+n+1)$:

$$\begin{aligned} & (\vartheta_{m, n}(G) \circ (Im_m(\eta))_n) \left(\gamma_{n, m+n}^{f, \bullet}(k) \underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes} F([m, id_{m+n+1}](w)) \right) \\ &= \gamma_{m, m+n}^{f, \bullet}(k) \underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes} \eta_{m+n+1}(F([m, id_{m+n+1}](w))) \\ &= ((\mathbf{LM}\tau_m(\eta))_n \circ \vartheta_{m, n}(F)) \left(\gamma_{n, m+n}^{f, \bullet}(k) \underset{\mathbb{K}[\mathbf{F}_{m+n}]}{\otimes} F([m, id_{m+n+1}](w)) \right). \end{aligned}$$

□

The following proposition is thus a direct consequence of this lemma and Proposition 4.9.

Proposition 4.11. *Let m be a natural integer. For a coherent quadruplet $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$, there is an isomorphism in the category $\mathbf{Fct}(\mathcal{U}\beta\text{-}\mathfrak{Mod}, \mathcal{U}\beta\text{-}\mathfrak{Mod})$:*

$$\delta_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \cong \Upsilon_m^{(\bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \bigoplus \frac{\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \circ \tau_m}{Im_m}.$$

We may refine this proposition using the following lemma.

Lemma 4.12. *Let m be a natural integer and $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$ be a coherent quadruplet. There is an isomorphism in the category $\mathbf{Fct}(\mathcal{U}\beta\text{-}\mathfrak{Mod}, \mathcal{U}\beta\text{-}\mathfrak{Mod})$:*

$$\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \circ \delta_m \cong \frac{\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \circ \tau_m}{Im_m}.$$

Proof. Let F be a $\mathcal{U}\beta$ -module. Let us consider the exact sequence 1 of Lemma 2.7 applied to F :

$$0 \longrightarrow \kappa_m(F) \xrightarrow{\Omega_m(F)} F \xrightarrow{i_m(F)} \tau_m(F) \xrightarrow{\Delta_m(F)} \delta_m(F) \longrightarrow 0.$$

Since \mathbf{LM} is an exact functor (see Proposition 3.18), we deduce that the following sequence is exact:

$$0 \longrightarrow \mathbf{LM}(\kappa_m F) \xrightarrow{\mathbf{LM}(\Omega_m(F))} \mathbf{LM}(F) \xrightarrow{\mathbf{LM}(i_m(F))} \mathbf{LM}(\tau_m(F)) \xrightarrow{\mathbf{LM}(\Delta_m(F))} \mathbf{LM}(\delta_m(F)) \longrightarrow 0.$$

Therefore, because of the exactness of this sequence and Lemma 4.10, for all natural integer n :

$$\ker(\mathbf{LM}(\Delta_m(F))_n) = Im(\mathbf{LM}(i_m(F))_n) = Im_m(F)(n).$$

We consider the following commutative diagram, where the first row is a short exact sequence.

$$\begin{array}{ccccccc}
0 & \longrightarrow & Im_m(F)(n) & \xrightarrow{\vartheta_{m,n}(F)} & \mathbf{LM}(\tau_m F)(n) & \xrightarrow{\pi_n(F)} & \frac{\mathbf{LM}(\tau_m F)(n)}{Im_m(F)(n)} \longrightarrow 0 \\
& & & \searrow 0 & \downarrow \mathbf{LM}(\Delta_m(F))_n & & \\
& & & & \mathbf{LM}(\delta_m(F)) & &
\end{array}$$

The universal property of the cokernel ensures that there exists a unique \mathbb{K} -module morphism $\Psi_{n,F} : \frac{\mathbf{LM}(\tau_m F)(n)}{Im_m(F)(n)} \longrightarrow \mathbf{LM}(\delta_m F)(n)$ such that the following diagram commutes.

$$\begin{array}{ccc}
\mathbf{LM}(\tau_m(F))(n) & \xrightarrow{\mathbf{LM}(\Delta_m(F))_n} & \mathbf{LM}(\delta_m(F))(n) \\
\pi_n(F) \downarrow & \nearrow \exists! \Psi_{n,F} & \\
\frac{\mathbf{LM}(\tau_m F)(n)}{Im_m(F)(n)} & &
\end{array}$$

Moreover, since $\ker(\mathbf{LM}(\Delta_m(F))_n) = Im_m(F)(n)$, we know that $\Psi_{n,F}$ is an isomorphism of \mathbb{K} -modules.

