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| Luck Darnière. Semi-algebraic triangulation over p-adically closed fields. 2017. hal-01469754v1

**HAL Id: hal-01469754**

**<https://hal.science/hal-01469754v1>**

Preprint submitted on 16 Feb 2017 (v1), last revised 8 Nov 2018 (v2)

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# Semi-algebraic triangulation over $p$ -adically closed fields

Luck Darnière

February 16, 2017

## Abstract

We prove a triangulation theorem for semi-algebraic sets over a  $p$ -adically closed field, quite similar to its real counterpart. We derive from it several applications like the existence of flexible retractions and splitting for semi-algebraic sets.

## Contents

1	Introduction . . . . .	1
2	Prerequisites and notation . . . . .	4
3	Applications . . . . .	14
4	Largely continuous cell decomposition . . . . .	20
5	Cellular complexes . . . . .	25
6	Cellular monoplexes . . . . .	29
7	Cartesian morphisms . . . . .	40
8	Triangulation . . . . .	47

## 1 Introduction

Our knowledge of geometric objects in affine spaces over  $p$ -adic fields, that is the field  $\mathbf{Q}_p$  of  $p$ -adic numbers or a finite extension of it, has grown up intensively in the past decades. Several remarkable analogies have emerged with real geometry, in spite of the striking differences between the real and the  $p$ -adic metrics. The present paper raises a new such analogy: we prove a triangulation theorem over  $p$ -adically closed fields, quite similar to its real counterpart. Let us first recall the classical results in  $p$ -adic geometry which will be used here.

Semi-algebraic sets over a field  $K$  are the finite unions of set defined by finitely many conditions “ $f(x) = 0$ ” or “ $f$  has a non-zero  $N$ -th root in  $K$ ”, where  $f(x)$  is a polynomial function with  $m$  variables. Of course we can restrict to  $N = 2$  if  $K$  is real closed (and to  $N = 1$  if it is algebraically closed). Macintyre has proved in [Mac76] that semi-algebraic sets over  $\mathbf{Q}_p$  are stable by boolean combinations *and* projection (from  $\mathbf{Q}_p^m$  to  $\mathbf{Q}_p^{m-1}$ , for every  $m$ ). This is a  $p$ -adic version of Tarski’s theorem for real closed fields (and of Chevalley’s theorem for

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AMS Classification: 12J12, 51H20, 57N80.

Keywords:  $p$ -adic geometry, semi-algebraic sets, cell decomposition, triangulation, retraction, simplicial complex

algebraically closed fields). Prestel and Roquette (see [PR84]) have generalized it to arbitrary  $p$ -adically closed fields (a  $p$ -adic version of real closed fields).

Denef has proved in [Den84] a cell decomposition theorem for  $p$ -adic semi-algebraic sets very similar to its real counterpart, and derived from it the rationality of Poincaré series. Another important application of the cell decomposition is that it provides a good dimension theory for semi-algebraic sets (see [SvdD88]). All this material has been later widely generalized, first to arbitrary  $p$ -adically closed fields and then to richer structures on these fields (see [DvdD88] and [Clu04] for subanalytic sets, [HM97], [CKL16], [CKDL15] and [DH15] for definable sets in  $P$ -minimal and  $p$ -optimal structures).

By means of Denef's cell decomposition, Cluckers has proved in [Clu01] that for every two semi-algebraic sets  $A, B$  over a  $p$ -adically closed field, there is a semi-algebraic bijection from  $A$  to  $B$  if and only if  $A, B$  have the same dimension and the same number of isolated points. This has been later generalized to subanalytic sets [Clu04] and  $p$ -optimal sets [DH15]. Note that it applies for example to the valuation ring  $\mathbf{Z}_p$  and to  $\mathbf{Z}_p \setminus \{0\}$ . Since the former is compact and the latter is not, they do not belong to the same class of homeomorphism. So the question of classifying semi-algebraic sets up to semi-algebraic homeomorphism is left open by Cluckers' theorem.

In real algebraic geometry, this latter question can be addressed by triangulation. We prove here that, in spite of the lack of Euler characteristic implied by Clucker's result, there exists a  $p$ -adic triangulation quite similar to the real one. We build it for semi-algebraic sets and maps over an arbitrary  $p$ -adically closed field  $K$ , which is fixed throughout all this paper.

**Remark 1.1.** Most of our proofs remain valid for subanalytic sets and maps, and more generally for definable sets and maps in any  $p$ -optimal structure over a  $p$ -adic field (see [DH15]). Unfortunately there is one exception: the argument given in Theorem 4.6 to build a cell decomposition with largely continuous (see below) centers and bounds, by means of small deformations, only works for semi-algebraic sets. This is the single obstruction for a generalization to other structures, and there is good hope that a work-around can be found in the future.

Our triangulation theorem requires numerous non-trivial prerequisites, so we state it here as a foretaste and refer the reader to Section 2 for precise definitions.

**Theorem (Triangulation  $\mathbf{T}_m$ ).** *Given a finite family  $(\theta_i : A_i \subseteq K^m \rightarrow K)_{i \in I}$  of semi-algebraic functions and integers  $n, N \geq 1$ , for some integers  $e, M$  which can be made arbitrarily large<sup>1</sup>, there exists a simplicial complex  $\mathcal{T}$  of index  $M$  and a semi-algebraic homeomorphism  $\varphi : \biguplus \mathcal{T} \rightarrow \bigcup_{i \in I} A_i$  such that for every  $i$  in  $I$ :*

1.  $\{\varphi(T) : T \in \mathcal{T} \text{ and } \varphi(T) \subseteq A_i\}$  is a partition of  $A_i$ .
2.  $\forall T \in \mathcal{T}$  such that  $\varphi(T) \subseteq A_i$ ,  $\theta_i \circ \varphi|_T$  is  $N$ -monomial mod  $U_{e,n}$ .

We call the pair  $(\mathcal{T}, \varphi)$  given by  $\mathbf{T}_m$  a **triangulation** of the  $\theta_i$ 's with **parameters**  $(n, N, e, M)$ . When a finite family  $(A_i)_{i \in I}$  of semi-algebraic sets is

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<sup>1</sup>The exact meaning of “ $e, M$  can be made arbitrarily large” is a bit special here: it says that for any given integers  $e_* \geq 1$  and  $M_* \geq 1$ , the integers  $e, M$  can be chosen so that  $e_*$  divides  $e$  and  $M_* \leq M$ .

given, the result of the application of  $\mathbf{T}_m$  to the indicator functions of the  $A_i$ 's is called a triangulation of  $(A_i)_{i \in I}$ .

**Remark 1.2.** The elements of the simplicial complex appearing in the conclusion of  $\mathbf{T}_m$  are not contained in  $K^m$  but in finitely many copies of  $K^q$  for various  $q$ , usually much larger than  $m$ . This is the main, but harmless, structural difference with the triangulation in the real case.

In section 3 we derive from  $\mathbf{T}_m$  the next four applications.

**Theorem (Lifting).** *For every semi-algebraic function  $f : X \subseteq K^m \rightarrow K$  such that  $|f|$  is continuous, there is a continuous semi-algebraic function  $h : X \rightarrow K$  such that  $|f| = |h|$ .*

The real counterpart of the above result is quite obvious. At the contrary, the two next results do not hold true in real geometry. In the same vein as Clucker's result on classification of semi-algebraic sets up to semi-algebraic bijection [Clu01], they confirm the intuition that the lack of connectedness and of "holes" (in the sense of algebraic topology, see below) makes semi-algebraic sets over  $p$ -adically closed fields much more flexible than over real closed fields.

Recall that a **retraction** of a topological space  $X$  onto a subspace  $Y$  is a continuous map  $\sigma : X \rightarrow Y$  such that  $\sigma(y) = y$  for every  $y \in Y$ . When such a retraction exists on a Hausdorff space  $X$ , then necessarily  $Y$  is closed in  $X$ .

Over the reals, the main obstruction for the existence of retractions is the existence of "holes" which are detected by homotopy. This does not work over  $p$ -adic fields. Indeed, given a non-empty semi-algebraic set  $X \subseteq K^m$  and a point  $a \in X$ , the semi-algebraic function  $H : X \times R \rightarrow X$  defined by  $H(x, s) = x$  if  $s \in R^\times$  and  $H(x, s) = a$  if  $s \in \pi R$  is obviously continuous. In that sense, with the unit ball  $R$  replacing in  $K$  the unit interval  $[0, 1]$  in the reals, the identity function on  $X$  is homotopic to a constant function, that is  $X$  is "contractile". But it is another story to prove that retractions actually exist.

**Theorem (Retraction).** *For every non-empty semi-algebraic sets  $Y \subseteq X \subseteq K^m$ , there is a semi-algebraic retraction of  $X$  onto  $Y$  if and only if  $Y$  is closed in  $X$ .*

It is worth mentioning that it is the next Splitting Theorem, already conjectured in [Dar06], which has motivated all the present paper. Here  $\partial X$  denotes the topological frontier of  $X$ , see Section 2.

**Theorem (Splitting).** *Let  $X$  be a relatively open non-empty semi-algebraic subset of  $K^m$  without isolated points, and  $Y_1, \dots, Y_s$  a collection of closed semi-algebraic subsets of  $\partial X$  such that<sup>2</sup>  $Y_1 \cup \dots \cup Y_s = \partial X$ . Then there is a partition of  $X$  in non-empty<sup>3</sup> semi-algebraic sets  $X_1, \dots, X_s$  such that  $\partial X_i = Y_i$  for  $1 \leq i \leq s$ .*

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<sup>2</sup>Note that  $Y_1, \dots, Y_s$  are not assumed to be disjoint. All of them can be equal to  $\partial X$ , for example.

<sup>3</sup>A **partition** of a set  $X$  is for us a family of two-by-two disjoint subsets of  $X$  covering  $X$ . We do not assume that the pieces must be non-empty. So when it happens by exception, like here, that this property is required and does not follow from the context, we explicitly mention it.

The trivial remark that every ball  $B \subseteq K^m$  is disconnected can be seen as a very special case of the above Splitting Property (applied to  $X = B$  with  $Y_1 = Y_2 = \emptyset$ ). This property is actually (in a sense which can be made precise, see [Dar06]) the strongest possible form of disconnectedness that can be observed in a topological space whose points are closed. It is a versatile property which we encountered in different contexts with minor changes (see [Dar06], [DJ10]). In the present paper, it plays a key role in the induction step.

A **limit value** for a function  $f : X \subseteq K^m \rightarrow K$  at a point  $x$  adherent to  $X$ , is a value  $l \in K$  such that  $(x, l)$  is adherent to the graph of  $f$ . Of course  $f$  tends to  $l$  at  $x$  if and only if  $l$  is the unique limit value of  $f$  at  $x$ . Let us say that  $f$  is **largely continuous** on a subset  $A$  of  $X$  if the restriction of  $f$  to  $A$  has a unique limit value at every point adherent to  $X$ , that is if  $f$  extends to a continuous function on the topological closure of  $A$ . If  $A$  is not mentioned it simply means that  $f$  is largely continuous on its domain  $X$ . Finally  $f$  is **piecewise largely continuous** if there exists a finite partition of  $X$  in semi-algebraic pieces on which  $f$  is largely continuous. Of course in that case  $f$  has finitely many limit values at every point adherent to  $X$ .

**Theorem** (Largely Continuous Splitting). *Let  $f : X \subseteq K^m \rightarrow K$  be a semi-algebraic bounded function with bounded<sup>4</sup> domain. If  $f$  has finitely many limit values at every point adherent to  $X$  then  $f$  is piecewise largely continuous.*

The real counterpart of this result is easily seen to be true, by means of a triangulation and the trivial remark that every real simplex is connected (see Section 3). This last argument is no longer valid in the  $p$ -adic case but, as we will see, the existence of retractions allows us to bypass this problem and recover the full result in the  $p$ -adic context.

This paper is a continuation of [Dar16], where  $p$ -adic simplexes were introduced and studied. However all the results of [Dar16] used here are recalled in Section 2, as well as all the classical prerequisites needed. The above applications are then derived from  $\mathbf{T}_m$  in Section 3. In Section 4 we derive from Denef’s cell decomposition and  $\mathbf{T}_m$  a “largely continuous cell decomposition up to a small deformation” for semi-algebraic functions in  $m + 1$  variables (Theorem 4.6). Sections 5 to 7 are then devoted to our main constructions, which are summarized in Lemma 6.1 and Lemma 7.10. In Section 8, we finally derive  $\mathbf{T}_{m+1}$  from  $\mathbf{T}_m$  by means of these two technical lemmas. This finishes the proof of our  $p$ -adic triangulation theorem for every  $m$ .

**Acknowledgement.** The idea of the proof of the Good Direction Lemma 4.4 was given to me by my colleague Daniel Naie (Univ. Angers, France).

## 2 Prerequisites and notation

We let  $\mathbf{N}$  denote the set of positive integers and  $\mathbf{N}^* = \mathbf{N} \setminus \{0\}$ . For every integers  $k, l$  we let  $\llbracket k, l \rrbracket$  be the set of integers  $i$  such that  $k \leq i \leq l$  (hence an

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<sup>4</sup>These boundedness assumptions could easily be removed if we add to  $K$  a point at infinity and require that  $f$  has finitely limit values in  $\hat{K} = K \cup \{\infty\}$  at every point of the closure of  $X$  in  $\hat{K}^m$ , using the same construction as in the preparation of the proof of Lemma 3.3.

empty set if  $k > l$ ).

A **Z-group** is a linearly ordered group  $\mathcal{Z}$  such that, with additive notation,  $\mathcal{Z}$  has a least strictly positive element 1 and  $\mathcal{Z}/n\mathcal{Z}$  has exactly  $n$  elements for every integer  $n \geq 1$ . A  **$p$ -adically closed field** (see [PR84]) is a Henselian valued field of characteristic zero whose residue field is a finite field with characteristic  $p$  and whose value group is a **Z-group**. The most prominent examples of such fields, which are the  $p$ -adic counterpart of real closed fields, are obviously the field  $\mathbf{Q}_p$  of  $p$ -adic numbers and all its finite extensions. But there are other natural examples such as the algebraic closure of  $\mathbf{Q}$  inside  $\mathbf{Q}_p$  (which is not complete), the  $t$ -adic completion of the field  $\bigcup_{n \geq 1} \mathbf{Q}((t^{1/n}))$  of Puiseux series over  $\mathbf{Q}_p$  (whose value group is not  $\mathbf{Z}$ , but  $\mathbf{Z} \times \mathbf{Q}$  lexicographically ordered), and many others (every ultraproduct of  $p$ -adically closed fields is still  $p$ -adically closed).

Throughout all this paper we consider a  $p$ -adically closed field  $K$  fixed once for all. We let  $v$  denote its (unique)  $p$ -adic valuation,  $R$  its valuation ring,  $R^\times$  the multiplicative group of invertible elements of  $R$ , and  $\pi$  a fixed generator of the maximal ideal of  $R$ .

**Remark 2.1.** From now on the letter  $p$  will usually denote various integers unrelated to the characteristic of the residue field of  $v$ . The only exception is when it appears in expressions such as “the field  $\mathbf{Q}_p$ ”, “a  $p$ -adically closed field” or “a  $p$ -adic simplex”, which refer to  $p$ -adic analogues of notions coming from real algebraic geometry.

We let  $\mathcal{Z}$  denote the value group of  $K$ , with additive notation. It is extended by a largest element  $+\infty$  for  $v(0)$  and we let  $\Gamma = v(K) = \mathcal{Z} \cup \{+\infty\}$ . We let  $\mathcal{Q}$  denote the divisible hull of  $\mathcal{Z}$  and  $\Omega = \mathcal{Q} \cup \{+\infty\}$ . As an ordered group,  $\mathbf{Z}$  identifies naturally to the smallest non-trivial convex subgroup of  $\mathcal{Z}$ . We consider  $\mathbf{Z}$  and  $\mathbf{Q}$  as embedded into  $\mathcal{Q}$  *via* this identification.

For every subset  $X$  of  $K$  we let  $X^* = X \setminus \{0\}$ . However, if  $X^*$  is a subgroup of the multiplicative group of  $K$ , we denote it  $X^\times$  in order to remind this property (so  $R^* = R \setminus \{0\} \neq R^\times$  but  $K^* = K \setminus \{0\} = K^\times$ ). For every subgroup  $G$  of  $K^\times$  we let  $xG = \{xg : g \in G\}$  for every  $x \in K$ , and  $K/G = \{xG : x \in K\}$ . Abusing the notation,  $0G = \{0\}$  will be denoted  $0$  whenever the context makes it unambiguous.

We let  $|a| = aR^\times$  for every  $a \in K$ , and  $|K| = K/R^\times = \{|a| : a \in K\}$ . The latter is naturally ordered by inclusion, and isomorphic to  $\Gamma$  with the reverse order :  $|a| \leq |b|$  if and only if  $v(a) \geq v(b)$ . So  $|a|$  is just a multiplicative notation for  $v(a)$ : we have  $|ab| = |a| \cdot |b|$  and  $|a + b| \leq \max(|a|, |b|)$ , and of course  $|a| = 0$  if and only if  $a = 0$ .

In order to ease the notation, given  $a \in K^m$ ,  $A \subseteq K^m$  and  $f : X \rightarrow K^m$  we will often write  $va$  for  $v(a)$ ,  $vA$  for the direct image  $v(A)$ ,  $vf$  for the composite  $v \circ f$ , and similarly for  $|A|$  and  $|f|$ .

At some rare places it will be convenient to add to  $K$  a new element  $\infty$  (and to  $\Gamma$  and  $|K|$  new elements  $-\infty$  and  $+\infty$  respectively) with the natural convention that  $0^{-1} = \infty$ ,  $\infty^{-1} = 0$ ,  $v(\infty) = -\infty$ ,  $|\infty| = +\infty$ , and  $a \cdot \infty = \infty$  for every  $a \in K^\times$ . We also let  $0 \cdot \infty = 1$  and  $0^0 = 1$  when needed.

## 2.a Topology and coordinate projections

When an  $m$ -tuple  $a$  is given, it is understood that  $(a_1, \dots, a_m)$  are its coordinates, except if otherwise specified. For every  $a \in K^m$  we let

$$va = (va_1, \dots, va_m) \quad \text{and} \quad |a| = (|a_1|, \dots, |a_m|).$$

This has not to be confused with  $\|a\| = \max(|a_1|, \dots, |a_m|)$ . For  $r \in K^\times$  the (clopen) **ball** of center  $a$  and radius  $r$  is defined as

$$B(a, r) = \{x \in K^m : \|x - a\| \leq |r|\}.$$

The valuation induces a topology on  $K$ , which is inherited by  $|K|$  and  $\Gamma$ . The topology generated on  $\Omega$  by the open intervals and the intervals  $]a, +\infty]$  for  $a \in \mathcal{Q}$ , extends the topology of  $\Gamma$ . The direct products of these topological spaces endow the product topology. For every subset  $X$  of any of these spaces,  $\overline{X}$  denotes the topological closure of  $X$ . In particular  $\overline{\mathcal{Z}} = \Gamma$  and  $\overline{\mathcal{Q}} = \Omega$ . Note that  $\Gamma$  is closed in  $\Omega$ . The **specialisation preorder** on the subsets of  $X$  is defined by  $B \leq A$  iff  $B \subseteq \overline{A}$ .

We let  $\partial X = \overline{X} \setminus X$  denote the **frontier** of  $X$ . We say that  $X$  is **relatively open** if it is open in  $\overline{X}$ , that is if  $\partial X = \overline{X} \setminus X$ .

When a function  $f$  is largely continuous (see Section 1) we usually denote  $\overline{f}$  the continuous extension of  $f$  to the closure of its domain. At the contrary, the restriction of  $f$  to some subset  $A$  of its domain is denoted  $f|_A$ .

The **support** of  $a \in K^m$ , denoted  $\text{Supp } a$ , is the set of indexes  $k$  such that  $a_k \neq 0$ . The support of an element of  $|K|^m$  or  $\Gamma^m$  is defined accordingly, so that

$$\text{Supp } |a| = \text{Supp } v(a) = \text{Supp } a.$$

For every subset  $S$  of  $K^m$  and every  $I \subseteq \{1, \dots, m\}$  we let

$$F_I(S) = \{a \in \overline{S} : \text{Supp } a = I\}.$$

When  $F_I(S) \neq \emptyset$  we call it the **face of  $S$  with support  $I$** . Faces of subsets of  $|K|^m$  or  $\Gamma^m$  are defined accordingly. The **coordinate projection** of  $K^m$  (resp.  $|K|^m, \Gamma^m$ ) onto its face with support  $I$  will be denoted  $\pi_I$ . So  $\pi_I(a)$  is the unique point  $b$  with support  $I$  such that  $b_i = a_i$  for every  $i \in I$ .

For every  $a \in K^{m+1}$  (resp.  $|K|^{m+1}$  or  $\Gamma^{m+1}$ ) we let  $\widehat{a}$  denote the tuple of the first  $m$  coordinates of  $a$ , so that  $a = (\widehat{a}, a_{m+1})$ . If  $A$  is a set of  $(m+1)$ -tuples we let  $\widehat{A} = \{\widehat{a} : a \in A\}$ , and if  $\mathcal{A}$  is a family of such sets we let

$$\widehat{\mathcal{A}} = \{\widehat{A} : A \in \mathcal{A}\}.$$

We call  $\widehat{A}$  (resp.  $\widehat{\mathcal{A}}$ ) the **socle** of  $A$  (resp.  $\mathcal{A}$ ).

Given two families  $\mathcal{H}, \mathcal{A}$  of subsets of  $K^{m+1}$  we say that  $\mathcal{H}$  is **finer** than  $\mathcal{A}$  if every  $H \in \mathcal{H}$  which meets a set  $A \in \mathcal{A}$  is contained in  $A$ . If moreover  $\mathcal{H}$  is a partition of  $\bigcup \mathcal{A}$  we say that  $\mathcal{H}$  **refines**  $\mathcal{A}$ . We will often distinguish between “*horizontal refinements*” for which  $\widehat{\mathcal{H}} = \widehat{\mathcal{A}}$ , and “*vertical refinement*” for which  $\mathcal{H}$  is the family of  $A \cap (X \times K)$  where  $A$  ranges over  $\mathcal{A}$  and  $X$  over a refinement of the socle of  $\mathcal{A}$ .

## 2.b Semi-algebraic sets and formulas

For every integer  $N \geq 1$  let

$$\mathbf{P}_N = \{a \in K : \exists x \in K, a = x^N\}.$$

$\mathbf{P}_N^\times = \mathbf{P}_N \setminus \{0\}$  is a clopen subgroup of  $K^\times$  with finite index, and  $\mathbf{P}_1 = K^\times$ . Hence a subset  $K^m$  is a **semi-algebraic set** if it is a boolean combination of finitely many sets  $S_i$  defined by conditions

$$f_i(x) \in \mathbf{P}_{N_i} \tag{1}$$

where the  $f_i$ 's are  $m$ -ary polynomial functions. A **semi-algebraic map** is a function whose graph is semi-algebraic. Rational functions, root functions and monomial functions (see below) are semi-algebraic, among many others.

Abusing a little bit the terminology, we also say that a subset  $S$  of  $K^m \times |K|^n$  is semi-algebraic if  $\{(x, t) \in K^{m+n} : (x, |t|) \in S\}$  is semi-algebraic. Similarly a function  $f : X \subseteq K^m \rightarrow |K|^n$  is semi-algebraic if its graph is. When a map  $\varphi$  is defined on the disjoint union of finitely many semi-algebraic sets  $A_i$  living in different copies of  $K^m$ , we say that  $\varphi$  is semi-algebraic if its restriction to each  $A_i$  is semi-algebraic in the classical sense.

**Remark 2.2.** If  $N'$  divides  $N$  then  $\mathbf{P}_N^\times$  is a clopen subgroup of  $\mathbf{P}_{N'}^\times$ , with finite index. For this reason, all the integers  $N_{i,j}$  appearing in (1) can be replaced by any common multiple  $N$ . Note also that  $0 \in \mathbf{P}_N$  is an empty condition, equivalent to  $1 \in \mathbf{P}_N$ , hence all the  $f_{i,j}$ 's can be assumed to be non-zero polynomials.

The so-called first order formulas in the language of rings with parameters in  $K$ , **formulas** for short, are defined inductively as follows:

1. Every equation " $f(x) = 0$ " with  $f$  an  $n$ -ary polynomial with coefficients in  $K$  is a formula with  $n$  free variables.
2. Every combination of formulas with  $n$  free variables using the logical connectives "and", "or", "not" (denoted  $\wedge$ ,  $\vee$ ,  $\neg$ ) is a formula with  $n$  free variables.
3. If  $\varphi(x)$  is a formula with an  $n$ -tuple  $x$  of free variables  $x_1, \dots, x_n$ , then  $\exists x_n \varphi(x)$  and  $\forall x_n \varphi(x)$  are formulas with  $n - 1$  free variables.

In the last notation above it is understood that the quantified variable  $x_n$  only concerns elements of  $K$ . Thus a formula  $\varphi(x)$  with  $n$  free variables always states a property of the elements of  $K^n$ . When a tuple  $a \in K^n$  has this property we say that  $\varphi(x)$  is **satisfied** by  $a$  in  $K$  and denote it  $K \models \varphi(a)$ . A set  $S$  is **definable** over  $K$  if there exists a formula  $\varphi(x)$  with  $n$  free variables such that

$$S = \{a \in K^n : K \models \varphi(a)\}.$$

A family  $(C_a)_{a \in A}$  of semi-algebraic subsets of  $K^n$  is **uniformly definable** if  $A \subseteq K^m$  is definable and there is a formula  $\varphi(x, y)$  with  $m + n$  free variables such that  $C_a = \{b \in K^n : K \models \varphi(a, b)\}$  for every  $a \in A$ .



Given  $m$ -ary definable functions  $f, g$ , the set of points in  $K^m$  satisfying the condition “ $|f(x)| \leq |g(x)|$ ” is known to be definable<sup>5</sup>. Thus we will consider these expressions as formulas as well (more exactly as abbreviations of some first order formulas in the language of rings stating the same property). Similarly, if  $\varphi(x, y)$  is a formula with  $m + n$  variables and  $S \subseteq K^n$  is definable by a formula  $\psi(y)$  then we will consider  $\exists y \in S, \varphi(x, y)$  as a formula since it is an abbreviation for the genuine formula  $\exists y, \psi(y) \wedge \varphi(x, y)$ . We refer the reader to [Hod97] for more information on this topic.

**Theorem 2.3** (Macintyre). *The definable subsets of  $K^m$  are exactly the semi-algebraic ones.*

**Remark 2.4.** This fundamental theorem says that, in this paper, “semi-algebraic” and “definable” are synonymous. In particular a subset of  $K^m$  is semi-algebraic whenever it can be defined by a formula. This is an extremely convenient criterion for being semi-algebraic, which we will use everywhere without more comments.

Another important consequence of Macintyre’s theorem is that every  $p$ -adically closed field is elementarily equivalent to a finite extension of  $\mathbf{Q}_p$  (see [PR84]). With other words, there is a finite extension  $L$  of  $\mathbf{Q}_p$  such that  $K$  and  $L$  satisfy exactly the same formulas. The following semi-algebraic version of the so-called “théorème des fermés emboîtés” transfers from  $L$  to  $K$  by means of this elementary equivalence.

**Theorem 2.5.** *Let  $(C_\alpha)_{\alpha \in R^+}$  be a uniformly definable family of non-empty, closed and bounded subsets of  $K^n$ , such that  $|\beta| \leq |\alpha|$  implies that  $C_\beta \subseteq C_\alpha$ . Then  $\bigcap_{\alpha \in R^+} C_\alpha$  is non-empty.*

The next classical properties can easily be derived from this theorem (or transferred from  $L$  to  $K$  by elementary equivalence).

**Theorem 2.6.** *For every continuous semi-algebraic function  $f : X \subseteq K^m \rightarrow K^n$  whose domain  $X$  is closed and bounded,  $f(X)$  is closed and bounded. As a consequence:*

1.  $\|f\|$  is bounded and attains its bounds.
2. If  $f$  is injective then it is a homeomorphism from  $X$  to  $f(X)$ .

**Corollary 2.7.** *For every bounded semi-algebraic subset  $X$  of  $K^m$  which is non-empty, there is an element  $x \in X$  such that  $\|x\|$  is maximal on  $X$ .*

Another crucial property of  $p$ -adically closed fields is the existence of so called “built-in Skolem functions” (see [vdD84], or the appendix of [DvdD88] for a more constructive proof). Basically, it says that for every semi-algebraic subset  $A$  of  $K^{m+n}$ , the coordinate projection of  $A$  onto  $K^m$  has a *semi-algebraic* section.

**Theorem 2.8** (Skolem functions). *Let  $X \subseteq K^m$  be semi-algebraic set and  $\varphi(x, t)$  a formula with  $m + n$  free variables. If, for every  $a \in X$  there is  $b \in K^n$  such that  $K \models \varphi(a, b)$ , then there exists a semi-algebraic function  $\xi : X \rightarrow K^n$  (called a Skolem function) such that  $K \models \varphi(x, \xi(x))$  for every  $x \in X$ .*

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<sup>5</sup>This follows from the non-trivial fact that  $R$  is definable by means of the Kochen operator (see [PR84]).

