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On function field Mordell-Lang: the semiabelian case and the socle theorem

Franck Benoist* Elisabeth Bouscaren* Anand Pillay†

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Abstract

In this paper we complete our “second generation” model-theoretic account of the function field Mordell-Lang conjecture, where we avoid appeals to dichotomy theorems for (generalized) Zariski geometries. In the current paper we reduce the semiabelian case to the abelian case using model-theoretic tools. We also use our results from [2] to prove modularity of the “Manin kernels” (over $\mathbb{F}_p(t)^{sep}$ in positive characteristic and over $\mathbb{C}(t)^{alg}$ in characteristic 0).

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1 Introduction

This paper is the third in a series of papers by the authors, of which the first two are [1], [2], revisiting the model-theoretic approach to function field Mordell-Lang initiated by Hrushovski [11], and where positive characteristic is the main case of interest. In [1] we gave, among other things, a counterexample to a claim, implicit in [11], that a key model-theoretic object G^\sharp attached to a semiabelian variety G , had finite relative Morley rank. In [2] and the current paper, the main aim is to give a model-theoretic proof of the

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main theorem (function field Mordell-Lang) avoiding the appeal in [11] to results around Zariski geometries. Our reasons are (i) this appeal to Zariski geometries is something of a black box, which is difficult for model theorists and impenetrable for non model-theorists and (ii) in the positive characteristic case, it is “type-definable” Zariski geometries which are used and for which there is no really comprehensive exposition, although the proofs in [11] are correct. In [2] we concentrated on the abelian variety case, and found a new model-theoretic proof when the ground field K is $\mathbb{F}_p(t)^{sep}$, by reduction to a Manin-Mumford statement. The new contributions of the current paper are the following:

(i) We give a reduction of function field Mordell-Lang for semiabelian varieties to the case of traceless abelian varieties in all characteristics and without any assumptions on the ground field. This result, together with [2], gives a self-contained model theoretic proof of function field Mordell-Lang for semiabelian varieties over $\mathbb{F}_p(t)^{sep}$, by reduction to a Manin-Mumford statement (and avoids appeal to Zariski geometries).

(ii) We prove the “weak socle theorem” from [11] but in the general form that is really needed both in [11] and in the current paper (namely the finite U -rank rather than finite Morley rank context).

(iii) We introduce a new algebraic-geometric object, in all characteristics, the algebraic K_0 -socle of a semiabelian variety G over K_1 , where $K_0 < K_1$ are algebraically closed fields. We use this socle to reduce function field Mordell-Lang for semiabelian varieties to the case of semiabelian varieties which are isogenous to a direct product of a torus and an abelian variety.

(iv) We give a new proof of the modularity (or 1-basedness) of A^\sharp when A is simple with trace 0 over the constants, for the cases from [2], where we derived Mordell-Lang for abelian varieties over $\mathbb{F}_p(t)^{sep}$ or over $\mathbb{C}(t)^{alg}$ without appealing to the dichotomy theorem for Zariski geometries.

Here is a brief description of what is covered, section-by-section. In Section 2 we state and prove the “weak socle theorem” for type-definable groups of finite U -rank. In Section 3 we apply this to the case where the type-definable group is G^\sharp for G a semiabelian variety, and we also introduce the algebraic socle. In Section 4 we reduce Mordell-Lang to the case of the algebraic socle, and then prove the main result of the paper. In Section 5 we discuss model theoretic properties of the socle of G^\sharp , in particular proving quantifier elimination and finite relative Morley rank. In the appendix we give proofs of *modularity* (or 1-basedness) of A^\sharp as mentioned earlier.

We assume familiarity with model theory, basic stability, as well as differ-

entially and separably closed fields. The book [13] is a reasonable reference, as well as [16] for more on stability theory. Definability means with parameters unless we say otherwise.

We finish the introduction with some more comments around the socle theorem. The socle is used, via the weak socle theorem in both characteristics by Hrushovski to prove function field Mordell-Lang directly for semiabelian varieties. For an abelian variety A , the socle of A^\sharp is A^\sharp itself, but still the weak socle theorem is needed in characteristic 0 for our model theoretic proofs for example in [2]. But this weak socle theorem is *not* needed for abelian varieties in characteristic p . It is needed though for semiabelian varieties even in characteristic p , and in fact in this paper we aim to explain why by showing how the use of the socle allows us to reduce to semiabelian varieties of a particular pleasant form.

We will see in particular that, in characteristic p , the “bad behavior” from the point of view of model theory of some semiabelian varieties G , which translates into “bad behavior” of G^\sharp , disappears when one replaces G^\sharp by its socle.

2 The model theoretic socle; the abstract case

2.1 Preliminaries

We suppose that $T = T^{eq}$ is a stable theory. We work in a big model of T , \mathcal{U} , saturated and homogeneous.

Some conventions: Let $S \subset \mathcal{U}^n$, we say that S is *type-definable* if S is an intersection of definable sets, or equivalently if S is a partial type, that is, given by a conjunction of formulas which can be infinite (of cardinality strictly smaller than that of \mathcal{U}). As usual, we will identify a type-definable $S(x)$ with its set of realizations in the big model \mathcal{U} . For $M \preceq \mathcal{U}$ a model of T , if S is type-definable in \mathcal{U}^n , with parameters from M , we denote by $S(M)$ the set of realizations of S in M , $S(M) := \{m \in M^n : M \models S(m)\}$.

When we say that S is a *type-definable minimal set*, we mean that for any definable set D , $S(\mathcal{U}) \cap D(\mathcal{U})$ is finite or cofinite in $S(\mathcal{U})$. In other words, every relatively definable subset of $S(\mathcal{U})$ is finite or cofinite. When we say that Q is a *minimal (complete) type* over a given set of parameters A , we mean that Q is minimal and is also a complete type over A , hence is precisely a complete stationary type of U -rank one over A .

Further conventions of notation: If H, G are groups, we will use the notation $H < G$ to mean that H is a (non necessarily proper) subgroup of G . Likewise, if K_0, K_1 are fields, $K_0 < K_1$ just means that K_0 is a subfield of K_1 .

We recall first the classic Indecomposability theorem, stated for the case of a group of finite U -rank. This is, in principle, a particularly simple case ($\alpha = 0$) of the Theorem of α -indecomposables for superstable groups, which generalizes the Zilber Indecomposability Theorem for groups of finite Morley Rank (Theorem 6.10 in [20] or Theorem 3.6.11 in [24]). However we are typically working with a type-definable group G rather than a definable group, so strictly speaking there may not be an ambient superstable theory in which G is type-definable. Nevertheless, the proofs in the references above easily adapt.

Definition 2.1 *A type-definable subset S of G is said to be indecomposable if for every relatively definable subgroup H of G , either S/H is infinite or all elements of S are in the same coset of H (denoted $|S/H| = 1$).*

Let us recall first:

Lemma 2.2 – *If Q is a complete stationary type in G , then Q is indecomposable. Moreover if a is any realization of the type Q in \mathcal{U} , then the type-definable set $aQ := \{x \in \mathcal{U} : a^{-1}x \models Q\}$ is also indecomposable.*

– *Any type-definable connected subgroup of G is indecomposable.*

Proof : Suppose G and Q are defined over \emptyset . Let H_b be any relatively definable subgroup of G , defined over some finite set b such that $|Q/H_b| = n > 1$. Let H_0 be the intersection of all the $H_{b'}$, with $b' \in \mathcal{U}$, realizing the same type as b over \emptyset . Then by stability of T and the Baldwin-Saxl condition (Prop. 1.4 in [20]), H_0 is the intersection of a finite number of the H_b . Then H_0 is definable over \emptyset and Q/H_0 is finite, strictly bigger than 1. This contradicts the stationarity of Q . The rest is obvious. \square

Proposition 2.3 *The Indecomposability Theorem. Let G be a type-definable connected group of finite U -rank, n .*

Let $\{S_i : i \in I\}$ be a family of type-definable indecomposable subsets of G , such that each S_i contains the identity of G . Then the subgroup H of G generated by the S_i is type-definable and connected. In fact, there is some $m \leq n$ such that $H = (S_{i_1} \dots S_{i_m})^2$.

Corollary 2.4 1. *If $(B_i)_{i \in I}$ are type-definable connected subgroups of G , then the group generated by the B_i is type-definable and connected. Furthermore it is the product of finitely many of the B_i .*

From now on, we will suppose that the group G is commutative, to simplify notation.

Definition 2.5 1. *Let G be a type-definable commutative connected group of finite U -rank. We say that G is almost generated by type-definable sets Q_1, \dots, Q_n if $G \subset \text{acl}(F \cup Q_1 \cup \dots \cup Q_n)$, where F is some finite set of parameters, by which we mean that $\mathcal{U}, G(\mathcal{U}) \subset \text{acl}(F \cup Q_1(\mathcal{U}) \cup \dots \cup Q_n(\mathcal{U}))$.*

When G is definable and almost generated by a single strongly minimal set, then G is usually called almost strongly minimal. If G is almost generated by a single minimal type-definable set Q , we say that G is almost minimal with respect to Q . In order to be able, further down, to treat uniformly all characteristics, we will make the convention that we will say that G is almost minimal with respect to Q , whether Q is a strongly minimal definable set or a minimal type-definable set.

If G is almost generated by minimal type-definable Q_1, \dots, Q_n we say that G is almost pluri-minimal.

2. *Let G be a type-definable connected group, defined over some $B = \text{acl}(B)$, G is said to be rigid if, passing to a saturated model, all connected type-definable subgroups of G are defined over $\text{acl}(B)$.*

The following is well-known, but follows from the more general Lemma 4.7 from Chapter 7 of [16].

Proposition 2.6 *Let G be a type-definable connected group, almost minimal with respect to a minimal type Q . If Q is not orthogonal to the type-definable set D , then there is a type-definable group $H \subset \text{dcl}(D)$ and a definable surjective homomorphism $h : G \rightarrow H$ with finite kernel.*

2.2 The Socle

One can find definitions and proofs for the existence of the socle and the “weak socle theorem” in the case of groups of finite Morley Rank, for example

in [11] or [4]. But there is no reference, as far as we know, for the case of type-definable groups of finite U -rank, although it is a result which has been used before. So we take this opportunity to give a complete presentation here.

Let again $T = T^{eq}$ be stable and G be a type-definable commutative group, connected and of finite U -rank, n . We work in a big model \mathcal{U} , over some algebraically closed set A over which G is defined.

Remark: We know that in the group G , because of finite U -rank, there are only a finite number of orthogonality classes of minimal types, hence we could work over some model M_0 over which they are all represented; on the other hand, the arguments below reprove that there are only finitely many such classes.

Lemma 2.7 *Let Q be a minimal type-definable set, contained in G . Then there is a unique maximal connected type-definable subgroup B_Q of G which is almost minimal with respect to Q . Furthermore, if Q' is a minimal type extending Q then Q'/B_Q is trivial.*

Proof: Let Q' be the unique *minimal type* extending Q over the same set of parameters as Q (i.e. complete stationary type of U -rank 1, extending Q).

First let $a \in \mathcal{U}$ realize the type Q' , and let $Q'_1 = Q' - \{a\} := \{x ; x = b - a, b \models Q'\}$, Q'_1 is an indecomposable type-definable set containing zero (2.2). So by the indecomposability theorem (2.3), the group generated by Q'_1 , $H(Q'_1)$, is type-definable and connected. Clearly $Q' \subset H(Q'_1) \cup (a + H(Q'_1))$, so $Q'/H(Q'_1)$ is finite and by indecomposability of Q' , of cardinality one.

Now consider the class of all connected type-definable subgroups of G , which are almost minimal with respect to Q , denoted by S_Q . By finite U -rank of G , one can choose such a group, B_Q , of maximal U -rank. Then, for any other connected type-definable group, H in S_Q , $B_Q + H$ is connected and type definable (by the Indecomposability Theorem), and is also almost minimal with respect to Q . As it contains B_Q it has bigger or equal rank. By maximality of the rank of B_Q , it has same rank, so by connectedness it is equal to B_Q and it follows that any such H is contained in B_Q . So B_Q is the unique type-definable subgroup of G which is maximal connected almost minimal with respect to Q . In particular the group $H(Q'_1)$ from above is contained in B_Q hence Q'/B_Q has cardinality one. \square

Proposition 2.8 *There is a type-definable connected subgroup of G , $S(G)$, which is the unique maximal almost pluri-minimal connected subgroup of G . Furthermore, $S(G) = B_{Q_1} + \dots + B_{Q_m}$, for some m , where each Q_i is a minimal type-definable set, and:*

- B_{Q_i} is the maximal type-definable connected subgroup of G which is almost minimal with respect to Q_i
- $Q_j \perp Q_i$ if $i \neq j$, so in particular for $i \neq j$, the intersection of B_{Q_i} and B_{Q_j} is finite
- for any other minimal type-definable Q in G , Q is non orthogonal to one of the Q_i .