Let us check the naturality on n . Let $n, n' \in \mathbb{N}$, such that $n' \geq n$, and $[n' - n, \sigma] \in Hom_{\mathcal{U}\beta}(n, n')$. The following square commutes.

$$\begin{array}{ccc}
\mathbf{LM}(\tau_m(F))(n) & \xrightarrow{\mathbf{LM}(\Delta_m(F))_n} & \mathbf{LM}(\delta_m(F))(n) \\
\downarrow \mathbf{LM}(\tau_m(F))[n' - n, \sigma] & & \downarrow \mathbf{LM}(\delta_m(F))[n' - n, \sigma] \\
\mathbf{LM}(\tau_m(F))(n') & \xrightarrow{\mathbf{LM}(\Delta_m(F))_{n'}} & \mathbf{LM}(\delta_m(F))(n')
\end{array}$$

Since the functor $Im_m(F)$ is a subfunctor of $\mathbf{LM}(\tau_m F)$ by Lemma 4.10, we define the quotient functor $\frac{\mathbf{LM}(\tau_m F)}{Im_m(F)}$ on morphisms by the universal property of cokernel:

$$\begin{array}{ccccccc}
0 & \longrightarrow & Im_m(F)(n) & \xrightarrow{\vartheta_{m,n}(F)} & \mathbf{LM}(\tau_m(F))(n) & \xrightarrow{\pi_n(F)} & \frac{\mathbf{LM}(\tau_m F)(n)}{Im_m(F)(n)} \longrightarrow 0 \\
Im_m(F)([n' - n, \sigma]) \downarrow & & & & \downarrow \mathbf{LM}(\tau_m(F))[n' - n, \sigma] & & \downarrow \exists! \frac{\mathbf{LM}(\tau_m F)}{Im_m(F)}([n' - n, \sigma]) \\
0 & \longrightarrow & Im_m(F)(n') & \xrightarrow{\vartheta_{m,n'}(F)} & \mathbf{LM}(\tau_m(F))(n') & \xrightarrow{\pi_{n'}(F)} & \frac{\mathbf{LM}(\tau_m F)(n')}{Im_m(F)(n')} \longrightarrow 0
\end{array}$$

Thus, the following diagram commutes.

$$\begin{array}{ccc}
\frac{\mathbf{LM}(\tau_m F)(n)}{Im_m(F)(n)} & \xrightarrow{\Psi_{n,F}} & \mathbf{LM}(\delta_m(F))(n) \\
\downarrow \frac{\mathbf{LM}(\tau_m F)}{Im_m(F)}([n' - n, \sigma]) & & \downarrow \mathbf{LM}(\delta_m(F))[n' - n, \sigma] \\
\frac{\mathbf{LM}(\tau_m F)(n')}{Im_m(F)(n')} & \xrightarrow{\Psi_{n',F}} & \mathbf{LM}(\delta_m(F))(n')
\end{array}$$

But for all natural integer n , $\pi_n(F)$ is an epimorphism. So, we obtain the commuting proving that $\Psi_F : \frac{\mathbf{LM}(\tau_m F)}{Im_m(F)} \xrightarrow{\cong} \mathbf{LM}(\delta_m F)$ is an isomorphism of $\mathbf{Fct}(\mathcal{U}\beta\text{-}\mathcal{M}od, \mathcal{U}\beta\text{-}\mathcal{M}od)$. It remains to show that we may define a natural isomorphism $\Psi : \frac{\mathbf{LM} \circ \tau_m}{Im_m} \xrightarrow{\cong} \mathbf{LM} \circ \delta_m$ in the category $\mathcal{U}\beta\text{-}\mathcal{M}od$. Let F and G be two $\mathcal{U}\beta$ -modules, and $\eta : F \rightrightarrows G$ a natural transformation. We naturally define the quotient functor $\frac{\mathbf{LM} \circ \tau_m}{Im_m}$ on natural transformations to be the quotient:

$$\begin{array}{ccc}
\mathbf{LM}(\tau_m(F))(n) & \xrightarrow{\pi_n(F)} & \frac{\mathbf{LM}(\tau_m F)(n)}{Im_m(F)(n)} \\
(\mathbf{LM}\tau_m(\eta))_n \downarrow & & \downarrow (\frac{\mathbf{LM} \circ \tau_m}{Im_m}(\eta))_n \\
\mathbf{LM}(\tau_m(G))(n) & \xrightarrow{\pi_n(G)} & \frac{\mathbf{LM}(\tau_m G)(n)}{Im_m(G)(n)}
\end{array}$$