For example, if a semi-algebraic function  $f : X \rightarrow K$  takes values in  $\mathbf{P}_N$ , then Theorem 2.8 applied the formula  $\varphi(x, t)$  saying that “ $f(x) = t^N$ ” gives a semi-algebraic function  $\xi : X \rightarrow K$  such that  $f = \xi^N$ .

## 2.c Root functions and monomial functions

Following Lemma 1.3 in [CL12] there is for each integer  $M > 0$  a unique group homomorphism  $\overline{ac}_M$  from  $K^\times$  to  $(R/\pi^M R)^\times$  such that  $\overline{ac}_M(\pi) = 1$  and  $\overline{ac}_M(u) = u + \pi^M R$  for every  $u \in R^\times$ . The construction of  $\overline{ac}_M$  given in [CL12] shows that for each integer  $N > 0$  the set

$$Q_{N,M} = \{0\} \cup \{x \in \mathbf{P}_N^\times \cdot (1 + \pi^M R) : \overline{ac}_M(x) = 1\}$$

is semi-algebraic.  $Q_{N,M}^\times = Q_{N,M} \setminus \{0\}$  is a clopen subgroup of  $K^\times$  with finite index. When  $v(K^\times) = \mathbf{Z}$ , it is worth mentioning that  $Q_{N,M}$  can be given a simpler description as follows:

$$Q_{N,M} = \{0\} \cup \bigcup_{k \in \mathbf{Z}} \pi^{kN} (1 + \pi^M R).$$

If  $M > 2v(N)$ , Hensel’s Lemma implies that  $1 + \pi^M R \subseteq \mathbf{P}_N$ , hence  $Q_{N,M}$  is contained in  $\mathbf{P}_N$ . The importance of  $Q_{N,M}$  comes from the following property, which also follows from Hensel’s lemma (see for example lemma 1 and corollary 1 in [Clu01]).

**Lemma 2.9.** *The function  $x \mapsto x^e$  is a group endomorphism of  $Q_{N,M}^\times$ . If  $M > v(e)$  this endomorphism is injective and its image is  $Q_{eN, v(e)+M}^\times$ .*

In particular  $x \mapsto x^e$  defines a continuous bijection from  $Q_{1, v(e)+1}$  to  $Q_{e, 2v(e)+1}$ . We let  $x \mapsto x^{1/e}$  denote the reverse continuous bijection. In particular it is defined on  $Q_{N,M}$  for every  $N, M$  such that  $e$  divides  $N$  and  $M > 2v(e)$ .

For every positive integers  $e, n$  we let

$$\mathbf{U}_e = \{x \in K : x^e = 1\} \quad \text{and} \quad U_{e,n} = \mathbf{U}_e \cdot (1 + \pi^n R).$$

Analogously to Landau’s notation  $\mathcal{O}(x^n)$  of calculus, we let  $\mathcal{U}_{e,n}(x)$  denote any semi-algebraic function in the multi-variable  $x$  with values in  $U_{e,n}$ . Any such function is the product of two semi-algebraic functions, with values in  $\mathbf{U}_e$  and  $1 + \pi^n R$  respectively. So, given a family of functions  $f_i, g_i$  on the same domain  $X$ , we write that  $f_i = \mathcal{U}_{e,n} g_i$  for every  $i$ , when there are semi-algebraic functions  $\omega_i : X \rightarrow R$  and  $\chi_i : X \rightarrow \mathbf{U}_e$  such that for every  $x$  in  $X$

$$f_i(x) = \chi_i(x) (1 + \pi^n \omega_i(x)) g_i(x).$$

$\mathcal{U}_{1,n}(x)$  is simply denoted  $\mathcal{U}_n(x)$ .

**Remark 2.10.** If  $f(x) = \mathcal{U}_n(x)$  for some  $n > 2v(e)$  then  $f^{1/e}$  is well defined and takes values in  $1 + \pi^{n-v(e)} R$ . Therefore we can write  $\mathcal{U}_n(x) = (\mathcal{U}_{n-v(e)}(x))^e$ .

A function  $g$  is  **$N$ -monomial** on  $S \subseteq K^q$  if either it is constantly equal to  $\infty$  or there exists  $\xi \in K$  and  $\beta_1, \dots, \beta_q \in \mathbf{Z}$  such that

$$\forall x = (x_1, \dots, x_q) \in S, \quad g(x) = \xi \prod_{i=1}^q x_i^{N\beta_i}.$$

In this definition we use when needed our convention that  $0^0 = 1$ . A function  $f$  is  **$N$ -monomial mod  $U_{e,n}$**  if  $f = \mathcal{U}_{e,n} g$  with  $g$  an  $N$ -monomial function.

## 2.d Cell decomposition and dimension theory

Given a clopen semi-algebraic subgroup  $\mathbf{G}$  of  $K^\times$  with finite index, a **presented cell**  $A \bmod \mathbf{G}$  in  $K^{m+1}$  is a tuple  $(c_A, \nu_A, \mu_A, G_A)$  with  $c_A$  a semi-algebraic function on a non-empty domain  $X \subseteq K^m$  with values in  $K$  (called the **center** of  $A$ ),  $\nu_A$  and  $\mu_A$  either semi-algebraic functions on  $X$  with values in  $K^\times$  or constant functions on  $X$  with values 0 or  $\infty$  (called the **bounds** of  $A$ ), and  $G_A$  an element of  $K/G$  (called the **coset** of  $A$ ), having the property that for every  $x \in X$  there is  $t \in K$  such that:

$$|\nu_A(x)| \leq |t - c_A(x)| \leq |\mu_A(x)| \quad \text{and} \quad t - c_A(x) \in G_A \quad (2)$$

The set of tuples  $(x, t) \in X \times K$  satisfying (2) is the **cellular set underlying**  $A$ . Abusing the notation we will also denote it  $A$  most often. The conditions enumerated above (2) ensure that the domain  $X$  of  $c_A, \mu_A, \nu_A$  is exactly the socle  $\widehat{A}$  of  $A$ . When two presented cells  $A$  and  $B$  have the same underlying cellular set we write it  $A \simeq B$ .

$A$  is of **type 0** if  $G_A = \{0\}$ , of type 1 otherwise. We say that  $A$  is **largely continuous** if its center and bounds are so. We say that it is **well presented** if either  $v\nu_A - v\mu_A$  is unbounded or  $\nu_A = \mu_A$ .

We call  $A$  a **fitting cell** if it has **fitting bounds** that is, for every  $x \in \widehat{A}$ :

$$|\mu_A(x)| = \sup\{|t - c_A(x)| : (x, t) \in A\}$$

$$|\nu_A(x)| = \inf\{|t - c_A(x)| : (x, t) \in A\}$$

Cellular sets mod  $K^\times$  and  $\mathbf{P}_N^\times$  have been first introduced in [Den84]. Cellular sets mod  $Q_{N,M}^\times$  appear implicitly in [Clu01], and explicitly in further papers of Cluckers. In this paper we will use the word “**cell**” mostly for *presented cells* but also very often for the *underlying cellular sets*, the difference being clear from the context. For instance we will freely talk of disjoint (presented) cells, of bounded (presented) cells, of (presented) cells partitioning some set and so on, meaning that the corresponding cellular sets have these properties.

For any  $Z \subseteq \widehat{A}$  we will write  $A \cap (Z \times K)$  both for this (cellular) set and for the presented cell  $(c_{A|Z}, \nu_{A|Z}, \mu_{A|Z}, G_A)$ . The latter will also be denoted  $(c_A, \nu_A, \mu_A, G_A)|_Z$ . Similarly  $\text{Gr } c_A$  both denotes the graph of  $c_A$  and the presented cell  $(c_A, 0, 0, \{0\})$ .

Sometimes it will be convenient to write  $G_A = \lambda_A \mathbf{G}$  for some  $\lambda_A \in G_A$ . We will always do this uniformly, so that  $\lambda_A = \lambda_B$  whenever  $G_A = G_B$ . To that end a set  $\Lambda_{\mathbf{G}}$  of representatives of  $K/\mathbf{G}$  is fixed once for all, and when we consider a presented cell  $A \bmod \mathbf{G}$  it is understood that  $\lambda_A$  is the unique element of  $G_A \cap \Lambda_{\mathbf{G}}$ . In addition, we require from this set of representatives that every  $\lambda \in \Lambda_{\mathbf{G}}$  has the smallest possible positive valuation. In particular if  $\mathbf{G} = P_N$  or  $Q_{N,M}$  and  $A$  is a cell mod  $\mathbf{G}$  of type 1 then  $0 \leq v\lambda_A < N$ .

For every family  $\mathcal{A}$  of presented cells in  $K^{m+1}$  we let  $\text{CB}(\mathcal{A})$  denote the family of all the functions  $c_A, \mu_A, \nu_A$  for  $A \in \mathcal{A}$ . Given another family  $\mathcal{D}$  of presented cells in  $K^{m+1}$  we say that:

1.  $\mathcal{D}$  belongs to  $\text{Vect } \mathcal{A}$  if for every  $D \in \mathcal{D}$ ,  $c_D, \nu_D$  are linear combinations (with coefficients in  $K$ ) of the restrictions to  $\widehat{D}$  of the centers and bounds of the cells  $A \in \mathcal{A}$  such that  $\widehat{D} \subseteq \widehat{A}$ , and either  $\mu_D$  is so or  $\mu_D = \infty$ .

2.  $\mathcal{D}$  belongs to  $\text{Alg}_n \mathcal{A}$  if  $\mathcal{D}$  is finer than  $\mathcal{A}$  and for every  $A \in \mathcal{A}$ , every  $D \in \mathcal{D}$  contained in  $A$  and every  $(x, t) \in D$ :
  - (a) either  $t - c_A(x) = \mathcal{U}_n(x, t)(t - c_D(x))$ ;
  - (b) or  $t - c_A(x) = \mathcal{U}_n(x, t)h_{D,A}(x)$  where  $h_{D,A} : \widehat{D} \rightarrow R$  is the product of (finitely many) linear combinations of functions  $c_{B|_{\widehat{D}}}$  such that  $B \in \mathcal{A}$  and  $\widehat{D} \subseteq \widehat{B}$ .

**Theorem 2.11** (Denef). *Given a semi-algebraic subgroup  $\mathbf{G}$  of  $K^\times$  with finite index, let  $\mathcal{A}$  be a finite family of presented cells mod  $\mathbf{G}$  in  $K^{m+1}$ . Then for every positive integer  $n$  there exists a finite family  $\mathcal{D}$  of fitting cells mod  $\mathbf{G}$  refining  $\mathcal{A}$  such that  $\widehat{\mathcal{D}}$  is a partition and  $\mathcal{D}$  belongs to  $\text{Vect } \mathcal{A}$  and to  $\text{Alg}_n \mathcal{A}$ .*

This is essentially theorem 7.3 of [Den84]. Indeed, for any given integer  $N$ , if  $n$  is large enough then  $1 + \pi^n R \subseteq \mathbf{P}_N \cap R^\times$ . Hence  $\mathcal{U}_n(x, t)$  in conditions (2a), (2b) of the definition of  $\text{Alg}_n \mathcal{A}$  can be written  $u(x, t)^N$  with  $u$  a semi-algebraic function from  $A$  to  $R^\times$  (thanks to Theorem 2.8). This is how the above result is stated in [Den84] with  $\mathbf{G} = K^\times$ . Our slightly more precise form, as well as the additional properties involving  $\text{Vect } \mathcal{A}$  and  $\text{Alg}_n \mathcal{A}$ , appear only in the proof of theorem 7.3 in [Den84] (still with  $\mathbf{G} = K^\times$ ). The generalization to fitting cells mod an arbitrary clopen semi-algebraic group  $\mathbf{G}$  with finite index in  $K^\times$  is straightforward.

The cell decomposition leads to a good dimension theory for semi-algebraic sets over  $p$ -adically closed fields, see [SvdD88] and [vdD89]. We will use repeatedly its following properties, for every semi-algebraic sets  $A, B$  and semi-algebraic map  $f$  defined on  $A$ . By convention  $\dim \emptyset = -\infty$ .

1.  $\dim A = 0$  if and only if  $A$  is finite non-empty.
2.  $\dim A \cup B = \max(\dim A, \dim B)$ .
3. If  $A \neq \emptyset$ ,  $\dim \partial A < \dim A$ .
4.  $\dim f(A) \leq \dim A$ .

The **local dimension** of a semi-algebraic set  $A \subseteq K^m$  at a point  $a \in A$  is the minimum of  $\dim U$ , for every semi-algebraic neighbourhood  $U$  of  $a$  in  $A$  (with respect to the relative topology, induced by  $K^m$  on  $A$ ).  $A$  is **pure dimensional** if it has the same local dimension at every point. Note that if a semi-algebraic set  $B$  is open in  $A$  and  $A$  is pure dimensional then  $B$  is so, and that a cell is pure dimensional if and only if its socle is. This last point, combined with Denef's Cell Decomposition Theorem 2.11 and a straightforward induction, shows that every semi-algebraic set  $A$  is the union of finitely many pure dimensional ones.

## 2.e Discrete and $p$ -adic simplexes

We say that  $f : S \subseteq F_I(\Gamma^m) \rightarrow \Omega$  is **affine** if either it is constantly equal to  $+\infty$ , or there are elements  $\alpha_0 \in \mathcal{Q}$  and  $\alpha_i \in \mathbf{Q}$  for  $i \in I$  such that

$$\forall x \in S, f(x) = \alpha_0 + \sum_{i \in I} \alpha_i x_i.$$

In [Dar16] we introduced discrete polytopes in  $\Gamma^q$  as “largely continuous precells mod  $N$ ”, for an arbitrary  $q$ -tuple  $N$  of positive integers. In this paper  $N = (1, \dots, 1)$  will not play any role so we can remove it in the next definition.

The only polytope in  $\Gamma^0$  is  $\Gamma^0$  itself (which is a one-point set). For every  $I \subseteq \llbracket 1, q+1 \rrbracket$ , a subset  $A$  of  $F_I(\Gamma^{q+1})$  is a **discrete polytope** of  $\Gamma^{q+1}$  if  $\widehat{A}$  is a discrete polytope of  $\Gamma^q$  and if there is a pair  $(\mu, \nu)$  of *largely continuous* affine maps from  $\widehat{A}$  to  $\Omega$ , called a **presentation** of  $A$ , such that  $0 \leq \mu \leq \nu$  and

$$A = \left\{ a \in F_I(\Gamma^q) : \widehat{a} \in \widehat{A} \text{ and } \mu(\widehat{a}) \leq a_{q+1} \leq \nu(\widehat{a}) \right\}.$$

**Example 2.12.**

- $A = \{(x, y) \in \mathbf{Z}^2 : 0 \leq y \leq x/2\}$  is a discrete simplex, with proper faces  $F_{\{2\}}(A) = \{(+\infty, y) \in \{+\infty\} \times \mathbf{Z} : 0 \leq y\}$  and  $F_\emptyset(A) = \{(+\infty, +\infty)\}$ .
- $B = \{(x, y, z) \in \mathbf{Z}^3 : (x, y) \in A \text{ and } z = 2y - x\}$  is a subset of  $\mathbf{Z}^3$  defined by linear inequalities, whose proper faces  $F_{\{3\}}(B) = \{+\infty\}^2 \times \mathbf{N}$  and  $F_\emptyset(B)$  are linearly ordered by specialization. However it is not a polytope, because the linear map  $\varphi(x, y) = 2y - x$  is not largely continuous on  $A$ : it has no limit when  $(x, y)$  tends to  $(+\infty, +\infty)$  in  $A$ .

All the references in the next proposition are taken from [Dar16].

**Proposition 2.13.** *Let  $q \geq 1$  and  $A \subseteq F_I(\Gamma^q)$  be a discrete polytope. Let  $(\mu, \nu)$  be a largely continuous presentation of  $A$ , let  $J$  be a subset of  $I$ , and  $\widehat{J} = J \setminus \{m\}$ . Finally let  $Y = F_{\widehat{J}}(\widehat{A})$ . Then  $F_J(A) \neq \emptyset$  if and only if either  $m \in J$  and  $\bar{\mu} < +\infty$  on  $Y$ , or  $m \notin J$  and  $\bar{\nu} = +\infty$  on  $Y$  (Proposition 3.11). When this happens:*

1.  $F_J(A) = \pi_J(A)$  (Proposition 3.3).
2. The socle of  $F_J(A)$  is a face of  $\widehat{A}$ :  $\widehat{F_J(A)} = F_{\widehat{J}}(\widehat{A}) = Y$  (Proposition 3.7).
3.  $F_J(A)$  is a discrete polytope and  $(\bar{\mu}|_Y, \bar{\nu}|_Y)$  is a presentation of it (Proposition 3.11):

$$F_J(A) = \{b \in F_J(\Gamma^m) : \widehat{b} \in Y, \text{ and } \bar{\mu}(\widehat{b}) \leq b_m \leq \bar{\nu}(\widehat{b})\}.$$

We will also use the next result (Proposition 3.5 in [Dar16]).

**Proposition 2.14.** *Let  $A \subseteq \Gamma^q$  be a discrete polytope,  $f : A \rightarrow \Omega$  be an affine map and  $B = F_J(A) = \pi_J(A)$  a face of  $A$ . Assume that  $f$  extends to a continuous map  $f^* : A \cup B \rightarrow \Omega$ . Then  $f^*$  is affine and if  $f^* \neq +\infty$  then  $f = f^*|_B \circ \pi_{J|A}$ . In particular if  $f^* \neq +\infty$  then  $f(A) = f^*(B)$ .*

A **discrete simplex** is a discrete polytope whose faces are linearly ordered by specialization. This is a “monohedral largely continuous precell mod  $(1, \dots, 1)$ ” in [Dar16]. Of course every face of a simplex is a simplex (see Remark 3.12 of [Dar16]).

For every  $M \geq 1$  we let  $D^M R^q = (R \cap Q_{1,M})^q$  and define  **$p$ -adic simplexes of index  $M$**  as the inverse images of discrete simplexes by the restriction of the valuation to  $D^M R^q$ . The faces of a simplex  $S$  of index  $M$  are obviously the pre-images in  $D^M R^q$  of the faces of  $vS$ . In particular they are linearly ordered by specialization.  $S$  is closed if and only if  $vS$  is a singleton in  $\Gamma^q$ . If  $S$  is not closed, its largest proper face  $T$  is called its **facet** and  $\partial S = \overline{T}$ .

**Remark 2.15.** With the notation of Proposition 2.14, if  $S = v^{-1}(A) \cap D^M R^q$  and  $T = v^{-1}(B) \cap D^M R^q$  then  $T = F_J(S)$ , and so by Proposition 2.14  $T = \pi_J(S)$ . We will sometimes refer to the restriction of  $\pi_J$  to  $A$  (resp.  $S$ ) as to “the coordinate projection of  $A$  onto  $B$  (resp. of  $S$  onto  $T$ )”.

## 2.f Simplicial complexes

We will have to consider complexes of sets, of cells and of simplexes. All of them are finite families of subsets of a topological space  $X$ , organised in a such a way that one controls how the closures of these sets intersect.

Recall first that an ordered set  $\mathcal{A}$  is a **tree** if for every  $A$  in  $\mathcal{A}$ , the set of elements in  $\mathcal{A}$  smaller than  $A$  is linearly ordered. It is a **rooted tree** if it has one smallest element. A **lower subset** of  $\mathcal{A}$  is a subset  $\mathcal{B}$  of  $\mathcal{A}$  such that whenever an element of  $\mathcal{A}$  is smaller than an element of  $\mathcal{B}$ , it belongs to  $\mathcal{B}$ .

Now, given a finite family  $\mathcal{A}$  of pairwise disjoint subsets of  $X$ , we call  $\mathcal{A}$  a **closed complex** if every  $A \in \mathcal{A}$  is relatively open and if its frontier  $\partial A$  is a union of elements of  $\mathcal{A}$ . The specialization preorder is then an order on  $\mathcal{A}$ . If  $\mathcal{A}$ , ordered by specialization, is a tree (resp. a rooted tree) we call it a (resp. **rooted**) **closed monoplex**. A **complex** (resp. **monoplex**) is then an arbitrary subfamily of a closed complex (resp. closed monoplex). Of course a complex  $\mathcal{A}$  is a closed complex if and only if  $\bigcup \mathcal{A}$  is closed.

**Remark 2.16.** Using that every semi-algebraic set is the disjoint union of finitely many pure dimensional ones, and that  $\dim \partial A < A$  for every semi-algebraic set  $A$ , a straightforward induction shows that every finite family of semi-algebraic subsets of  $K^m$  refines in a complex of pure dimensional semi-algebraic sets.

A **simplicial complex** in  $D^M R^q$  (resp. in  $\Gamma^q$ ) is a complex of simplexes in  $D^M R^q$  (resp. in  $\Gamma^q$ ). We say that  $\mathcal{S}$  is **well dispatched** if for every  $S, S' \in \overline{\mathcal{S}}$ ,  $S' \leq S$  if and only if  $\text{Supp } S' \subseteq \text{Supp } S$ .

Let  $\mathcal{S}$  be a finite family of simplexes in  $D^M R^q$  (or  $\Gamma^q$ ). Then  $\mathcal{S}$  is a simplicial complex if and only if for every  $S, T \in \mathcal{S}$ ,  $\overline{S} \cap \overline{T}$  is the union of the common faces of  $S$  and  $T$ . When this happens:

1.  $\mathcal{S}$  is a monoplex;
2. every subset  $\mathcal{S}_0$  of  $\mathcal{S}$  in  $D^M R^q$  is again a simplicial complex;
3.  $\biguplus \mathcal{S}_0$  is closed in  $\biguplus \mathcal{S}$  if and only if  $\mathcal{S}_0$  is a lower subset of  $\mathcal{S}$ .

Let  $\overline{\mathcal{S}}$  denote the family of all the faces of the elements of  $\mathcal{S}$ . We call it the **closure** of  $\mathcal{S}$ , and say that  $\mathcal{S}$  is **closed** if  $\mathcal{S} = \overline{\mathcal{S}}$ . Note that  $\mathcal{S}$  is a complex (resp. a closed complex) if and only if  $\mathcal{S} \subseteq \overline{\mathcal{S}}$  (resp.  $\mathcal{S} = \overline{\mathcal{S}}$ ) and the elements of  $\overline{\mathcal{S}}$  are pairwise disjoint.

If  $\mathcal{S}$  is a simplicial complex, we say  $\mathcal{T}$  is a **simplicial subcomplex** of  $\mathcal{S}$  if  $\mathcal{T}$  is a simplicial complex such that  $\overline{\mathcal{T}}$  refines a decreasing subset of  $\overline{\mathcal{S}}$ , and  $\bigcup \mathcal{T}$  is a closed subset of  $\bigcup \mathcal{S}$ .

The following results are respectively Theorem 6.3 and Proposition 6.4 of [Dar16].

**Theorem 2.17** (Monotopic Division). *Let  $S$  be a simplex in  $D^M R^m$  and  $\mathcal{T}$  a simplicial complex in  $D^M R^m$  which is a partition of  $\partial S$ . Let  $\varepsilon : \partial S \rightarrow K^\times$  be a definable function such that the restriction of  $|\varepsilon|$  to every proper face of  $S$  is continuous. Then there exists a finite partition  $\mathcal{U}$  of  $S$  such that  $\mathcal{U} \cup \mathcal{T}$  is a simplicial complex in  $D^M R^m$ ,  $\mathcal{U}$  contains for every  $T \in \mathcal{T}$  a unique cell  $U$  with facet  $T$ , and moreover  $\|u - \pi_J(u)\| \leq |\varepsilon(\pi_J(u))|$  for every  $u \in U$ , where  $J = \text{Supp}(T)$ .*

**Proposition 2.18.** *Let  $A \subseteq D^M R^q$  be a relatively open set. Assume that  $A$  is the union of a simplicial complex  $\mathcal{A}$  in  $D^M R^q$ . Then for every integer  $n \geq 1$  there exists a finite partition of  $A$  in semi-algebraic sets  $A_1, \dots, A_n$  such that  $\partial A_k = \partial A$  for every  $k$ .*

Finally, a **simplicial complex of index  $M$**  is a collection  $\mathcal{S}$  of finitely many (possibly zero) rooted simplicial complexes  $\mathcal{S}_i$  in  $D^M R^{q_i}$ , for various integers  $q_i$  (indexed by some finite set  $\mathcal{I}$ ). Its **closure** is the collection of the closures of the  $\mathcal{S}_i$ 's. It is **well dispatched** if each  $\mathcal{S}_i$  is. We say that  $\mathcal{T} = (\mathcal{T}_i)_{i \in \mathcal{I}}$  is a **simplicial subcomplex** of  $\mathcal{S}$  if each  $\mathcal{T}_i$  is a simplicial subcomplex of  $\mathcal{S}_i$ .

Given a semi-algebraic homeomorphism  $\varphi$  from  $\bigsqcup \mathcal{S}$  to a subset  $X$  of  $K^m$ , we will let

$$\varphi(\mathcal{S}) = \{\varphi(S) : S \in \mathcal{S}\}.$$

If  $\mathcal{S}$  is closed,  $\varphi(\mathcal{S})$  is obviously a closed monoplex of pure dimensional semi-algebraic sets partitioning  $X$ .

**Remark 2.19.** With  $\varphi$  as above,  $\mathcal{S}$  is closed if and only if  $X$  is closed and bounded. Indeed, each  $\bigcup \mathcal{S}_i$  is clopen in  $\bigsqcup \mathcal{S}$ , hence its homeomorphic image  $X_i$  by  $\varphi$  is clopen in  $X$ . In particular  $X$  is closed and bounded in  $K^m$  if and only if each  $X_i$  is so. Let  $\varphi_i$  be the semi-algebraic homeomorphism from  $\bigcup \mathcal{S}_i$  to  $X_i$  induced by restriction of  $\varphi$ . Note that  $\bigcup \mathcal{S}_i$  is bounded (it is contained in  $R^{q_i}$ ). By Theorem 2.6 applied to  $\varphi_i$  and  $\varphi_i^{-1}$  it follows that  $\bigcup \mathcal{S}_i$  is closed in  $R^{q_i}$ , that is  $\mathcal{S}_i$  is closed, if and only if  $X_i$  is closed and bounded in  $K^m$ .

### 3 Applications

The proof of our main result  $\mathbf{T}_m$  goes by induction on  $m$ . Some of the applications given in this section are actually needed in the induction step, hence it is important to emphasize that all of them follow from  $\mathbf{T}_m$  for a fixed (throughout all this section) integer  $m$ .

**Theorem 3.1.** *If  $f : X \subseteq K^m \rightarrow K$  is semi-algebraic and  $|f|$  is continuous, then there exists a function  $h : X \rightarrow K$  semi-algebraic and continuous such that  $|f| = |h|$  on  $X$ .*

*Proof:*  $\mathbf{T}_m$  gives a triangulation  $(\mathcal{T}, \varphi)$  of  $f$  with parameters  $(1, 1, e, M)$ . On every  $T \in \mathcal{T}$ ,  $f \circ \varphi|_T = \mathcal{U}_{e,1}\psi$  with  $\psi : T \rightarrow K$  a 1-monomial function. Thus for some  $q_T$  such that  $T$  is contained in  $D^M R^{q_T}$ , there are  $\lambda_T$  in  $K$  and  $\alpha_{1,T}, \dots, \alpha_{q_T,T}$  in  $\mathbf{Z}$  such that:

$$\forall x \in T, |f \circ \varphi(x)| = \left| \lambda_T \prod_{i=1}^{q_T} x_i^{\alpha_{i,T}} \right| \quad (3)$$

Let  $\alpha_{0,T} = v\lambda_T$  and  $\xi_T : vT \rightarrow \Gamma$  be defined by:

$$\forall a \in vT, \xi_T(a) = \alpha_{0,T} + \sum_{i=1}^{q_T} \alpha_{i,T} a_i$$

By construction  $\xi_T(vx) = vf(\varphi(x))$  for every  $x \in T$  (in particular  $\xi_T$  only depends on  $f \circ \varphi$ , even if the coefficients  $\alpha_{i,T}$  in (3) are not uniquely determined by  $f \circ \varphi$  on  $T$ ). By assumption  $vf$  is continuous on  $X$  hence so is  $vf \circ \varphi$  on  $\bigsqcup \mathcal{T}$ . In particular  $\xi_T$  extends continuously to  $vS$  for every face  $S$  of  $T$  in  $\mathcal{T}$ , and the restriction to  $S$  of such an extension  $\bar{\xi}_T$  is precisely  $\xi_S$ . By proposition 2.14 it follows that if  $\xi_S \neq +\infty$  (that is if  $f \circ \varphi \neq 0$  on  $S$ ) then  $\xi_T = \xi_S \circ \pi_S$  where  $\pi_{vS}$  denotes the coordinate projection of  $vT$  to  $vS$  (see Remark 2.15).