We call $S(G)$ the (model-theoretic) socle of G .

Proof: Consider the class S of all the connected type-definable subgroups of G , H , which are such that, for some finite F , and some k , $H \subset \text{acl}(F \cup Q_1 \cup \dots \cup Q_k)$, for F finite and Q_i minimal type-definable in G .

The exact same arguments as in Lemma 2.7 show that if one chooses a group B in S having maximal U-rank, it will, as above, be maximal and unique. We denote it $S(G)$.

Let R be the class of all subgroups (connected and type-definable by the Indecomposability Theorem (2.3)) of the form $H = B_{Q_1} + \dots + B_{Q_n}$, where B_Q denotes the unique maximal subgroup of G almost minimal with respect to Q , given by the previous lemma.

Choose a group H in R with maximal U-rank. As above, H is maximal and unique.

Claim 2.9 $H = S(G)$.

Proof of the Claim: Of course, by maximality of $S(G)$, $H \subset S(G)$. Suppose for contradiction that H and $S(G)$ are not equal, then by connectedness of $S(G)$, $S(G)/H$ is infinite of unbounded cardinality. Let M_0 be a model such that $G, S(G)$ and B_{Q_1}, \dots, B_{Q_n} (and hence also H) are defined over M_0 . As $S(G)/H$ is infinite, if c realizes the generic type of $S(G)$ over M_0 , then the coset $\hat{c} := c + H$ is not algebraic over M_0 .

By definition of $S(G)$, there are a finite set F (which we can suppose included in M_0), and b_1, \dots, b_k , each realizing a minimal type over M_0 , such that $c \in \text{acl}(F, b_1, \dots, b_k)$, so $c \in \text{acl}(M_0, b_1, \dots, b_k)$. Choose k minimal. Let $B = \{b_1, \dots, b_k\}$ and $B_j := B - \{b_j\}$. By minimality of k , for any j , $c \notin \text{acl}(M_0 B_j)$. Now \hat{c} is algebraic over $M_0 c$, hence over $M_0 B$. Let $B' \subset B$ be

maximal such that \hat{c} is not algebraic over M_0B' but is algebraic over $M_0B'b_j$, for some $b_j \in B \setminus B'$ (this exists because \hat{c} is not algebraic over M_0). So \hat{c} and b_j are equialgebraic over M_0B' . It follows that c is algebraic over $M_0B_j\hat{c}$. Now c has the same U -rank as b_j over M_0B_j , that is, it has U -rank one. Let Q denote $tp(c/acl(M_0B_j))$. Then we have seen earlier that $|Q/B_Q| = 1$ and, as by maximality of H , $B_Q \subset H$, it follows that $|Q/H| = 1$ also. Hence \hat{c} is in fact algebraic over M_0B_j , and so $c \in acl(M_0B_j)$, which contradicts the minimality of k . \square

Claim 2.10 $S(G) = B_{Q_1} + \dots + B_{Q_n}$, where the minimal Q_i are pairwise orthogonal and any other minimal Q is non orthogonal to one of the Q_i 's.

Proof of the Claim : If Q_1 and Q_2 are non orthogonal, then $B_{Q_1} = B_{Q_2}$: If they are non orthogonal, they might still be weakly orthogonal, but, over some finite set F , Q_1 and Q_2 are equialgebraic so each B_{Q_i} is almost minimal with respect to the other.

If Q_1 and Q_2 are orthogonal, then $B_{Q_1} \cap B_{Q_2}$ is contained in the algebraic closure of a finite set, hence, as $B_{Q_1} \cap B_{Q_2}$ is type-definable, by compactness, it is finite.

Suppose some Q were orthogonal to all the Q_i 's. Then B_Q could not be included in $S(G)$, contradicting the maximality. \square

This finishes the proof of the proposition. \square

Note that this reproves that in G there are only finitely many non orthogonality classes of minimal types.

If G is a definable group with finite Morley rank, then we can work with definable minimal sets, namely strongly minimal sets, and the groups B_{Q_i} are almost strongly minimal (see for example [11] or [20]). Recall also that in the finite Morley rank case, every type-definable subgroup of G is in fact definable.

So in that case, Proposition 2.8 gives the following:

Proposition 2.11 The finite Morley Rank case: *Let G be a definable connected commutative group of finite Morley rank. Then there is a definable connected subgroup of G , $S(G)$ which is the unique maximal connected almost pluri-minimal subgroup of G .*

Furthermore, $S(G) = B_{Q_1} + \dots + B_{Q_m}$, for some m , where each Q_i is a strongly minimal set and:

- B_{Q_j} is the maximal definable connected subgroup of G which is almost minimal with respect to Q_i
- $Q_j \perp Q_i$ if $i \neq j$, so in particular for $i \neq j$, the intersection of B_{Q_i} and B_{Q_j} is almost finite (contained in the algebraic closure of a finite set)
- for any other strongly minimal set Q in G , Q is non orthogonal to one of the Q_i .

In this section there has been a blanket assumption that the ambient theory T is stable. But in fact in Proposition 2.11 for example the assumption is not required, noting that finite Morley rank of the definable group G is in the sense of the ambient theory, and implies stable embeddability of G .

Finally some remarks about sets of definition for $S(G)$: In the finite Morley rank case, the finite number of non orthogonality classes of strongly minimal sets in G are represented over any model over which G is defined. In the finite U -rank case, as mentioned above, in order to have representatives for all the finitely many non orthogonality classes of complete minimal types, one might need to work over a slightly saturated model. But, in both cases, if G is defined over some set A , then by uniqueness and maximality, each of the type-definable groups B_{Q_i} and the type-definable group $S(G)$ itself will be left invariant by any A -automorphism. It follows that they are all in fact defined also over A .

2.3 The Weak Socle Theorem

Here we will state and prove what we call the “weak socle theorem” for type-definable groups of finite U -rank. This was originally proved by Hrushovski in [11], although, in the paper, he only gave the proof for the finite Morley rank case. There are a few steps where one has to be a little more careful when dealing with type-definable groups and finite U -rank. The reason we call it the weak socle theorem is as in [2], namely that a stronger statement which is sometimes referred to as the socle theorem (Theorem 2.1 of [18]) was proved in the context of algebraic D -groups

We work in a saturated model \mathcal{U} of a stable theory $T = T^{eq}$, G will be a connected type-definable group of finite U -rank. We will assume G to be commutative, and will use additive notation, but the theorem works in general. Let $S(G)$ be the model-theoretic socle of G as we have defined it in the previous section. Assume G , so also $S(G)$, are type-defined over \emptyset . We

will be making use of the stability-theoretic fact that if D is a type-definable set (over parameters A say) and a is a finite tuple, then there is a small subtuple c from D such that $tp(a/A, c)$ implies $tp(a/D, A)$. It can be proved by using the fact that $tp(a/D, A)$ is definable (Proposition 2.19 of [15]).

Theorem 2.12 *Let $p(x)$ be a complete stationary type (over \emptyset say) of an element of G . Assume $\text{Stab}_G(p)$ is finite. Assume also that every connected type-definable subgroup of $S(G)$ is type-defined over $\text{acl}(\emptyset)$ (ie $S(G)$ is rigid). Then there is a coset (translate) C of $S(G)$ in G such that all realizations of p are contained in C .*

Proof : Without loss work over $\text{acl}(\emptyset)$, so $\emptyset = \text{acl}(\emptyset)$. By stability ([16], Lemma 6.18) $S(G)$ is an intersection of relatively definable subgroups L_i of G , all over \emptyset . As G has finite U -rank, there is L_i such that $S(G)$ is the connected component of L_i .

Note that if the set X of realizations of p is contained in a single coset of L_i , then it is also contained in a single coset of $S(G)$: such a coset C of L_i will be defined over \emptyset (it must be left invariant by every automorphism), and as $S(G)$ has bounded index in L_i and p is stationary, X will also be contained in a single coset of $S(G)$.

So now we write L for L_i . As L is relatively definable in G , G/L is also a type-definable group (in T^{eq}). The group L is not connected, but our assumptions imply that every connected type-definable subgroup of L is still type-defined over \emptyset (as it is contained in the connected component of L , $S(G)$).

Claim 1. If X is contained in $\text{acl}(L(\mathcal{U}))$, then X is contained in a single coset of $S(G)$.

Proof of Claim 1. If $X \subset \text{acl}(L(\mathcal{U}))$, as X is a complete type over \emptyset , there is a formula $\phi(x, y)$, with $y = (y_1, \dots, y_k)$, which is algebraic for all y , such that $p(x) \vdash \exists y_1 \dots \exists y_k (\bigwedge_i (y_i \in L \wedge y_i \in Y_i) \wedge \phi(x, y))$, and Y_i is a complete type over \emptyset . As $S(G)$ is the connected component of L , L is the union of a bounded number of cosets of $S(G)$ hence all elements of the complete type Y_i are in the same coset of $S(G)$. Let d_i be an element of Y_i , then $X \subset \text{acl}(d_1, \dots, d_k, S(G))$, so in particular X is semi-pluriminimal. As X is a stationary type, it is indecomposable (2.2). Let $a \in X$, then $X - a$ is indecomposable and contains the identity, so generates a connected infinitely definable group, which is almost pluri-minimal (as X is), and hence must be included in $S(G)$. So $X - a$ is contained in $S(G)$. \square

Let $\pi : G \rightarrow G/L$ be the canonical projection. Let a realize p and let $b = \pi(a)$. So $tp(b)$ is stationary too. If it is algebraic then by stationarity, it is a single point, whereby X is contained in a single coset of L , hence, as noted above, also in a single coset of $S(G)$ as required. Otherwise $tp(b)$ is nonalgebraic, and we aim for a contradiction.

Let $G_b = \pi^{-1}(b)$ and $X_b = X \cap G_b$. Note that G_b is a coset of L so L acts on G_b by addition. We will show, in Claims 2 and 3 below, together with the assumption that $Stab(p)$ is finite, that in fact $G_b \subset acl(L, b)$ (even in the case when $tp(b)$ is algebraic). The argument is in hindsight a Galois-theoretic argument (finiteness of a certain definable automorphism group). When $tp(b)$ is *not* algebraic, we get our contradiction by showing that the socle of G has to properly contain $S(G)$.

Claim 2. There is a type-definable subgroup H of L such that each orbit of H contained in G_b is the set of realizations of a complete type over $Lb := L \cup \{b\}$ and moreover these are precisely the complete types over Lb realized in G_b .

Proof of Claim 2. Consider any a' in X_b . As remarked just before the statement of the Theorem, there is a small tuple c from L such that $r := tp(a'/c, b)$ implies $tp(a'/L, b)$. Let Z be the set of realizations of r .

As Z is also a complete type over Lb , for each $s \in L$, $s + Z$ is the set of realizations of a complete type over Lb . Moreover $s + Z \subset G_b$ and for any $x \in G_b$, there is some $s \in L$ such that $x - a' = s$, so $x \in L + Z$, hence G_b is the union of the $s + Z$, which are pairwise disjoint or equal, as they are complete types over Lb .

It remains to see that Z (and so each of its L -translates) is an H -orbit for some type-definable subgroup H of L . Let $H = \{h \in L : h + a' \models r\}$. Then H is a type-definable subset of S , which is clearly independent of the choice of a' realizing r (as any two realizations of r have the same type over Lb). It follows that $H = \{h \in S : h + Z = Z\}$ is the stabilizer of the complete type r , and is hence a type-definable group (over cb). It follows also that $Z = a'' + H$ for any $a'' \in Z$: as $s + Z$ and Z are either equal or disjoint, for any $a_1, a_2 \in Z$, $a_1 - a_2 \in H$.

Now for $s \in L$, $s + Z = s + a + H$, so is an H -orbit. Hence G_b is a union of H -orbits which are each a complete type over Lb . \square

As X_b is a type-definable subset of G_b (over b), it is also a union of such H -orbits. Let H^0 be the connected component of H , so by assumption H^0 is type-definable over \emptyset . As H itself is a (bounded) union of H^0 -orbits, so is X_b .

Claim 3. H^0 is contained in $Stab(p)$.