Moreover, by naturality in objects of $\mathcal{U}\beta\text{-}\mathcal{M}\text{od}$ of the exact sequence 1 of Lemma 2.7 and because of the exactness of the Long-Moody functor (see Proposition 3.18), the natural transformation $\mathbf{LM} \circ \delta_m(\eta)$ is defined to make the following square commutative for all natural integer n .

$$\begin{array}{ccc} \mathbf{LM}(\tau_m(F))(n) & \xrightarrow{\mathbf{LM}(\Delta_m(F))_n} & \mathbf{LM}(\delta_m(F))(n) \\ (\mathbf{LM}\tau_m(\eta))_n \downarrow & & \downarrow (\mathbf{LM}\circ\delta_m(\eta))_n \\ \mathbf{LM}(\tau_m(G))(n) & \xrightarrow{\mathbf{LM}(\Delta_m(G))_n} & \mathbf{LM}(\delta_m(G))(n) \end{array}$$

Then, by definition of $\Psi_{n,F}$, we deduce that the following diagram is commutative for all natural integer n .

$$\begin{array}{ccc} \frac{\mathbf{LM}(\tau_m F)(n)}{Im_m(F)(n)} & \xrightarrow{\Psi_{n,F}} & \mathbf{LM}(\delta_m(F))(n) \\ \left(\frac{\mathbf{LM}\circ\tau_m}{Im_m}(\eta)\right)_n \downarrow & & \downarrow (\mathbf{LM}\circ\delta_m(\eta))_n \\ \frac{\mathbf{LM}(\tau_m G)(n)}{Im_m(G)(n)} & \xrightarrow{\Psi_{n,G}} & \mathbf{LM}(\delta_m(G)) \end{array}$$

□

Lemma 4.12 leads therefore to the following result.

Theorem 4.13. *Let m be a natural integer and $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$ let be a coherent quadruplet. There is a natural isomorphism in the category $\mathcal{U}\beta\text{-}\mathcal{M}\text{od}$:*

$$\delta_m \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \cong \gamma_m \bigoplus \left(\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \circ \delta_m \right).$$

Furthermore, we may determine the behaviour of the evanescence functor.

Theorem 4.14. *Let m be a natural integer and $(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$ be a coherent quadruplet. Then, the endofunctor κ_m commutes with the endofunctor $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$ of the category $\mathbf{Fct}(\mathcal{U}\beta\text{-}\mathcal{M}\text{od}, \mathcal{U}\beta\text{-}\mathcal{M}\text{od})$. In other words, there is a natural isomorphism in the category $\mathbf{Fct}(\mathcal{U}\beta\text{-}\mathcal{M}\text{od}, \mathcal{U}\beta\text{-}\mathcal{M}\text{od})$:*

$$\kappa_m \circ \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \cong \mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})} \circ \kappa_m.$$

Proof. Since the Long-Moody functor is exact (see Proposition 3.18), we have the following exact sequence.

$$0 \longrightarrow \mathbf{LM} \circ \kappa_m \xrightarrow{\mathbf{LM} \circ \Omega_m} \mathbf{LM} \xrightarrow{\mathbf{LM} \circ i_m} \mathbf{LM} \circ \tau_m \xrightarrow{\mathbf{LM} \circ \Delta_m} \mathbf{LM} \circ \delta_m \longrightarrow 0$$

Therefore, the image functor $Im(\mathbf{LM} \circ i_m)$ is isomorphic to the kernel functor $Ker(\mathbf{LM} \circ \Delta_m)$. Applying the exact sequence 1 of Lemma 2.7 to $\mathbf{LM}(F)$, we obtain the following diagram, where the first line is an exact sequence and the square is commutative.

$$\begin{array}{ccccccc} 0 \longrightarrow \kappa_m \circ \mathbf{LM} & \xrightarrow{\Omega_m \circ \mathbf{LM}} & \mathbf{LM} & \xrightarrow{i_m \circ \mathbf{LM}} & \tau_m \mathbf{LM} & \xrightarrow{\Delta_m \circ \mathbf{LM}} & \delta_m \mathbf{LM} \longrightarrow 0 \\ & & & \downarrow \cong & \downarrow \cong & & \downarrow \cong \text{ by Theorem 4.13} \\ & & & \gamma_m \bigoplus (\mathbf{LM} \circ \tau_m) & \xrightarrow[\text{by Lemma 4.10}]{id \oplus (\mathbf{LM} \circ \Delta_m)} & \gamma_m \bigoplus (\mathbf{LM} \circ \delta_m) & \end{array}$$