Now, for every  $T$  in  $\mathcal{T}$  let  $g_T : T \rightarrow K$  be defined (by induction on  $\mathcal{T}$  ordered by specialisation) as follows:

1. If  $f \circ \varphi = 0$  on  $T$ ,  $g_T = 0$ .
2. If  $T$  is minimal (with respect to the specialisation preorder) among the simplexes in  $\mathcal{T}$  on which  $f \circ \varphi \neq 0$  then for every  $x \in T$ :

$$g_T(x) = \pi^{\alpha_{0,T}} \prod_{i=1}^{q_T} x_i^{\alpha_{i,T}}$$

3. Otherwise  $g_T = g_S \circ \pi_S$  where  $\pi_S$  is the coordinate projection (see Remark 2.15) of  $T$  onto its larger proper face  $S$  in  $\mathcal{T}$  on which  $f \circ \varphi \neq 0$ .

By construction  $vg_T(x) = \xi_T(vx)$  for every  $x \in T$  hence  $|g_T| = |f|$  on  $T$ . Moreover for every  $S \leq T$  and  $y \in S$ ,  $g_T(x)$  tends to  $g_S(y)$  as  $x$  tends to  $y$  in  $T$  (because  $g_T(x) = g_S(\pi_S(x))$  if  $g_S \neq 0$ , and otherwise because  $|g_T| = |f \circ \varphi|$  on  $T$ ,  $|g_S| = |f \circ \varphi| = 0$  on  $S$  and  $|f \circ \varphi| = |f| \circ \varphi$  is continuous by assumption).

The function  $h : X \rightarrow K$  defined by  $h = g_T \circ \varphi^{-1}$  on every  $\varphi(T)$  with  $T$  in  $\mathcal{T}$ , is clearly semi-algebraic. By construction  $|f| = |h|$  on  $X$ , and by the above argument  $h$  is continuous on  $X$ .

■

**Theorem 3.2.** *For every non-empty semi-algebraic sets  $Y \subseteq X \subseteq K^m$ , there is a semi-algebraic retraction of  $X$  onto  $Y$  if and only if  $Y$  is closed in  $X$ .*

*Proof:* One direction is general. For the converse we assume that  $Y$  is closed in  $X$ . Let  $(\mathcal{S}, \varphi)$  be a triangulation of  $X$ ,  $Y$  given by  $\mathbf{T}_m$ , and  $\mathcal{T}$  be the family of simplexes  $T$  in  $\mathcal{S}$  such that  $\varphi(T) \subseteq Y$ . It suffices to construct a continuous retraction of  $\bigsqcup \mathcal{S}$  onto  $\bigsqcup \mathcal{T}$ .

Let  $\mathcal{S}_0 = \mathcal{T}$  and  $\sigma_0$  be the identity map on  $\bigsqcup \mathcal{T}$ . Because  $Y$  is closed in  $X$ ,  $\mathcal{T}$  is a lower subset of  $\mathcal{S}$ . Let  $k$  be a positive integer and assume that there is a lower subset  $\mathcal{S}_{k-1}$  of  $\mathcal{S}$  containing  $\mathcal{T}$ , and a retraction  $\sigma_{k-1}$  of  $\bigsqcup \mathcal{S}_{k-1}$  to  $\mathcal{T}$ . If  $\mathcal{S}_{k-1} = \mathcal{S}$  we are done. Otherwise let  $S$  be a minimal element (with respect to the specialisation order) in  $\mathcal{S} \setminus \mathcal{S}_{k-1}$ , and let<sup>6</sup>  $\mathcal{S}_k = \mathcal{S}_{k-1} \cup \{S\}$ . It only remains

<sup>6</sup>We are abusing the notation here:  $\mathcal{S}$  is a finite collection of simplicial simplexes  $\mathcal{S}_i$  in  $D^M R^{q_i}$  for various  $q_i$ ,  $\mathcal{S}_k$  is a collection of lower subsets  $\mathcal{S}_{k,i}$  of  $\mathcal{S}_i$ , there is an index  $i_0$  such that  $S$  belongs to  $\mathcal{S}_{i_0}$ , and what we have denoted abusively  $\mathcal{S}_{k-1} \cup \{S\}$  is actually the collection of all the  $\mathcal{S}_{k,i}$ 's for  $i \neq i_0$  and of  $\mathcal{S}_{k,i_0} \cup \{S\}$ .



to build a retraction  $\tau$  of  $\biguplus \mathcal{S}_k$  onto  $\biguplus \mathcal{S}_{k-1}$ . Indeed  $\sigma_{k-1} \circ \tau$  will then be a continuous retraction of  $\mathcal{S}_k$  onto  $\mathcal{T}$ , and the result will follow by induction.

If  $S$  has no proper face in  $\mathcal{S}$  then it is clopen in  $\biguplus \mathcal{S}_k$ . So the map  $\tau$  which is the identity map on  $\biguplus \mathcal{S}_k$  and which sends every point of  $S$  to an arbitrary given point of  $\biguplus \mathcal{S}_{k-1}$  is continuous on  $\mathcal{S}_k$ , and a retraction of  $\biguplus \mathcal{S}_k$  onto  $\mathcal{S}_{k-1}$ .

Otherwise let  $T$  be the largest proper face of  $S$  in  $\mathcal{S}$ . By minimality of  $S$ ,  $T$  belongs to  $\mathcal{S}_{k-1}$ . Let  $\pi_T$  be the coordinate projection of  $S$  onto  $T$ . The frontier of  $S$  inside  $\biguplus \mathcal{S}_k$  is the closure of  $T$  in  $\biguplus \mathcal{S}_k$ , hence the function  $\tau$  which coincides with the identity map on  $\biguplus \mathcal{S}_{k-1}$  and with  $\pi_T$  on  $S$  is continuous. It is then a retraction  $\biguplus \mathcal{S}_k$  onto  $\mathcal{S}_{k-1}$ , which finishes the proof.  $\blacksquare$

The Splitting Theorem 3.4 is a strengthening of the next lemma using retractions.

**Lemma 3.3.** *Let  $X \subseteq K^m$  be a relatively open semi-algebraic set without isolated points and  $n \geq 1$  an integer. Then there exists a partition of  $X$  in semi-algebraic sets  $X_k$  for  $1 \leq k \leq n$  such that  $\partial X_k = \partial X$  for every  $k$ .*

We are going to prove Lemma 3.3 by using a triangulation  $(\mathcal{U}, \varphi)$  of  $(X, \partial X)$  and applying Proposition 2.18 to  $\varphi^{-1}(X)$ . In order to ensure that this set is still relatively open, we reduce first to the case where  $X$  is bounded by means of the following construction.

Let  $\hat{K} = K \cup \{\infty\}$  and for every  $I \subseteq \{1, \dots, m\}$  let  $K_I^m = \hat{K}_I^m \cap K^m$  where

$$\hat{K}_I^m = \{x \in \hat{K}^m : x_k \in R \iff k \in I\}.$$

Let  $R_I^m = \{x \in R^m : \forall k \notin I, x_k \neq 0\}$ , and for every  $x \in R_I^m$  let  $\psi_I(x) = (y_k)_{1 \leq k \leq m}$  be defined by  $y_k = x_k$  if  $k \in I$ , and  $y_k = 1/(\pi x_k)$  otherwise. Clearly  $\psi_I$  is semi-algebraic homeomorphism from  $R_I^m$  to  $K_I^m$  which extends uniquely to a homeomorphism  $\hat{\psi}_I$  from  $R^m$  to  $\hat{K}_I^m$ .

*Proof:* Note first that if it is given a partition of  $X$  in finitely many semi-algebraic pieces  $U_1, \dots, U_r$  which are clopen in  $X$ , then it suffices to prove the result separately for each  $U_j$ . Indeed, each  $U_j$  will then be relatively open with  $\partial U_j \subseteq \partial X$  (because  $U_j$  is clopen in  $X$ ), and  $\bigcup_{j \leq r} \partial U_j = \partial X$  (because  $\partial U_j = \overline{U_j} \setminus X$  and  $\bigcup_{j \leq r} \overline{U_j} = \overline{X}$ ). So, if a partition of each  $U_j$  in semi-algebraic pieces  $(U_{j,k})_{1 \leq k \leq n}$  is found such that  $\partial U_{j,k} = \partial U_j$  for every  $k$ , then the union  $X_k$  of  $U_{j,k}$  for  $1 \leq j \leq r$  defines a partition of  $X$  in semi-algebraic pieces and we have  $\partial X_k = \bigcup_{j \leq r} \partial U_{j,k}$  (same argument as above) hence  $\partial X_k = \bigcup_{j \leq r} \partial U_j = \partial X$ .

Now, as  $I$  ranges over the subsets of  $\{1, \dots, m\}$ , the sets  $\overline{X} \cap K_I^m$  form a partition of  $X$  in semi-algebraic sets clopen in  $X$ . By the argument above we can deal separately with each of these sets, which reduces to the case where  $X \subseteq K_I^m$  for some  $I$ .

Let  $Y = \psi_I^{-1}(X)$  and  $\hat{X}$  be the closure of  $X$  in  $\hat{K}_I^m$ . Note that  $\hat{\psi}_I(\overline{Y}) = \hat{X}$ . The fact that  $\overline{X} \setminus X$  is closed in  $K_I^m$ , hence in  $K^m$ , implies that  $\hat{X} \setminus X$  is closed in  $\hat{K}_I^m$ . It follows that its image by  $\hat{\psi}_I^{-1}$ , which is precisely  $\overline{Y} \setminus Y$ , is closed in  $R^m$ , hence in  $K^m$ . Thus  $Y$  is relatively open. It then suffices to prove the result for  $Y$ , that is we can assume that  $X = Y$  is bounded. Of course we can assume as well that  $\partial X$  is non-empty (otherwise  $X_1 = X$  and  $X_k = \emptyset$  for  $2 \leq k \leq n$  is obviously a solution).

$\mathbf{T}_m$  gives a triangulation  $(\mathcal{U}, \varphi)$  of  $(X, \partial X)$ .  $\mathcal{U}$  is the disjoint union of finitely many simplicial complexes  $\mathcal{U}_j$  in  $D^M R^{q_j}$  for  $1 \leq j \leq r$ . Let  $U_j = \varphi(\bigcup \mathcal{U}_j) \cap X$  for every  $j$ , this defines a partition of  $X$  in semi-algebraic sets clopen in  $X$ . By using again the initial remark in this proof, it suffices to check the result for each  $U_j$  separately. So we can assume that  $\mathcal{U}$  itself is a simplicial complex in  $D^M R^q$  for some  $q$ .

By construction  $\overline{X}$  is semi-algebraic, closed and bounded, and  $\varphi^{-1}$  is semi-algebraic and continuous, so  $\overline{\varphi^{-1}(Y)} = \overline{\varphi^{-1}(Y)}$  for every semi-algebraic<sup>7</sup>  $Y \subseteq \overline{X}$ . Let  $A = \varphi^{-1}(X)$ , we have  $\overline{A} = \varphi(\overline{X})$  hence  $\overline{A} \setminus A = \varphi(\overline{X} \setminus X)$  is closed, that is  $A$  is relatively open. Proposition 2.18 then applies to  $A$  and gives a partition of  $A$  in semi-algebraic sets  $A_1, \dots, A_n$  such that  $\partial A_k = \partial A$  for every  $k$ .

For  $1 \leq k \leq n$  let  $X_k = \varphi(A_k)$ . This semi-algebraic sets form a partition of  $X$ , because  $A_1, \dots, A_n$  form a partition of  $A$ . Moreover, since  $\bigcup \mathcal{U}$  is semi-algebraic, closed and bounded, we have  $\varphi(\overline{B}) = \overline{\varphi(B)}$  for every semi-algebraic<sup>8</sup> set  $B$  contained in  $\bigcup \mathcal{U}$ . It follows that for  $1 \leq k \leq n$  we have  $\partial X_k = \varphi(\partial A_k) = \varphi(\partial A) = \partial X$ , which proves the result.  $\blacksquare$

**Theorem 3.4.** *Let  $X$  be a relatively open non-empty semi-algebraic subset of  $K^m$  without isolated points, and  $Y_1, \dots, Y_s$  a collection of closed semi-algebraic subsets of  $\partial X$  such that  $Y_1 \cup \dots \cup Y_s = \partial X$ . Then there is a partition of  $X$  in non-empty semi-algebraic sets  $X_1, \dots, X_s$  such that  $\partial X_i = Y_i$  for  $1 \leq i \leq s$ .*

*Proof:*  $X$  is non-empty and has no isolated point, hence is infinite. The result is obvious for  $s = 0$  (there is nothing to prove) and  $s = 1$  (take  $X_1 = X$ ). By induction it suffices to prove it for  $s = 2$ . Indeed, if  $s \geq 3$  and the result is proved for  $s - 1$ , then the result for  $s = 2$  applied to  $X$  with  $Z_1 = Y_1 \cup \dots \cup Y_{s-1}$  and  $Z_2 = Y_s$  gives a partition in two pieces  $X'_1, X'_2$  such that  $\partial X'_l = Z_l$  for  $l = 1, 2$ , and the induction hypothesis applied to  $X'_1$  with  $Y_1, \dots, Y_s$  gives a partition of  $X'_1$  in pieces  $X_1, \dots, X_{s-1}$  such that  $\partial X_i = Y_i$  for  $1 \leq i \leq s$ . The conclusion follows, by taking  $X_s = X'_2$ . So from now on we assume that  $s = 2$ .

It suffices to prove the weaker result that a partition  $(X'_1, X'_2)$  exists with all the required properties for  $(X_1, X_2)$  except possibly the condition that they are non empty. Indeed, if such a partition is found and for example  $X'_2 = \emptyset$  then necessarily  $Y_2 = \partial X'_2 = \emptyset$ . In that case pick any  $x \in X$ , and choose a clopen neighbourhood  $V$  of  $x$  such that  $V \cap \partial X$  is empty (this is possible because  $X$  is relatively open). Then  $X_1 = X \setminus V$  and  $X_2 = X \cap V$  give the conclusion.

Let  $\rho : \overline{X} \rightarrow \partial X$  be a continuous retraction of  $\overline{X}$  onto  $\partial X$  given by Theorem 3.2. Let  $V \subseteq \partial X$  be any semi-algebraic set open in  $\partial X$ ,  $Z$  its closure and  $A = \rho^{-1}(Z) \cap X$ . We are claiming that  $\partial A = Z$ . Note that  $A$  is closed in  $X$  by continuity of  $\rho$ , because  $A$  is the inverse image of the closed set  $Z$  by  $\rho|_X$ . So it suffices to prove that  $\overline{A} \cap \partial X = Z$ , or equivalently that  $\overline{A} \cap \partial X$  contains  $V$  and is contained in  $Z$ . For the first inclusion let  $y$  be any element of  $V$ , and  $W$  any neighbourhood of  $y$ . We have to prove that  $W \cap A \neq \emptyset$ . By continuity of  $\rho$  at  $y = \rho(y)$  there is a neighbourhood  $U$  of  $y$  such that  $U \cap \overline{X}$  is contained in

<sup>7</sup>For every continuous map  $f : X \subseteq K^q \rightarrow K^r$  and every  $Y \subseteq X$ , if  $X$  is closed then  $f(\overline{Y}) \subseteq \overline{f(Y)}$ . The reverse inclusion holds true if  $X$  is compact, or if  $f, Y, X$  are semi-algebraic and  $X$  is closed and bounded (see Theorem 2.6).

<sup>8</sup>See footnote 7

$\rho^{-1}(W \cap V)$ . In particular

$$U \cap W \cap X \subseteq U \cap \overline{X} \subseteq \rho^{-1}(W \cap V) \subseteq \rho^{-1}(V) = A$$

so  $U \cap W \cap A = U \cap W \cap X$ . On the other hand,  $U \cap W \cap X \neq \emptyset$  because  $U \cap W$  is a neighbourhood of  $y$  and  $y \in V \subseteq \overline{X}$ . *A fortiori*  $W \cap A$  is non-empty. This proves that  $y \in \overline{A}$ , hence that  $V \subseteq \overline{A} \cap \partial X$ . Conversely, if  $y'$  is any element of  $\partial X \setminus Z$ , there is a neighbourhood  $W'$  of  $y'$  such that  $W' \cap \partial X$  is disjoint from  $Z$ . By continuity of  $\rho$ ,  $\rho^{-1}(W')$  is then a neighbourhood of  $y'$  in  $\overline{X}$ . It is disjoint from  $\rho^{-1}(Z) = A$  hence  $y' \notin \overline{A}$ . So  $\overline{A}$  is disjoint from  $\partial X \setminus Z$ . That is  $\overline{A} \cap \partial X \subseteq Z$ , which proves our claim.

For  $k = 1, 2$  let  $Z_k$  be the closure of  $Y_k \setminus Y_{2-k} = \partial X \setminus Y_{2-k}$  and  $A_k = \rho^{-1}(Z_k)$ . Let  $Z_0$  be the closure of  $\partial X \setminus (Z_1 \cup Z_2)$  and  $A_0 = \rho^{-1}(Z_0)$ . The above claim gives that  $\partial A_k = Z_k$  for  $0 \leq k \leq 2$ . Let  $B_0$  be the set of non-isolated points of  $A_0$ . Clearly  $\partial B_0 = \partial A_0 = Z_0$  since  $A_0 \setminus B_0$  is finite. In particular  $B_0$  is relatively open, and Lemma 3.3 gives two semi-algebraic sets  $B_1, B_2$  partitioning  $B_0$  such that  $\partial B_1 = \partial B_2 = Z_0$ . So if we set  $X_1 = A_1 \cup B_1$  and  $X_2 = A_2 \cup B_2 \cup (A_0 \setminus B_0)$  we get the conclusion.

■

**Theorem 3.5.** *Let  $f : X \subseteq K^m \rightarrow K$  be a semi-algebraic function with bounded graph (that is  $f$  is a bounded function on a bounded domain). If it has finitely many limit values at every point of  $\overline{X}$  then  $f$  is piecewise largely continuous.*

Note that the counterpart of Theorem 3.5 for real-closed fields holds true. Indeed, by triangulation we are reduced to the case of a continuous function  $f$  on a simplex  $S \subseteq K^m$ . The assumption that  $f$  has finitely many limit values at every point of  $\overline{S}$  then implies directly that  $f$  is largely continuous. Indeed, this follows easily from the fact that over real-closed fields the direct image by a continuous semi-algebraic map of any semi-algebraically connected set (such as  $S \cap B$  with  $B$  a ball centered at any point of  $\overline{S}$ ) is again semi-algebraically connected.

At the contrary,  $p$ -adic simplexes are not at all semi-algebraically connected and it can happen that a function satisfying all these assumptions on a  $p$ -adic simplex is not largely continuous. For example on the simplex  $S = D^1 R^*$  the semi-algebraic function  $f$  defined by  $f(x) = 0$  if  $v(x) \in 2\mathcal{Z}$  and  $f(x) = 1$  otherwise is a continuous, bounded function having two distinct limit values at 0. Thus  $f$  is not largely continuous. It is obviously piecewise largely continuous, though.

*Proof:* Every semi-algebraic function is piecewise continuous (see for example [Mou09]). So, replacing  $f$  by its restriction to the pieces of an appropriate partition of  $X$  is necessary, we can assume that  $f$  is continuous. Removing  $X \cap \partial X$  if necessary (using a straightforward induction on  $\dim X$  and the fact that  $\dim \partial X < \dim X$ ) we can even assume that  $X$  is relatively open. The proof then goes by induction on the lexicographically ordered tuples  $(e, e')$  where  $e = \dim X$  and  $e' = \dim \partial X$ . If  $\partial X$  is empty, that is  $X$  is closed, then  $f$  is largely continuous and the result is obvious. So let us assume that  $e' \geq 0$  (hence  $e \geq 1$ ) and the result is proved for smaller tuples  $(e, e')$ .

Let  $D = (\partial X \times K) \cap \overline{\text{Gr} f}$ . The projection of  $D$  onto  $\partial X$  has finite fibers hence  $D$  is a union of cells of type 0. The number of these cells, say  $N$ , then

bounds the cardinality of these fibers, that is the number of limit values of  $f$  at every point of  $\partial X$ . For every  $a \in \partial X$  let  $D_a = \{t \in K : (a, t) \in D\}$ . We first show that  $\widehat{D} = \partial X$ , that is  $D_a \neq \emptyset$  for every  $a \in \partial X$ . For every  $\varepsilon \in R^*$  let  $C_\varepsilon = (B(a, \varepsilon) \times K) \cap \overline{\text{Gr } f}$ . This is a uniformly semi-algebraic family of closed and bounded semi-algebraic subsets of  $K^n$ . Each of them is non-empty because  $C_\varepsilon$  contains  $(x, f(x))$  for any  $x \in B(a, \varepsilon) \cap X$  (which is non-empty since  $a \in \partial X$ ). Obviously  $C_{\varepsilon_1} \subseteq C_{\varepsilon_2}$  whenever  $|\varepsilon_1| \leq |\varepsilon_2|$ , so  $\bigcap_{\varepsilon \in R^*} C_\varepsilon$  is non-empty by Theorem 2.5. The latter is equal to  $D_a$ , which proves our claim.

For  $1 \leq i \leq N$  let  $W_i$  be the set of  $a \in \partial X$  such that  $D_a$  has exactly  $i$  elements. These sets  $W_i$  form a partition of  $D$  in semi-algebraic pieces. By Theorem 2.8 (and a straightforward induction) there are semi-algebraic functions  $f_{i,j} : W_i \rightarrow K$  such that  $D_a = \{f_{i,j}(a)\}_{1 \leq j \leq i}$  for every  $a \in D_a$ . Since  $\dim \partial X < \dim X$ , by induction hypothesis these functions  $f_{i,j}$  are piecewise largely continuous. This gives a partition of  $\partial X$  in semi-algebraic pieces  $V_k$  for  $1 \leq k \leq r$ , and a family of largely continuous semi-algebraic functions  $g_{k,l} : V_k \rightarrow K$  for  $1 \leq l \leq s_k$  such that  $V_k \subseteq W_{s_k}$  and  $D$  is the union of the graphs of all these functions  $g_{k,l}$ .

Theorem 3.4 applied to  $X$  and the sets  $\overline{V_k}$  for  $1 \leq k \leq r$  gives a partition of  $X$  in semi-algebraic pieces  $X_k$  such that  $\partial X_k = \overline{V_k}$ . It suffices to prove that the restrictions of  $f$  to each  $X_k$  is piecewise largely continuous. So we can assume that  $r = 1$  and  $X = X_1$ . That is, we have a semi-algebraic set  $V = V_1$  dense in  $\partial X$  and largely continuous functions  $g_l = g_{1,l} : V \rightarrow K$  for  $1 \leq l \leq s = s_1$  such that  $D_a = \{g_l(a)\}_{1 \leq l \leq s}$  has  $s$  elements for every  $a \in D_a$ . Replacing  $V$  by  $V \setminus \partial V$  if necessary we can assume that  $V$  is relatively open.

Let  $\rho : \overline{X} \rightarrow \overline{V}$  be a continuous retraction given by Theorem 3.2. For  $1 \leq l \leq s$  let

$$U_l = \{x \in X : \forall k \neq l, |f(x) - \bar{g}_l(\rho(x))| < |f(x) - \bar{g}_k(\rho(x))|\}.$$

Each  $U_l$  is open in  $X$  by continuity of  $f$ ,  $\rho$  and the  $\bar{g}_k$ 's. Their complement  $X' = X \setminus \bigcup_{k=1}^l U_k$  is closed in  $X$ , hence  $\partial X' \subseteq \partial X$ . Moreover, for every  $a \in V$ , the limit values  $g_k(a)$  of  $f$  at  $a$  being distinct by construction, there exists  $\varepsilon \in R^*$  such that every point of  $B(a, \varepsilon) \cap X$  belongs to one of the  $U_k$ 's. With other words  $B(a, \varepsilon) \cap X' = \emptyset$  hence  $a$  does not belong to the closure of  $X'$ . So  $\partial X \subseteq \partial X \setminus V = \partial V$ , in particular  $\dim \partial X' < \dim V = \dim \partial X$  hence the induction hypothesis applies to the restriction of  $f$  to  $X'$ .

It only remains to check that the restrictions of  $f$  to each  $U_l$  is piecewise largely continuous. We are claiming that  $f$  has only one limit value at every point  $a$  of  $\overline{U_l} \setminus \partial V$ . Note that  $\overline{U_l}$  is the disjoint union of  $\overline{U_l} \cap X$  and  $\overline{U_l} \cap \partial X$ , and that  $\partial X = V \cup \partial V$ . Obviously, if  $a \in \overline{U_l} \cap X$  then by continuity of  $f$ ,  $f(x)$  tends to  $f(a)$  as  $x$  tends to  $a$  in  $U_l$ . Now if  $a \in (\overline{U_l} \setminus \partial V) \setminus X$  then  $a \in V$ ,  $\rho(a) = a$  and  $\bar{g}_k(\rho(a)) = g_k(a)$  for every  $k$ . Hence by definition of  $U_l$ ,  $f(x)$  is closer to  $g_l(a)$  than to every other  $g_k(a)$ , so  $g_l(a)$  is the only possible limit value of  $f(x)$  as  $x$  tends to  $a$  in  $U_l$ , which proves our claim. So the semi-algebraic function  $g$  which coincides with  $f$  on  $\overline{U_l} \cap X$  and with  $g_l$  on  $V$  is continuous. The frontier of its domain is contained in  $\overline{U_l} \cap \partial X \subseteq \partial X = V \cup \partial V$  and is disjoint from  $V$ , hence is contained in  $\partial V$ . By induction hypothesis,  $g$  is then piecewise largely continuous, hence so is  $f|_{U_l}$  since  $f$  and  $g$  coincide on  $U_l$ .

■

## 4 Largely continuous cell decomposition

This section recalls the main theorem of [Den84] in order to emphasize some details which appear only in its proof. These details are important for us because they ensure that the center and bounds of the cells involved in the conclusions inherit certain properties, defined below, from the center and bounds of the cells in the assumptions. Using them we finally derive from  $\mathbf{T}_m$  a new preparation theorem for semi-algebraic functions “up to a small deformation” (Theorem 4.6). The point is that after such a deformation, we get a preparation theorem involving only largely continuous cells.

Given a basic function  $f$ , we say that a function  $h : X \subseteq K^m \rightarrow K$  belongs to  $\text{coalg}(f)$  if there exists a finite partition of  $X$  into definable pieces  $H$ , on each of which the degree in  $t$  of  $f(x, t)$  is constant, say  $e_H$ , and such that the following holds. If  $e_H \leq 0$  then  $h(x)$  is identically equal to 0 on  $H$ . Otherwise there is a family  $(\xi_1, \dots, \xi_{r_H})$  of  $K$ -linearly independent elements in an algebraic closure of  $K$  and a family of definable functions  $b_{i,j} : H \rightarrow K$  for  $1 \leq i \leq e_H$  and  $1 \leq j \leq r_H$ , and  $a_{e_H} : H \rightarrow K^*$  such that for every  $x$  in  $H$

$$f(x, T) = a_{e_H}(x) \prod_{1 \leq i \leq e_H} \left( T - \sum_{1 \leq j \leq r_H} b_{i,j}(x) \xi_j \right)$$

and

$$h(x) = \sum_{1 \leq i \leq e_H} \sum_{1 \leq j \leq r_H} \alpha_{i,j} b_{i,j}(x)$$

with the  $\alpha_{i,j}$ 's in  $K$ . If  $\mathcal{F}$  is any family of basic functions we let  $\text{coalg}(\mathcal{F})$  denote the set of linear combinations of functions in  $\text{coalg}(f)$  for  $f$  in  $\mathcal{F}$ .