Proof of Claim 3. Let $h \in H^0$ independent from a over \emptyset , then $h+a \in X_b$ by the previous claim, so $h+a$ also realizes p . As p is stationary, then for any d realizing p and independent from h over \emptyset , $h+d$ will also realize p . \square

It follows from Claim 3 and our assumption that p has finite stabilizer, that H^0 is finite, so trivial, and so H is finite. Thus also the translate Z of H is finite. So Z , the set of realizations of $r = tp(a/b, c)$, is defined by an algebraic formula $\phi(x, b, c)$.

Now we make use of the assumption that $tp(b)$ is nonalgebraic. Note that $tp(b/c)$ remains non algebraic. If not, then $Z \subset acl(c)$, so $a \in acl(L)$, which was ruled out in Claim 1.

Let Y be the set of realizations of $tp(b/c)$. We now work over c . Let Z' be the union of all the sets defined by formulas $\phi(x, b', c)$ where $b' \in Y$. Then Z' is a type-definable (over c) subset of G which projects finite-to-one on Y . Let Q be a complete minimal type extending Z' over some finite $F \supset c$. Then Q/L must be infinite as $Q \cap \pi^{-1}(b')$ is finite for every $b' \in Y$. Let $d \in Q$, then $Q - d$ is indecomposable and contains the identity, so it generates a connected type-definable almost minimal subgroup in G , which must hence be contained in $S(G) \subset L$, contradicting that Q/L is infinite. \square

3 The Socle for semiabelian varieties

We study the model-theoretic socle for groups of the form G^\sharp (described below) where G is a semiabelian variety, as part of reducing Mordell-Lang for semiabelian varieties to Mordell-Lang for abelian varieties. In [2] our new proof of Mordell-Lang for abelian varieties in characteristic 0 made use of the weak socle theorem in its finite Morley rank form. But our new proof in the same paper of Mordell-Lang for abelian varieties over $\mathbb{F}_p(t)^{sep}$ did NOT make use of the weak socle theorem. In the positive characteristic case, it is only for semiabelian varieties that the weak socle comes into the picture. We also insist here that we do everything without appealing to the trichotomy theorem (for U -rank 1-types in differentially closed or separably closed fields). In fact in the appendix we give new proofs that A^\sharp is 1-based for a traceless simple abelian variety, in those cases where we know how to prove Mordell-Lang without appeal to the trichotomy theorem.

3.1 Analyzing the socle of G^\sharp

We will now analyze the model theoretic socle in the context of the groups used for the model-theoretic proofs of function field Mordell-Lang, in particular the original proof of Hrushovski ([11]) and then use this analysis to deduce Mordell-Lang for all semiabelian varieties from Mordell-Lang for abelian varieties. This will be done in Section 4.

So we will consider, in characteristic 0, a differentially closed field (K, ∂) , and k its field of constants and in characteristic p , K a separably closed field of non-zero finite degree of imperfection, which we will assume to be \aleph_1 -saturated, and $k = K^{p^\infty} := \bigcap_n K^{p^n}$, the biggest algebraically closed subfield of K . We will in both cases refer to k as the *field of constants* of K . This terminology makes sense also in the positive characteristic case, as K can be endowed with a family of iterative Hasse derivations, and K^{p^∞} is then the field of common constants (see for example [25] or [1]). In the present paper, we do not directly use the Hasse derivations framework. At some point, we will however use the formalism of λ -functions (for a fixed p -basis), which give the coordinates of elements of K viewed as a K^{p^n} -vector space. The analogue of the Kolchin topology is the λ -topology, which is not Noetherian but is the limit of the λ_n -topologies, which are Noetherian. Quantifier elimination in this language implies that definable sets are λ_n -constructible for some n (see [11] and [5] for details).

We let G be a semiabelian variety over K , and the finite U -rank or Morley rank connected type-definable groups we will consider are the subgroups G^\sharp . We recall below briefly the main definitions and facts about G^\sharp which we will be using. For more details about definitions and properties of G^\sharp we refer the reader to [1](in particular Section 3 for the basic properties) and [2].

From now on G is a semiabelian variety over K , we work in a big saturated model \mathcal{U} , $K \preceq \mathcal{U}$, the field of constants of \mathcal{U} will be denoted $C := C(\mathcal{U})$, and $k = C(K)$.

Recall that if A is an abelian variety over some field K_1 , and $K_0 < K_1$, K_0 algebraically closed, the K_1/K_0 -trace of A is a final object in the category of pairs (B, f) , consisting of an abelian variety B over K_0 , equipped with a K -map of abelian varieties $f : B_{K_1} \mapsto A$, where B_{K_1} denotes the scalar extension $B \times_{K_0} K_1$. We will just need here the property that some abelian variety A has K_1/K_0 -trace zero, which we will just call *having K_0 -trace zero*, if and only if A has no non trivial subabelian variety A_0 isogenous to some abelian variety over K_0 .

Definition 3.1 The group G^\sharp .

(i) In characteristic 0, G^\sharp denotes the “Kolchin closure of the torsion”, namely the smallest definable (in the sense of differentially closed fields) subgroup of $G(\mathcal{U})$ which contains the torsion subgroup (so note that G^\sharp is definable over K).

(ii) In positive characteristic, G^\sharp denotes $p^\infty(G(\mathcal{U})) =_{\text{def}} \bigcap_n p^n(G(\mathcal{U}))$, a type-definable subgroup over K , it is the biggest divisible subgroup of $G(\mathcal{U})$.

Remark 3.2 In characteristic 0, the smallest definable subgroup containing the torsion subgroup exists by ω -stability of the theory DCF_0 . In positive characteristic, G^\sharp is only type-definable, it is also the smallest type-definable subgroup of $G(\mathcal{U})$ which contains the prime-to- p torsion of G . Moreover, as noted, G^\sharp is defined over K , and $G^\sharp(K)$ is also the biggest divisible subgroup of $G(K)$.

Some basic facts which hold in all characteristics:

Fact 3.3 (i) G^\sharp is also the smallest Zariski dense type-definable subgroup of $G(\mathcal{U})$.

(ii) G^\sharp is connected (no relatively definable subgroup of finite index), and of finite U -rank in char. p , and finite Morley rank in char. 0.

(iii) If A is a simple abelian variety, A^\sharp has no proper infinite definable subgroup. It follows that A^\sharp is almost minimal. If A is any abelian variety, then $A^\sharp = S(A^\sharp)$.

(iv) If G is the sum of a finite number of G_i then G^\sharp is the sum of the G_i^\sharp .

(v) If G is a semiabelian variety over the field of constants $k = C(K)$, then $G^\sharp = G(C)$.

Proposition 3.4 (i) In characteristic p , every connected type-definable subgroup of G^\sharp is of the form H^\sharp , for some closed connected subgroup (i.e. a semiabelian subvariety) H of G . It follows that G^\sharp is rigid.

(ii) In characteristic 0, if A is an abelian variety, every connected definable subgroup of A^\sharp is of the form B^\sharp for some abelian subvariety B of A . It follows that A^\sharp is rigid.

Proof: (i) Let R be a connected type-definable subgroup of G^\sharp . By U -rank inequalities, and because for each n , the n -torsion is finite, R is divisible (see for example [6]). Let $H := \overline{R}$, the Zariski closure of R in G . Then as H^\sharp is the biggest divisible subgroup of H , $R \subset H^\sharp$. But (Fact 3.3 (i)) H^\sharp is

also the smallest type-definable subgroup of H which is Zariski dense in H . Hence $R = H^\sharp$. Now every connected closed subgroup of G is defined over K (by rigidity of semiabelian varieties), so H is defined over K and hence so is $R = H^\sharp$, and it follows that G^\sharp is rigid.

(ii) Let R be a connected definable subgroup of A^\sharp . Then $B := \overline{R}$, the Zariski closure of R in A , is an abelian subvariety of A . By exactness of the \sharp -functor for abelian varieties in characteristic 0 (Proposition 5.23 in [1]), $B^\sharp = B(\mathcal{U}) \cap A^\sharp$, and so $B^\sharp \supset R$. But (Fact 3.3 (i)) B^\sharp is the smallest Zariski-dense definable subgroup of $B(\mathcal{U})$, hence $R = B^\sharp$. As in (i) above, it follows that A^\sharp is rigid. \square

Remark 3.5 In characteristic 0, it is not true that for G , any semiabelian variety over K , every connected definable subgroup of G^\sharp is of the form H^\sharp . We have shown in [1](Section 3 and Cor. 5.22) that there are cases when, for G of the form: $0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0$, the induced sequence $T^\sharp \rightarrow G^\sharp \rightarrow A^\sharp$ is not exact, which is equivalent to the fact that $R := G^\sharp \cap T(\mathcal{U})$, which is a connected definable group, strictly contains T^\sharp . But if $R = H^\sharp$, for some H , then $\overline{R} = H < T$ but as $T^\sharp \subset R$, $\overline{R} = T = H$. Contradiction. Note that R/T^\sharp is a finite-dimensional vector space over the constants, so when it has Morley rank ≥ 2 , G^\sharp will not be rigid. The same example also gives, in characteristic p , an induced \sharp -sequence which is not exact, hence such that $R \neq T^\sharp$, but, in contrast to the characteristic 0 case, R will not be connected (in fact T^\sharp will be the connected component of R).

Now as the group G^\sharp has finite U -rank in characteristic p , and finite Morley rank in characteristic 0, we know by the preceding section, that it has a (model-theoretic) socle, the maximal almost pluri-minimal subgroup $S(G^\sharp)$. We know that there are a finite number Q_0, \dots, Q_n of pairwise orthogonal minimal types in the characteristic p case, and of pairwise orthogonal strongly minimal sets in the characteristic 0 case, such that $S(G^\sharp) = B_{Q_0} + \dots + B_{Q_n}$ where each $B_{Q_j} \subset \text{acl}(F_j Q_j)$ is the maximal connected type-definable subgroup of G^\sharp which is almost minimal with respect to Q_j . Furthermore, any other minimal (or strongly minimal) Q must be non orthogonal to one of the Q_j .

Remark 3.6 Whether in characteristic 0 or p , there is at most one minimal type (resp. strongly minimal set) amongst the Q_j which is non orthogonal

to the field of constants in \mathcal{U} , C : recall that, in the characteristic p case, $C = U^{p^\infty}$ is a pure type-definable algebraically closed field, hence with a unique minimal generic type, and that in characteristic 0, C is a pure definable algebraically closed field, hence a strongly minimal set. This particular minimal type (or strongly minimal set) will be denoted, in what follows, by Q_0 .

Proposition 3.7 *In all characteristics:*

1. *If G is a semiabelian variety over K and not an abelian variety, or if G is an abelian variety over K , with k -trace non zero, then, there is in G^\sharp a minimal type (or a strongly minimal set in characteristic 0), Q , non orthogonal to the field of constants C .*
2. *Let R be any definable connected subgroup of $G(\mathcal{U})$ of finite Morley rank in characteristic 0, or any connected type-definable subgroup of $G(\mathcal{U})$ of finite U -rank in characteristic p . Let Q be a strongly minimal set (or a minimal type-definable set) in R , non orthogonal to C , and let $R_Q \subset R$ be any connected type-definable subgroup of R which is almost minimal with respect to Q . Then there exists a semiabelian variety $H < G$, $R_Q \subset H(\mathcal{U})$, $H^\sharp = R_Q$, and a semiabelian variety J over k with an isogeny f from J onto H such that $f^{-1}(H^\sharp) = J(C)$.*
3. *Let R be any definable connected subgroup of $G(\mathcal{U})$ of finite Morley rank in characteristic 0, or any connected type-definable subgroup of $G(\mathcal{U})$ of finite U -rank in characteristic p . Let Q be a strongly minimal set (or a minimal type) in R orthogonal to C and let $R_Q \subset R$ be any connected type-definable subgroup of R , almost minimal with respect to Q . Then $A := \overline{R_Q}$ is an abelian variety with k -trace zero and $R_Q = A^\sharp$.*

Proof :

1: If G is a semiabelian variety over K , and not an abelian variety, G has a non trivial closed connected subgroup T which is a torus, that is, isomorphic to a product of \mathbb{G}_m (defined over C). Then T^\sharp is isomorphic to $(\mathbb{G}_m)^n(C)$. It is hence almost minimal with respect to the strongly minimal field C in characteristic 0, or to the minimal generic type of the field C in characteristic p . There must then be in T^\sharp , hence in G^\sharp , a strongly minimal set or a minimal type, Q , which is not orthogonal to the field C .

If G is an abelian variety with k -trace non zero, then there is some $A < G$, A isogenous to some abelian variety B over $k = C(K)$, it follows that A^\sharp is definably isogenous to $B(C)$ (by [1] Lemmas 3.3 and 3.4). Hence, $A^\sharp = S(A^\sharp)$,

must be non orthogonal to the field C .