By the exactness property, we deduce:

$$Im(\mathbf{LM} \circ i_m) \cong Ker(\mathbf{LM} \circ \Delta_m) \cong Im(i_m \circ \mathbf{LM}).$$

As the functor $\kappa_m \circ \mathbf{LM}$ (resp. $\mathbf{LM} \circ \kappa_m$) is defined to be the kernel of $\mathbf{LM} \xrightarrow{i_m \circ \mathbf{LM}} \tau_m \mathbf{LM}$ (resp. of $\mathbf{LM} \xrightarrow{\mathbf{LM} \circ i_m} \mathbf{LM} \circ \tau_m$), by the unicity up to isomorphism of the kernel, we conclude that $\kappa_m \circ \mathbf{LM} \cong \mathbf{LM} \circ \kappa_m$. □

4.3 Increasing of the polynomial degree

Proposition 4.15. *Let m be a natural integer and F be a $\mathfrak{U}\beta$ -module. If the functor F is strong (resp. very strong) polynomial of degree n , then the functor $\Upsilon_m^{(\bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$ sends F to a strong polynomial functor of degree n and $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$ sends F to a strong (resp. very strong) polynomial functor of degree $n+1$.*

Proof. The result for a strong polynomial functor is trivial for Υ_m by Corollary 4.7. We proceed by induction for \mathbf{LM} . For a strong polynomial functor of degree 0, by Theorem 4.13, we obtain that:

$$\delta_m \mathbf{LM}(F) \cong \Upsilon_m(F).$$

Therefore $\mathbf{LM}(F)$ is a degree 1 strong polynomial functor. Now, suppose that F is a strong polynomial functor of degree n . By Theorem 4.13:

$$\delta_m \mathbf{LM}(F) \cong \mathbf{LM}(\delta_m)(F) \bigoplus \Upsilon_m(F).$$

By the induction hypothesis and the result on Υ_m , we deduce that $\mathbf{LM}(F)$ is a strong polynomial functor of degree $n+1$. Moreover, if we assume that F is a very strong polynomial functor of degree n , the result follows directly from Theorem 4.14. \square

Corollary 4.16. *The endofunctor $\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}$ of $\mathfrak{U}\beta\text{-Mod}$ increases the degree of a (very) strong polynomial functor by 1. For all natural integer n , it restricts to a functor:*

$$\mathbf{LM}_{(\bullet_a, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}^{\text{Pol}, n} : \text{Pol}_n(\mathfrak{U}\beta, \mathbb{K}\text{-Mod}) \longrightarrow \text{Pol}_{n+1}(\mathfrak{U}\beta, \mathbb{K}\text{-Mod})$$

which sends every degree n polynomial functor to a degree $n+1$ polynomial functor.

Example 4.17. A first interesting example of application of the Long-Moody construction is what happens when one applies it to $\chi_{n+1} : \mathbf{B}_{n+1} \rightarrow \mathbb{K}$ a one-dimensional complex representation of a braid group. We assume additionally that for all $i \in \{1, \dots, n\}$, $\chi_{n+1}(\sigma_i) \in \mathbb{K}^\times$. Because of the braid relation $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$, we obtain that:

$$\exists x \in \mathbb{K}^*, \forall i \in \{1, \dots, n\}, \chi_{n+1}(\sigma_i) = x.$$

Let $(1, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})$ be a coherent quadruplet where $a_{n, \bullet} = a_{n, 1}$. Applying the Long-Moody construction, we obtain:

$$\forall i \in \{1, \dots, n-1\}, \text{lm}_{(1, \bullet_\varsigma, \bullet_{\gamma^b}, \bullet_{\gamma^f})}(\chi_{n+1})(\sigma_i) = \begin{bmatrix} x & 0 & & \cdots & & 0 \\ 0 & \ddots & & & & \\ & & x & & & \vdots \\ & & \ddots & X(x) & \ddots & \\ \vdots & & & & x & \\ & & & & & \ddots & 0 \\ 0 & \cdots & & & & 0 & x \end{bmatrix}$$

where, for $x^2 = t$, $X(x) = x \begin{bmatrix} 0 & t \\ 1 & (1-t) \end{bmatrix}$. Therefore, we recover a variation of the Burau representation of \mathbf{B}_n . We denote by $\mathfrak{X} : \mathfrak{U}\beta \rightarrow \mathbb{K}\text{-Mod}$ the functor defined by:

- *Objects:* $\forall n \in \mathbb{N}, \mathfrak{X}(n) = \mathbb{K}$.
- *Morphisms:*
 - *Automorphisms:* for all natural integer n , for every Artin generator σ_i of \mathbf{B}_n (with $i \in \{1, \dots, n-1\}$):

$$\mathfrak{X}(\sigma_i) = \chi_n(\sigma_i) = x.$$

- General morphisms: Let $n, n' \in \mathbb{N}$, such that $n' \geq n$, and $[n' - n, \sigma_i] \in \text{Hom}_{\mathfrak{U}\beta}(n, n')$. We define:

$$\mathfrak{X}([n' - n, \sigma_i]) := \mathfrak{X}(\sigma_i) \circ i_{\mathbb{C} \hookrightarrow \mathbb{C}} = \mathfrak{X}(\sigma_i).$$

Example 4.18. By Proposition 2.21, \mathfrak{X} is a degree 0 strong polynomial functor. Now applying the Long-Moody functor, we obtain the functor $\mathbf{LM}(\mathfrak{X}) : \mathfrak{U}\beta \longrightarrow \mathbb{K}\text{-Mod}$ defined by:

- Objects: $\forall n \in \mathbb{N}$, $\mathbf{LM}(\mathfrak{X})(n) \cong \mathbb{K}^{\oplus n}$.
- Morphisms:

- Automorphisms: for all natural integer n , for every Artin generator σ_i of \mathbf{B}_n (with $i \in \{1, \dots, n-1\}$):

$$\mathbf{LM}_{(1, \bullet_{\varsigma}, \bullet_{\gamma b}, \bullet_{\gamma f})}(\mathfrak{X})(\sigma_i) = x \begin{bmatrix} 1 & 0 & & \cdots & 0 \\ 0 & \ddots & & & \\ & & 1 & & \vdots \\ & & \ddots & \tilde{B}(t) & \ddots \\ \vdots & & & & 1 \\ 0 & \cdots & & & \ddots & 0 \\ & & & & 0 & 1 \end{bmatrix}$$

where $x^2 = t$ and:

$$\tilde{B}(t) = \begin{bmatrix} 0 & 1 \\ t & 1-t \end{bmatrix}.$$

- General morphisms: Let $n, n' \in \mathbb{N}$, such that $n' \geq n$, and $[n' - n, \sigma_i] \in \text{Hom}_{\mathfrak{U}\beta}(n, n')$. We define:

$$\mathbf{LM}_{(1, \bullet_{\varsigma}, \bullet_{\gamma b}, \bullet_{\gamma f})}(\mathfrak{X})([n' - n, \sigma_i]) = \mathbf{LM}_{(1, \bullet_{\varsigma}, \bullet_{\gamma b}, \bullet_{\gamma f})}(\mathfrak{X})(\sigma_i) \circ i_{\mathbb{K}^{\oplus n} \hookrightarrow \mathbb{K}^{\oplus n'}}.$$

The functor $\mathbf{LM}_{(1, \bullet_{\varsigma}, \bullet_{\gamma b}, \bullet_{\gamma f})}(\mathfrak{X})$ is very similar to \mathfrak{Bur} . For $y \in \mathbb{K}^*$, let $\mathfrak{Bur}_y : \mathfrak{U}\beta \longrightarrow \mathbb{K}\text{-Mod}$ be the functor defined in the same way as \mathfrak{Bur} , except that for all natural integer n , for every Artin generator σ_i of \mathbf{B}_n (with $i \in \{1, \dots, n-1\}$): $\mathfrak{Bur}_y(\sigma_i) = y\mathfrak{Bur}(\sigma_i)$. Then, we define the natural transformation $\mathbf{LM}_{(1, \bullet_{\varsigma}, \bullet_{\gamma b}, \bullet_{\gamma f})}(\mathfrak{X}) \xrightarrow{r_n'''} \mathfrak{Bur}_x$ with for all integer n , $r_n''' : \mathbb{K}^{\oplus n} \hookrightarrow \mathbb{K}[t^{\pm 1}]^{\oplus n}$ being the matrix

$$\overbrace{\begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}}^n.$$

Hence, $\mathbf{LM}_{(1, \bullet_{\varsigma}, \bullet_{\gamma b}, \bullet_{\gamma f})}(\mathfrak{X})$ is a subfunctor of \mathfrak{Bur}_x (which is clearly very strong polynomial by Proposition (2.29)), and a fortiori by Proposition 2.19, $\mathbf{LM}(\mathfrak{X})$ is a very strong polynomial functor of degree 1.

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