**Theorem 4.1** (Denef). *Let  $\mathcal{F} \subseteq K[X, T]$  be a finite family of polynomials, with  $X$  an  $m$ -tuple of indeterminates and  $T$  one more indeterminate. Let  $N \geq 1$  be an integer and  $\mathcal{A}$  a family of boolean combinations of subsets of the form  $f^{-1}(\mathbf{P}_N)$  with  $f \in \mathcal{F}$ . For every integer  $n \geq 1$  there is a finite family of fitting cells mod  $\mathbf{P}_N^\times$  refining  $\mathcal{A}$ , with center and bounds in  $\text{coalg}(\mathcal{F})$ , and for every such cell  $H$  a positive integer  $\alpha_{f,H}$  and a semi-algebraic function  $h_{f,H} : \widehat{H} \rightarrow K$  such that for every  $(x, t) \in H$ :*

$$f(x, t) = \mathcal{U}_n(x, t) h_{f,H}(x) (t - c_H(x))^{\alpha_{f,H}}.$$

*Proof:* W.l.o.g. we can assume that every  $f$  in  $\mathcal{F}$  is non constant and that  $n$  is large enough so that  $1 + \pi^n R \subseteq \mathbf{P}_N^\times$ . Theorem 7.3 in [Den84] gives a finite family of cells  $B$  mod  $K^\times$  partitioning  $K^m$ , and for each of them a positive integer  $\alpha_{f,B}$  and semi-algebraic functions  $u_{f,B} : B \rightarrow R^\times$  and  $h_{f,B} : \widehat{B} \rightarrow K$  such that:

$$\forall (x, t) \in B, f(x, t) = u_{f,B}(x, t)^N h_{f,B}(x) (t - c_B(x))^{\alpha_{f,B}} \quad (4)$$

Moreover it appears along the line of the proofs of lemma 7.2 and theorem 7.3 in [Den84] that these functions  $u_{f,B}^N$  are precisely of the form  $1 + \pi^n \omega_{f,B}$  for some semi-algebraic function  $\omega_{f,B}$  on  $B$ , and that the center and bounds of  $B$  belong to  $\text{coalg}(\mathcal{F})$ . Refining the socle of  $B$  if necessary we can ensure that  $h_{f,B}(x) \mathbf{P}_N^\times$  is constant as  $(x, t)$  ranges over  $B$ . On the other hand  $B$  splits into

finitely many cells mod  $\mathbf{P}_N^\times$ , with the same center and bounds as  $B$ , because  $\mathbf{P}_N^\times$  as finite index in  $K^\times$ . On each of these cells  $H$ ,  $f(x, t)\mathbf{P}_N^\times$  is constant by (4). Hence  $H$  is either contained or disjoint from  $A$ , for every  $A \in \mathcal{A}$ . So the family of all these cells  $H$  which are contained in  $\bigcup \mathcal{A}$  gives the conclusion. ■

Using that every semi-algebraic function is piecewise continuous, the cells mod  $\mathbf{P}_N^\times$  given by Theorem 4.1 can easily be chosen with continuous center and bounds. However it is not possible to ensure that they are largely continuous (think to the case where  $\mathcal{A}$  consists in a single semi-algebraic set which is itself the graph of a semi-algebraic function which is not largely continuous). Our aim, in the remaining of this section, is to find a work-around. We are going to prove that it can be done, not exactly for  $\theta$  but for a function  $\theta \circ u_\eta$  where  $\eta \in K^m$  can be chosen arbitrarily small and  $u_\eta$  is the linear automorphism of  $K^{m+1}$  defined by:

$$\forall (x, t) \in K^m \times K, \quad u_\eta(x, t) = (x + t\eta, t). \quad (5)$$

**Remark 4.2.** The smaller is  $\eta$ , the closer is  $u_\eta$  from the identity map since  $\|\eta\|$  is also the norm (in the usual sense for linear maps) of  $u_\eta - \text{Id}$ . So the functions  $\theta \circ u_\eta$  can be considered as “arbitrarily small deformations” of  $\theta$ .

In [vdD98] a good direction for a subset  $S$  of  $K^{m+1}$  is defined as a non-zero vector  $x = (x_1, \dots, x_{m+1}) \in K^{m+1}$  such that every line directed by  $x$  has finite intersection with  $S$ . It is more convenient to identify collinear such vectors hence we redefine **good directions** for  $S$  as the points  $x = [x_1, \dots, x_{m+1}]$  in the projective space  $\mathbf{P}^m(K)$  such that every affine line in  $K^{m+1}$  directed by  $x$  has finite intersection with  $S$ .

Analogously we call  $x \in \mathbf{P}^m(K)$  a **geometrically good direction** for a family  $\mathcal{F}$  of polynomials in  $K[X, T]$  if for every algebraic extension  $F$  of  $K$  and every  $f \in \mathcal{F}$ ,  $x$  is a good direction for the zero set of  $f$  in  $F^{m+1}$ .

**Remark 4.3.** With the above notation,  $[\eta, 1]$  is a good direction for  $S$  if and only if the projection of  $u_\eta^{-1}(S)$  onto  $K^m$  has finite fibers. Indeed for every  $a \in K^m$  and every  $t \in K$  we have:

$$(a, 0) + t(\eta, 1) \in S \iff (a + t\eta, t) \in S \iff (a, t) \in u_\eta^{-1}(S)$$

Therefore  $[\eta, 1]$  is a geometrically good direction for  $\mathcal{F}$  if and only if for every algebraic extension  $F$  of  $K$  and every  $f \in \mathcal{F}$ , the projection onto  $F^m$  of the zero set of  $f \circ u_\eta$  in  $F^{m+1}$  has finite fibers.

**Lemma 4.4** (Good Direction). *For every finite family  $\mathcal{F}$  of non-zero polynomial in  $K[X, T]$ , the set of geometrically good directions for  $\mathcal{F}$  contains a non empty Zariski open subset of  $\mathbf{P}^m(K)$ . In particular, for every non-zero  $\varepsilon \in R$  there is  $\eta \in R^m$  such that  $\|\eta\| \leq |\varepsilon|$  and  $[\eta, 1]$  is a good direction for  $\mathcal{F}$ .*

*Proof:* Let  $p_{\mathcal{F}}$  be the product of the polynomials in  $\mathcal{F}$ , and  $d$  its total degree. Then  $p_{\mathcal{F}}$  writes  $p_{\mathcal{F}} = p_{\mathcal{F}}^{\circ} - q_{\mathcal{F}}$  with  $p_{\mathcal{F}}^{\circ}$  a non zero homogeneous polynomial of degree  $d$  and  $q_{\mathcal{F}}$  a polynomial of total degree  $< d$ .

Let  $b \in K^{m+1}$  be non-zero and  $x$  be the corresponding point in  $\mathbf{P}^m(K)$ . It is not a geometrically good direction for  $\mathcal{F}$  if and only if for some algebraic extension  $F$  of  $K$  and some  $a \in F^m$  the line  $a + F.b$  is contained in the zero

set of  $p_{\mathcal{F}}$  in  $F^{m+1}$ , that is  $p_{\mathcal{F}}(a + tb) = 0$  for every  $t \in F$  or equivalently  $p_{\mathcal{F}}^{\circ}(a + Tb) = q_{\mathcal{F}}(a + Tb)$ . This implies that the degree in  $T$  of  $p_{\mathcal{F}}^{\circ}(a + Tb)$  is  $< d$ . In particular the coefficient of  $T^d$  in  $p_{\mathcal{F}}^{\circ}(a + Tb)$  is zero. A straightforward computation shows that this coefficient is just  $p_{\mathcal{F}}^{\circ}(b)$ .

So every element in  $\mathbf{P}^m(K)$  which is outside the zero set of  $p_{\mathcal{F}}^{\circ}$  is a geometrically good direction for  $\mathcal{F}$ . This proves the main point. Now if  $K^m$  is identified to its image in  $\mathbf{P}^m(K)$  by the mapping  $a \mapsto [a, 1]$  then every ball in  $K^m$  is Zariski dense in  $\mathbf{P}^m(K)$ , so the last point. ■

**Lemma 4.5.** *Assume  $\mathbf{T}_m$ . Let  $\eta \in K^m$  be such that  $[\eta, 1]$  is a geometrically good direction for  $\mathcal{F}$ . Let  $u_{\eta}$  be as in (5) and  $\mathcal{F}_{\eta} = \{f \circ u_{\eta} : f \in \mathcal{F}\}$ . Then every function in  $\text{coalg}(\mathcal{F}_{\eta})$  whose graph is bounded is piecewise largely continuous.*

*Proof:* The functions in  $\text{coalg}(\mathcal{F}_{\eta})$  are linear combinations of functions in  $\text{coalg}(f_{\eta})$  for  $f \in \mathcal{F}$ , hence it suffices to fix any  $f$  in  $\mathcal{F}$  and prove the result for  $\text{coalg}(f_{\eta})$ . Let  $d$  be the degree in  $T$  of  $f$ , and  $F$  a Galois extension of  $K$  in which every polynomial in  $K[T]$  of degree  $\leq d$  splits in linear factor. Given a basis  $\mathcal{B} = (\xi_1, \dots, \xi_r)$  of  $F$  over  $K$ , for each integer  $e \leq d$  let  $a_e \in K[X]$  be the coefficient of  $T^e$  in  $f_{\eta}$ , let  $A_e \subseteq K^m$  be the set of elements  $x \in K^m$  such that  $d_T^{\circ} f_{\eta}(x, T) = e$ , and choose a family of semi-algebraic functions  $b_{i,j} : A_e \rightarrow K$  such that for every  $x \in A_e$

$$f_{\eta}(x, T) = a_e(x) \prod_{i \leq e} \left( T - \sum_{j \leq r} b_{i,j}(x) \xi_j \right). \quad (6)$$

Let  $Z_F(f_{\eta})$  denote the zero set of  $f_{\eta}$  in  $F$ , and  $\sigma_1, \dots, \sigma_r$  be the list of  $K$ -automorphisms of  $F$ . Fix an integer  $i \leq e$ , and for every  $x \in A_e$  let

$$\lambda_i(x) = \sum_{j \leq r} b_{i,j}(x) \xi_j.$$

For every  $k \leq r$  we have

$$\sigma_k(\lambda_i(x)) = \sum_{j \leq r} b_{i,j}(x) \sigma_k(\xi_j).$$

Inverting the matrix  $(\sigma_k(\xi_j))_{j \leq r, k \leq r}$  gives for every  $j \leq r$  the function  $b_{i,j}$  as a linear combination of  $\sigma_k \circ \lambda_i$  for  $k \leq r$ . By construction  $\text{Gr } \sigma_k \circ \lambda_i$  is contained in  $Z_F(f_{\eta})$ . The latter is closed, hence  $\overline{\text{Gr } \sigma_k \circ \lambda_i}$  is contained in  $Z_F(f_{\eta})$  too.

The projection of  $Z_F(f_{\eta})$  onto  $F^m$  has finite fibers since  $\eta$  is a good direction for  $\mathcal{F}$  (see Remark 4.3). So the same holds true for the closure of the graph of  $\sigma_k \circ \lambda_i$ . This means that each  $\sigma_k \circ \lambda_i$  has finitely many different limit values at every point of  $\overline{A_e}$ . Obviously each  $b_{i,j}$  inherit this property, hence so does every  $h \in \text{coalg } f_{\eta}$ . If moreover the graph of  $h$  is bounded, it then follows from Theorem 3.5 (using  $\mathbf{T}_m$ ) that  $h$  is piecewise largely continuous. ■

Now we can turn to the “largely continuous cell decomposition up to small deformation” which was the aim of this section. We obtain it by combining the above construction based on good directions and the classical cell preparation theorem for semi-algebraic functions from Denef (Corollary 6.5 in [Den84]) revisited by Cluckers (Lemma 4 in [Clu01]).

**Theorem 4.6.** *Assume  $\mathbf{T}_m$ . Let  $(\theta_i : A_i \subseteq K^{m+1} \mapsto K)_{i \in I}$  be a finite family of semi-algebraic functions whose domains  $A_i$  are bounded. Then for some integer  $e \geq 1$  and every integers  $n, N \geq 1$  there exists a tuple  $\eta \in K^m$ , an integer  $M_0 > 2v(e)$ , an integer  $N_0$  divisible by  $eN$ , and a finite family  $\mathcal{D}$  of largely continuous fitting cells mod  $Q_{N_0, M_0}^\times$ , such that  $\mathcal{D}$  refines  $\{u_\eta^{-1}(A_i) : i \in I\}$  and such that for every  $i \in I$ , every  $D \in \mathcal{D}$  contained in  $u_\eta^{-1}(A_i)$  and every  $(x, t) \in D$*

$$\theta_i \circ u_\eta(x, t) = \mathcal{U}_{e,n}(x, t) h_{i,D}(x) [\lambda_D^{-1}(t - c_D(x))]^{\frac{\alpha_{i,D}}{e}}$$

where  $u_\eta$  is as in (5),  $h_{i,D} : \widehat{D} \rightarrow K$  is a semi-algebraic function and  $\alpha_{i,D} \in \mathbf{Z}$ .

Moreover the set of  $\eta \in K^m$  having this property is Zariski dense (in particular  $\eta$  can be chosen arbitrarily small), and the integers  $e, M$  can be chosen arbitrarily large (in the sense of footnote 1).

**Remark 4.7.** The above expression of  $\theta_i \circ u_\eta$  is well defined because  $e$  divides  $N_0, M_0 > 2v(e)$  and  $\lambda_D^{-1}(t - c_D(x))$  belongs to  $Q_{N_0, M_0}$  for every  $(x, t) \in D$  (see the definition of  $x \mapsto x^{\frac{1}{e}}$  on  $Q_{N_0, M_0}$  after Lemma 2.9). Of course if  $D$  is of type 0, then  $\lambda_D = t - c_D(x) = 0$  and we use our conventions that  $0^{-1} = \infty$  and  $\infty \cdot 0 = 1$ .

If we were only interested in the existence of such a preparation theorem with largely continuous cells for  $\theta_i \circ u_\eta$ , the integer  $N$  would be of no use and could be taken equal to 1. However it will be convenient to allow different values of  $N$  when we will use Theorem 4.6 in the proof the Triangulation Theorem.

*Proof:* Let  $e_*, M_* \geq 1$  be arbitrary integers. Corollary 6.5 in [Den84] applied to each  $\theta_i$  gives an integer  $e_i \geq 1$  and a family  $\mathcal{A}_i$  of semi-algebraic sets partitioning  $A_i$  such that for every  $A$  in  $\mathcal{A}_i$  and every  $(x, t)$  in  $A$ :

$$\theta_i^{e_i}(x, t) = u_{i,A}(x, t) \frac{f_{i,A}(x, t)}{g_{i,A}(x, t)} \quad (7)$$

where  $u_{i,A}$  is a semi-algebraic function from  $A$  to  $R^\times$  and  $f_{i,A}, g_{i,A}$  are polynomial functions such that  $g_{i,A}(x, t) \neq 0$  on  $A$ . Replacing if necessary each  $e_i$  by a common multiple  $e$  of them and of  $e_*$ , we can assume that  $e_i = e$  for every  $i$  and  $e$  is divisible by  $e_*$ . Let  $\mathcal{A}$  be a refinement of  $\bigcup_{i \in I} \mathcal{A}_i$ 's.

Fix any two integers  $n, N \geq 1$  and any integer  $n_0$  such that  $n_0 \geq n + v(e)$  and  $n_0 > 2v(e)$ . Since  $D^{n_0}R^\times$  is a subgroup of  $R^\times$  with finite index, every  $A \in \mathcal{A}$  splits into finitely many semi-algebraic pieces on each of which  $u_{i,A}$  is constant modulo  $D^{n_0}R^\times$  (for every  $i \in I$  such that  $A \subseteq A_i$ ). Thus, refining  $\mathcal{A}$  if necessary, (7) can be replaced, for every  $A$  in  $\mathcal{A}$  contained in  $A_i$  and every  $(x, t)$  in  $A$ , by

$$\theta_i^e(x, t) = \mathcal{U}_{n_0}(x, t) \tilde{u}_{i,A} \frac{f_{i,A}(x, t)}{g_{i,A}(x, t)} \quad (8)$$

with  $\tilde{u}_{i,A} \in R^\times$ .

Macintyre's Theorem 2.3, applied separately to each atom of the finite boolean algebra generated by  $\mathcal{A}$  in the power set of  $K^{m+1}$ , gives a finite family  $\mathcal{B}$  of semi-algebraic sets refining  $\mathcal{A}$ , an integer  $N_0 \geq 1$  and a finite list  $\mathcal{F}$  of non-zero polynomials in  $m + 1$  variables such that every element of  $\mathcal{B}$  is a boolean combinations of sets  $f^{-1}(\mathbf{P}_{N_0})$  with  $f \in \mathcal{F}$ . By Remark 2.2,  $N_0$  can be



chosen divisible by  $eN$ . Expanding  $\mathcal{F}$  if necessary, we can assume that all the polynomials  $f_{i,A}$  and  $g_{i,A}$  in (8) also belong to  $\mathcal{F}$  (except of course those which are null).

Lemma 4.4 gives  $\eta \in K^{m+1}$  such that  $[\eta, 1]$  is a geometrically good direction for  $\mathcal{F}_\eta$ , where  $\mathcal{F}_\eta = \{f \circ u_\eta : f \in \mathcal{F}\}$ . Note that every set in  $\mathcal{A}_\eta = \{u_\eta^{-1}(A) : A \in \mathcal{A}\}$  is a boolean combination of sets  $f_\eta^{-1}(\mathbf{P}_{N_0})$  with  $f_\eta \in \mathcal{F}_\eta$ . Denef's Theorem 4.1 applied to  $\mathcal{F}_\eta$  gives a finite family  $\mathcal{C}$  of fitting cells mod  $\mathbf{P}_{N_0}^\times$  which refines  $\mathcal{A}_\eta$  and whose center and bounds belong to  $\text{coalg } \mathcal{F}_\eta$ , such that for every  $f \in \mathcal{F}$ , every  $C \in \mathcal{C}$  and every  $(x, t) \in C$

$$f_\eta(x, t) = \mathcal{U}_{n_0}(x, t)h_{f,C}(x)(t - c_C(x))^{\alpha_{f,C}} \quad (9)$$

where  $h_{f,C} : \widehat{C} \rightarrow K$  is a semi-algebraic function and  $\alpha_{f,C}$  is a positive integer. We removed the zero polynomial from  $\mathcal{F}$ , but obviously (9) holds true for  $f = 0$  as well, by taking  $h_{f,C} = 0$  in that case. Each  $A_i$  is bounded hence so is their union  $\bigcup \mathcal{A}$  as well as  $\bigcup \mathcal{A}_\eta$ . So the center and bounds of every cell in  $\mathcal{C}$  must be bounded functions with bounded domain. By Lemma 4.5 (assuming  $\mathbf{T}_m$ ) these functions are piecewise largely continuous. Refining the socle of  $\mathcal{C}$  if necessary, and  $\mathcal{C}$  accordingly, we are then reduced to the case where every cell in  $\mathcal{C}$  is largely continuous. Note that  $\mathcal{U}_{n_0} \circ u_\eta = \mathcal{U}_{n_0}$ , so by combining (8) and (9) we get that for every  $i \in I$ , every  $C \in \mathcal{C}$  contained in  $u_\eta^{-1}(A_i)$  and every  $(x, t) \in C$

$$\theta_{i,\eta}(x, t)^e = \mathcal{U}_{n_0}(x, t)h_{i,C}(x)(t - c_C(x))^{\alpha_{i,C}} \quad (10)$$

where  $\theta_{i,\eta} = \theta_i \circ u_\eta$ ,  $h_{i,C} : \widehat{C} \rightarrow K$  is a semi-algebraic function and  $\alpha_{i,C} \in \mathbf{Z}$ . For any integer  $M_0 > 2v(e)$ ,  $Q_{N_0, M_0}^\times$  is a subgroup with finite index in  $\mathbf{P}_{N_0}^\times$  hence every such cell  $C$  mod  $\mathbf{P}_{N_0}^\times$  splits into finitely many cells  $D$  mod  $Q_{N_0, M_0}^\times$  with the same center, bounds and type as  $C$ . The integer  $M_0$  can be chosen arbitrarily large, in particular greater than  $M_*$ . Let  $\mathcal{D}$  be the family of all these cells  $D$ . From (10) and Lemma 2.9 we derive that for every  $i \in I$ , every  $D \in \mathcal{D}$  contained in  $u_\eta^{-1}(A_i)$  and every  $(x, t) \in D$

$$\theta_{i,\eta}(x, t)^e = \mathcal{U}_{n_0}(x, t)\tilde{h}_{i,D}(x)\left([\lambda_D^{-1}(t - c_D(x))]^{\frac{\alpha_{i,D}}{e}}\right)^e \quad (11)$$

where  $\tilde{h}_{i,D} = h_{i,C}$  and  $\alpha_{i,D} = \alpha_{i,C}$  with  $C$  the unique cell in  $\mathcal{C}$  containing  $D$ . The factor  $\mathcal{U}_{n_0}$  in (11) can be written  $\mathcal{U}_{n_0-v(e)}^e$  by Remark 2.10. Thus (11) implies that  $\tilde{h}_{i,D}$  takes values in  $\mathbf{P}_e$ . So by Theorem 2.8 there is a semi-algebraic function  $h_{i,D}$  such that  $\tilde{h}_{i,D} = h_{i,D}^e$ . As a consequence, from (11) it follows that there is a semi-algebraic function  $\chi_{i,D}$  with values in  $\mathbf{U}_e$  such that for every  $(x, t) \in D$

$$\theta_{i,\eta}(x, t) = \chi_{i,D}(x, t)\mathcal{U}_{n_0-v(e)}(x, t)h_{i,D}(x)[\lambda_D^{-1}(t - c_D(x))]^{\frac{\alpha_{i,D}}{e}} \quad (12)$$

By construction  $n_0 - v(e) \geq n$  hence the factor  $\mathcal{U}_{n_0-v(e)}$  can *a fortiori* be replaced by  $\mathcal{U}_n$ . Then  $\chi_{i,D}\mathcal{U}_n$  (which is just  $\mathcal{U}_{e,n}$ ) replaces  $\chi_{i,D}\mathcal{U}_{n_0-v(e)}$  in (12), which proves the result.

■

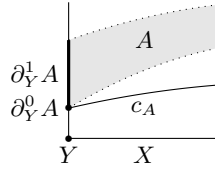
## 5 Cellular complexes

For this and the next section, let  $\mathbf{G}$  be a fixed semi-algebraic clopen subgroup of  $K^\times$  with finite index. Then  $v\mathbf{G}$  is a subgroup of  $\mathcal{Z}$  with finite index, hence  $v\mathbf{G} = N_0\mathcal{Z}$  for some integer  $N_0 \geq 1$ . Our aim in these two sections is to prove that every finite family of bounded largely continuous fitting cells mod  $\mathbf{G}$ , such as the one given by Theorem 4.6, can be refined in a complex of cells mod  $\mathbf{G}$  satisfying certain restrictive assumptions defined below.

**Notation.** For every largely continuous fitting cell  $A$  mod  $\mathbf{G}$  in  $K^{m+1}$  with socle  $X$ , and every semi-algebraic set  $Y$  contained in  $\overline{X}$  let:

- $\partial_Y^0 A = (\bar{c}_A, 0, 0, \{0\})|_Y$  if  $\bar{\nu}_A = 0$  on  $Y$ ,  $\partial_Y^0 A = \emptyset$  otherwise;
- $\partial_Y^1 A = (\bar{c}_A, \bar{\nu}_A, \bar{\mu}_A, G_A)|_Y$  if  $\bar{\mu}_A \neq 0$  on  $Y$ ,  $\partial_Y^1 A = \emptyset$  otherwise.

Provided that on  $Y$ ,  $\bar{\nu}_A$  and  $\bar{\mu}_A$  either take values in  $K^\times$  or are constant,  $\partial_Y^0 A$  and  $\partial_Y^1 A$  are (if non-empty) largely continuous fitting cells mod  $\mathbf{G}$  contained in  $\overline{A} \cap (Y \times K)$ . Intuitively, we can represent them (when  $\nu_A \neq 0$  and  $\bar{\nu}_A = 0 < \bar{\mu}_A$  on  $Y$ ) like this:



**Remark 5.1.** If  $\mathcal{X}$  is a partition of  $\overline{X}$ , the family of non-empty  $\partial_Y^i A$  for  $i \in \{0, 1\}$  and  $Y \in \mathcal{X}$  form a partition of  $\overline{A}$ .

Given two cells  $A, B$  in  $K^{m+1}$  and an integer  $n \geq 1$ , we write  $B \triangleleft^n A$  if  $B \subseteq A$  and if there exists  $\alpha \in \{0, 1\}$  and a semi-algebraic function  $h_{B,A} : \widehat{B} \rightarrow K$  such that for every  $(x, t)$  in  $B$ :

$$t - c_A(x) = \mathcal{U}_n(x, t) h_{B,A}(x)^\alpha (t - c_B(x))^{1-\alpha}$$

We call  $h_{B,A}$  a  $\triangleleft^n$ -**transition** for  $(B, A)$ . If  $\mathcal{A}, \mathcal{B}$  are families of cells in  $K^{m+1}$  we write  $\mathcal{B} \triangleleft^n \mathcal{A}$  if  $B \triangleleft^n A$  for every  $B \in \mathcal{B}$  and  $A \in \mathcal{A}$  such that  $B$  meets  $A$ . A  $\triangleleft^n$ -**system** for  $(\mathcal{B}, \mathcal{A})$  is then the data of one  $\triangleleft^n$ -transition for each possible  $(B, A)$  in  $\mathcal{B} \times \mathcal{A}$ .

**Remark 5.2.** For any two finite families  $\mathcal{A}, \mathcal{B}$  of cells mod  $\mathbf{G}$ , if  $\mathcal{B}$  refines  $\mathcal{A}$  and belongs to  $\text{Alg}_n \mathcal{A}$  then  $\mathcal{B} \triangleleft^n \mathcal{A}$ .

A **closed  $\triangleleft^n$ -complex** of cells mod  $\mathbf{G}$  is a finite family  $\mathcal{A}$  of largely continuous fitting cells mod  $\mathbf{G}$  such that  $\bigcup \mathcal{A}$  is closed, the socle of  $\mathcal{A}$  is a complex of sets and for every  $A, B \in \mathcal{A}$  if  $B$  meets  $\overline{A}$  then for some  $i \in \{0, 1\}$ ,  $\partial_Y^i A$  is a cell<sup>9</sup> and  $B \triangleleft^n \partial_Y^i A$ , with  $Y = \widehat{B}$ . If moreover  $B = \partial_Y^i A$  we call  $\mathcal{A}$  a **closed cellular complex** mod  $\mathbf{G}$ . As the terminology suggests, we are going to prove that closed  $\triangleleft^n$ - and cellular complexes are complexes of sets in the general sense of

<sup>9</sup>The condition  $\partial^i Y A$  is a cell means that on  $Y$ ,  $\bar{\mu}_A$  and  $\bar{\nu}_A$  either take values in  $K^\times$  or are constant.

Section 2 (see Proposition 5.3). Any subset of a closed  $\triangleleft^n$ -complex (resp. closed cellular complex) is a  $\triangleleft^n$ -**complex** (resp. a **cellular complex**). As usually we call them **monoplexes** if they form a tree with respect to the specialization order.

When  $\mathcal{A}$  is a  $\triangleleft^n$ -complex of cells mod  $\mathbf{G}$ , for every  $Y \in \widehat{\mathcal{A}}$  and every cells  $A, B$  in  $\mathcal{A}$  such that  $B$  meets  $\overline{A}$ , there is an integer  $\alpha \in \{0, 1\}$  and a semi-algebraic function  $h_{B,A} : \widehat{B} \rightarrow K$  such that for every  $(x, t)$  in  $B$ :

$$t - \bar{c}_A(x) = \mathcal{U}_n(x, t) h_{B,A}(x)^\alpha (t - c_B(x))^{1-\alpha}.$$

An **inner  $\triangleleft^n$ -system** for  $\mathcal{A}$  is the data of one function  $h_{B,A}$  as above for every possible  $A, B \in \mathcal{A}$ .

**Proposition 5.3.** *Let  $\mathcal{A}$  be a closed  $\triangleleft^n$ -complex of cells mod  $\mathbf{G}$ . Then  $\mathcal{A}$  is a closed complex of sets. Moreover, for every  $A, B \in \mathcal{A}$  and every  $Z \in \widehat{\mathcal{A}}$  if  $B$  meets  $\partial_Z^0 A$  then  $B = \partial_Z^0 A = \text{Gr } \bar{c}_{A|Z}$ .*

*Proof:* By assumption the socle of every cell  $A$  in  $\mathcal{A}$  is relatively open and pure dimensional. Thanks to the restrictions we made on the bounds in our definition of presented cells, it follows that  $A$  is also relatively open and pure dimensional.

In order to show that  $\mathcal{A}$  is a partition, let  $A, B$  be two cells in  $\mathcal{A}$  which are not disjoint and let  $X = \widehat{A}$ . Both  $\widehat{B}$  and  $X$  belong to  $\widehat{\mathcal{A}}$  and are not disjoint, hence  $\widehat{B} = X$ . Since  $B$  meets  $A \subseteq \overline{A}$ , by assumption  $B$  is contained in  $\partial_X^i A$  with  $i = \text{tp } B$ . But then  $\partial_X^i A$  meets  $A$ , hence obviously is equal to  $A$ . So  $B \subseteq A$ , and equality holds by symmetry.

Now let  $A$  be any cell in  $\mathcal{A}$  and  $X = \widehat{A}$ . Since  $\bigcup \mathcal{A}$  is closed, every point of  $\overline{A}$  belongs to a unique  $B \in \mathcal{A}$ . Since  $B$  meets  $\overline{A}$ , by assumption  $B \subseteq \partial_Y^i A$  with  $Y = \widehat{B}$  and  $i = \text{tp } B$ . In particular  $B \subseteq \overline{A}$ , which proves that  $\overline{A}$  is a union of cells in  $\mathcal{A}$  (hence so is  $\partial A$  since  $\mathcal{A}$  is a partition and  $\partial A$  is disjoint from  $A$ ). This proves that  $\mathcal{A}$  is closed complex of sets.

The last point follows. Indeed, if  $B$  meets  $\partial_Z^0 A \subseteq \overline{A}$  then it is contained in  $\partial_Y^i A$  for some  $i \in \{0, 1\}$ , with  $Y = \widehat{B}$ . In particular  $\partial_Z^0 A$  meets  $\partial_Y^i A$ . They are two pieces of a partition of  $\overline{A}$  (see Remark 5.1) hence  $\partial_Z^0 A = \partial_Y^i A$ . Therefore  $Y = Z$  and  $i = 0$ , so  $B \subseteq \partial_Z^0 A$ . That is,  $B$  is of type 0 and  $c_B = \bar{c}_A$  on  $\widehat{B} = Z$ , so  $B = \text{Gr } \bar{c}_{A|Z} = \partial_Z^0 A$ . ■

**Proposition 5.4.** *Let  $\mathcal{A}$  be a finite family of largely continuous fitting cells mod  $\mathbf{G}$  and  $n \geq 1$  an integer. There exists a  $\triangleleft^n$ -complex  $\mathcal{D}$  of cells mod  $\mathbf{G}$  refining  $\mathcal{A}$  such that  $\mathcal{D} \triangleleft^n \mathcal{A}$ .*

In the next section we will prove that one can even require that  $\mathcal{D}$  is a cellular monoplex mod  $G$ .