2: First we apply Proposition 2.6, so there is a type-definable group $N \subset dcl(C)$ and a definable surjective homomorphism $h : R_Q \mapsto N$, with finite kernel. As in [11] (see also [4] or [6]), it follows from the “pureness” of the algebraically closed field of constants and from the model theoretic version of Weil’s theorem, that N is definably isomorphic to $J(C)$ for some commutative connected algebraic group J over the field C . As R_Q is a divisible group and the n -torsion is finite for each n , there is a definable “dual” isogeny, f from $J(C)$ onto R_Q . By quantifier elimination in the algebraically closed field C , the definable map f is given locally by rational functions (in characteristic p , a priori f is given locally by p -rational functions, that is might include some negative powers of the Frobenius, but by changing J one gets also to a locally rational map).

Taking Zariski closures, extend f to a surjective morphism \tilde{f} of algebraic groups from J onto $H := \overline{R_Q}$. By Chevalley’s theorem, there is a connected unipotent algebraic subgroup of J over C , N , such that J/N is the maximal semiabelian quotient of J . A priori, \tilde{f} is no longer an isogeny, but by maximality, as H itself is a semiabelian variety, $N < Ker \tilde{f}$ and as f was an isogeny in $J(C)$, $N(C)$ must be finite, hence trivial, as N is connected. It follows that J itself is a semiabelian variety. But then the connected component of $Ker \tilde{f}$ which is a connected closed subgroup of J is defined over C , so must be trivial, hence \tilde{f} is also an isogeny.

Now as J is defined over C , $J^\sharp = J(C)$ and as f is dominant, $f(J^\sharp) = H^\sharp$ ([1] Lemma 3.4) and so $R_Q = H^\sharp$.

3: If $\overline{R_Q} = H$, then $H^\sharp \subset R_Q \subset H(\mathcal{U})$ (as H^\sharp is the smallest definable subgroup Zariski dense in H). But then it is also the case that H^\sharp is almost minimal with respect to Q and must be orthogonal to the field C , hence by (i) H must be an abelian variety of k -trace zero. Then we know by Proposition 3.4, that R_Q is of the form B^\sharp for some $B < H$ and we must have $B = H$. \square

Corollary 3.8 (i) *In characteristic 0: (a) If R is a connected definable subgroup of $S(G^\sharp)$, then $R = H^\sharp$ for some semiabelian variety $H < G$.*

(b) *If L is a connected definable subgroup of $G(\mathcal{U})$ of finite Morley rank, and $L \supset G^\sharp$, then $S(L) = S(G^\sharp)$.*

(ii) *In all characteristics: $S(G^\sharp)$ and all its connected type-definable subgroups*

are defined over K and rigid.

Before beginning the proof, let us make immediately some remarks: In characteristic p (a) is true more generally for all connected type-definable subgroups of G^\sharp itself (Fact 3.4), but we saw that in characteristic 0 this may be false if G is not an abelian variety (Remark 3.5). Secondly (b) is vacuous in characteristic p as any connected type-definable subgroup of $G(\mathcal{U})$ with finite U -rank must be divisible, hence contained in G^\sharp which is the biggest such.

Proof : (i)(a) If $R \subset S(G^\sharp)$, then R is almost pluri-minimal, and hence $R = S(R)$. It follows that $R = R_{Q_0} + \dots + R_{Q_n}$, where the Q_j are pairwise orthogonal strongly minimal sets and R_{Q_j} is a connected definable group, almost minimal with respect to Q_j . It follows from Proposition 3.7 that, for each j , $R_{Q_j} = H_j^\sharp$ for some $H_j < G$ and so $R = H_0^\sharp + \dots + H_n^\sharp = H^\sharp$ for $H = H_0 + \dots + H_n$, using the definition of the \sharp -group in characteristic zero as the definable closure of the torsion.

(i)(b) Recall that G^\sharp is the smallest definable subgroup of G which contains all the torsion of G . It follows that for any $H < G$, H^\sharp , the smallest definable group containing all the torsion of H , must be contained in G^\sharp . Now by Proposition 3.7, $S(L) = H_0^\sharp + \dots + H_n^\sharp$ where $H_j < G$, hence $S(L) \subset G^\sharp$. By maximality of $S(G^\sharp)$, $S(L) \subset S(G^\sharp)$ and as $G^\sharp \subset L$, $S(G^\sharp) \subset S(L)$.

(ii) follows from the rigidity of G , from (i)(a) and from Proposition 3.4. \square

We can now completely describe $S(G^\sharp)$:

Proposition 3.9 *If G is semiabelian over K , we have that $S(G^\sharp) = H^\sharp = H_0^\sharp + H_1^\sharp + \dots + H_n^\sharp$, where:*

- $H = H_0 + \dots + H_n$, where for $i \neq j$, H_i and H_j have finite intersection,
- for each j , H_j^\sharp is almost strongly minimal in characteristic 0 and almost minimal in characteristic p
- $H_0 < H$ is the unique maximal semiabelian subvariety of G which is isogenous to some semiabelian variety defined over k . Note that H_0 will be trivial iff G is an abelian variety with k -trace zero
- for $j > 0$, $H_j < H$ is an abelian variety with k -trace zero
- for $i \neq j$, H_j^\sharp and H_i^\sharp are orthogonal and have finite intersection
- $A_G := H_1 + \dots + H_n$ is the unique maximal abelian subvariety of G with k -trace zero.

Proof : Following the notation from the previous section, we know that $S(G^\sharp) = B_{Q_0} + B_{Q_1} + \dots + B_{Q_n}$, where the Q_j are minimal type-definable sets (definable in the char. 0 case), and where we decide that Q_0 denotes the (generic type of) the field of constants. Each B_{Q_j} is the maximal type-definable connected subgroup of G^\sharp which is almost minimal with respect to Q_j . The Q_j are pairwise orthogonal and any other minimal Q in G^\sharp is non orthogonal to one of the Q_j .

By Proposition 3.7, $B_{Q_0} = 0$ if and only if G is an abelian variety with k -trace zero.

By Proposition 3.4 and Corollary 3.8, $S(G^\sharp) = H^\sharp$ for some $H < G$, and for each j , $B_{Q_j} = H_j^\sharp$ for some $H_j < H$. It follows that $H = H_0 + H_1 + \dots + H_n$.

By Proposition 3.7, $H_0 = \overline{B_{Q_0}}$ is isogenous to some D defined over k . We claim that it is maximal such in G , that is that any other $J < G$, isogenous to some semiabelian variety E over k must be contained in H_0 : indeed then J^\sharp is definably isogenous to $E^\sharp = E(C)$. It follows that E^\sharp is almost minimal with respect to Q_0 , and as $J^\sharp = f^{-1}(E(C))$ and f has finite kernel, J^\sharp is also almost minimal with respect to Q_0 . But then, by maximality of B_{Q_0} , we have that $J^\sharp \subset B_{Q_0}$ and passing to Zariski closures, $J < H_0$.

As for $j > 0$, Q_j is orthogonal to Q_0 hence to the field of constants, by Proposition 3.7, each $H_j := \overline{B_{Q_j}}$ must be an abelian variety of k -trace zero.

Let $A_G := H_1 + \dots + H_n$, A_G is an abelian variety with k -trace zero. We claim that it is maximal such in G : Suppose there is another one, R . Then as $S(R^\sharp) = R^\sharp$, $R^\sharp = R_1 + \dots + R_m$, where each R_i is almost minimal with respect to a (strongly) minimal P_i . For each i , P_i must be non orthogonal to one of the minimal Q_j . It cannot be Q_0 because R has k -trace zero, hence for some $j > 1$, by maximality of B_{Q_j} , $R_i \subset B_{Q_j} = H_j^\sharp$, and $R^\sharp \subset A_G^\sharp$. Passing to Zariski closures, $R < A_G$.

Finally, we know that for $i \neq j$, Q_j and Q_i are orthogonal. It follows that, in characteristic *zero*, the two almost strongly minimal groups B_{Q_j} and B_{Q_i} have finite intersection. In characteristic p we can a priori conclude only that the two almost minimal groups B_{Q_j} and B_{Q_i} have an intersection which is contained in the algebraic closure of a finite set. But as this intersection is also type-definable, it will have to be finite (by compactness). \square

We will give some examples of socles of G^\sharp in the case of semiabelian varieties in 5.2,

But for now we summarize what we will need in the next section:

Corollary 3.10 *If G is semiabelian over K , $S(G^\sharp) = H^\sharp = G_0^\sharp + A_G^\sharp$, where $G_0 < G$ is the unique maximal semiabelian variety isogenous to one over the constants k , and A_G is the unique maximal abelian subvariety $A_G < G$ with k -trace zero and $H = H_0 + A_G$. Furthermore, G_0 and A_G have finite intersection and G_0^\sharp and A_G^\sharp are orthogonal.*

3.2 The algebraic socle

We introduce here a possibly new algebraic-geometric notion, building on and motivated by our analysis of the model-theoretic socle of G^\sharp in the previous section. So given a pair $K_0 < K_1$ of algebraically closed fields, and a semiabelian variety G over K_1 , we will define below the K_0 -algebraic socle of G , a semiabelian subvariety of G . In Section 4 we will show how (function field) Mordell-Lang for G reduces to (function field) Mordell-Lang for its K_0 -algebraic socle, and then use this to reduce further to (function field) Mordell-Lang for abelian varieties. Both reductions use model theory and the weak socle theorem.

Definition 3.11 *Let $K_0 < K_1$ be two algebraically closed fields in any characteristic and G be a semiabelian variety over K_1 . We define the algebraic K_0 -socle of G as follows:*

$$S_{K_0}(G) := G_0 + A_G$$

where G_0 is the maximal closed connected algebraic subgroup of G which is isogenous to $H \times_{K_0} K_1$, for some H defined over K_0 (note that in characteristic zero, one can replace isogenous by isomorphic, but not in characteristic p); A_G is the maximal abelian subvariety of G with K_0 -trace zero.

The following lemma should be clear but we give the proof for completeness.

Lemma 3.12 *(i) G_0 and A_G are well defined, and if G is an abelian variety, then $S_{K_0}(G) = G$.*

(ii) G_0 and A_G have finite intersection.

(iii) If $G = G_1 + G_2$, then $S_{K_0}(G) = S_{K_0}(G_1) + S_{K_0}(G_2)$.

(iv) If G is defined over K_1 , $K_0 < K_1 < L_1$, $K_0 < L_0 < L_1$ algebraically closed fields such that L_0 and K_1 are linearly disjoint over K_0 , then $(S_{K_0}(G)) \times_{K_1}$

$L_1 = S_{L_0}(G \times_{K_1} L_1)$.

(v) If G is defined over some K , $K_0 < K$, K separably closed, then G_0 , A_G and $S_{K_0}(G)$ are also defined over K .

Proof : (i) It suffices to check that if $H_1, H_2 < G$ are closed connected subgroups isogenous to groups defined over K_0 , then so is $H_1 + H_2$, which is itself closed and connected, and that if A_1, A_2 are two abelian varieties in G with K_0 -trace zero, then the abelian variety $A_1 + A_2$ also has K_0 -trace zero.

If G is itself an abelian variety, then G is a finite sum of simple abelian varieties, G_i . Each G_i is either isogenous to some H defined over K_0 , or has K_0 -trace zero.

(ii) is clear: if $G_0 \cap A_G$ was infinite, then its connected component would give A_G a non zero K_0 -trace.

(iii) Let $S_{K_0}(G_1) = H_1 + A_1$ and $S_{K_0}(G_2) = H_2 + A_2$. Then it is easy to check that $H_1 + H_2$ is the maximal connected subgroup of $G_1 + G_2$ which is isogenous to some H over K_0 and that $A_1 + A_2$ is the maximal abelian variety in $G_1 + G_2$ with K_0 -trace zero.

(iv) It suffices to show that if $H < G \times_{K_1} L_1$ and there is some D over L_0 and an isogeny from $D \times_{L_0} L_1$ onto H , then there is some D_0 over K_0 and an isogeny f from $D_0 \times_{K_0} L_1$ onto H . Note that H itself is in fact defined over K_1 (a connected closed subgroup of G), hence it is of the form $H' \times_{K_1} L_1$, for H' over K_1 .

There is a rather direct proof using model theory: the linear disjointness condition means exactly that, in the language of pairs of algebraically closed fields, the pair (K_0, K_1) is an elementary substructure of the pair (L_0, L_1) (see [21] for model-theory of pairs of algebraically closed fields). One can then argue as in [11] or [4], expressing the existence of an isogeny between H and a commutative algebraic group over the small model of the pair by a first order formula in the big pair, formula which by elementary substructure will also be true in the small pair.

For purely algebraic proofs, in the case of characteristic 0, one can suppose that the isogeny is in fact an isomorphism and the conclusion follows classically using linear disjointness.

In characteristic p one cannot replace f by an isomorphism. For the case of abelian varieties, the result follows for example from more general results about the behaviour of the Trace under field extensions, shown for example in [8](Theorem 6.8).

(v) This follows directly from the fact that if K is separably closed and G is defined over K , then every connected closed subgroup of G is also defined over K . \square

It follows from what we saw in section 3 that, if K is differentially closed or separably closed (sufficiently saturated), if k denotes the field of constants of K , and if G is semiabelian over K , then:

Proposition 3.13 1. $S(G^\#) = (S_k(G))^\# = G_0^\# + A_G^\#$.

Passing to Zariski closures, $S(G^\#) = S_k(G)$.

2. If $K' \preceq K$ are two differentially closed fields in characteristic 0 (\preceq here denotes elementary extension in the language of differential fields) or separably closed fields of finite imperfection degree in characteristic p (\preceq then refers to a suitable language for separably closed fields, considered either as Hasse differential fields or fields with λ -functions), if k', k are their respective fields of constants, and if G is defined over K' , then $S_k(G \times_{K'} K) = S_{k'}(G) \times_{K'} K$.

Proof: 1. is clear.

2. follows from 3.12 and the linear disjointness of k and K' over k' . If $S_k(G \times_{K'} K) = G_0 + A_G$, as above, note that both G_0 and A_G are defined over K' , so of the form $G_0 = (G'_0) \times_{K'} K$ and $A_G = A' \times_{K'} K$. Then A_G is also the maximal abelian variety in $G \times_{K'} K$ with k' -trace 0, A' is the maximal abelian variety in G with k' -trace zero, G_0 is also the maximal semiabelian variety in $G \times_{K'} K$ isogenous to one defined over k' and G'_0 is the maximal semiabelian variety in G isogenous to one over k' . \square

4 Mordell-Lang

We first recall the general statement of function field Mordell-Lang for semiabelian varieties (for more on the conjecture and equivalent statements see for example [4] and [10]):

Statement of Function field Mordell Lang for semiabelian varieties, all characteristics: Let $K_0 < K_1$ be two algebraically closed fields, G be a semiabelian variety over K_1 , X an irreducible subvariety of G over K_1 , and $\Gamma \subset G(K_1)$, a finite rank subgroup. Suppose that $\Gamma \cap X$ is Zariski dense in X and that the stabilizer of X in G is finite. Then there is a semiabelian

subvariety H_0 of G , a semiabelian variety, S_0 over K_0 , an irreducible subvariety X_0 of S_0 defined over K_0 and a bijective morphism h from S_0 onto H_0 such that $h(X_0) = a + X$ for some $a \in G(K_1)$.

Remark: Note that equivalently, the conclusion above can be stated as there is a semiabelian subvariety H_0 of G , a semiabelian variety S_0 over K_0 , an irreducible subvariety X_0 of S_0 defined over K_0 and an isogeny f from H_0 onto S_0 such that $X = a + f^{-1}(X_0)$ for some $a \in G(K_1)$.

Note also that if G is an abelian variety over K_1 with K_0 -trace zero, then the conclusion is that $X = \{a\}$ for some $a \in G(K_1)$.

Recall that we say that a subgroup Γ of $G(K_1)$ has “finite rank” if: in characteristic zero, $\Gamma \otimes \mathbb{Q}$ has finite dimension (equivalently Γ is contained in the divisible hull of some finitely generated group $\Gamma_0 \subset G(K_1)$), and in characteristic p , $\Gamma \otimes \mathbb{Z}_{(p)}$ has finite rank as a module (equivalently Γ is contained in the p' -divisible hull of some finitely generated group $\Gamma_0 \subset G(K_1)$). Note that, in characteristic p , this is a restrictive notion of “finite rank”. It is still open whether the same statement holds, in characteristic p , with the more general assumption that $\Gamma \otimes \mathbb{Q}$ has finite dimension.

4.1 Recalling the model-theoretic setting

We recall the reduction to the differential model-theoretical setting, as in Hrushovski’s original proof in [11] (see also [4] or [2]):

- In characteristic 0, we can replace the pair $K_0 < K_1$ by a pair $k < K$ where (K, δ) is a differentially closed field (a model of the theory DCF_0) and k is the field of constants of K , $K_0 < k$, $K_1 < K$ and K_1 and k are linearly disjoint over K_0 . Then there exists a definable (in the sense of DCF_0) connected group H , with finite Morley rank, such that H contains both Γ and G^\sharp . It follows that $X \cap H$ is Zariski dense in X . By Proposition 3.8, the (model theoretic) socle of the group H is equal to the socle of G^\sharp .

- In characteristic p , we can replace the pair $K_0 < K_1$ by a pair $k < K$, where K is an ω_1 -saturated separably closed field of finite degree of imperfection (a model of the theory $SCF_{p,e}$, with e finite and non-zero), and $k = K^{p^\infty} := \bigcap_n K^{p^n}$ (we also call k , which is the biggest algebraically closed subfield of K , the field of constants of K). We have that $K_0 < k$ and K_1 and k are linearly disjoint over K_0 , G is defined over K , and $\Gamma \subset G(K)$. Then we can replace Γ by the connected type-definable group $G^\sharp = p^\infty G(K) :=$

$\bigcap_n p^n G(K)$, which has finite U -rank, and is such that for some $a \in G(K)$, $(a + X) \cap G^\sharp$ is Zariski dense in $a + X$. Without loss of generality, by translating, we can suppose that $X \cap G^\sharp$ is Zariski dense in H and, by changing Γ to another finite rank group, that it is still true that $X \cap \Gamma$ is Zariski dense in X .

Claim 4.1 (i) *In characteristic 0: There is a complete stationary type q_X in $X \cap H$ such that $q_X(K)$ is Zariski dense in X and $\text{Stab}(q_X)$ in H is finite.*
(ii) *In characteristic p : There is a complete stationary type q_X in $X \cap G^\sharp$ such that $q_X(K)$ is Zariski dense in X and $\text{Stab}(q_X)$ in G^\sharp is finite.*

Proof:

In characteristic 0: Consider the set $Z := X \cap H$ which is closed in the δ -topology in $G(K)$. By Noetherianity of the topology, Z is a union of finitely many irreducible components, so one of them, Z_1 is also Zariski dense in X . Let q_X be the complete DCF_0 -type which is the topological generic of Z_1 . Then $Y := q_X(K)$ is also Zariski dense in X : indeed \bar{Y} , the Zariski closure of Y , contains the δ -closure of Y , which is equal to Z_1 . So $\bar{Y} = \bar{Z}_1 = X$.

Now suppose $g \in G$ stabilizes the type q_X , that is, $(g + q_X(K)) \cap q_X(K)$ contains $q_{X|g}(K)$. But the δ -closure of $q_{X|g}(K)$ is still equal to Z_1 , so $(g + q_X(K)) \cap q_X(K)$ is also Zariski dense in X and it follows that $g + X = X$. So the stabilizer of the type q_X must be finite.

In characteristic p : $X \cap G^\sharp = \bigcap Y_i$, where the Y_i are a decreasing sequence of λ_i -closed sets. Each of the Y_i is closed in the λ_i -topology, which is Noetherian, so has a finite number of irreducible components for this topology. Every component of Y_{i+1} is contained in a component of Y_i . So we obtain a tree of λ -closed irreducible sets of finite type, branching finitely. On a given branch we obtain an irreducible component of $X \cap G^\sharp$. As the tree branches finitely, at each level one of the closed sets must be Zariski dense in X . Taking the intersection, by ω_1 -saturation of K , we obtain an irreducible component Z_1 of $X \cap G^\sharp$ which is Zariski dense in X . Then, as in the characteristic 0 case, we take q_X to be the complete $SCF_{p,1}$ -type which is the topological generic of Z_1 . Then $q_X(K)$ is also Zariski dense in X . The same arguments as above show that the stabilizer of q_X must also be finite. \square

4.2 Reduction to the algebraic socle

We have, in the previous section, defined what we called, for any pair of algebraically closed fields, $K_0 < K_1$, and any semiabelian G over K_1 , the

K_0 -algebraic socle of G .

The following reduction follows from the “weak socle theorem”:

Proposition 4.2 *In order to prove function field Mordell-Lang for all semiabelian varieties, it suffices to prove it for semiabelian varieties of the form $G_0 + A$, where G_0 is a semiabelian variety isogenous to one over K_0 and A is an abelian variety of K_0 -trace zero.*

Indeed, going back to the “weak socle theorem” (2.12) and its application to the algebraic situation (section 3.2), we see that, passing to Zariski closures, what we have proved at the algebraic level is the following:

Proposition 4.3 *Let $K_0 < K_1$ be any pair of algebraically closed fields, G a semiabelian variety over K_1 , X an irreducible subvariety of G over K_1 such that the stabilizer of X in G is finite. Let Γ be a subgroup of $G(K_1)$ of finite rank, if $X \cap \Gamma$ is dense in X , then for some $a \in G(K_1)$ $a + X \subset S_{K_0}(G)$.*

Proof : We pass to the model theoretic setting described in the previous section.

In both characteristics, we apply the weak socle theorem (Theorem 2.12) to the type q_X obtained from Claim 4.1 and to the group H in characteristic zero and to the group G^\sharp in characteristic p . As we know that in characteristic zero, the socle of H is equal to the socle of G^\sharp (3.8), in both cases we conclude that a translate of the type q_X is contained in the socle of G^\sharp , $S(G^\sharp)$. Taking Zariski closures, it follows that a translate of X is contained in the algebraic k -socle of G , $S_k(G)$. Now as G and X were defined over the algebraically closed field K_1 , by Lemmas 3.12 and 3.13, a translate of X will be contained in $S_{K_0}G$. \square

Remark 4.4 The condition that Γ is a group of finite rank is used to pass to a group of finite rank in the sense of model theory, in order to use the model theoretic weak socle theorem. One might ask whether a purely algebraic-geometric statement suffices, for example whether if X is an irreducible subvariety of G with finite stabilizer, then X is contained in a translate of $S_{K_0}(G)$. But it is not the case, as the following simple example shows: Let A be a simple abelian variety over K_1 , of dimension > 1 and with K_0 -trace 0. Let G be a non almost split extension of A by \mathbb{G}_m (see Section 5.2). Then $S_{K_0}(G) = \mathbb{G}_m$. If X is a curve in A , and Y is a curve in G projecting onto X , then $Stab_G(Y)$ is finite, but Y is not contained in a translate of \mathbb{G}_m .

We can now prove Proposition 4.2:

Proof : Suppose function field Mordell-Lang is true for “socle-like” semiabelian varieties (i.e. as in the hypothesis of Proposition 4.2), $K_0 < K_1$ are algebraically closed fields, and G is any semiabelian variety over K_1 . Let Γ be a finite rank subgroup of $G(K_1)$, and X an irreducible subvariety of G over K_1 , such that $X \cap \Gamma$ is dense in X and the stabilizer of X in G is finite. By 4.3, for some $a \in G(K_1)$, $a + X \subset S_{K_0}(G)$. Let Γ_0 be the finitely generated subgroup such that Γ is contained in the (p') -divisible hull of Γ_0 . Let $\Gamma_1 := \langle a, \Gamma_0 \rangle \cap S_{K_0}(G)(K_1)$. Then Γ_1 is a finitely generated subgroup, let Γ' be the (p') -divisible hull of Γ_1 , then $\Gamma' \subset S_{K_0}(G)(K_1)$. Note that $(a + \Gamma) \cap S_{K_0}(G)(K_1) \subset \Gamma'$. We claim that $\Gamma_1 \cap (a + X)$ is dense in $a + X$: Let O be any open set (over K_1) in $a + X$, then $O - a$ is an open set of X . As $\Gamma \cap X$ is dense in X , there is some element of $X \cap \Gamma$ in $O - a$, g . Then $a + g \in (a + X) \cap (a + \Gamma) \subset S_{K_0}(G)$. One can check that $a + g \in \Gamma'$. So $\Gamma' \cap (a + X)$ is dense in $a + X$, and of course the stabilizer of $a + X$ in $S_{K_0}(G)$ is finite. Hence by our assumptions, there is H_0 some semiabelian subvariety of $S_{K_0}(G)$, a semiabelian variety S_0 over K_0 , an irreducible subvariety X_0 of S_0 defined over K_0 and an isogeny h from H_0 to S_0 such that $h^{-1}(X_0) = b + a + X$ for some $b \in S_{K_0}G(K_1)$. This also gives the result (Mordell-Lang) for G itself. \square

Remark. This reduction is a purely algebraic result, but our proof goes through model-theory and once one has the differential setting and the weak socle theorem, can be deduced quite easily. To our knowledge this reduction as stated is new and we do not know if it has a direct algebraic proof.