*Proof:* The proof goes by induction on  $d = \dim \bigcup \widehat{\mathcal{A}}$ . If a  $\triangleleft^n$ -complex  $\mathcal{D}^+$  is found which proves the result for a family  $\mathcal{A}^+$  of cells mod  $\mathbf{G}$  containing  $\mathcal{A}^+$  then obviously the family  $\mathcal{D}$  of cells in  $\mathcal{D}^+$  contained in  $\bigcup \mathcal{A}$  proves the result for  $\mathcal{A}$ . Thus, enlarging  $\mathcal{A}$  if necessary, we can assume that  $\bigcup \mathcal{A}$  and  $\bigcup \widehat{\mathcal{A}}$  are closed. By Denef's Theorem 2.11 and Remark 5.2 there is a finite family  $\mathcal{B}$  of largely continuous fitting cells mod  $\mathbf{G}$  refining  $\mathcal{A}$  such that  $\mathcal{B} \triangleleft^n \mathcal{A}$ . Replacing  $\mathcal{A}$  by this refinement if necessary we can also assume that  $\mathcal{A}$  is a partition.

If  $\mathcal{B}$  is any vertical refinement of  $\mathcal{A}$  then obviously  $\mathcal{B} \triangleleft^n \mathcal{A}$ . Thus, by taking if necessary a finite partition  $\mathcal{X}$  refining  $\widehat{\mathcal{A}}$  and replacing  $\mathcal{A}$  by the corresponding vertical refinement (that is the family of all cells  $A \cap (X \times K)$  with  $A \in \mathcal{A}$  and  $X \in \mathcal{X}$  contained in  $\widehat{\mathcal{A}}$ ), we can assume that  $\widehat{\mathcal{A}} = \mathcal{X}$  is a partition. By the same argument we can assume as well that for every  $A \in \mathcal{A}$  and every  $X \in \mathcal{X}$  contained in  $\partial X$ , the restrictions of  $\bar{\mu}_A$  and  $\bar{\nu}_A$  to  $X$  take values in  $K^\times$  or are constant, hence  $\partial_X^0 A$  (resp.  $\partial_X^1 A$ ) is a cell with socle  $X$  whenever it is non-empty. By Remark 2.16 we can even assume that it is a complex of pure dimensional sets. Let  $\mathcal{X}_d$  be the family of  $X \in \mathcal{X}$  with dimension  $d$ . Note that every  $X \in \mathcal{X}_d$  is open in  $\bigcup \mathcal{X}$  because  $\mathcal{X}$  is a complex and  $\dim \bigcup \mathcal{X} = d$ .

For every  $X \in \mathcal{X}_d$  let  $\mathcal{A}_X$  be the family of cells in  $\mathcal{A}$  with socle  $X$ . For every cell  $A \in \mathcal{A}_X$  of type 1 such that  $\nu_A = 0$ ,  $\text{Gr } c_A$  is contained in  $\overline{A}$  hence in  $\bigcup \mathcal{A}_X$  since  $\bigcup \mathcal{A}$  is closed and  $\widehat{\mathcal{A}}$  is a partition. It may happen that  $\text{Gr } c_A$  does not belong to  $\mathcal{A}$ . With Proposition 5.3 in view we have to remedy this. Every point  $(x, c_A(x))$  in  $\text{Gr } c_A$  belongs to some cell  $B$  in  $\mathcal{A}_X$ . This cell must be of type 0 otherwise the fiber  $B_x = \{t \in K : (x, t) \in B\}$  would be open, hence it would contain a neighbourhood  $V$  of  $c_A(x)$  and so  $\{x\} \times V$  would be contained in  $B$  and meet  $A$ , which implies that  $B \subseteq A$  since  $\mathcal{A}_X$  is partition, in contradiction with the fact  $B$  meets  $\text{Gr } c_A$ . So there is a finite partition  $\mathcal{Y}_A$  of  $X$  in semi-algebraic pieces  $Y$  on each of which there is a unique cell  $B \in \mathcal{A}_X$  of type 0 whose center coincides with  $c_A$  on  $Y$ . Repeating the same argument for every  $A \in \mathcal{A}_X$  and every  $X \in \mathcal{X}_d$  gives a finite partition  $\mathcal{Y}$  of  $\bigcup \mathcal{X}_d$  finer than every such  $\mathcal{Y}_A$ . Let  $\mathcal{X}'$  be a complex of pure dimensional semi-algebraic sets refining  $\mathcal{X} \cup \mathcal{Y}$ . Replacing if necessary  $\mathcal{A}$  by the vertical refinement defined by  $\mathcal{X}'$ , we can then assume from now on that for every  $X$  in  $\mathcal{X}_d$  and every  $A \in \mathcal{A}$  with socle  $X$ , if  $\nu_A = 0$  then  $\text{Gr } c_A$  belongs to  $\mathcal{A}$ .

Let  $\mathcal{A}_d = \{A \in \mathcal{A} : \widehat{A} \in \mathcal{X}_d\}$  and  $\mathcal{B}$  be the union of  $\mathcal{A} \setminus \mathcal{A}_d$  and of the family of non-empty  $\partial_Y^i A$  for  $i \in \{0, 1\}$ ,  $A \in \mathcal{A}_d$  and  $Y \in \mathcal{X} \setminus \mathcal{X}_d$ . Clearly  $\dim \bigcup \widehat{\mathcal{B}} < d$  so the induction hypothesis gives a  $\triangleleft^n$ -complex  $\mathcal{C}$  of cells mod  $\mathbf{G}$  refining  $\mathcal{B}$  such that  $\mathcal{C} \triangleleft^n \mathcal{B}$ . *A fortiori*  $\mathcal{C} \triangleleft^n (\mathcal{A} \setminus \mathcal{A}_d)$  because the latter is contained in  $\mathcal{B}$ . So if we let  $\mathcal{D} = \mathcal{C} \cup \mathcal{A}_d$ , then  $\mathcal{D}$  refines  $\mathcal{A}$  and  $\mathcal{D} \triangleleft^n \mathcal{A}$ . It only remains to check that  $\mathcal{D}$  is a  $\triangleleft^n$ -complex, and first that  $\widehat{\mathcal{D}}$  is a complex of sets.

Note that  $\widehat{\mathcal{A}}_d = \mathcal{X}_d$  hence  $\widehat{\mathcal{D}} = \widehat{\mathcal{C}} \cup \widehat{\mathcal{A}}_d = \widehat{\mathcal{C}} \cup \mathcal{X}_d$  is a partition, and every set in  $\widehat{\mathcal{D}}$  is pure dimensional and relatively open (by induction hypothesis for  $\widehat{\mathcal{C}}$  and by construction for  $\mathcal{X}_d$ ). For every  $X \in \widehat{\mathcal{D}}$ , we have to prove that  $\partial X$  is a union of sets in  $\widehat{\mathcal{C}} \cup \mathcal{X}_d$ . If  $X \in \widehat{\mathcal{C}}$  this is clear because  $\widehat{\mathcal{C}}$  is a complex. Otherwise  $X \in \mathcal{X}_d$  hence  $\partial X$  is a union of sets in  $\mathcal{X}$  (because  $\mathcal{X}$  is a complex). All these sets have dimension  $< d = \dim X$  hence belong to  $\mathcal{X} \setminus \mathcal{X}_d$ . But  $\mathcal{C}$  refines  $\mathcal{B}$ , which contains  $\mathcal{A} \setminus \mathcal{A}_d$ , whose socle is  $\mathcal{X} \setminus \mathcal{X}_d$ , hence  $\widehat{\mathcal{C}}$  refines  $\mathcal{X} \setminus \mathcal{X}_d$ . Thus  $\partial X$  is also the union of sets in  $\widehat{\mathcal{C}}$ , hence of  $\widehat{\mathcal{D}}$ .

Now let  $D, E \in \mathcal{D}$  be such that  $E$  meets  $\overline{D}$ , let  $X = \widehat{D}$  and  $Y = \widehat{E}$ . By construction  $\partial_Y^0 D$  and  $\partial_Y^1 D$  are cells (if non-empty) and cover  $\overline{D} \cap (Z \times K)$ . So there is  $i \in \{0, 1\}$  such that  $\partial_Y^i D$  is a cell which meets  $E$ . We have to prove that  $E \triangleleft^n \partial_X^i D$ . Note that  $Y$  meets the socle of  $\overline{D}$ , which is contained in  $\overline{X}$ , hence  $Y = X$  or  $Y \subseteq \partial X$  because  $\widehat{\mathcal{D}}$  is a complex. So, if  $\dim X < d$  then also  $\dim Y < d$  hence  $D, E \in \mathcal{C}$ . In that case  $E \triangleleft^n \partial_X^i D$  because  $\mathcal{C}$  is a  $\triangleleft^n$ -complex. Thus we can assume that  $\dim X = d$ , that is  $\widehat{D} \in \mathcal{A}_d$ . We know that  $Y = X$  or  $Y \subseteq \partial X$ . In the first case  $Y = X$  hence  $\partial_X^i D \in \mathcal{A}_d \subseteq \mathcal{D}$  by construction, so  $E = \partial_X^i D$  because  $\mathcal{D}$  is a partition. In the second case  $Y \in \widehat{\mathcal{C}}$  hence  $E \in \mathcal{C}$ .

Now  $Y$  is contained in some  $Z \in \mathcal{X} \setminus \mathcal{X}_d$  because  $\widehat{\mathcal{C}}$  refines  $\mathcal{X} \setminus \mathcal{X}_d$ , and  $E$  meets  $\partial_Z^i D$ . By construction  $\partial_Z^i D$  belongs to  $\mathcal{B}$ . Since  $\mathcal{C} \triangleleft^n \mathcal{B}$  it follows that  $E \triangleleft^n \partial_Z^i D$  hence *a fortiori*  $E \triangleleft^n \partial_Y^i D$  because  $E \subseteq Y \times K$  and  $\partial_Y^i D = \partial_Z^i D \cap (Y \times K)$ .

■

Before entering in more complicated constructions, let us mention here two elementary properties of fitting cells which will be of some use latter.

**Proposition 5.5.** *Let  $A \subseteq K^{m+1}$  be a cell mod  $\mathbf{G}$  of type 1. Then:*

- $\mu_A$  is a fitting bound if and only if  $\mu_A = \infty$  or  $v\mu_A(\widehat{A}) \subseteq vG_A$ .
- $\nu_A$  is a fitting bound if and only if  $\nu_A = 0$  or  $v\nu_A(\widehat{A}) \subseteq vG_A$ .

*Proof:* The case where  $\mu_A = \infty$  being trivial, we can omit it. If  $\mu_A \neq \infty$  is a fitting bound then obviously  $v\mu_A(\widehat{A}) \subseteq vG_A$  because  $v(t - c_A(x)) \in vG_A$  for every  $(x, t) \in A$ . Conversely assume that  $v\mu_A(\widehat{A}) \subseteq vG_A$ . Let  $x$  be any element of  $\widehat{A}$ . We have to prove that  $|\mu_A(x)| = \max\{|d| : d \in D_x\}$  where  $D_x = \{t - c_A(x) : (x, t) \in A\}$ .  $D_x$  is bounded since  $\mu_A \neq \infty$ , hence by Corollary 2.7 it contains an element  $d$  of maximal norm. By construction  $|d| \leq |\mu_A(x)|$ . Assume for a contradiction that  $|d| < |\mu_A(x)|$ , that is  $v(d/\mu_A(x)) > 0$ . By construction  $v(d)$  and  $v\mu_A(x)$  belong to  $vG_A = v\lambda_A + v\mathbf{G}$  hence  $v(d/\mu_A(x)) \in v\mathbf{G} = N_0\mathcal{Z}$ . Thus  $v(d/\mu_A(x)) \geq N_0$ , that is  $|d| \leq |\pi^{N_0}\mu_A(x)|$ . Pick any  $g \in \mathbf{G}$  such that  $v(g) = N_0$  and let  $t' = c_A(x) + d/g$ . We have  $t' - c_A(x) = d/g \in G_A$ ,  $|\nu_A(x)| \leq |d| \leq |d/g|$  and  $|d/g| \leq |\mu_A(x)|$ , hence  $(x, t') \in A$ . So  $t' - c_A(x) \in D_x$  and  $|d| < |t' - c_A(x)|$ , a contradiction. The proof for  $\nu_A$  is similar and left to the reader.

■

**Proposition 5.6.** *For every fitting cell  $A$  mod  $Q_{N_0, M_0}$  in  $K^{m+1}$ , if  $A \subseteq R^{m+1}$  then  $v\mu_A \geq -M_0$ .*

Since  $A \subseteq R^{m+1}$ , one may naively expect that  $|\mu_A| \leq 1$ , that is  $v\mu_A \geq 0$ . The presented cell  $A = (-\pi^{-M_0}, \pi^{-M_0}, \pi^{-M_0}, Q_{N_0, M_0})$  is a counterexample in  $K$ : it is contained in  $R$  (it is actually equal to  $R$ ) and  $v\mu_A = -M_0 < 0$ .

*Proof:* Assume the contrary, that is  $v\mu_A(x) < -M_0$  for some  $x \in \widehat{A}$ . Since  $A$  is a fitting cell there is  $t \in K$  such that  $(x, t) \in A$  and  $v(t - c_A(x)) = v\mu_A(x)$ . Since  $A \subseteq R^{m+1}$ ,  $t \in R$  hence  $v(t - c_A(x)) < 0 = v(t)$  implies that  $vc_A(x) = v(t - c_A(x)) = v\mu_A(x)$ . So there are  $a \in R$  and  $g \in Q_{N, M}$  such that  $c_A(x) = a\pi^{M_0+1}$  and  $t - c_A(x) = \lambda_A g$ . In particular  $v(\lambda_A g) = v(t - c_A(x)) = v\mu_A(x) < -M_0$  so  $\pi^{M_0}\lambda_A g \notin R$ . Now let  $t' = t + \pi^{M_0}\lambda_A g$ , then  $t' \notin R$  since  $t \in R$  and  $\pi^{M_0}\lambda_A g \notin R$ . On the other hand  $1 + \pi^{M_0} \in Q_{N_0, M_0}$  and

$$t' - c_A(x) = t - c_A(x) + \pi^{M_0}\lambda_A g = \lambda_A g + \pi^{M_0}\lambda_A g = \lambda_A(1 + \pi^{M_0})g.$$

So  $t' - c_A(x) \in \lambda_A Q_{N_0, M_0}$  and  $v(t' - c_A(x)) = v(\lambda_A(1 + \pi^{M_0})g) = v(\lambda_A g) = v\mu_A(x)$ . Thus  $(x, t') \in A$ , a contradiction since  $t' \notin R$  and  $A \subseteq R^{m+1}$ .

■

## 6 Cellular monoplexes

We keep as in Section 5 a semi-algebraic clopen subgroup  $\mathbf{G}$  of  $K^\times$  with finite index, and  $N_0 \geq 1$  an integer such that  $v\mathbf{G} = N_0\mathcal{Z}$ . Lemma 6.1 below (together with Lemma 7.10) is the technical heart of this paper. All this section is devoted to its proof.

**Lemma 6.1.** *Assume  $\mathbf{T}_m$ . Let  $\mathcal{A}$  be a finite set of bounded, largely continuous, fitting cells mod  $\mathbf{G}$  in  $K^{m+1}$ . Let  $\mathcal{F}_0$  be a finite family of definable functions with domains in  $\widehat{\mathcal{A}}$ . Let  $n, N \geq 1$  be a pair of integers. For some integers  $e, M > 2v(e)$  which can be made arbitrarily large (in the sense of footnote 1), there is a tuple  $(\mathcal{V}, \varphi, \mathcal{D}, \mathcal{F}_{\mathcal{D}})$  such that:*

- $\mathcal{D}$  is a cellular monoplex mod  $\mathbf{G}$  refining  $\mathcal{A}$  such that  $\mathcal{D} \triangleleft^n \mathcal{A}$ .
- $\mathcal{F}_{\mathcal{D}}$  is a  $\triangleleft^n$ -system for  $(\mathcal{D}, \mathcal{A})$ .
- $(\mathcal{V}, \varphi)$  is a triangulation of<sup>10</sup>  $\mathcal{F}_0 \cup \mathcal{F}_{\mathcal{D}} \cup \text{CB}(\mathcal{D})$  with parameters  $(n, N, e, M)$ , such that  $\widehat{\mathcal{D}} = \varphi(\mathcal{V})$ .

Note that, in order to obtain this result, it does not suffice to find a continuous monoplex  $\mathcal{D}$  of well presented cells mod  $\mathbf{G}$  refining  $\mathcal{A}$  such that  $\mathcal{D} \triangleleft^n \mathcal{A}$ , and then to select an arbitrary  $\triangleleft^n$ -system  $\mathcal{F}_{\mathcal{D}}$  for  $(\mathcal{D}, \mathcal{A})$  and to apply  $\mathbf{T}_m$  to  $\mathcal{F}_0 \cup \mathcal{F}_{\mathcal{D}} \cup \text{CB}(\mathcal{D})$ . Indeed, this will give a triangulation  $(\mathcal{V}, \varphi)$  of  $\mathcal{F}_0 \cup \mathcal{F}_{\mathcal{D}} \cup \text{CB}(\mathcal{D})$ . But  $\varphi(\mathcal{V})$  will then be a refinement of  $\widehat{\mathcal{D}}$ , not  $\widehat{\mathcal{D}}$  itself. It is then tempting to refine vertically  $\mathcal{D}$ , that is to replace  $\mathcal{D}$  by the family  $\mathcal{E}$  of cells  $D \cap (\varphi(V) \times K)$  for  $D \in \mathcal{D}$  and  $V \in \mathcal{V}$  such that  $\varphi(V) \subseteq \widehat{\mathcal{D}}$ . This ensures that  $\widehat{\mathcal{E}} = \varphi(\mathcal{V})$  and  $\mathcal{E}$  is a cellular complex such that  $\mathcal{E} \triangleleft^n \mathcal{A}$ . But  $\mathcal{E}$  is no longer a monoplex.

In order to break this vicious circle we have to build  $\mathcal{V}$ ,  $\mathcal{D}$  and  $\mathcal{F}_{\mathcal{D}}$  simultaneously. The remaining of this section is devoted to this construction. It is divided in three parts: (6.a) preparation, (6.b) vertical refinement, (6.c) horizontal refinement.

### 6.a Preparation

Given a family  $\mathcal{X}$  of subsets of  $K^m$ , we let  $\mathcal{F}_{0|\mathcal{X}}$  denote the family of all the restrictions  $f|_X$  with  $f \in \mathcal{F}_0$  and  $X \in \mathcal{X}$  contained in the domain of  $f$ . By the same argument as in the beginning of the proof of Proposition 5.4, we can assume that  $\bigcup \mathcal{A}$  is closed. Finally, replacing if necessary  $\mathcal{A}$  by a refinement  $\mathcal{D}$  given by Proposition 5.4 and  $\mathcal{F}_0$  by  $\mathcal{F}_{0|\mathcal{X}}$  with  $\mathcal{X} = \widehat{\mathcal{D}}$ , we are reduced to the case where  $\mathcal{A}$  is a closed  $\triangleleft^n$ -complex of bounded cells mod  $\mathbf{G}$ . Enlarging  $\mathcal{F}_0$  if necessary, we can, and will, assume that it contains  $\text{CB}(\mathcal{A})$  and an inner  $\triangleleft^n$ -system for  $\mathcal{A}$ . For some integers  $e, M > 2v(e)$  which can be made arbitrarily large,  $\mathbf{T}_m$  gives a triangulation  $(\mathcal{S}, \varphi)$  of  $\mathcal{F}_0$  with parameters  $(n, N, e, M)$ . For every  $A \in \mathcal{A}$  we let  $S_A = \varphi^{-1}(\widehat{A})$ .

Since  $\bigcup \mathcal{A}$  is bounded and closed in  $K^{m+1}$ , its image  $\bigcup \widehat{\mathcal{A}}$  by the coordinate projection is closed in  $K^m$  by Theorem 2.6. Now  $\varphi$  is a homeomorphism from  $\biguplus \mathcal{S}$  to  $\bigcup \widehat{\mathcal{A}}$ , hence  $\mathcal{S}$  is closed by Remark 2.19.

Let  $\mathcal{A}'$  be the family of cells  $A \cap (\varphi(S) \times K)$  for  $A \in \mathcal{A}$  and  $S \in \mathcal{S}$  such that  $\varphi(S) \subseteq \widehat{A}$ , and let  $\mathcal{F}'_0 = \mathcal{F}_{0|\varphi(\mathcal{S})}$ . Since every cell in  $\mathcal{A}'$  has the same center

<sup>10</sup>Recall that  $\text{CB}(\mathcal{D})$  denotes the family of center and bounds of the cells in  $\mathcal{D}$ .

and bounds as the unique cell in  $\mathcal{A}$  which contains it, clearly  $\mathcal{A}'$  is still a closed  $\triangleleft^n$ -complex,  $\mathcal{F}'_0$  contains  $\text{CB}(\mathcal{A}')$  and an inner  $\triangleleft^n$ -system for  $\mathcal{A}'$ , and  $(\mathcal{S}, \varphi)$  is still a triangulation of  $\mathcal{F}'_0$ . Thus, replacing  $(\mathcal{A}, \mathcal{F}_0)$  by  $(\mathcal{A}', \mathcal{F}'_0)$  if necessary, we can assume that  $\varphi(\mathcal{S}) = \widehat{\mathcal{A}}$ , that is  $S_A \in \mathcal{S}$  for every  $A \in \mathcal{A}$ .

A **preparation** for  $(\mathcal{S}, \varphi, \mathcal{A}, \mathcal{F}_0)$  is a tuple  $(\mathcal{T}, \mathcal{B}, \mathcal{F}_\mathcal{B})$  such that:

**(P1)**  $\mathcal{T}$  is a simplicial subcomplex of  $\mathcal{S}$ . We let  $\mathcal{S}_{|\mathcal{T}} = \{S \in \mathcal{S} : S \subseteq \bigsqcup \mathcal{T}\}$ , and  $\mathcal{A}_{|\mathcal{T}}$  be the family of cells  $A \in \mathcal{A}$  such that  $S_A \in \mathcal{S}_{|\mathcal{T}}$ . Note that:

- $\mathcal{T}$  is closed because  $\bigsqcup \mathcal{T}$  is closed in  $\bigsqcup \mathcal{S}$ .
- By Remark 2.19 it follows that the image by  $\varphi$  of  $\bigcup \mathcal{T}$ , that is the socle of  $\mathcal{A}_{|\mathcal{T}}$ , is closed too.
- Hence  $\mathcal{A}_{|\mathcal{T}}$  is closed because  $\bigcup \mathcal{A}_{|\mathcal{T}}$  is the inverse image of its socle by the (continuous) coordinate projection of  $\bigcup \mathcal{A}$  onto  $\bigcup \widehat{\mathcal{A}}$ .

**(P2)**  $\mathcal{B}$  is a cellular monoplex mod  $\mathbf{G}$  refining  $\mathcal{A}_{|\mathcal{T}}$  such that  $\varphi(\mathcal{T}) = \widehat{\mathcal{B}}$ . For every  $B \in \mathcal{B}$  we let  $T_B = \varphi^{-1}(\widehat{B})$ . Note that  $\mathcal{B}$  is closed because  $\bigcup \mathcal{B} = \bigcup \mathcal{A}_{|\mathcal{T}}$ .

**(P3)**  $\mathcal{B} \triangleleft^n \mathcal{A}_{|\mathcal{T}}$  and  $\mathcal{F}_\mathcal{B}$  is a  $\triangleleft^n$ -system for  $(\mathcal{B}, \mathcal{A}_{|\mathcal{T}})$ .

**(P4)**  $\mathcal{T}$  together with the restriction of  $\varphi$  to  $\bigsqcup \mathcal{T}$ , which we will denote  $\varphi_{|\mathcal{T}}$ , is a triangulation of  $\mathcal{F}_\mathcal{B} \cup \text{CB}(\mathcal{B})$  with parameters  $(n, N, e, M)$ . Note that, since  $\mathcal{T}$  refines  $\mathcal{S}_{|\mathcal{T}}$  and  $(\mathcal{S}, \varphi)$  is a triangulation of  $\mathcal{F}_0$ ,  $(\mathcal{T}, \varphi_{|\mathcal{T}})$  is also a triangulation of  $\mathcal{F}_{0|\mathcal{X}}$  with  $\mathcal{X} = \varphi(\mathcal{T})$ .

**Remark 6.2.** Obviously  $(\emptyset, \emptyset, \emptyset)$  is preparation for  $(\mathcal{S}, \varphi, \mathcal{A}, \mathcal{F}_0)$ . Given an arbitrary preparation  $(\mathcal{T}, \mathcal{B}, \mathcal{F}_\mathcal{B})$  for  $(\mathcal{S}, \varphi, \mathcal{A}, \mathcal{F}_0)$  such that  $\bigcup \mathcal{T} \neq \bigcup \mathcal{S}$ , and  $S$  a minimal element in  $\mathcal{S} \setminus \mathcal{S}_{|\mathcal{T}}$ , it suffices to build from it a preparation  $(\mathcal{U}, \mathcal{C}, \mathcal{F}_\mathcal{C})$  such that  $\bigsqcup \mathcal{U} = \bigsqcup \mathcal{T} \cup S$ . Indeed,  $\mathcal{S}_{|\mathcal{U}}$  contains one more element of  $\mathcal{S}$  than  $\mathcal{S}_{|\mathcal{T}}$  thus, starting from  $(\emptyset, \emptyset, \emptyset)$  and repeating the process inductively we will finally get a preparation  $(\mathcal{V}, \mathcal{D}, \mathcal{F}_\mathcal{D})$  such that  $\bigsqcup \mathcal{V} = \bigsqcup \mathcal{S}$ , hence  $\mathcal{A}_{|\mathcal{V}} = \mathcal{A}$ . (P4) then implies that  $(\mathcal{V}, \varphi)$  is a triangulation of  $\mathcal{F}_0 \cup \mathcal{F}_\mathcal{B} \cup \text{CB}(\mathcal{B})$  with parameters  $(n, N, e, M)$ . So the tuple  $(\mathcal{V}, \varphi, \mathcal{D}, \mathcal{F}_\mathcal{D})$  satisfies the conclusion of Lemma 6.1, which finishes the proof.

So from now on, let  $(\mathcal{T}, \mathcal{B}, \mathcal{F}_\mathcal{B})$  be a given preparation for  $(\mathcal{S}, \varphi, \mathcal{A}, \mathcal{F}_0)$  such that  $\bigsqcup \mathcal{T} \neq \bigsqcup \mathcal{S}$ . Let  $S$  be a minimal element in  $\mathcal{S} \setminus \mathcal{S}_{|\mathcal{T}}$  and  $\mathcal{A}_S = \{A \in \mathcal{A} : \widehat{A} = \varphi(S)\}$ . The minimality of  $S$  ensures that every proper face of  $S$  belongs to  $\mathcal{S}_{|\mathcal{T}}$ , hence  $\bigsqcup \mathcal{S}_{|\mathcal{T}} \cup S$  and  $\bigcup(\mathcal{A}_{|\mathcal{T}} \cup \mathcal{A}_S)$  are closed.

**Claim 6.3.** *Let  $A$  be a cell of type 1 in  $\mathcal{A}_S$ ,  $T$  a simplex in  $\mathcal{T}$  contained in  $\overline{S}$ , and  $Y = \varphi(T)$ . If  $\bar{\nu}_A = 0$  on  $Y$  then  $\text{Gr } \bar{c}_{A|Y} = \partial_Y^0 A$  belongs to  $\mathcal{B}$ . If moreover  $\bar{\mu}_A \neq 0$  on  $Y$  then  $\partial_Y^1 A$  is covered by the cells in  $\mathcal{B}$  that it meets, and among them there is a unique cell  $B_T^1$  whose closure meets  $\partial_Y^0 A$ . More precisely:*

$$B_T^1 = (\bar{c}_{A|Y}, 0, \mu_{B_T^1}, G_A)$$

and either  $|\mu_{B_T^1}| = |\bar{\mu}_A|$  on  $Y$ , or  $|\mu_{B_T^1}| \leq |\pi^{N_0} \bar{\mu}_A|$  on  $Y$ . In particular the closure of  $B_T^1$  contains  $\partial_Y^0 A$ .