The same question arises for the reduction of Mordell-Lang for semiabelian varieties to the case of abelian varieties of trace zero which we present in the next section.

4.3 Function field Mordell-Lang for abelian varieties implies function field Mordell-Lang for semiabelian varieties

The material in this section is close to Hrushovski’s original proof via the weak socle theorem, but replacing the use of one-basedness by the assumption that (function field) Mordell-Lang is true for abelian varieties. More precisely, we show how to derive function field Mordell-Lang for semiabelian varieties

from:

Mordell-Lang for trace zero abelian varieties: Let $K_0 < K_1$ be two algebraically closed fields, A an abelian variety over K_1 with K_0 -trace zero, X an irreducible subvariety of A over K_1 , and $\Gamma \subset G(K_1)$, a finite rank subgroup. Suppose that $\Gamma \cap X$ is Zariski dense in X and that the stabilizer of X in G is finite. Then $X = \{a\}$ for some $a \in A(K_1)$.

Proposition 4.5 *In all characteristics, function field Mordell-Lang for trace zero abelian varieties implies function field Mordell-Lang for all semiabelian varieties.*

Proof: First, by Proposition 4.2, we can suppose that $G = H_0 + A$, where H_0 is a semiabelian variety isogenous to one defined over K_0 and A is an abelian variety with K_0 -trace 0.

Again we pass to the model theoretic differential setting as described in the previous section, and so we have K a (sufficiently saturated) separably closed field (of non-zero finite degree of imperfection) in characteristic p and a differentially closed field in characteristic 0; k denotes the constant field of K , and $G = H_0 + A$, where H_0 is isogenous to a semiabelian variety over k , and A has k -trace zero (by Lemmas 3.12 and 3.13). We have $\Gamma \subset G(K)$, X an irreducible subvariety of G with finite stabilizer in G , such that $X \cap G^\sharp$ is Zariski dense in X , and also $\Gamma \cap X$ is dense in X . As in Claim 4.1, there is a complete type q_X in $X \cap G^\sharp$, with finite stabilizer, and which is Zariski dense in X , that is, $\overline{q_X(K)} = X$.

Remark that because of the ‘‘socle-like’’ form of G , $G^\sharp = H_0^\sharp + A^\sharp$ is already an almost pluri-minimal group, and so $S(G^\sharp) = G^\sharp$ (see Lemma 3.13), and H_0^\sharp and A^\sharp are orthogonal. By orthogonality, it follows that there are two complete orthogonal types p_{H_0} in H_0^\sharp , and p_A in A^\sharp such that $q_X = p_{H_0} + p_A$. We let $V := \overline{p_{H_0}(K)}$ and $W := \overline{p_A(K)}$.

Claim 4.6 *$X = V + W$ and the stabilizer of W in A is finite.*

Proof of the Claim: First we show that $V + W \subset X$. Fix any b realizing the type p_{H_0} and consider the set $\{c \in G : b + c \in X\}$. This is closed in G , contains $p_A(K)$, hence contains $W = \overline{p_A(K)}$. So, for any b realizing the type p_{H_0} , $b + W \subset X$. Now consider the set $\{b \in G : b + W \subset X\} = \bigcap_{w \in W} X - w$. This is an intersection of closed sets, so it is closed, and it contains $p_{H_0}(K)$, so contains also $V = \overline{p_{H_0}(K)}$. Hence $V + W \subset X = \overline{p_{H_0}(K)} + \overline{p_A(K)}$. Now

in fact $V + W$ is closed: indeed if f is the natural map from $H_0 \times A$ onto $H_0 + A_0$, as H_0 and A_0 have finite intersection, f is an isogeny of algebraic groups, hence it is closed, and $f(V \times W) = V + W$ is closed.

It follows that $X = V + W$.

Let $a \in A(K)$ be such that $a + W = W$. Then $a + X = a + V + W = V + (a + W) = V + W = X$, so a also stabilizes X . \square

Now let $\pi : G \rightarrow G/H_0$. As $H_0 \cap A$ is finite, π restricted to A , which we denote π_A , is an isogeny. Let $A' := \pi_A(A)$, then A' is also an abelian variety, and π_A is a closed morphism, so $W' := \pi_A(W)$ is closed in A' .

Claim 4.7 *The stabilizer of W' in A' is finite.*

Proof of the Claim : This follows from the fact that π_A is an isogeny. Indeed, let $b \in \text{Stab}_{A'} W'$, $b = \pi_A(a)$; then $a \in \text{Stab}_G W + \text{Ker } \pi_A$: $\pi_A(a + W) = b + W' = W'$. So $a + W \subset \pi_A^{-1}(W') = W + \text{Ker } \pi_A = a_1 + W \cup \dots \cup a_n + W$. By irreducibility of W , for some i , $a + W = a_i + W$, so $a - a_i \in \text{Stab}_G W$, which is finite. \square

Claim 4.8 *There is a finite rank subgroup in $A'(K)$, Γ' , such that $\Gamma' \cap W'$ is Zariski dense in W' .*

Proof of the Claim : Let $\Gamma' := \pi_A(\Gamma)$. Then we claim that $\overline{\Gamma' \cap W'} = W'$. Indeed if F is a closed set in G/H_0 such that $F \supset \Gamma' \cap W'$, then $\pi^{-1}(F) \supset (W + H_0) \cap \Gamma \supset X \cap \Gamma$. And as $\pi^{-1}(F)$ is closed, $\pi^{-1}(F) \supset \overline{X \cap \Gamma} = X$. Then $\pi(\pi^{-1}(F)) = F \supset \pi(X) = \pi(V + W) = \pi(W) = \pi_A(W) = W'$. \square

As the abelian variety A' is isogenous to A , it also has k -trace zero. By Mordell-Lang applied to A' , W' and Γ' , $W' = \{a\}$. It follows that W itself is also reduced to a point, $W = \{b\}$. Hence $q_X = p_{H_0} + b$. So the Zariski closure of $q_X(K)$, X , is equal to $V + b$. Hence $X - b \subset V \subset H_0$. As H_0 is isogenous to some S_0 over the algebraically closed field K_0 , let h be the isogeny from $S_0 \times_{K_0} K$ onto H_0 and let X_0 be the Zariski closure of $h^{-1}(p_{H_0}(K))$. Then as required $X = h(X_0) + b$, and as $h^{-1}(p_{H_0}(K))$ must be contained in $S_0^\sharp = S_0(k)$, X_0 is defined over k . By linear disjointness of k and K_1 over K_0 , it is defined over K_0 , as required. \square

5 G^\sharp versus $S(G^\sharp)$

We study the relationship between G^\sharp and $S(G^\sharp)$. We know from [1] that in the general semiabelian case, G^\sharp and more generally the \sharp -functor may not be so well-behaved. Here we point out ways in which the socle $S(G^\sharp)$ is better behaved.

5.1 Characteristic p case, quantifier elimination for the socle of G^\sharp

Let G be a semiabelian variety over K (sufficiently saturated) separably closed of characteristic $p > 0$ and let $k := K^{p^\infty}$ denote the field of constants. In this section we will see that the socle of G^\sharp recovers all the good properties of A^\sharp when A is an abelian variety, properties that can be lost for semiabelian varieties: in particular quantifier elimination and relative Morley rank (see [1], [2]).

Recall the notation from the previous sections: $S(G^\sharp) = (S_k(G))^\sharp = G_0^\sharp + A_G^\sharp$, where G_0 is the maximal closed connected algebraic subgroup of G which is isogenous to $H \times_k K$, for some H defined over k ; A_G is the maximal abelian subvariety of G with k -trace zero.

Furthermore, G_0 and A_G have finite intersection, and G_0^\sharp and A_G^\sharp are orthogonal.

The first “good” properties of the socle of G^\sharp follow easily from its definition.

Proposition 5.1 *1. Suppose that G is defined over $K_1 < K$ where K_1 is the separable closure of $\mathbb{F}_p^{alg}(t)$. Then $S(G^\sharp)(K_1)$ is contained in the torsion of G .*

2. $S(G^\sharp)$ has finite relative Morley Rank.

Proof: 1. Let K_0 denote \mathbb{F}_p^{alg} , the field of constants of the separably closed field K_1 . Then without loss of generality, $K_0 < k < K$, $K_0 < K_1 < K$ and K_1 and $k = C(K)$ are linearly disjoint over $K_0 = C(K_1)$.

As we noted in Lemma 3.12, as G is defined over K_1 then both the K_0 -algebraic socle of G , $S_{K_0}(G)$, and the model theoretic socle $S(G^\sharp) = (S_k(G \times_{K_1} K))^\sharp$ are also defined over K_1 , as are all their connected type-definable (or connected closed, in the algebraic case) subgroups. Also if $(S_k(G))^\sharp = (G_0 \times_{K_1} K)^\sharp + (A_G \times_{K_1} K)^\sharp$ as above, then G_0 is also the biggest

closed connected algebraic subgroup of G which is isogenous to $H \times_{K_0} K_1$, for some H defined over K_0 and A_G is the biggest abelian subvariety with K_0 -trace zero.

Then $S(G^\sharp)(K_1) = G_0^\sharp(K_1) + A_G^\sharp(K_1)$.

By the result for abelian varieties ([22]), $A_G^\sharp(K_1)$ is contained in the torsion of A_G . As G_0 is isogenous to some S_0 defined over K_0 , the field of constants of K_1 , it follows that $G_0^\sharp(K_1)$ is definably isogenous to $S_0^\sharp(K_1) = S_0(K_0) = S_0(\mathbb{F}_p^{alg})$ which is exactly the group of torsion elements of S_0 .

2. By definition of the model theoretic socle, $S(G^\sharp) \subset acl(F \cup Q_1 \cup \dots \cup Q_k)$ for some finite set F and some (pairwise orthogonal) minimal types Q_i in G^\sharp , and is of the form $B_{Q_1} + \dots + B_{Q_n}$, where each B_{Q_j} is almost minimal with respect to Q_j . Then each B_{Q_j} has finite relative Morley rank (see for example [1], section 2.3 and Fact 3.8), hence so does $S(G^\sharp)$. \square

We will now show that the induced structure on the socle, in contrast with G^\sharp itself, always has quantifier elimination. For precise definitions and basic facts about induced structures, see for example [2], where the result is proved when G is an abelian variety (it was proved there in the case of degree of imperfection 1, but the proof is also valid for any non-zero finite degree of imperfection). A counterexample to quantifier elimination of the induced structure on G^\sharp is given in [14]: it is shown there that in the canonical example given in [2] (Proposition 4.9 and Corollary 4.14) of a semiabelian G such that G^\sharp does not have relative Morley rank (taking the abelian part of G to be an elliptic curve), then G^\sharp does not eliminate quantifiers.

We will need a general lemma about quantifier elimination.

Lemma 5.2 *In an ambient, stable and sufficiently saturated structure, let Γ and Δ be type-definable groups, and $f : \Gamma \rightarrow \Delta$ a relatively definable isogeny (i.e. surjective with finite kernel). Assume that Γ , with the structure induced by relatively definable subsets, has quantifier elimination. Then the same holds for Δ .*

Proof: Write $\Gamma = \bigcap_i \Gamma_i$ as the intersection of a decreasing family of definable groups. We may assume that f is the restriction to Γ of a definable group homomorphism of domain Γ_0 , denoted by \tilde{f} . Since $\text{Ker}(f)$ is definable, because finite, we know by compactness that there is some i_0 such that $\text{Ker}(f) = \text{Ker}(\tilde{f}) \cap \Gamma_{i_0}$.

Now consider a relatively definable subset $D \subseteq \Delta^{r+1}$, and $\text{pr}_\Delta : \Delta^{r+1} \rightarrow$

Δ^r a projection map; we have to prove that $\text{pr}_\Delta(D)$ is relatively definable. Let us denote by $f^r : \Gamma^r \rightarrow \Delta^r$ the cartesian power of f , then $\text{pr}_\Delta(D) = f^r(\text{pr}_\Gamma((f^{r+1})^{-1}(D)))$. By quantifier elimination for Γ , we know that $F := \text{pr}_\Gamma((f^{r+1})^{-1}(D))$ is relatively definable in Γ^r ; write it $F = \tilde{F} \cap \Gamma^r$ with \tilde{F} definable.