*Proof:* Note first that for every  $i \in \{0, 1\}$ ,  $\partial_Y^i A$  is contained in  $\overline{A}$  hence in  $\bigcup \mathcal{A}$  since it is closed by assumption. Every cell in  $\mathcal{A}$  which meets  $\partial_Y^i A$  is contained in it since  $\mathcal{A}$  is a  $\triangleleft^n$ -complex, and belongs to  $\mathcal{A}_{|\mathcal{T}}$  (otherwise its socle would not meet  $Y$  since  $Y \subseteq \bigcup \varphi(\mathcal{T})$ ). Since  $\mathcal{B}$  refines  $\mathcal{A}_{|\mathcal{T}}$  it follows that  $\partial_Y^i A$  is the union of the cells  $B$  in  $\mathcal{B}$  which it contains.

In particular, if  $\bar{\nu}_A = 0$  on  $A$  then  $\partial_Y^0 A \neq \emptyset$  hence it contains a cell  $B \in \mathcal{B}$ . Necessarily  $B$  is of type 0 since  $\partial_Y^0 A$  is so, and thus  $B = \partial_Y^0 A$  since they have the same socle  $Y$ . This proves the first point.

For the second point, since  $\bar{\nu}_A = 0 \neq \bar{\mu}_A$  on  $Y$  both  $\partial_Y^0 A$  and  $\partial_Y^1 A$  are non-empty. Now  $\partial_Y^0 A$  is contained in the closure of  $\partial_Y^1 A$ , which is the union of the closure of the cells in  $\mathcal{B}$  contained in  $\partial_Y^1 A$ . Hence necessarily the closure of at least one of them, say  $B$ , meets  $\partial_Y^0 A$ .

$\widehat{B}$  meets  $Y$  and both of them belong to  $\widehat{\mathcal{B}}$  so  $\widehat{B} = Y$ . Since  $\overline{B} \cap (Y \times K)$  meets  $\partial_Y^0 A$  and  $B \subseteq \partial_Y^1 A$  is disjoint from  $\partial_Y^0 A$ ,  $B$  must be of type 1 with  $\nu_B = 0$  because otherwise  $B$  would be closed in  $Y \times K$ . It follows that  $\overline{B} \cap (Y \times B)$  is the union of  $B$  and  $\partial_Y^0 B$ , and the latter meets  $\partial_Y^0 A$ . By the first point  $\partial_Y^0 A \in \mathcal{B}$ . By Proposition 5.3 applied to  $\mathcal{B}$ ,  $\partial_Y^0 B \in \mathcal{B}$ . Thus  $\partial_Y^0 B = \partial_Y^0 A$ , in particular they have the same center so  $c_B = \bar{c}_{A|Y}$ . Pick any  $(x, t) \in B$ , so that  $t - c_B(x) \in G_B$ .  $B$  is contained in  $\partial_Y^1 A$  hence  $t - \bar{c}_A(x) \in G_A$ . Since  $c_B(x) = \bar{c}_A(x)$  it follows that  $G_B \cap G_A \neq \emptyset$  hence  $G_A = G_B$ .

This proves that  $B = (\bar{c}_{A|Y}, 0, \mu_B, G_A)$ . The uniqueness of  $B$  follows. Indeed if  $B'$  is any cell in  $\mathcal{B}$  contained in  $\partial_Y^1 A$  whose closure meets  $\partial_Y^0 A$ , the same argument shows that  $B' = (\bar{c}_{A|Y}, 0, \mu_{B'}, G_A)$ . This implies that for any  $t' \in K$  such that  $t' - \bar{c}_A(x)$  is small enough and belongs to  $G_A$ , the point  $(x, t')$  will belong both to  $B$  and  $B'$ , so  $B = B'$ .

If  $|\mu_B| = |\bar{\mu}_{A|Y}|$  we are done, so let us assume the contrary. Then  $|\mu_B(x)| \neq |\bar{\mu}_A(x)|$  for some  $x \in Y$ .  $B$  is a fitting cell so let  $t \in K$  be such that  $(x, t) \in B$  and  $|t - c_B(x)| = |\mu_B(x)|$ . We have  $|t - \bar{c}_A(x)| \leq |\bar{\mu}_A(x)|$  because  $(x, t) \in \partial_Y^1 A$ , so  $|\mu_B(x)| < |\bar{\mu}_A(x)|$ . We are going to show that  $|\mu_B| < |\bar{\mu}_A|$  on  $Y$ . Since  $\partial_Y^1 A$  is a fitting cell it follows that  $\partial_Y^1 A$  is not contained in  $B$ , so there is at least one other cell  $C$  in  $\mathcal{B}$  contained in  $\partial_Y^1 A$ . Now  $\widehat{C}$  is contained in  $Y$  and both of them belong to  $\widehat{\mathcal{B}}$  so  $\widehat{B} = Y$ . For each  $y$  in  $Y$  fix  $t_y$  in  $K$  such that  $(y, t_y) \in C$ . Since  $C$  is contained in  $\partial_Y^1 A$  we have:

$$0 \leq |t_y - \bar{c}_A(y)| \leq |\bar{\mu}_A(y)| \text{ and } t_y - \bar{c}_A(y) \in G_A$$

Necessarily  $|\mu_B(y)| < |t_y - \bar{c}_A(y)|$  because otherwise  $(y, t_y)$  would belong both to  $C$  and  $B$ , a contradiction. Hence *a fortiori*  $|\mu_B(y)| < |\bar{\mu}_A(y)|$ . By Proposition 5.5 this implies that  $|\mu_B(y)| \leq |\pi^{N_0} \bar{\mu}_A(y)|$  because  $B$  and  $A$  are fitting cells mod  $\mathbf{G}$ ,  $\nu_{\mathbf{G}} = N_0 \mathcal{Z}$  and  $G_B = G_A$ . So  $|\mu_B| \leq |\pi^{N_0} \bar{\mu}_{A|Y}|$  in that case, which proves our claim.  $\blacksquare$

We can start now our construction of a preparation  $(\mathcal{U}, \mathcal{C}, \mathcal{F}_{\mathcal{C}})$  for  $(\mathcal{S}, \varphi, \mathcal{A}, \mathcal{F}_0)$  such that  $\biguplus \mathcal{U} = \biguplus \mathcal{T} \cup S$ . We are going to refine  $\mathcal{A}_S$  twice. First “vertically”, according to the image by  $\varphi$  of a certain partition of  $S$  which, together with  $\mathcal{T}$ , forms a simplicial subcomplex  $\mathcal{U}$  of  $\mathcal{S}$  refining  $\mathcal{S}_{|\mathcal{T}} \cup \{S\}$  (Claim 6.5). Then “horizontally” by enlarging the cells in  $\mathcal{B}$  contained in the closure of  $\bigcup \mathcal{A}_S$  in such a way that the family of these new cells, together with  $\mathcal{B}$ , forms a cellular monoplex  $\mathcal{C}$  mod  $\mathbf{G}$  refining  $\mathcal{A}_{|\mathcal{T}} \cup \mathcal{A}_S = \mathcal{A}_{|\mathcal{U}}$  such that  $\mathcal{C} \triangleleft^n \mathcal{A}_{|\mathcal{U}}$ .



The point of the construction is to ensure that  $\mathcal{C}$  comes with a  $\triangleleft^n$ -system  $\mathcal{F}_{\mathcal{C}}$  for  $(\mathcal{C}, \mathcal{A}_{|\mathcal{U}})$  such that  $(\mathcal{U}, \mathcal{C}, \mathcal{F}_{\mathcal{C}})$  is a preparation for  $(S, \varphi, \mathcal{A}, \mathcal{F}_0)$ .

## 6.b Vertical refinement

Let  $\mathcal{T}_S$  be the list of proper faces of  $S$ . We first deal with the case when  $S$  is not closed, that is  $\mathcal{T}_S \neq \emptyset$ . For every  $A$  in  $\mathcal{A}_S$  let:

$$(c_A^\circ, \nu_A^\circ, \mu_A^\circ) = (c_A \circ \varphi, \nu_A \circ \varphi, \mu_A \circ \varphi)|_S$$

For every  $T \in \mathcal{T}_S$  and every  $A \in \mathcal{A}_S$  let  $\Phi_{T,A}(t, \varepsilon)$  be the formula saying that  $(t, \varepsilon) \in T \times R^*$  and that one of the following conditions hold, with  $n_1 = \max(n, 1 + 2vN)$ :

**(A1)** $_{t,\varepsilon}$ :  $\bar{\nu}_A^\circ(t) \neq 0$  and for every  $s \in S$  such that  $\|s - t\| \leq |\varepsilon|$ :

$$|c_A^\circ(s) - \bar{c}_A^\circ(t)| \leq |\pi^{n_1} \bar{\nu}_A^\circ(t)|$$

$$\text{and } |\nu_A^\circ(s)| = |\bar{\nu}_A^\circ(t)| \text{ and } |\mu_A^\circ(s)| = |\bar{\mu}_A^\circ(t)|$$

**(A2)** $_{t,\varepsilon}$ :  $\bar{\nu}_A^\circ(t) = 0$ ,  $\bar{\mu}_A^\circ(t) \neq 0$  and for every  $s \in S$  such that  $\|s - t\| \leq |\varepsilon|$ :

$$|c_A^\circ(s) - \bar{c}_A^\circ(t)| \leq |\pi^{n_1 - N_0} \bar{\mu}_B^\circ(t)|$$

and  $|\nu_A^\circ(s)| \leq |\mu_B^\circ(t)| \leq |\bar{\mu}_A^\circ(t)| = |\mu_A^\circ(s)|$  where  $B$  is the cell  $B_T^1$  given by Claim 6.3.

**(A3)** $_t$ :  $\bar{\nu}_A^\circ(t) = \bar{\mu}_A^\circ(t) = 0$ .

Let  $\Phi(t, \varepsilon)$  be the conjunction of the (finitely many)  $\Phi_{T,A}(t, \varepsilon)$ 's as  $T$  ranges over  $\mathcal{T}_S$  and  $A$  over  $\mathcal{A}_S$ . Finally let  $\Psi(t, \varepsilon)$  be the formula saying that that  $|\varepsilon|$  is maximal among the elements  $\varepsilon'$  in  $R^*$  such that  $K \models \Phi(t, \varepsilon')$ . Obviously  $\Psi(t, \varepsilon)$  implies  $\Phi(t, \varepsilon)$ .

By continuity of the center and bounds of  $A$ , for every  $t \in T$  there exists  $\varepsilon_{t,T,A} \in R^*$  such that  $K \models \Phi_{T,A}(t, \varepsilon_{t,T,A})$ . Hence for every  $t \in \partial S$  there is  $\varepsilon \in R^*$  such that  $K \models \Phi(t, \varepsilon)$  (every  $\varepsilon \in R^*$  such that  $|\varepsilon| \leq |\varepsilon_{t,T,A}|$  for every  $(T, A) \in \mathcal{T}_S \times \mathcal{A}_S$  is a solution). For every  $t \in \partial S$ , the set  $E_t$  of elements  $\varepsilon$  of  $R^*$  such that  $K \models \Phi(t, \varepsilon)$  is semi-algebraic, bounded and non-empty. So by Corollary 2.7 there is  $\varepsilon_t \in E_t$  such that  $|\varepsilon_t|$  is maximal in  $|E_t|$ , that is  $K \models \Psi(t, \varepsilon_t)$ . Theorem 2.8 then gives a semi-algebraic function  $\varepsilon : \partial S \rightarrow R^*$  such that  $K \models \Psi(t, \varepsilon(t))$  for every  $t \in \partial S$ , hence *a fortiori*:

$$\forall t \in \partial S, K \models \Phi(t, \varepsilon(t)). \quad (13)$$

**Claim 6.4.** *Let  $\varepsilon : \partial S \rightarrow R^*$  be the semi-algebraic function defined above. Then the restriction of  $|\varepsilon|$  to every proper face  $T$  of  $S$  is continuous.*

*Proof:* Note first that if  $K \models \Phi_{T,A}(t, \varepsilon')$  for some  $t \in T$  and  $\varepsilon' \in R^*$  then  $K \models \Phi_{T,A}(t', \varepsilon')$  for every  $t' \in T \cap B$  where  $B = B(t, \varepsilon')$  is the ball with center  $t$  and radius  $\varepsilon'$ .

Indeed, assume for example that  $\bar{\nu}_A^\circ(t) \neq 0$ , hence (A1) $_{t,\varepsilon'}$  holds true. It asserts that for every  $s \in S \cap B$

$$|c_A^\circ(s) - \bar{c}_A^\circ(t)| \leq |\pi^{n_1} \bar{\nu}_A^\circ(t)| \quad (14)$$

and  $|\nu_A(s)| = |\bar{\nu}_A^\circ(t)|$  and  $|\mu_A(s)| = |\bar{\mu}_A^\circ(t)|$ . Now  $T \cap B \subseteq \overline{S \cap B}$  hence, as  $s$  tends in  $S \cap B$  to any given  $t' \in T \cap B$  we get

$$|c_A^\circ(t') - \bar{c}_A^\circ(t)| \leq |\pi^{n_1} \bar{\nu}_A(t)| \quad (15)$$

and  $|\nu_A(t')| = |\bar{\nu}_A^\circ(t)|$  and  $|\mu_A(t')| = |\bar{\mu}_A^\circ(t)|$ . By combining (14) and (15) with the triangle inequality we obtain that for every  $s \in S \cap B$

$$|c_A^\circ(s) - \bar{c}_A^\circ(t')| \leq |\pi^{n_1} \bar{\nu}_A(t)| = |\pi^{n_1} \bar{\nu}_A(t')|$$

and  $|\mu_A(s)| = |\bar{\mu}_A^\circ(t')|$ , that is (A1) $_{t', \varepsilon'}$ .

Assume now that  $\bar{\mu}_A^\circ(t) = 0$ , hence  $\bar{\nu}_A^\circ(t) = 0$ , that is (A3) $_t$  holds true. Then  $\varphi(T) \in \widehat{A}$  and  $\bar{\mu}_A(\varphi(t)) = 0$  imply that  $\bar{\mu}_A = 0$  on  $\varphi(T)$  because  $\mathcal{A}$  is a closed  $\triangleleft^n$ -complex (see footnote 9). So  $\bar{\mu}_A^\circ = \bar{\nu}_A^\circ = 0$  on  $T$ , and (A3) $_{t'}$  follows.

The intermediate case (A2) $_{t, \varepsilon}$  where  $\bar{\nu}_A^\circ(t) = 0$  and  $\bar{\mu}_A^\circ(t) \neq 0$  is similar, and left to the reader.

Now it follows that if  $K \models \Psi(t, \varepsilon')$  and  $\|t' - t\| \leq |\varepsilon'|$ , then  $K \models \Psi(t, \varepsilon'')$  if and only if  $|\varepsilon'| = |\varepsilon''|$ . So  $|\varepsilon(t)| = |\varepsilon(t')|$  for every  $t, t' \in T$  such that  $\|t - t'\| \leq |\varepsilon(t)|$ . Thus  $|\varepsilon|$  is locally constant, hence continuous on  $T$ .

■

Theorem 2.17 applies to  $S$ ,  $\mathcal{T}_S$  and the function  $\varepsilon$ . It gives a partition  $\mathcal{U}_S$  of  $S$  such that  $\mathcal{U}_S \cup \mathcal{T}_S$  is a simplicial complex, for each  $T \in \mathcal{T}_S$  there is a unique  $U \in \mathcal{U}_S$  with facet  $T$ , and for every  $u \in U$ :

$$\|u - \pi_U(u)\| \leq |\varepsilon(\pi_U(u))| \quad (16)$$

where  $\pi_U$  is the coordinate projection of  $U$  onto  $T$  (see Remark 2.15). On  $\varphi(U)$  let  $\sigma_U = \varphi \circ \pi_U \circ \varphi^{-1}$ . This is a continuous retraction of  $\varphi(U)$  onto  $\varphi(T)$ .

For every  $U \in \mathcal{U}_S$  and every  $A \in \mathcal{A}_S$  let

$$A_U = A \cap (\varphi(U) \times K). \quad (17)$$

Let  $T_U = \emptyset$  if  $U$  is closed, and  $T_U \in \mathcal{T}_S$  be the facet of  $U$  otherwise. Finally let  $\mathcal{U} = \mathcal{U}_S \cup \mathcal{T}_S$ .

**Claim 6.5.** *With the notation above,  $\mathcal{U}$  is a simplicial subcomplex of  $\mathcal{S}$  refining  $\mathcal{S}_{|\mathcal{T} \cup \{S\}}$  and containing  $\mathcal{T}$ . For every  $U \subseteq S$  in  $\mathcal{U}$  and every  $A \in \mathcal{A}_S$ ,  $A_U$  is a largely continuous fitting cell mod  $\mathbf{G}$ . Moreover if  $U$  is not closed then:*

1. *If  $|\bar{\nu}_A| \neq 0$  on  $\varphi(T_U)$ , then for every  $x \in \widehat{A}_U$ :*

$$|c_A(x) - \bar{c}_A(\sigma_U(x))| \leq |\pi^{n_1} \bar{\nu}_A(\sigma_U(x))| \quad (18)$$

$$|\nu_A(x)| = |\bar{\nu}_A(\sigma_U(x))| \quad \text{and} \quad |\mu_A(x)| = |\bar{\mu}_A(\sigma_U(x))| \quad (19)$$

2. *If  $|\bar{\nu}_A| = 0 < |\bar{\mu}_A|$  on  $\varphi(T_U)$ , then for every  $x \in \widehat{A}_U$ :*

$$|c_A(x) - \bar{c}_A(\sigma_U(x))| \leq |\pi^{n_1 - N_0} \mu_B(\sigma_U(x))| \quad (20)$$

$$|\nu_A(x)| \leq |\mu_B(\sigma_U(x))| \leq |\bar{\mu}_A(\sigma_U(x))| = |\mu_A(x)| \quad (21)$$

where  $B$  is the cell  $B_T^1$  given by Claim 6.3.

*Proof:* By construction  $\mathcal{U}$  is clearly a simplicial complex refining  $\mathcal{T} \cup \{S\}$ , hence refining  $\mathcal{S}_{\mathcal{T} \cup \{S\}}$  since  $\mathcal{T}$  refines  $\mathcal{S}_{\mathcal{T}}$ . For every  $U \subseteq S$  in  $\mathcal{U}$  and every  $A \in \mathcal{A}_S$ ,  $A_U$  is a largely continuous fitting cell mod  $\mathbf{G}$  by (17), because  $A$  is so. If moreover  $U$  is not closed let  $T = T_U \in \mathcal{T}_S$  be its facet, let  $x$  be any element of  $\widehat{A}_U = \varphi(U)$ ,  $s = \varphi^{-1}(x) \in U$  and  $t = \pi_U(s) \in T$ , where  $\pi_U$  is the coordinate projection of  $U$  onto  $T$  (see Remark 2.15). Note that  $\sigma_U(x) = \varphi \circ \pi_U(s) = \varphi(t)$  hence  $\bar{c}_A(\sigma_U(x)) = \bar{c}_A^\circ(t)$ , and similarly for  $\bar{\nu}_A(\sigma_U(x))$  and  $\bar{\mu}_A(\sigma_U(x))$ . By (13) we have  $K \models \Phi(t, \varepsilon(t))$ .

If  $|\bar{\nu}_A| \neq 0$  on  $\varphi(T)$  then  $\bar{\nu}_A^\circ(t) = \bar{\nu}_A(\sigma_U(x)) \neq 0$  hence  $\Phi(t, \varepsilon(t))$  says that  $(A1)_{t, \varepsilon(t)}$  holds for  $t$ . By (16),  $\|s - t\| \leq |\varepsilon(t)|$  so (18) and (19) follow from  $(A1)_{t, \varepsilon(t)}$ . Similarly, if  $|\bar{\nu}_A| = 0 < |\bar{\mu}_A|$  on  $\varphi(T)$  then (20) and (21) follow from  $(A2)_{t, \varepsilon(t)}$ .

■

This finishes the construction of the vertical refinement of  $\mathcal{A}_S$  if  $S$  is not closed. When  $S$  is closed we simply take  $\mathcal{U} = \mathcal{S}_{\mathcal{T} \cup \{S\}}$ . Claim 6.5 holds true in this case too, for the trivial reason that there is no non-closed  $U \subseteq S$  in  $\mathcal{U}$ .

**Remark 6.6.** For every  $U \in \mathcal{U}_S$ , if  $\nu_{A_U} = 0$  then  $\text{Gr } c_{A_U} = B_U$  for some  $B \in \mathcal{A}_S$ . Indeed  $\nu_{A|_{\varphi(U)}} = \nu_{A_U} = 0$  implies that  $\nu_A = 0$  (thanks to our definition of presented cells) hence  $\text{Gr } c_A = \partial_{\varphi(S)}^0 A$  belongs to  $\mathcal{A}$ : it is contained in  $\overline{A}$ , hence in  $\bigcup \mathcal{A}$  since the latter is closed, in particular it meets at least one cell  $B$  in  $\mathcal{A}$ , and the last point of Proposition 5.3 then gives that  $B = \text{Gr } c_A$ . Thus  $B = \text{Gr } c_A \in \mathcal{A}_S$ , and clearly  $\text{Gr } c_{A_U} = B_U$ .

## 6.c Horizontal refinement

For every  $A \in \mathcal{A}_S$  we are going to construct for each  $U \in \mathcal{U}_S$  a partition  $\mathcal{E}_{A,U}$  of  $A_U$ , and for each  $E$  in  $\mathcal{E}_{A,U}$  a semi-algebraic function  $h_{E,A_U} : \varphi(U) \rightarrow K$  such that:

**(Pres)**  $\widehat{E} = \varphi(U) = \widehat{A_U}$  and  $E$  is a largely continuous fitting cell mod  $\mathbf{G}$ .

**(Fron)** One of the following holds:

**(Ø1)**  $\partial E = \emptyset$ .

**(Ø2)**  $\partial E = \overline{\text{Gr } c_E}$  and  $\text{Gr } c_E \in \mathcal{E}_{C,U}$  for some  $C \in \mathcal{A}_S$ .

**(Ø3)**  $\partial E = \overline{B}$  for some  $B \in \mathcal{B}$ , in which case  $U$  is not closed,  $\widehat{B} = \varphi(T_U)$  and:

$$(c_B, \nu_B, \mu_B) = (\bar{c}_E, \bar{\nu}_E, \bar{\mu}_E)|_{\varphi(T_U)}.$$

**(Out)**  $E \triangleleft^n A_U$  and  $h_{E,A_U}$  is a  $\triangleleft^n$ -transition for  $(E, A_U)$ .

**(Mon)**  $c_E \circ \varphi|_U, \mu_E \circ \varphi|_U, \nu_E \circ \varphi|_U$  and  $h_{E,A_U} \circ \varphi|_U$  are  $N$ -monomial mod  $U_{e,n}$ .

This last construction will finish the proof of Lemma 6.1. Indeed, assuming that it is done, let  $\mathcal{C}$  be the union of  $\mathcal{B}$  and all the cells  $E$  in  $\mathcal{E}_{A,U}$  for  $A \in \mathcal{A}_S$  and  $U \in \mathcal{U}_S$ . Let  $\mathcal{F}_{\mathcal{C}}$  be the union of the family of the corresponding functions  $h_{E,A_U}$  and of  $\mathcal{F}_{\mathcal{B}}$ . By Claim 6.5,  $\mathcal{U}$  is a simplicial subcomplex of  $\mathcal{S}$  such that  $\biguplus \mathcal{U} = \biguplus \mathcal{T} \cup S$ . The assumption (P2) for  $\mathcal{B}$ , together with (Pres) and (Fron), give that  $\mathcal{C}$  is a cellular monoplex mod  $\mathbf{G}$  refining  $\mathcal{A}_{\mathcal{T} \cup \mathcal{A}_S}$  and that  $\varphi(\mathcal{U}) = \widehat{\mathcal{C}}$ . The assumption (P3) for  $\mathcal{B}$  and  $\mathcal{F}_{\mathcal{B}}$ , together with (Out) above, give that

$\mathcal{C} \triangleleft^n \mathcal{A}|_{\mathcal{U}}$  and  $\mathcal{F}_{\mathcal{C}}$  is a  $\triangleleft^n$ -system for  $(\mathcal{C}, \mathcal{A}|_{\mathcal{U}})$ . Finally the assumption (P4) for  $(\mathcal{T}, \varphi|_{\mathcal{T}})$  together with (Mon) ensure that  $(\mathcal{U}, \varphi|_{\mathcal{U}})$  is a triangulation of  $\mathcal{F}_{\mathcal{C}} \cup \text{CB}(\mathcal{C})$  with parameters  $(n, N, e, M)$ . So  $(\mathcal{U}, \mathcal{C}, \mathcal{F}_{\mathcal{C}})$  is a preparation of  $(\mathcal{S}, \varphi, \mathcal{A}, \mathcal{F}_0)$ , and since  $\biguplus \mathcal{U} = \biguplus \mathcal{T} \cup S$  we conclude by Remark 6.2.

So let  $A \in \mathcal{A}_S$  and  $U \in \mathcal{U}_S$  be fixed once for all in the remaining.

**Remark 6.7.** Recall that  $(\mathcal{S}, \varphi)$  is a triangulation of  $\mathcal{F}_0$ , and  $\mathcal{F}_0$  contains  $\text{CB}(\mathcal{A})$ . In particular  $c_A \circ \varphi|_S$  is  $N$ -monomial mod  $U_{e,n}$  hence *a fortiori*  $c_A \circ \varphi|_U$  is so. By (17)  $c_{A_U} = c_{A|_{\varphi(U)}}$  hence  $c_{A_U} \circ \varphi|_U = c_A \circ \varphi|_U$ . Thus  $c_{A_U} \circ \varphi|_U$  is  $N$ -monomial mod  $U_{e,n}$ , and so are  $\mu_{A_U} \circ \varphi|_U$  and  $\nu_{A_U} \circ \varphi|_U$  by the same argument.

Let us first assume that  $U$  is closed. We distinguish two elementary cases.

**Case 1.1:**  $\mu_{A_U} = 0$  or  $\nu_{A_U} \neq 0$ .

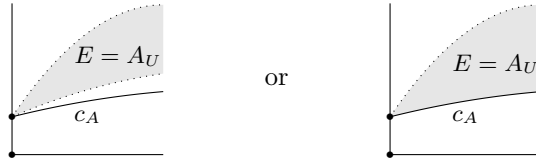
Then  $A_U$  is closed. We let  $\mathcal{E}_{A,U} = \{A_U\}$  and  $h_{A_U, A_U} = 1$ . (Pres),  $(\partial 1)$ , (Out) and (Mon) are obvious (using Remark 6.7 for the latter).

**Case 1.2:**  $0 = |\nu_{A_U}| < |\mu_{A_U}|$ .

We let  $\mathcal{E}_{A,U} = \{A_U\}$  and  $h_{A_U, A_U} = 1$ . Again (Pres), (Out) and (Mon) are obvious (same as Case 1.1). Moreover  $\partial A_U = \text{Gr } c_{A_U} = \text{Gr } c_A$ , which belongs to  $\mathcal{A}_S$  by Remark 6.6. If we let  $B = \text{Gr } c_A$ , we have  $B_U = \text{Gr } c_{A_U}$  and  $\mu_{B_U} = 0$  hence  $\mathcal{E}_{B,U} = \{B_U\}$  by the previous case. So  $\partial A_U = \text{Gr } c_A$  and  $\text{Gr } c_A \in \mathcal{E}_{B,U}$ , which proves that  $(\partial 2)$  holds true.

These cases being solved, we assume in the remaining that  $U$  is not closed. Recall that  $T_U$  is then the facet of  $U$  and belongs to  $\mathcal{T}$ . By construction  $\partial \varphi(U) = \varphi(\partial U) = \varphi(\overline{T_U}) = \overline{\varphi(T_U)}$ . For the convenience of the reader, each of the following cases is illustrated by a geometric representation of its conditions (almost like if we were dealing with a cell  $A$  over a real closed field, except that the vertical intervals representing the fibres of  $A$  over  $\widehat{A}$  can be clopen). In this figures  $A_U$  is represented by a gray area in  $K^2$ , its bounds by dotted lines, its socle  $\varphi(U)$  by the horizontal axe,  $\partial \varphi(U) = \varphi(T_U)$  by a dot on the left bound of  $\varphi(U)$ , and  $\overline{A} \cap (\varphi(T_U) \times K)$  by a thick line or dot on the vertical axe above  $\varphi(T_U)$ .

**Case 2.1:**  $|\bar{\mu}_{A_U}| = 0$  on  $\varphi(T_U)$ .

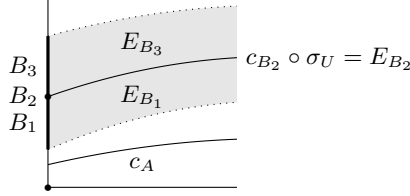


We let  $\mathcal{E}_{A,U} = \{A_U\}$  and  $h_{A_U, A_U} = 1$ . (Pres), (Out) and (Mon) are obvious as in the previous cases.

- *Sub-case 2.1.a:*  $\nu_{A_U} \neq 0$  or  $\mu_{A_U} = \nu_{A_U} = 0$ . Then  $\partial A_U$  is the closure of  $\text{Gr } \bar{c}_{A_U|_{\varphi(T_U)}} = \text{Gr } \bar{c}_{A|_{\varphi(T_U)}}$ . The latter belongs to  $\mathcal{B}$  by Claim 6.3, which proves  $(\partial 3)$ .