Now $f^r(F) = \{(d_1, \dots, d_r) \in \Delta^r; \exists g_1 \dots \exists g_r \in \Gamma, \bigwedge_j f(g_j) = d_j \wedge (g_1, \dots, g_r) \in \tilde{F}\}$ is contained in $X = \{(d_1, \dots, d_r) \in \Delta^r; \exists g_1 \dots \exists g_r \in \Gamma_{i_0}, \bigwedge_j \tilde{f}(g_j) = d_j \wedge (g_1, \dots, g_r) \in \tilde{F}\}$.

Note that X is a relatively definable subset of Δ^r . If $(d_1, \dots, d_r) \in X$, then for each j , $d_j = \tilde{f}(\tilde{g}_j)$, with $\tilde{g}_j \in \Gamma_{i_0}$ and $(\tilde{g}_1, \dots, \tilde{g}_r) \in \tilde{F}$; but since $f : \Gamma \rightarrow \Delta$ is surjective, we also have $d_j = f(g_j)$ with $g_j \in \Gamma$. So for each j , $\tilde{g}_j - g_j \in \text{Ker}(\tilde{f}) \cap \Gamma_{i_0} = \text{Ker}(f) \subset \Gamma$. Hence in fact $\tilde{g}_j \in \Gamma$ and $f^r(F) = X$, so $\text{pr}_\Delta(D)$ is relatively definable. \square

We are now back to G defined over K . To simplify notation, we now denote $S(G)$ by S , G_0 by H and A_G by A in the rest of this section, hence $S(G^\sharp) = S^\sharp = H^\sharp + A^\sharp$.

Proposition 5.3 *The type-definable group S^\sharp , with the induced structure from G , has quantifier elimination.*

Proof :

We need to prove that, in S^\sharp , the projection of a relatively definable subset is still relatively definable.

Claim 5.4 *Both H^\sharp and A^\sharp are relatively definable in S^\sharp .*

Proof of the Claim : This follows directly from the fact that S^\sharp has finite relative Morley rank: By [1] Lemma 3.6, H^\sharp is the connected component of $S^\sharp \cap H$. As S^\sharp has finite relative Morley rank, it follows that H^\sharp has finite index in $S^\sharp \cap H$, hence H^\sharp is a finite union of coset of the relatively definable group $S^\sharp \cap H$. Similarly for A^\sharp .

There is an alternative short argument to show that the index is finite, which only uses the fact that $H \cap A$ is finite: Consider any coset of H^\sharp in $S^\sharp \cap H$, $s + H^\sharp$. As $s \in S^\sharp \cap H$, $s = h + a$, with $h \in H^\sharp$ and $a \in A^\sharp \cap H$, so $s + H^\sharp = a + H^\sharp$, and $a \in A \cap H$, which is finite. Same argument for A^\sharp . \square

Since the multiplication map gives an isogeny $H^\sharp \times A^\sharp \rightarrow H^\sharp + A^\sharp = S^\sharp$, we know by Lemma 5.2 that it suffices to prove quantifier elimination for

$H^\sharp \times A^\sharp$. The semiabelian variety H is isogenous to $B \times_k K$ for some B defined over the field of constants k , hence H^\sharp is isogenous to $B^\sharp = B(k)$. As the field k is a pure algebraically closed subfield in K , $B(k)$ has quantifier elimination, and H^\sharp also by Lemma 5.2 again. And we know from [2] that A^\sharp has quantifier elimination (this is true for all abelian varieties).

Now, by orthogonality of A^\sharp and H^\sharp , if F is a relatively definable subset of $(H^\sharp \times A^\sharp)^r$, then $F = \bigcup_{1 \leq i \leq r} R_i \times T_i$, where R_i is a relatively definable subset of $H^{\sharp r}$, and T_i a relatively definable subset of $A^{\sharp r}$. It follows easily that in $(H^\sharp \times A^\sharp)^r$, the projection of any relatively definable subset is also relatively definable. \square

5.2 Some examples

In the following, we consider fields $k \subset K$, with k algebraically closed, and K separably closed, in arbitrary characteristic; G is a semiabelian variety over K .

We first deal with the extreme cases. If G is isogenous to a semiabelian variety over k , then $S_k(G) = G$. If G is an abelian variety, $S_k(G) = G$, as we already noted in lemma 3.12. It is a consequence of the decomposition of abelian varieties into a sum of simple abelian varieties, hence a consequence of the Poincaré reducibility Theorem. And in fact, the gap between G and $S_k(G)$ “witnesses” the defect in Poincaré’s reducibility for semiabelian variety, up to descent to k .

Let G be an extension of an abelian variety A by a torus $T = \mathbb{G}_m^s$, all defined over K . Let us recall that such extensions are parametrized, up to isomorphism, by $Ext(A, T) \simeq \hat{A}^s$, where \hat{A} is the dual abelian variety of A . We will say that the extension $0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0$ is *almost split* if there is a semiabelian subvariety B of G such that $G = T + B$ and $T \cap B$ is finite. Note that in this case, B must be isogenous to A , and it follows that G is almost split if and only if G is isogenous to the direct product $T \times A$ (which corresponds to the neutral element in $Ext(A, T)$).

Recall that we say G *descends* to k if G is *isomorphic* to a semiabelian variety $H \times_k K$, where H is defined over k . In characteristic 0, if G is isogenous to some $H \times_k K$, where H is defined over k , then G descends to k but in characteristic p , this is not always true.

The following facts are undoubtedly quite classical. For lack of an easy reference, we reprove them in Appendix II.

Fact 5.5 1. G is almost split if and only if G is a torsion point in $\text{Ext}(A, T)$.
2. Assume that A is defined over k . If G is isogenous to $H \times_k K$, where H is a semiabelian variety over k , then G descends to k .

Now let us summarize the situation.

Proposition 5.6 1. Assume that G is almost split. Then $S_k(G) = G$.

2. Conversely:

(a) If A has k -trace 0, $S_k(G) = G$ if and only if G is almost split.

(b) If A is isogenous to an abelian variety over k , then $S_k(G) = G$ if and only if G is isogenous to a semiabelian variety over k . If we assume moreover that A is simple, then $S_k(G) \neq G$ if and only if $S_k(G) = T$.

(c) If A descends to k , then $S_k(G) = G$ if and only if G descends to k .

Remark 5.7 The previous proposition enables us to cover all the cases: If the extension is given by the map $f : G \rightarrow A$, and if $A = A_1 + \dots + A_r$ is the decomposition of A into a sum of simple abelian varieties, then $G = G_1 + \dots + G_r$, where $G_i = f^{-1}(A_i)$ and $S_k(G) = S_k(G_1) + \dots + S_k(G_r)$. And, for A_i a simple abelian variety, either A_i has k -trace 0, or A_i is isogenous to some $B \times_k K$, for B an abelian variety over k .

Proof:

1. If $G = T + B$, where B is an abelian subvariety of G , we know from the extreme cases that $S_k(G) = S_k(T) + S_k(B) = T + B = G$.
2. (a) Assume that A has k -trace 0 and $S_k(G) = G_0 + A_G = G$. Since A has k -trace 0, G_0 has a trivial abelian part, i.e. $G_0 = T$. Hence G is almost split.
(b) One direction is one of the extreme cases. Now assume that G is not isogenous to a semiabelian variety over k , and write $S_k(G) = G_0 + A_G$. The map $f : G \rightarrow A$ induces a map $A_G \rightarrow A$, with finite kernel. But A_G has k -trace 0, and A is isogenous to an abelian variety over k , hence $A_G = 0$. Now $S_k(G) = G_0 \neq G$, since G_0 is isogenous to some H_K , for H a semiabelian variety over k , contrary to G . If we assume furthermore that A is simple, we know that $T \subseteq G_0$, and the abelian part of G_0 is 0 since it can not be A . Hence $S_k(G) = T$.

- (c) We can apply (b) for the remaining implication: if $S_k(G) = G$, then G is isogenous to a semiabelian variety over k . But using Fact 5.5.2, this implies that G descends to k .

□

Suppose now that K is differentially closed of characteristic 0, or K is a (sufficiently saturated) model of $SCF_{p,1}$, and let $k = C(K)$ be the field of constants. Let $0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0$ be a semiabelian variety over K . We discussed in [1] the question of whether the induced sequence

$$(\sharp) \quad 0 \rightarrow T^\sharp \rightarrow G^\sharp \rightarrow A^\sharp \rightarrow 0$$

is exact or not. It relates to the current question:

Proposition 5.8 *If $S_k(G) = G$, then the sequence (\sharp) is exact.*

Proof: Suppose that the sequence (\sharp) is not exact. Since $S_k(G)^\sharp = S(G^\sharp)$, it suffices to prove that $S(G^\sharp) \neq G^\sharp$.

In characteristic p , we have shown (Proposition 5.1) that $S(G^\sharp)$ always has finite relative Morley rank. We had shown in [1] that G^\sharp had finite relative Morley rank if and only if the sequence \sharp is exact.

In characteristic 0, we know that every connected definable subgroup of $S(G^\sharp)$ can be written as H^\sharp for some semiabelian subvariety H (see Corollary 3.8), and it is not the case for G^\sharp (see Remark 3.5). □

Unfortunately, the only known cases at the moment where the sequence (\sharp) is not exact are already covered by Proposition 5.6: it is when A descends to k (and is ordinary) but G does not descend to k .

For the other direction, there are two different kinds of examples which show that the \sharp sequence can be exact without $S_k(G) = G$:

In characteristic p , take A an abelian variety over k with p -rank 0 (which implies $A[p^\infty](K) = 0$), and G an extension of $A \times_k K$ by T which does not descend to k , then the sequence (\sharp) is exact (see Corollary 4.15 in [1]) but $S_k(G) \neq G$.

In characteristic 0, take A an elliptic curve which does not descend to k , and G a non almost split extension of A by T . Following [1], we know that U_A , the maximal unipotent D -subgroup of the universal extension \tilde{A} of A by a vector group (which is endowed with its unique D -group structure), is 0. But in this case, the map $U_G \rightarrow U_A$, induced by the map $G \rightarrow A$, is surjective, from which we deduce that the sequence (\sharp) is exact (see Proposition 5.21 in [1]). But we know from Proposition 5.6 that $S_k(G) \neq G$.

6 Appendix I: modularity

The key argument in Hrushovski's proof of the Mordell-Lang conjecture was a model-theoretic result, more precisely a dichotomy result for Zariski types. The question whether one can go in the other direction and deduce this dichotomy result for abelian varieties from the Mordell-Lang statement has been open since. More precisely the dichotomy result one wants to deduce has the following form, where as before K is a differentially closed field in characteristic 0 and a separably closed field in characteristic p , and k is the field of constants of K :

- Let A be an abelian variety over K with k -trace 0. Then*
- (i) A^\sharp is 1-based.*
 - (ii) If moreover A is simple, then A^\sharp is strongly minimal.*

Remark: The reader might think that this dichotomy follows as in [17] where the truth of Mordell-Lang is shown to be equivalent to the 1-basedness of Γ in the expansion $(K, +, \times, \Gamma)$ by the relevant finite rank subgroup Γ of the relevant (semi)abelian variety. However we are working here with A^\sharp with all of its induced structure from the model K of DCF_0 or $SCF_{p,1}$, so additional arguments are needed.

We are able to deduce the dichotomy statement from Mordell-Lang, but only in the cases (from [2]) when we know how to prove Mordell-Lang without going through Zariski structures, deducing it from Manin-Mumford.

Here also we will in fact deduce the statement from the related Manin-Mumford statement:

Theorem 6.1 (MM) *Let K be an algebraically closed field of characteristic 0, and G a commutative algebraic group over K ; or let $k < K$ be algebraically closed fields of characteristic p , and G an abelian variety over K with k -trace 0.*

Let X be an algebraic subvariety of G defined over K .

Then there are a finite number of torsion points t_1, \dots, t_m , and algebraic subgroups G_1, \dots, G_m over K , such that

$$X(K) \cap G(K)_{\text{torsion}} = \bigcup_{i=1}^m t_i + G_i(K)_{\text{torsion}}.$$

Remark 6.2 *In characteristic 0, the statement with this level of generality has been proved by Hindry in [9]. We will use it for G an extension of the abelian variety A by a vector group. In characteristic p , the statement was proved by Pink and Rössler in [19]. The hypothesis of k -trace 0 is mandatory since for an abelian variety over $\mathbb{F}_p^{\text{alg}}$, every $\mathbb{F}_p^{\text{alg}}$ -rational point is a torsion point.*

As in [2], we will assume that $K_0 = (\mathbb{C}(t))^{\text{alg}}$ considered as a differential field with the natural extension of d/dt and $K = K_0^{\text{diff}}$ its differential closure in characteristic 0, and $K_0 = K = \mathbb{F}_p(t)^{\text{sep}}$, and \mathcal{U} an \aleph_1 -saturated elementary extension of K in characteristic p . In this situation, we can use the “theorem of the Kernel”:

Theorem 6.3 *Let A be an abelian variety over K_0 , of $C(K_0)$ -trace 0. Then every point in $A^\sharp(K)$ is a torsion point.*

Remark 6.4 *The characteristic 0 case is proved in [3], and the characteristic p case in [22]. The trace 0 hypothesis is only mandatory in characteristic 0.*

Recall (from [16]) that a type-definable connected commutative group G (in a sufficiently saturated model of a stable theory) is 1-based if every relatively definable subset of G^n is a Boolean combination of translates of relatively definable subgroups (over the same algebraically closed set of parameters as G). In the present situation, where A^\sharp is defined over the algebraically closed set of parameters K , it suffices that every relatively K -definable subset of $A^{\sharp n}$ is a Boolean combination of translates of relatively K -definable subgroups.

In the characteristic p case, we proved in [2] that the structure \mathcal{A} , whose underlying set is $A^\sharp = A^\sharp(\mathcal{U})$, and whose basic definable sets are the relatively K -definable subsets of the cartesian powers $A^{\sharp n}$, has quantifier elimination. It follows that A^\sharp is one-based iff \mathcal{A} is. But we are now in a definable setting (rather than type-definable), for which the saturation hypothesis is no longer relevant. Since we also know from [2] that $A^\sharp(K)$, with the same basic definable sets as above, is an elementary substructure of \mathcal{A} , we are allowed to replace $A^\sharp(\mathcal{U})$ by $A^\sharp(K)$.

Proposition 6.5 *Let K_0 and K be as above, and $k = C(K) = C(K_0)$ the subfield of constants. Let A be an abelian variety over K_0 with k -trace 0. Then*

(i) A^\sharp is 1-based.

(ii) If moreover A is simple, then A^\sharp is strongly minimal.

Proof :

(i) By replacing A by a cartesian power, we just have to consider (relatively) definable subset of A^\sharp . By quantifier elimination in the ambient structure, a definable set is a boolean combination of Kolchin-closed sets in characteristic 0, or of λ_n -closed sets for some n in characteristic p , so we can reduce to those cases.

In characteristic 0, we use the description of A^\sharp given in [3], with the same notations: there is an extension $\pi : \bar{A} \rightarrow A$ of A by a vector group, such that \bar{A} is endowed with a unique D -group structure, and such that π induces an isomorphism (in the category of differential algebraic groups) between \bar{A}^∂ and A^\sharp . If X is a Kolchin-closed set in A^\sharp , $\pi^{-1}(X)$ is a Kolchin-closed set in \bar{A}^∂ , which can be written as $Y \cap \bar{A}^\partial$ for some algebraic subvariety Y of \bar{A} since, in \bar{A}^∂ , the derivation coincides with a regular algebraic map. Furthermore, as π has a torsion free kernel, it induces an isomorphism between torsion points of \bar{A} and torsion points in A . Now we can use MM for Y in \bar{A} : $Y(K) \cap \bar{A}(K)_{torsion}$ is of the required form, and so is its isomorphic image by π , which is $X(K) \cap A(K)_{torsion} = X(K)$.

In characteristic p , we use the description of A^\sharp and its λ_n -closed relatively definable subsets given in Section 3 of [2]. There is an isogeny of abelian varieties over K , $\pi : A_n \rightarrow A$, and a definable map $\lambda_n : A(K) \rightarrow A_n(K)$, which induce an isomorphism from $A_n(K)$ to $p^n A(K) < A^\sharp$, with inverse isomorphism induced by λ_n . Furthermore, for X a relatively definable λ_n -closed subset of A^\sharp , there is an algebraic subvariety X_n of A_n such that $X = \lambda_n^{-1}(X_n)$. Now we use MM for X_n in A_n (which is still an abelian variety with $C(K_0)$ -trace 0, see [8] for the basic properties of the trace): $X_n(K) \cap A_n(K)_{torsion}$ is of the required form, and so is X , its image by π_n .

(ii) If A is simple, then A^\sharp is g -minimal, i.e. has no infinite (relatively) definable proper subgroup. So by part (i), every (relatively) definable subset of A^\sharp is finite or cofinite. \square

7 Appendix II: $Ext(A, T)$

We will work with two fields $k \subset K$, k algebraically closed, in arbitrary characteristic. We use results from [23] about $Ext(A, T)$, the moduli space

for extensions of A by T , for given abelian variety A and torus $T = \mathbb{G}_m^s$ over k , that we recall now.

The mapping $(A, T) \mapsto \text{Ext}(A, T)$ can be made into a bifunctor, covariant in T and contravariant in A , in the following way.

If $G \in \text{Ext}(A, T)$ and $f : T \rightarrow T'$ is a homomorphism, $f_*(G)$ is characterized as the unique element in $\text{Ext}(A, T')$ such that there is a homomorphism $u : G \rightarrow f_*(G)$ making the diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0 \\ & & f \downarrow & & u \downarrow & & \text{id} \downarrow & & \\ 0 & \longrightarrow & T' & \longrightarrow & f_*(G) & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Similarly, for a homomorphism $g : A' \rightarrow A$, $g^*(G)$ is characterized as the unique element in $\text{Ext}(A', T)$ such that there is a homomorphism $u : g^*(G) \rightarrow G$ making the diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & g^*(G) & \longrightarrow & A' & \longrightarrow & 0 \\ & & \text{id} \downarrow & & u \downarrow & & g \downarrow & & \\ 0 & \longrightarrow & T & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

We will also use that the existence of a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0 \\ & & f \downarrow & & u \downarrow & & g \downarrow & & \\ 0 & \longrightarrow & T' & \longrightarrow & G' & \longrightarrow & A' & \longrightarrow & 0 \end{array}$$

is equivalent to the equality $f_*(G) = g^*(G')$ in $\text{Ext}(A, T')$.

These constructions are used to put a commutative group law on $\text{Ext}(A, T)$, with the trivial extension $T \times A$ as neutral element. With these group laws, the maps f_* and g^* are now homomorphisms. We will describe below the multiplication by n for this group law. Now, the functor $K \mapsto \text{Ext}(A \times_k K, T \times_k K)$, from field extensions of k into groups, is represented by \hat{A}^s , where \hat{A} is the dual abelian variety of A . In other words, the extensions (implicitly: defined over K) of $A \times_k K$ by $T \times_k K$, up to isomorphism, are parametrized by $\hat{A}^s(K)$.

Let $0 \rightarrow T \rightarrow G \xrightarrow{f} A \rightarrow 0$ be an element in $\text{Ext}(A, T)$. Via the map f^n , G^n is an element of $\text{Ext}(A^n, T^n)$ ($n \geq 1$). For any algebraic group H , let

$d_{nH} : H \rightarrow H^n$ be the diagonal map and $m_{nH} : H^n \rightarrow H$ be the multiplication map. Then, in $\text{Ext}(A, T)$, $[n]G = d_{nA}^* m_{nT}^* G^n = m_{nT}^* d_{nA}^* G^n$.

Now consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T & \longrightarrow & G & \xrightarrow{f} & A & \longrightarrow & 0 \\
& & d_n \downarrow & & d_n \downarrow & & f^n & & \downarrow d_n \\
0 & \longrightarrow & T^n & \longrightarrow & G^n & \longrightarrow & A^n & \longrightarrow & 0 \\
& & m_n \downarrow & & m_n \downarrow & & f & & \downarrow m_n \\
0 & \longrightarrow & T & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

The composition on the columns is $[n]$. From the upper half of the diagram, we get that $d_{nA}^* G^n = d_{nT}^* G$ in $\text{Ext}(A, T^n)$. From the lower half, we get that $m_{nA}^* G = m_{nT}^* G^n$ in $\text{Ext}(A^n, T)$. Hence we get

$$[n]G = d_{nA}^* m_{nT}^* G^n = d_{nA}^* m_{nA}^* G = (m_{nA} d_{nA})^* G = [n]_A^* G$$

and also

$$[n]G = m_{nT}^* d_{nA}^* G^n = m_{nT}^* d_{nT}^* G = (m_{nT} d_{nT})_* G = [n]_{T^*} G.$$

Proposition 7.1 *G is an almost split extension of A by T if and only if it is a torsion point in $\text{Ext}(A, T)$.*

Proof : One direction uses the general fact that G is isogenous to $[n]G$: Indeed, since $[n]G = [n]_{T^*} G$, there is an homomorphism u such that the following diagram commutes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T & \longrightarrow & G & \xrightarrow{f} & A & \longrightarrow & 0 \\
& & [n]_T \downarrow & & u \downarrow & & \text{id} \downarrow & & \\
0 & \longrightarrow & T & \longrightarrow & [n]G & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

Clearly, u is an isogeny because $[n]_T$ and id are. We could have made the same argument using that $[n]G = [n]_A^* G$. Hence, if $[n]G = T \times A$, G is almost split.

Conversely, if $\phi : G \rightarrow T \times A$ is an isogeny, ϕ must map the linear part of G into the linear part of $T \times A$, hence it induces a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & T & \longrightarrow & G & \longrightarrow & A \longrightarrow 0 \\
& & \phi_T \downarrow & & \phi \downarrow & & \phi_A \downarrow \\
0 & \longrightarrow & T & \longrightarrow & T \times A & \longrightarrow & A \longrightarrow 0
\end{array}$$

Such a diagram implies that $\phi_{T*}G = \phi_A^*(T \times A) = T \times A$, since ϕ_A^* is a group homomorphism. Now, take $\theta : T \rightarrow T$ such that $\theta \circ \phi_T = [n]_T$ for some $n \geq 1$. We obtain $[n]G = [n]_{T*}G = \theta_*\phi_{T*}G = \theta_*(T \times A) = T \times A$, i.e. G is a torsion point. \square

Proposition 7.2 *Recall that A is defined over k . Let $G \in \text{Ext}(A \times_k K, T \times_k K)$ be defined over K . If G is isogenous to some $H \times_k K$ for H a semiabelian variety over k , then G descends to k .*

Proof: Write $H \in \text{Ext}(A', T)$ for some abelian variety A' , necessarily over k . We assume that there is an isogeny $\phi : G \rightarrow H_K$; since ϕ must map the linear part of G into the linear part of H_K , it induces isogenies $\phi_T : T_K \rightarrow T_K$ and $\phi_A : A_K \rightarrow A'_K$, such that the following diagram commutes

$$\begin{array}{ccccccc}
0 & \longrightarrow & T_K & \longrightarrow & G & \longrightarrow & A_K \longrightarrow 0 \\
& & \phi_T \downarrow & & \phi \downarrow & & \phi_A \downarrow \\
0 & \longrightarrow & T_K & \longrightarrow & H_K & \longrightarrow & A'_K \longrightarrow 0
\end{array}$$

Since A and A' are abelian varieties over k , we know from Chow ([7]) that there is an isogeny $\psi : A \rightarrow A'$ such that $\phi_A = \psi_K$ is obtained by base change. It follows that, in $\text{Ext}(A_K, T_K)$, $\phi_{T*}G = \phi_A^*H_K = \psi_K^*H_K = (\psi^*H)_K$. Since ψ^*H is over k , it is a k -rational point of \hat{A}^s . Furthermore, there is an isogeny $\theta : T_K \rightarrow T_K$ such that $\theta \circ \phi_T = [n]_T$ for some integer $n \geq 1$. Note that ϕ_T and θ are actually defined over k . Hence $[n]G = [n]_{T*}G = \theta_*\phi_{T*}G \in \hat{A}^s(k)$, with k algebraically closed, and then $G \in \hat{A}^s(k)$, that is, G descends to k . \square

Remark 7.3 *In characteristic 0, the argument is much simpler and more general (it works for any semiabelian variety): since there is no separability issue, we are allowed to take the quotient by the kernel of an isogeny. But in characteristic p , we have to restrict to particular cases, since we know examples of abelian varieties A over K , which are isogenous to some B_K for some B over k , but such that A does not descend to k (see [1]).*

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