- *Sub-case 2.1.b:*  $\nu_{A_U} = 0 \neq \mu_{A_U}$ . Then  $\partial A_U$  is the closure of  $\text{Gr } c_{A_U}$ . By Remark 6.6, there is a cell  $C \in \mathcal{A}_S$  such that  $C_U = \text{Gr } c_{A_U}$ . Then  $\mu_{C_U} = 0$  (because  $A_U$  is a fitting cell) hence  $\mathcal{E}_{C,U} = \{C_U\}$  by the previous sub-case. So  $\partial A_U = \overline{\text{Gr } c_{A_U}}$  and  $\text{Gr } c_{A_U} \in \mathcal{E}_{C,U}$ , which proves that  $(\partial 2)$  holds true.

**Case 2.2:**  $0 < |\bar{\nu}_A|$  on  $\varphi(T_U)$ .



In this case, by Claim 6.3,  $\partial_{\varphi(T_U)}^1 A = \bar{A} \cap (\varphi(T_U) \times K)$  is the union of the cells  $B \in \mathcal{B}$  which it contains. For every such  $B$ ,  $\widehat{B} = \varphi(T_U)$  (because  $\widehat{B} = \varphi(\mathcal{T})$  and  $\widehat{B} \subseteq \varphi(T_U)$ ) and we let:

$$E_B = (c_B \circ \sigma_U, \nu_B \circ \sigma_U, \mu_B \circ \sigma_U, G_B)$$

These  $E_B$ 's form a family  $\mathcal{E}_{A,U}$  of two by two disjoint largely continuous cells because the various cells  $B$  involved are so and:

$$(x, t) \in E_B \iff x \in \widehat{A_U} \text{ and } (\sigma_U(x), t) \in B. \quad (22)$$

Each  $E_B$  has socle  $\varphi(U) = \widehat{A_U}$  and for every  $x \in \varphi(U)$ ,  $\sigma_U(x)$  belongs to  $\varphi(T_U) = \widehat{B}$ . If  $B$  is of type 0, then so is  $E_B$  and  $\mu_B(\sigma_U(x)) = 0$  (because  $B$  is a fitting cell of type 0) hence  $E_B$  is a fitting cell. If  $B$  is of type 1, then  $\mu_B(\sigma_U(x)) \in vG_B$  by Proposition 5.5 (because  $B$  is a fitting cell of type 1). That is  $\mu_{E_B}(x) \in G_{E_B}$  hence  $\mu_{E_B}$  is a fitting bound by Proposition 5.5. Similarly  $\nu_{E_B}$  is a fitting bound, so  $E_B$  is a fitting cell. This proves (Pres), and one can easily derive from (22) that  $\partial E_B = \bar{B}$  so that  $(\partial 3)$  holds true. Note also that  $c_{E_B} \circ \varphi|_U$  is  $N$ -monomial mod  $U_{e,n}$  because  $c_B \circ \varphi|_{T_U}$  is so and  $c_{E_B} \circ \varphi|_U = c_B \circ \sigma_U \circ \varphi|_U = c_B \circ \varphi \circ \pi_U = c_B \circ \varphi|_{T_U}$ . The same reasoning applies to  $\nu_{E_B}$  and  $\mu_{E_B}$ . So the next claim finishes to prove that  $\mathcal{E}_{A,U}$  is a partition of  $A_U$  and that (out), (Mon) hold true.

**Claim 6.8.**  $E_B \triangleleft^n A_U$  and there is a semi-algebraic  $\triangleleft^n$ -transition  $h_{E_B, A_U}$  for  $(E_B, A_U)$  such that  $h_{E_B, A_U} \circ \varphi|_U$  is  $N$ -monomial mod  $U_{e,n}$ .

*Proof:* For every  $(x, t)$  in  $E_B$ , let us prove that  $(x, t)$  belongs to  $A_U$ . Since  $x \in \widehat{A_U}$  it suffices to prove that  $(x, t) \in A$ . By construction  $(\sigma_U(x), t)$  belongs to  $B$  hence to  $\partial_{\varphi(T_U)}^1 A$  so:

$$|\bar{\nu}_A(\sigma_U(x))| \leq |t - \bar{c}_A(\sigma_U(x))| \leq |\bar{\mu}_A(\sigma_U(x))| \text{ and } t - \bar{c}_A(\sigma_U(x)) \in G_A \quad (23)$$

By (19)  $|\nu_A(x)| = |\bar{\nu}_A(\sigma_U(x))|$  and  $|\mu_A(x)| = |\bar{\mu}_A(\sigma_U(x))|$ . Moreover by (18):

$$\begin{aligned} |(t - c_A(x)) - (t - \bar{c}_A(\sigma_U(x)))| &= |c_A(x) - \bar{c}_A(\sigma_U(x))| \\ &\leq |\pi^{n_1} \bar{\nu}_A(\sigma_U(x))| \\ &< |t - \bar{c}_A(\sigma_U(x))| \end{aligned} \quad (24)$$

Thus  $|t - c_A(x)| = |t - \bar{c}_A(\sigma_U(x))|$  and by (23):

$$|\nu_A(x)| \leq |t - c_A(x)| \leq |\mu_A(x)|$$

Moreover by (24):

$$\left| \frac{t - c_A(x)}{t - \bar{c}_A(\sigma_U(x))} - 1 \right| \leq \left| \pi^{n_1} \frac{\bar{\nu}_A(\sigma_U(x))}{t - \bar{c}_A(\sigma_U(x))} \right| \leq |\pi^{n_1}| \quad (25)$$

$n_1 > 2vN$  hence  $1 + \pi^{n_1}R \subseteq \mathbf{P}_N$  by Hensel's lemma. Since  $t - \bar{c}_A(\sigma_U(x)) \in G_A$  by (23) and  $G_A \in K^\times/\mathbf{P}_N^\times$ , it follows that  $t - c_A(x) \in G_A$ . So  $(x, t) \in A$  which proves that  $E_B \subseteq A$ .

It remains to check that  $E_B \triangleleft^n A_U$ , and to find a  $\triangleleft^n$ -transition for  $(E_B, A_U)$ . For every  $(x, t) \in E_B$  let:

$$\omega_B(x, t) = \pi^{-n} \left( \frac{t - c_A(x)}{t - \bar{c}_A(\sigma_U(x))} - 1 \right)$$

By (25)  $\omega_B$  takes values in  $\pi^{n_1-n}R$  hence in  $R$  since  $n_1 \geq n$ , thus for every  $(x, t) \in E_B$ :

$$t - c_A(x) = \mathcal{U}_n(x, t)(t - \bar{c}_A(\sigma_U(x))) \quad (26)$$

with  $\mathcal{U}_n = 1 + \pi^n \omega_B$  in this case. We have  $B \subseteq \partial_{\varphi|_{T_U}}^1 A$  and by (P3)  $\mathcal{B} \triangleleft^n \mathcal{A}|_{\mathcal{T}}$ . Since  $\mathcal{A}$  is a closed  $\triangleleft^n$ -complex this implies that for some  $A' \in \mathcal{A}$  we have  $B \triangleleft^n A' \triangleleft^n \partial_{\varphi|_{T_U}}^1 A$ . Let  $h_0 \in \mathcal{F}_0$  be a  $\triangleleft^n$ -transition function for  $(A', \partial_{\varphi|_{T_U}}^1 A)$ , and  $h_1 \in \mathcal{F}_B$  a  $\triangleleft^n$ -transition function for  $(B, A')$ . Then for some  $\alpha_0, \alpha_1 \in \{0, 1\}$  and every  $(x', t')$  in  $B$  we have

$$t' - \bar{c}_A(x') = \mathcal{U}_n(x', t') h_0^{\alpha_0}(x')(t' - c_{A'}(x'))^{1-\alpha_0}$$

and

$$t' - c_{A'}(x') = \mathcal{U}_n(x', t') h_1^{\alpha_1}(x')(t' - c_B(x'))^{1-\alpha_1}$$

hence  $t' - \bar{c}_A(x') = \mathcal{U}_n(x', t') h(x')^\alpha (t' - c_B(x'))^{1-\alpha}$  with  $h = h_0^{1-\alpha_0} h_1^{(1-\alpha_0)\alpha_1}$  and  $\alpha = \alpha_0 + \alpha_1 - \alpha_0\alpha_1$ . So  $h$  is a  $\triangleleft^n$ -transition function for  $(B, \partial_{\varphi|_{T_U}}^1 A)$ . Moreover  $h_0 \circ \varphi|_{T_U}$  and  $h_1 \circ \varphi|_{T_U}$  are  $N$ -monomial mod  $U_{e,n}$  by (P4), hence so is  $h \circ \varphi|_{T_U}$ . For every  $(x, t)$  in  $E_B$ ,  $(\sigma_U(x), t) \in B$  so

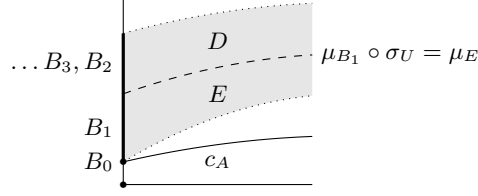
$$t - \bar{c}_A(\sigma_U(x)) = \mathcal{U}_n(x, t) h(\sigma_U(x))^\alpha [t - c_B(\sigma_U(x))]^{1-\alpha}.$$

Combining this with (26) and the definition of  $c_{E_B} = c_B \circ \sigma_U$  we get

$$t - c_A(x) = \mathcal{U}_n(x, t) h(\sigma_U(x))^\alpha [t - c_{E_B}(x)]^{1-\alpha}.$$

So  $E_B \triangleleft^n A_U$  and  $h \circ \sigma_U$  is a  $\triangleleft^n$ -transition for  $(E_B, A_U)$ . Moreover  $h \circ \sigma_U \circ \varphi|_U = h \circ \varphi \circ \pi_U$  by definition of  $\sigma_U$ . The coordinate projection  $\pi_U$  of  $U$  onto  $T_U$  is obviously 1-monomial, and  $h \circ \varphi|_{T_U}$  is  $N$ -monomial mod  $U_{e,n}$  by construction. So  $h \circ \sigma_U \circ \varphi$  is also  $N$ -monomial mod  $U_{e,n}$  and we can take  $h_{E_B, A_U} = h \circ \sigma_U$ .  $\blacksquare$

**Case 2.3:**  $0 = |\bar{\nu}_A| < |\bar{\mu}_A|$  on  $\varphi(T_U)$  and  $\nu_A \neq 0$ .



Let  $B_0 = B_{T_U}^0$  and  $B_1 = B_{T_U}^1$  the two cells in  $\mathcal{B}$  given by claim 6.3. Let:

$$E = (c_A, \nu_A, \mu_{B_1} \circ \sigma_U, G_A)|_{\varphi(U)}$$

If  $|\mu_{B_1}| = |\bar{\mu}_A|$  on  $\varphi(T_U)$  then  $|\mu_{B_1} \circ \sigma_U| = |\mu_A|$  on  $\varphi(U)$  by (21). Thus  $E$  and  $A_U$  have the same underlying set. In this case we let  $\mathcal{E}_{A,U} = \{E\}$  and properties (Pres), (Mon),  $(\partial 3)$  are trivially true. So is (out), using Remark 6.7 for  $c_A \circ \varphi|_U$ ,  $\nu_A \circ \varphi|_U$ , and (P4) for  $\mu_{B_1} \circ \sigma_U \circ \varphi|_U = \mu_{B_1} \circ \varphi|_{T_U}$ .

Otherwise  $|\mu_{B_1}| < |\bar{\mu}_A|$  on  $\varphi(T_U)$  by Claim 6.3 and we let:

$$D = (c_A, \pi^{-N_0} \mu_{B_1} \circ \sigma_U, \mu_A, G_A)|_{\varphi(U)}$$

$|\mu_{B_1}| \leq |\pi^{N_0} \bar{\mu}_A|$  on  $\varphi(T_U)$  by Claim 6.3,  $|\bar{\mu}_A \circ \sigma_U| = |\mu_A|$  on  $\varphi(U)$  by (21), so  $|\nu_D| = |\pi^{-N_0} \mu_{B_1} \circ \sigma_U| \leq |\bar{\mu}_A \circ \sigma_U| \leq |\mu_A| = |\mu_D|$  on  $\varphi(U)$ . Moreover  $A$  is a fitting cell hence for every  $x \in \varphi(U)$  there is  $t \in K$  such that  $(x, t) \in A$  and  $|t - c_A(x)| = |\mu_A(x)|$ , so  $(x, t) \in D$ . Thus  $D$  is indeed a cell, with socle  $\varphi(U)$ . It is actually a largely continuous cell, and  $\mu_D = \mu_A$  is a fitting bound. Let us check that  $\nu_D = \pi^{-N_0} \mu_{B_1} \circ \sigma_U$  is a fitting bound too.  $B_1$  is a fitting cell of type 1 with socle  $\varphi(T_U)$  hence  $\mu_{B_1}(\varphi(T_U)) \subseteq vG_{B_1}$  by Proposition 5.5. But  $G_{B_1} = G_A$  by Claim 6.3,  $G_A = G_D$  and  $\varphi(T_U) = \sigma_U(\varphi(U))$  by construction, and  $N_0 \in v\mathbf{G}$  so  $\nu_D(\varphi(U)) \subseteq vG_D$ . Thus  $\nu_D$  is indeed a fitting bound by Proposition 5.5.

Clearly  $A_U$  is the disjoint union of  $E$  and  $D$ . Moreover the cells in  $\mathcal{B}$  contained in  $\bar{D} \cap (\varphi(T_U) \times K)$  are exactly those contained in  $\bar{A} \cap (\varphi(T_U) \times K)$  except  $B_0$  and  $B_1$ . Thus the construction that we have done for  $A_U$  in case 2.2 applies to  $D$  because  $\bar{\nu}_D = \pi^{-N_0} \mu_{B_1} \neq 0$  on  $\varphi(T_U)$  and because the analogues of conditions (18) and (19) that we used for  $A_U$  in case 2.2 hold for  $D$  in the present case. Indeed by (20) we have

$$|c_{A_U}(x) - \bar{c}_{A_U}(\sigma_U(x))| \leq |\pi^{n_1 - N_0} \bar{\mu}_{B_1}(\sigma_U(x))|.$$

This is just condition (18) for  $D$  since  $c_D = c_{A_U}$  and  $\nu_D = \pi^{-N_0} \mu_{B_1}$ . Moreover condition (19) for  $D$  is:

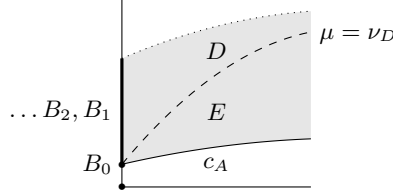
$$|\nu_D| = |\bar{\nu}_D \circ \sigma_U| \quad \text{and} \quad |\mu_D| = |\bar{\mu}_D \circ \sigma_U|$$

The first equality is true by definition of  $\nu_D$  as  $\pi^{-N_0} \mu_{B_1} \circ \sigma_U$ . The second one is true because  $\mu_D = \mu_A$  and because of (21).

So the construction of Case 2.2 gives a partition  $\mathcal{E}'$  of  $D$  and for each  $E' \in \mathcal{E}'$  a semi-algebraic function<sup>11</sup>  $h_{E',A_U} : \varphi(U) \rightarrow K$  satisfying conditions (Pres),  $(\partial 3)$ , (out) and (Mon). Since  $E$  also has these properties (with  $h_{E,A_U} = 1$  since  $c_E = c_A$  on  $\varphi(U)$ ) we can take  $\mathcal{E}_{A,U} = \{E\} \cup \mathcal{E}'$ .

<sup>11</sup>Case 2.2 applied to  $D$  actually gives for each  $E' \in \mathcal{E}'$  a  $\triangleleft^n$ -transition  $h_{E',D}$  for  $(E', D)$ . But  $D \subseteq A_U$  and  $c_D = c_{A_U}$  so  $h_{E',D}$  is also a  $\triangleleft^n$ -transition for  $(E', A_U)$  and we can set  $h_{E',A_U} = h_{E',D}$ .

**Case 2.4:**  $\bar{\mu}_A \neq 0$  on  $\varphi(T_U)$  and  $\nu_A = 0$ .



Let again  $B_1 = B_{T_U}^1$  be the cell given by claim 6.3. We are going to split  $A_U$  in two cells  $E$  and  $D$  to which previous cases apply. In order to do so, choose any  $i \in \text{Supp } U \setminus \text{Supp } T_U$ . For every  $u \in U$  let  $\xi_i(u) = u_i$ , the  $i$ -th coordinate of  $u$ . Clearly  $\xi_i$  is largely continuous and  $\bar{\xi}_i = 0$  on  $\partial U = \bar{T}_U$ . So the function:

$$\mu = (\xi_i \circ \varphi^{-1})^N \cdot (\mu_{B_1} \circ \sigma_U)$$

is largely continuous on  $\widehat{A_U} = \varphi(U)$  and  $\bar{\mu} = 0$  on  $\varphi(T_U)$ , hence also on  $\overline{\varphi(T_U)} = \partial\varphi(U)$ . Note that  $\mu \circ \varphi|_U$  is  $N$ -monomial mod  $U_{e,n}$ . Let:

$$E = (c_A, 0, \pi^{N_0} \mu, G_A)|_{\varphi(U)}$$

$$D = (c_A, \mu, \mu_A, G_A)|_{\varphi(U)}$$

$E$  and  $D$  are largely continuous fitting cells mod  $\mathbf{G}$  which define a partition of  $A_U$ . (Here we use that  $A$  is a fitting cell: for every  $x \in \varphi(U)$  there is  $t \in K$  such that  $(x, t) \in A$  and  $|t - c_A(x)| = |\mu_A(x)|$  so  $(x, t) \in D$ , which proves that  $D$  is really a cell. That  $D, E$  are fitting cells and  $A_U = E \cup D$  then follows from Proposition 5.5.) In particular  $E$  satisfies condition (Pres). Since  $\nu_E = 0$  and  $\bar{\mu}_E = \pi^{N_0} \bar{\mu} = 0$  on  $\partial\varphi(U)$ , we have  $\partial E = \overline{\text{Gr } c_E}$ . By Remark 6.6,  $\text{Gr } c_{A_U} = C_U$  for some  $C \in \mathcal{A}_S$ , and by Sub-case 2.1.1 applied to  $C_U$ ,  $\text{Gr } c_{A_U} \in \mathcal{E}_{C,U}$ . This proves  $(\partial 2)$  for  $E$  since  $c_E = c_{A_U}$ . Let  $h_{E,A_U} = 1$ , this is a  $\triangleleft^n$ -transition for  $(E, A_U)$  since they have the same center, so  $E$  satisfies (out). It also satisfies (Mon), thanks to Remark 6.7 for  $c_E = c_{A_U}$  and because  $\mu_E \circ \varphi|_U = \pi^{N_0} \mu \circ \varphi|_U$  is  $N$ -monomial mod  $U_{e,n}$ .

Case 2.3 applies to  $D$  because  $\nu_D = \mu \neq 0$ ,  $|\bar{\nu}_D| = |\bar{\mu}| = 0$  on  $\varphi(T_U)$  and  $|\bar{\mu}_D| = |\bar{\mu}_A| \neq 0$  on  $\varphi(T_U)$ , and because the analogues of conditions (20) and (21) that we used for  $A_U$  in case 2.3 hold for  $D$  in the present case. Indeed (20) holds for  $D$  because it holds for  $A_U$ , and because  $D$  and  $A_U$  have the same center. Condition (21) for  $D$  is:

$$|\nu_D| \leq |\mu_{B_1} \circ \sigma_U| \leq |\bar{\mu}_D \circ \sigma_U| = |\mu_D|$$

The first inequality is true because  $|\nu_D| = |\mu| \leq |\mu_{B_1} \circ \sigma_U|$  by construction, the second one is true by claim 6.3 and because  $\mu_D = \mu_A$ , and the last equality is true because it is true for  $A_U$  by (21) and because  $\mu_D = \mu_{A_U} = \mu_A|_{\varphi(U)}$ .

So the construction of case 2.3 gives a partition  $\mathcal{E}'$  of  $D$  and for each  $E' \in \mathcal{E}'$  a semi-algebraic function<sup>12</sup>  $h_{E',A_U} : \varphi(U) \rightarrow K$  satisfying conditions (Pres), (Fron), (out) and (Mon). Since  $E$  also has these properties we can take  $\mathcal{E}_{A,U} = \{E\} \cup \mathcal{E}'$ .

<sup>12</sup>Same remark as in footnote 11.



## 7 Cartesian morphisms

Let  $\mathcal{A}$  be a cellular monoplex mod  $\mathbf{G}$  such that  $\bigcup \mathcal{A}$  is a closed subset of  $R^{m+1}$ . Let  $(\mathcal{U}, \psi)$  be a triangulation of  $\text{CB}(\mathcal{A})$  with parameters  $(n, N, e, M)$  such that for every  $A \in \mathcal{A}$ ,  $\psi^{-1}(A) \in \mathcal{U}$  (we will denote it  $U_A$ ). Note that this is essentially the data given by the conclusion of Lemma 6.1. The aim of this section is to build a triangulation  $(\mathcal{S}, \varphi)$  of  $\mathcal{A}$  with the same parameters  $(n, N, e, M)$ , together with a continuous projection  $\Phi : \bigsqcup \mathcal{S} \rightarrow \bigsqcup \mathcal{U}$  such that the following diagram is commutative.

$$\begin{array}{ccc} \bigcup \mathcal{A} & \xleftarrow{\varphi} & \bigsqcup \mathcal{S} \\ \downarrow & & \downarrow \Phi \\ \bigcup \widehat{\mathcal{A}} & \xleftarrow{\psi} & \bigsqcup \mathcal{U} \end{array}$$

In order to do so we will make the assumption that  $\mathbf{G} = Q_{N, M'}$  with  $M' = M + v(N)$  and  $M > v(N)$ . In addition we temporarily assume that  $\mathcal{A}$  is a rooted tree, and  $\mathcal{U}$  a simplicial complex in  $D^M R^{q_1}$  for some  $q_1$ . We keep these data and assumptions until the end of this section, where we finally state our result in a more precise and slightly more general form.

The construction is done below through a series of claims whose guiding lines is the following. We first book some place  $F_{H(A)}(D^M R^{q_A})$  for each  $A \in \mathcal{A}$ . So we consider  $\mathcal{A}$  and  $\mathcal{U}$  as trees, ordered by specialization, and build a pair of trees of finite subsets of  $\mathbf{N}^*$  ordered by inclusion,  $\mathcal{H} = (H(A))_{A \in \mathcal{A}}$  and  $\mathcal{P} = (P(A))_{A \in \mathcal{A}}$ , such that  $\mathcal{H}$  is isomorphic to  $\mathcal{A}$  and  $\mathcal{P}$  is isomorphic to  $\mathcal{U}$  (Claim 7.3). Then we allocate this place  $F_{H(A)}(D^M R^{q_A})$  to a simplex  $S_A$  which we build in it, together with a semi-algebraic isomorphism  $\varphi_A$  and a semi-algebraic projection  $\Phi_A$  such that the following diagram is commutative (Claim 7.7).

$$\begin{array}{ccc} A & \xleftarrow{\varphi_A} & S_A \\ \downarrow & & \downarrow \Phi_A \\ \widehat{A} & \xleftarrow{\psi|_{U_A}} & U_A \end{array}$$

It is then the shape of the trees  $\mathcal{H}$  and  $\mathcal{P}$  which controls how the various pieces glue together, so that  $\mathcal{S} = (S_A)_{A \in \mathcal{A}}$  is a simplicial complex (Claim 7.8) and the resulting maps  $\varphi, \Phi$  are continuous on  $\bigsqcup \mathcal{S}$  (Claims 7.5 and 7.9).

**Claim 7.1.** *The faces of  $U_A$  are exactly the sets  $U_B$  with  $B \leq A$  in  $\mathcal{A}$ .*

*Proof:* Let  $B \leq A$  in  $\mathcal{A}$ ,  $Y = \widehat{B}$  and  $i = \text{tp } B$ . Then, with the notation of Section 5,  $B = \partial_Y^i A$  because  $\mathcal{A}$  is a cellular complex. Since  $A$  is bounded, the socle of  $\widehat{A}$  is closed hence  $Y$  must be contained in it. Since  $\psi^{-1}(Y) = U_B$ , it follows that  $U_B$  is a face of  $U_A$ . Conversely for every face  $V$  of  $U_A$ , the set  $B = \partial_{\psi(V)}^0 A$  (resp.  $B = \partial_{\psi(V)}^1 A$ ) is non-empty if  $\bar{\nu}_{A|Y} = 0$  (resp.  $\bar{\mu}_{A|Y} \neq 0$ ) hence belongs to  $\mathcal{A}$ . One of these two cases necessarily happens (because  $|\bar{\nu}_A| \leq |\bar{\mu}_A|$  on  $Y$ ), which gives  $B \in \mathcal{A}$  such that  $U_B = V$ . ■

**Claim 7.2.** *Given any two cells  $B \leq A$  in  $\mathcal{A}$ ,  $B < A$  if and only if either  $U_B < U_A$  or  $\text{tp } B < \text{tp } A$ . In particular if  $B$  is the predecessor of  $A$  in  $\mathcal{A}$  then either  $U_B$  is the facet of  $U_A$ , or  $U_B = U_A$ , in which case  $\text{tp } B = 0$  and  $\text{tp } A = 1$*

*Proof:* Recall that  $B = \partial_Y^j A$  with  $Y = \widehat{B} = \psi(U_B)$  and  $j = \text{tp } B$ . In particular  $A = \partial_X^i A$  with  $X = \widehat{A}$  and  $i = \text{tp } A$ . Thus  $B \neq A$  if and only if  $U_B \neq U_A$  or  $\text{tp } B \neq \text{tp } A$ . Since  $U_B \leq U_A$  by the previous claim, and obviously  $\text{tp } B \leq \text{tp } A$  (otherwise  $\partial_Y^j A = \emptyset$ ) this proves the equivalence. In particular if  $U_B = U_A$  then  $\text{tp } B < \text{tp } A$  hence  $\text{tp } B = 0$  and  $\text{tp } A = 1$ .

If  $B$  is the predecessor of  $A$  in  $\mathcal{A}$  and  $U_B \neq U_A$ , then  $U_B < U_A$  by Claim 7.1. Let  $V$  be the facet of  $U_A$ . Then  $U_B \leq V < A$  hence  $B \leq \partial_{\psi(V)}^j A < A$ . On the other hand  $B$  is the predecessor of  $A$  in  $\mathcal{A}$ , hence  $B = \partial_{\psi(V)}^j A$ . So  $\widehat{B} = \psi(V)$  and finally  $U_B = V$ .

■

Given a strictly increasing map  $\sigma : I \rightarrow J$  with  $I \subseteq \llbracket 1, r \rrbracket$  and  $J \subseteq \llbracket 1, s \rrbracket$ , we let  $[\sigma] : K^s \rightarrow K^r$  be defined by  $[\sigma](y) = u$  where  $u_i = y_{\sigma(i)}$  if  $i \in I$ , and  $u_i = 0$  otherwise. We say that an application  $f : S \subseteq K^r \rightarrow K^s$  is a **Cartesian map** if for every  $I \subseteq \llbracket 1, r \rrbracket$  the restriction of  $f$  to  $S \cap F_I(K^r)$  is of that form, that is if there is  $J \subseteq \llbracket 1, s \rrbracket$  and a strictly increasing map  $\sigma : I \rightarrow J$  such that  $f(y) = [\sigma](y)$  for every  $y \in S$  with support  $I$ . If  $X$  is the disjoint union of finitely many sets  $X_k \subseteq K^{r_k}$  for various  $k$ , then a Cartesian map on  $X$  is simply the data of a Cartesian map on each  $X_k$ . A **Cartesian morphism** is a continuous Cartesian map.

**Claim 7.3.** *There exists a pair of applications  $H, P$  from  $\mathcal{A}$  to  $\mathcal{P}(\mathbf{N}^*)$  such that  $H$  is strictly increasing and for every  $B \leq A$  in  $\mathcal{A}$ :*

(C0) *If  $\text{tp } A = 0$  then  $H(A) = P(A)$ .*

(C1) *If  $\text{tp } A = 1$  then  $H(A) = P(A) \cup \{r_A\}$  for some  $r_A > \max P(A)$ .*

(C2)  $\text{Card } P(A) = \text{Card}(\text{Supp } U_A)$ .

(C3)  $P(B) = H(B) \cap P(A)$  (in particular  $P$  is increasing and  $P(B) \subseteq H(B)$ ).

(C4) *If  $\sigma_A : \text{Supp } U_A \rightarrow P(A)$  denotes the increasing bijection given by (C2) then  $\sigma_A(\text{Supp } U_B) = P(B)$ .*

**Remark 7.4.** Since  $\sigma_A$  and  $\sigma_B$  are strictly increasing, (C4) implies that  $\sigma_A(i) = \sigma_B(i)$  for every  $i \in \text{Supp } U_B$ .

*Proof:* The construction goes by induction in  $\text{Card } \mathcal{A}$ . For the root  $A$  of  $\mathcal{A}$  we let  $P(A) = \text{Supp } U_A$ ,  $H(A) = P(A)$  if  $\text{tp } A = 0$ , and  $H(A) = P(A) \cup \{q_1 + 1\}$  if  $\text{tp } A = 1$  (recall that  $\mathcal{U}$  is a simplicial complex in  $D^M R^{q_1}$ ). If  $\mathcal{A} = \{A\}$  we are done. Otherwise let  $A$  be a maximal element of  $\mathcal{A}$  and apply the induction hypothesis to  $\mathcal{A} \setminus \{A\}$ . This defines  $P(A')$ ,  $H(A')$  for every  $A' \in \mathcal{A} \setminus \{A\}$  so that  $H$  is strictly increasing on  $\mathcal{A} \setminus \{A\}$  and properties (C0) to (C4) hold true for every  $B' \leq A'$  in  $\mathcal{A} \setminus \{A\}$ .

Let  $B$  be the predecessor of  $A$  in  $\mathcal{A}$  and  $k = \text{Card}(\text{Supp } U_A \setminus \text{Supp } U_B) + 1$ . For every  $A' \in \mathcal{A} \setminus \{A\}$  let  $P_k(A') = \{ki\}_{i \in P(A')}$  and  $H_k(A') = \{ki\}_{i \in H(A')}$ . Clearly  $P_k$  and  $H_k$  inherit all the properties of  $P$  and  $H$ . Thus, replacing if necessary  $P$  and  $H$  by  $P_k$  and  $H_k$  we can assume that  $H(A') \subseteq k\mathbf{N}^*$  for every  $A' \in \mathcal{A} \setminus \{A\}$ .

Let  $q'$  be the maximum of the integers in all these sets  $H(A')$ . We have to define  $P(A)$  and  $H(A)$  so that the resulting maps  $P, H$  satisfy (C0) to (C4) for every  $B' \leq A'$  in  $\mathcal{A}$ ,  $H(B') \subseteq H(A')$  and  $H(B') \neq H(A')$  if  $B' \neq A'$ .

By induction hypothesis it suffices to check these properties when  $A' = A$  and  $B' = B$ .

We are going to build  $\sigma_A$  first, and then let  $P(A) = \sigma_A(\text{Supp } U_A)$ . Let  $j_1 < \dots < j_r$  be an enumeration of  $\text{Supp } U_B$ . Let  $j_0 = 0$  and  $j_{r+1} = q' + 1$ . For every  $i \in \text{Supp } U_A$  there is a unique  $l \in \llbracket 0, r \rrbracket$  such that  $j_l \leq i < j_{l+1}$ . We then let  $\sigma_A(i) = \sigma_B(j_l) + i - j_l$  (if  $j_l = j_0 = 0$  we let  $\sigma_B(j_l) = 0$  in this definition). Note that  $j_l + k \leq j_{l+1}$  and  $\sigma_B(j_{l+1}) \in P(B) \subseteq k\mathbf{N}^*$  hence

$$\sigma_A(j_l) \leq \sigma_A(i) < \sigma_B(j_l) + j_{l+1} - j_l \leq \sigma_B(j_l) + k \leq \sigma_B(j_{l+1}).$$

It follows immediately that  $\sigma_A$  is strictly increasing. Let  $P(A) = \sigma_A(\text{Supp } U_A)$ , by construction (C2) and (C4) hold true,  $P(A) \cap k\mathbf{N}^* = P(B)$  and  $P(B)$  is strictly contained in  $P(A)$  except if  $\text{Supp } U_A = \text{Supp } U_B$ . Note also that in any case  $q' + k \notin H(B) \cup P(A)$ . In order to define  $H(A)$  we distinguish four cases, given by by Claim 7.2.

**Case 1:**  $\text{tp } A = 0$ , hence  $\text{tp } B = 0$  and  $U_B$  is the facet of  $A$ . In particular  $\text{Supp } U_B$  is strictly contained in  $\text{Supp } U_A$ , hence so is  $P(B)$  in  $P(A)$ . By induction hypothesis (C0),  $H(B) = P(B)$ . Let  $H(A) = P(A)$ , then  $H(B) \subseteq H(A)$ ,  $H(B) \neq H(A)$  and (C0), (C3) are obvious.

**Case 2:**  $\text{tp } A = 1$ ,  $\text{tp } B = 0$  and  $U_B = U_A$ . Then  $P(B) = P(A)$  by construction, and  $P(B) = H(B)$  by induction hypothesis (C0). Let  $H(A) = H(B) \cup \{q' + k\}$ , then  $H(B) \subseteq H(A)$ ,  $H(B) \neq H(A)$  and (C1) are obvious because  $q' + k \notin H(B)$ , and  $H(B) \cap P(A) = P(B) \cap P(A) = P(B)$  which proves (C3).

**Case 3:**  $\text{tp } A = 1$ ,  $\text{tp } B = 0$  and  $U_B$  is the facet of  $U_A$ . We let  $H(A) = P(A) \cup \{q' + 1\}$ . By induction hypothesis (C0)  $H(B) = P(B)$ . By construction  $P(B) \subseteq P(A) \subseteq H(A)$ . So  $H(B) \subseteq H(A)$ ,  $H(B) \neq H(A)$  and (C1) are obvious because  $q' + k \notin H(B) \cup P(A)$ . As in Case 2,  $H(B) \cap P(A) = P(B) \cap P(A) = P(B)$  which proves (C3).

**Case 4:**  $\text{tp } A = \text{tp } B = 1$  and  $U_B$  is the facet of  $U_A$ . By induction hypothesis (C1),  $P(B)$  is strictly contained in  $P(A)$ . Let  $H(A) = P(A) \cup H(B)$ . Then  $H(B) \subseteq H(A)$ ,  $H(B) \neq H(A)$  because  $H(A) \setminus H(B) = P(A) \setminus k\mathbf{N}^* = P(A) \setminus P(B) \neq \emptyset$ ,  $H(A) \cap P(B) = (P(A) \cap P(B)) \cup (H(B) \cap P(B)) = P(B) \cup P(B) = P(B)$  which proves (C3), and (C1) follows because then  $H(A) \setminus P(A) = H(B) \setminus P(A) = H(B) \setminus P(B)$  is a singleton by induction hypothesis (C1).

■

With the notation of Claim 7.3, let  $q_2$  be the maximal element of  $\bigcup_{A \in \mathcal{A}} H(A)$  and  $\mathcal{S}^\dagger = \{F_{H(A)}(D^M R^{q_2})\}_{A \in \mathcal{A}}$ . For every  $A \in \mathcal{A}$  let  $\Phi_A = [\sigma_A] : F_{H(A)}(D^M R^{q_2}) \rightarrow D^M R^{q_1}$ . Finally let  $\Phi : \bigcup \mathcal{S}^\dagger \rightarrow D^M R^{q_1}$  be the resulting Cartesian map.

**Claim 7.5.**  $\Phi$  is continuous, hence a Cartesian morphism.

*Proof:* We have to show that for every  $T \leq S$  in  $\mathcal{S}^\dagger$  and every  $z \in T$ ,  $\Phi(y)$  tends to  $\Phi(z)$  as  $y$  tends to  $z$  in  $S$ . By construction there are  $A, B$  in  $\mathcal{A}$  such that  $H(A) = \text{Supp } S$ ,  $H(B) = \text{Supp } T$ ,  $\Phi(y) = [\sigma_A](y)$  and  $\Phi(z) = [\sigma_B](z)$ .

Since  $[\sigma_A]$  is obviously continuous, it tends to  $[\sigma_A](z)$  so we have to prove that  $[\sigma_A](z) = [\sigma_B](z)$ . Let  $u = [\sigma_A](z)$  and  $u' = [\sigma_B](z)$ . Recall that  $u, u' \in D^M R^{q_1}$  and for every  $i \in \llbracket 1, q_1 \rrbracket$ ,  $u_i = z_{\sigma_A(i)}$  if  $i \in \text{Supp } U_A$ ,  $u_i = 0$  otherwise,  $u'_i = z_{\sigma_B(i)}$  if  $i \in \text{Supp } U_B$ , and  $u'_i = 0$  otherwise.

Since  $T \leq S$  we have  $\text{Supp } T \leq \text{Supp } S$ , that is  $H(B) \leq H(A)$ , hence  $B \leq A$  since  $H$  is strictly increasing. In particular  $\text{Supp } U_B \subseteq \text{Supp } U_A$  hence for every  $i \in \llbracket 1, q_1 \rrbracket$ , we have  $u_i = u'_i = 0$  if  $i \notin \text{Supp } U_A$ , and by Remark 7.4  $z_{\sigma_A(i)} = z_{\sigma_B(i)}$  if  $i \in \text{Supp } U_B$ , that is  $u_i = u'_i$  in this case too. The remaining case occurs when  $i \in \text{Supp } U_A \setminus \text{Supp } U_B$ , so that  $u_i = z_{\sigma_A(i)}$  and  $u'_i = 0$ . We have to prove that  $z_{\sigma_A(i)} = 0$ , that is  $\sigma_A(i) \notin \text{Supp } z$ . By (C4) and the assumption on  $i$ ,  $\sigma_A(i) \in P(A) \setminus P(B)$ . By (C3),  $P(A) \setminus P(B) = P(A) \setminus H(B)$ . So  $\sigma_A(i) \notin H(B)$ , and we are done since  $\text{Supp } z = \text{Supp } T = H(B)$ .

■

For every  $A \in \mathcal{A}$ ,  $\mu_A \circ \psi$  is  $N$ -monomial mod  $U_{e,n}$  so there are  $\zeta \in K$  and some integers  $\beta_{i,A}$  for  $i \in \text{Supp } U_A$  such that for every  $u \in U_A$

$$\mu_A \circ \psi(u) = U_{e,n}(u) \cdot \zeta \cdot \prod_{i \in \text{Supp } U_A} u_i^{N\beta_{i,A}}.$$

If  $\mu_A \neq 0$  then  $v\mu_A(\widehat{A}) = vG_A = v\lambda_A + N\mathcal{Z}$  by Proposition 5.5, and by the above displayed equation  $v(\zeta) \equiv v(\lambda_A) [N]$ . So there is  $\beta_{0,A} \in \mathcal{Z}$  such that  $v(\zeta) = v(\lambda_A) + N\beta_{0,A}$ . Let  $\mu_A^v : vU_A \rightarrow \mathcal{Z}$  be defined by<sup>13</sup>  $\mu_A^v(a) = M' + \beta_{0,A} + \sum_{i \in \text{Supp } U_A} \beta_{i,A} a_i$ . If  $\mu_A = 0$  then we let  $\mu_A(a) = +\infty$  for every  $a \in vU_A$ . Define  $\nu_A^v$  accordingly. By construction, for every  $u \in U_A$  we have

$$v\mu_A(\psi(u)) = v\lambda_A + N\mu_A^v(vu) - NM' \quad (27)$$

$$v\nu_A(\psi(u)) = v\lambda_A + N\nu_A^v(vu) - NM' \quad (28)$$

In particular  $\mu_A^v$  (resp.  $\nu_A^v$ ) is uniquely determined by  $\mu_A$  (resp.  $\nu_A$ ), even if the coefficients  $\beta_{i,A}$  are not.

**Remark 7.6.** Since  $A$  is a fitting cell mod  $Q_{N,M'}$  contained in  $R$ ,  $v\mu_A + M' \geq 0$  by Proposition 5.6. On the other hand  $0 \leq v\lambda_A \leq N - 1$  (see Section 2). So, for every  $u \in U_A$  we have by (27):

$$\begin{aligned} \mu_A^v(vu) &= v\mu_A(\psi(u)) + NM' - v\lambda_A \\ &\geq -M' + NM' - (N - 1) \\ &= (N - 1)(M' - 1) \geq 0 \end{aligned}$$

Let  $S_A \subseteq D^M R^{q_2}$  be defined as follows.

- If  $\text{tp } A = 0$ ,  $S_A$  is the set of  $y \in F_{H(A)}(D^M R^{q_2}) = F_{P(A)}(D^M R^{q_2})$  such that  $\Phi(y) \in U_A$ .
- If  $\text{tp } A = 1$ ,  $S_A$  is the set of  $y \in F_{H(A)}(D^M R^{q_2})$  such that  $\Phi(y) \in U_A$  and  $\mu_A^v(v\Phi(y)) \leq v y_{r_A} \leq \nu_A^v(v\Phi(y))$ .

In both cases, for every  $y \in S_A$  let

$$\varphi_A(y) = (\psi \circ \Phi(y), c_A(\psi \circ \Phi(y)) + \pi^{-NM'} \lambda_A y_{r_A}^N)$$

<sup>13</sup>We remind the reader that  $A$  is a cell mod  $Q_{N,M'}$  with  $M' = M + v(N)$ .

where  $r_A = \max H(A)$  (if  $H(A) = \emptyset$ , which happens when  $A$  is a point, then  $r_A$  is not defined but in that case  $\text{tp } A = 0$ , hence  $\lambda_A = 0$  and we can let  $\lambda_A y_{r_A}^N = 0$  by convention).

**Claim 7.7.**  $\Phi(S_A) = U_A$  and  $\varphi_A$  is a bijection from  $S_A$  to  $A$ .

*Proof:* If  $\text{tp } A = 0$  the result is trivial because in that case  $H(A) = P(A)$  hence the restriction of  $\Phi$  to  $F_{H(A)}(D^M R^{q_2})$  is a bijection onto  $F_{\text{Supp } U_A}(D^M R^{q_1})$ . So from now we assume that  $\text{tp } A = 1$ , hence  $H(A) = P(A) \cup \{r_A\}$  and  $r_A \notin P(A)$  by (C1).

Let  $y, y' \in S_A$  be such that  $\varphi_A(y) = \varphi_A(y')$ . Then  $\psi(\Phi(y)) = \psi(\Phi(y'))$  and

$$c_A(\psi \circ \Phi(y)) + \pi^{-NM'} \lambda_A y_{r_A}^N = c_A(\psi \circ \Phi(y')) + \pi^{-NM'} \lambda_A y_{r_A}'^N$$

These two equations imply that  $y_{r_A}^N = y_{r_A}'^N$ . Since  $y_{r_A}, y_{r_A}' \in D^M R = Q_{1,M} \cap R$  and  $M > v(N)$  it follows that  $y_{r_A} = y_{r_A}'$  by Lemma 2.9. On the other hand  $\psi(\Phi(y)) = \psi(\Phi(y'))$  implies  $\Phi(y) = \Phi(y')$  (because  $\psi$  is one-to-one), that is  $y_i = y_i'$  for every  $i \in P(A)$  (because  $\Phi(y) = [\sigma_A](y)$  by construction). Thus  $y_i = y_i'$  for every  $i \in P(A) \cup \{r_A\} = H(A)$ , that is  $y = y'$  since  $\text{Supp } y = \text{Supp } y' = H(A)$ . This proves that  $\varphi_A$  is one-to-one.

Let us check now that  $A \subseteq \varphi_A(S_A)$ . Pick any  $(x, t) \in A$ , let  $u = \psi^{-1}(x)$  and  $\delta = t - c_A(x)$ . Since  $\delta \in \lambda_A Q_{N,M'}$  and  $\pi^{NM'} \in Q_{N,M'}$  we have  $\pi^{NM'} \delta \in \lambda_A Q_{N,M'}$ . Recall that  $M' = M + v(N)$ , by Lemma 2.9 there is a unique  $z \in Q_{1,M}$  such that  $\pi^{NM'} \delta = \lambda_A z^N$ , hence  $t = c_A(x) + \pi^{-NM'} \lambda_A z^N$ . On the other hand we have  $v\mu_A(\psi(u)) = v\mu_A(x) \leq v\delta$  so by (27)

$$vz = \frac{v(\pi^{NM'} \delta / \lambda_A)}{N} \geq \frac{NM' + v\mu_A(\psi(u)) - v\lambda_A}{N} = \mu_A^v(vu).$$

In particular  $vz \geq 0$  by Remark 7.6 so  $z \in Q_{1,M} \cap R = D^M R$ . Similarly  $vz \leq \nu_A^v(vu)$  by (28). Let  $y \in D^M R^{q_2}$  be such that  $y_i = u_{\sigma_A(i)}$  if  $i \in P(A)$ ,  $y_i = z$  if  $i = r_A$ ,  $y_i = 0$  otherwise. Then  $y \in F_{H(A)}(D^M R^{q_2})$ ,  $\Phi(y) = [\sigma_A](y) = u$  and  $\mu_A^v(vu) \leq v y_{r_A} \leq \nu_A^v(vu)$  since  $y_{r_A} = z$ , so  $y$  belongs to  $S_A$ . By construction  $\varphi_A(y) = (x, t) \in A$ , which proves that  $A \subseteq \varphi_A(S_A)$ .

We turn now to  $\Phi(S_A)$ . For every  $u \in U_A$ ,  $\psi(u) \in \widehat{A}$  so there is  $(x, t) \in A$  such that  $u = \psi^{-1}(x)$ . The above construction gives  $y \in S_A$  such that  $\varphi_A(y) = (x, t)$ . In particular  $\psi \circ \Phi(y) = x$ , so  $\Phi(y) = \psi^{-1}(x) = u$ , which proves that  $\Phi(S_A) \subseteq U_A$ . Since  $\Phi(S_A) \subseteq U_A$  by definition of  $S_A$  we get that  $\Phi(S_A) = U_A$ .

It only remains to show that  $\varphi_A(S_A) \subseteq A$ . Pick any  $y \in S_A$ , let  $(x, t) = \varphi_A(y)$  and  $\delta = t - c_A(x) = \pi^{-NM'} \lambda_A y_{r_A}^N$ . Since  $\Phi(y) \in \Phi(S_A) = U_A$ , we have  $x = \psi(\Phi(y)) \in \psi(U_A) = \widehat{A}$ . Since  $y_{r_A} \in D^M R = Q_{1,M} \cap R$ , by Lemma 2.9  $y_{r_A}^N \in Q_{N, M+v(N)} = Q_{N, M'}$ . Hence  $\delta = \pi^{-NM'} \lambda_A y_{r_A}^N$  belongs to  $\lambda_A Q_{N, M'}$ . We have  $\mu_A^v(v\Phi(y)) \leq v y_{r_A}$  by definition of  $S_A$ , so by (27)

$$v\mu_A(\psi(\Phi(y))) = v\lambda_A + N\mu_A^v(v\Phi(y)) - NM' \leq v\lambda_A + N v y_{r_A} - NM'$$

The left hand side is equal to  $v\mu_A(x)$ . For the right hand side we have

$$v\lambda_A + N v y_{r_A} - NM' = v(\pi^{-NM'} \lambda_A y_{r_A}^N) = v\delta.$$

So  $v\mu_A(x) \leq v\delta$ , that is  $|\delta| \leq |\mu_A(x)|$ . Similarly  $|\nu_A(x)| \leq |\delta|$  hence  $(x, t) \in A$ .

■

**Claim 7.8.**  $S_A$  is a simplex in  $D^M R^{q_2}$ , whose faces are exactly the sets  $S_B$  with  $B \leq A$  in  $\mathcal{A}$ .

*Proof:* Let  $q = \text{Card } P(A)$  and  $q' = \text{Card } H(A)$ . Let  $\tau_A$  (resp.  $\tau'_A$ ) be the strictly increasing map from  $P(A)$  to  $\llbracket 1, q \rrbracket$  (resp. from  $H(A)$  to  $\llbracket 1, q' \rrbracket$ ). By construction and by Claim 7.5 the following diagram is commutative (vertical arrows are the natural coordinate projections).

$$\begin{array}{ccccc}
D^M R^{q'} & \xrightarrow{[\tau'_A]} & \overline{F_{H(A)}(D^M R^{q_2})} & & \\
\downarrow & & \downarrow & \searrow^{[\sigma_A]=\Phi} & \\
D^M R^q & \xrightarrow{[\tau_A]} & \overline{F_{P(A)}(D^M R^{q_2})} & \xrightarrow{[\sigma_A]=\Phi} & \overline{F_{\text{Supp } U_A}(D^M R^{q_1})}
\end{array}$$

The horizontal arrows in this diagram are isomorphisms. All of them are obtained simply by renumbering the coordinates, hence they preserve the faces and the property of being a simplex. It will then be convenient here to identify isomorphic spaces, hence to consider  $U_A \subseteq D^M R^q$  and  $S_A \subseteq D^M R^{q'}$ . Since  $\Phi(S_A) = U_A$  by Claim 7.7, after this identification  $U_A$  is just the image of  $S_A$  by the coordinate projection of  $D^M R^{q'}$  to  $D^M R^q$ . Since  $H(A) = \text{Supp } S_A$  we identify also  $H(A)$  with  $\llbracket 1, q' \rrbracket$ , and  $P(A)$  with  $\llbracket 1, q \rrbracket$ .

If  $\text{tp } A = 0$  then  $H(A) = P(A)$ ,  $q' = q$  and the vertical arrows are identity maps. Thus  $S_A$  identifies with  $U_A$ . In particular  $S_A$  is a simplex. Every  $B \leq A$  is also of type 0 and  $S_B$  identifies to  $U_B$ . The conclusion follows by Claim 7.1.

From now on, let us assume that  $\text{tp } A = 1$ . Then  $q' = q + 1$  hence  $U_A$  is just the socle of  $S_A$ . Similarly,  $U_B$  is the socle of  $S_B$  for every  $B \leq A$  (if  $\text{tp } B = 0$  we have  $S_B = U_B \times \{0\}$ ). By construction  $S_A$  is the inverse image of  $vS_A$  by the valuation (restricted to  $D^M R^{q+1}$ ) and

$$vS_A = \{a \in \mathcal{Z}^{q+1} : \hat{a} \in vU_A \text{ and } \mu_A^v(\hat{a}) \leq a_{q+1} \leq \nu_A^v(\hat{a})\}.$$

Since  $\mu_A \circ \psi$  and  $\nu_A \circ \psi$  are largely continuous on  $U_A$ , (27) and (28) imply that  $\mu_A^v$  is largely continuous on  $vU_A$ . They are affine maps by definition. Since  $0 \leq \mu_A^v$  by Remark 7.6, and  $\mu_A^v \leq \nu_A^v$  because  $|\nu_A| \leq |\mu_A|$ , it follows that  $vS_A$  is a polytope in  $\Gamma^{q+1}$ . We are going to check that its faces are exactly the sets  $vS_B$  for  $B \leq A$  in  $\mathcal{A}$ . This will finish the proof since  $S_A$  will then have the expected faces, which implies that  $S_A$  is a simplex because these faces form a chain by specialisation (because  $\mathcal{A}$  is a tree).

**Step 1.** Let  $B \leq A$  in  $\mathcal{A}$ , then  $B = \partial_Y^i A$  with  $Y = \hat{B}$  and  $i = \text{tp } B$ . Let  $J = H(B) \subseteq H(A) = \llbracket 1, q+1 \rrbracket$  and  $\hat{J} = P(A) = J \setminus \{q+1\}$ . Since  $(\mu_B, \nu_B) = (\bar{\mu}_A, \bar{\nu}_A)|_Y$ , if  $\text{tp } B = 1$  we have by construction

$$vS_B = \{a \in F_J(\Gamma^{q+1}) : \hat{a} \in vU_B \text{ and } \bar{\mu}_A^v(\hat{a}) \leq a_{q+1} \leq \bar{\nu}_A^v(\hat{a})\}. \quad (29)$$

This remains true also if  $\text{tp } B = 0$  because in that case  $q+1 \notin J$  and  $\bar{\nu}_A^v = +\infty$  on  $vU_B$  (because  $\bar{\nu}_A = \nu_B = 0$  on  $Y$ ) so the right hand side is just  $vU_B \times \{+\infty\}$ , that is  $vS_B$  (because  $S_B = U_B \times \{0\}$  when  $\text{tp } B = 0$ ). In both cases we also have  $vU_B = F_{\hat{J}}(U_A)$ , because  $vU_B$  is a face of  $U_A$  by Claim 7.1 and  $\text{Supp } vU_B = P(B) = \hat{J}$ . So the description of  $vS_B$  given by (29) coincides with the description of  $F_J(vS_A)$  given by Proposition 2.13, which proves that  $vS_B = F_J(vS_A)$ .

**Step 2.** Conversely let  $F_J(vS_A) \neq \emptyset$  be a face of  $vS_A$ , for some  $J \subseteq \llbracket 1, q+1 \rrbracket$ , and let  $\hat{J} = J \setminus \{q+1\}$ . By Proposition 2.13 the socle of  $F_J(vS_A)$  is  $F_{\hat{J}}(vU_A)$  (because  $vU_A$  is the socle of  $vS_A$ ) and two cases can happen:  $q+1 \in J$  and  $\bar{\mu}_A^v < +\infty$  on  $F_{\hat{J}}(vU_A)$ , or  $q+1 \notin J$  and  $\bar{\nu}_A^v = +\infty$  on  $F_{\hat{J}}(vU_A)$ . In both cases

$$F_J(vS_A) = \{a \in F_J(\Gamma^{q+1}) : \hat{a} \in F_{\hat{J}}(vU_A) \text{ and } \bar{\mu}_A^v(\hat{a}) \leq a_{q+1} \leq \bar{\nu}_A^v(\hat{a})\}. \quad (30)$$

Since  $F_{\hat{J}}(vU_A)$  is a face of  $vU_A$ , by Claim 7.1 there is  $C \leq A$  in  $\mathcal{A}$  such that  $F_{\hat{J}}(vU_A) = vU_C$ . Let  $Y = \hat{C} = \psi(U_C)$ .

If  $q+1 \notin J$  then by Proposition 2.13,  $\bar{\nu}_A^v = +\infty$  on  $F_{\hat{J}}(vU_A) = vU_C$ . That is  $\bar{\nu}_A = 0$  on  $Y = \psi(U_C)$ , hence  $\partial_Y^0 A \in \mathcal{A}$ . Let  $B = \partial_Y^0 A$  and apply Step 1 to  $B$ . Since  $J = \hat{J}$  is the support of  $vU_C = vU_B$  and of  $S_B$  (because  $\text{tp } B = 0$ ), we deduce from (29) and (30) that  $vS_B = F_J(S_A)$ .

If  $q+1 \in J$  then by Proposition 2.13,  $\bar{\mu}_A^v \neq +\infty$  on  $F_{\hat{J}}(vU_A) = vU_C$ . That is  $\bar{\mu}_A \neq 0$  on  $Y = \psi(U_C)$ , hence  $\partial_Y^1 A \in \mathcal{A}$ . Let  $B = \partial_Y^1 A$  and apply Step 1 to  $B$ . Since  $\hat{J}$  is the support of  $vU_C = vU_B$  and  $J = \hat{J} \cup \{q+1\}$  is the support of  $S_B$  (because  $\text{tp } B = 1$ ), we deduce from (29) and (30) that  $vS_B = F_J(S_A)$ .

■

Finally let  $\varphi : \bigcup \mathcal{S} \rightarrow \bigcup \mathcal{A}$  be defined by  $\varphi|_{S_A} = \varphi_A$  on each  $S_A$ .

**Claim 7.9.**  $\varphi$  is a homeomorphism from  $\bigcup \mathcal{S}$  to  $\bigcup \mathcal{A}$ .

*Proof:* We already know by Claim 7.7 that  $\varphi$  is a bijection from  $\bigcup \mathcal{S}$  to  $\bigcup \mathcal{A}$ . It follows from Claim 7.8 that  $\bigcup \mathcal{S}$  is closed, and it is obviously bounded. Thus by Theorem 2.6 it suffices to show that  $\varphi$  is continuous. Since each  $\varphi_A$  is obviously continuous on  $S_A$ , we only have to prove that for every  $z \in \partial S_A$  and  $y \in S_A$ ,  $\varphi_A(y)$  tends to  $\varphi(z)$  as  $y$  tends to  $S_A$ . By Claim 7.8 there is  $B \leq A$  in  $\mathcal{A}$  such that  $z \in S_B$ , hence  $\varphi(z) = \varphi_B(z)$ . By Claim 7.5,  $\psi \circ \Phi(y)$  tends to  $\psi \circ \Phi(z)$ . By Claim 7.7,  $\psi \circ \Phi(y) \in \hat{A}$  and  $\psi \circ \Phi(z) \in \hat{B}$  hence  $c_A(\psi \circ \Phi(y))$  tends to  $\bar{c}_A(\psi \circ \Phi(z))$ , which is equal to  $c_B(\psi \circ \Phi(z))$  since  $\bar{c}_A|_{\hat{B}} = c_B$ . Thus it only remains to check that  $\lambda_A y_{r_A}^N$  tends to  $\lambda_B z_{r_B}^N$ .

If  $\text{tp } A = 0$  then also  $\text{tp } B = 0$  hence and we are done, since  $\lambda_A y_{r_A}^N = 0 = \lambda_B z_{r_B}^N$ . If  $\text{tp } B = 1$  then  $\lambda_B = \lambda_A$  (because  $\mathcal{A}$  is a cellular complex) and  $r_A = r_B$  (because by (C1) and (C3),  $H(B) \neq P(B) = H(B) \cap P(A)$  implies that  $H(B)$  is not contained in  $P(A)$ , hence  $r_A \in H(B)$  since  $H(B) \subseteq H(A) = P(A) \cup \{r_A\}$ ). Hence obviously  $\lambda_A y_{r_A}^N$  tends to  $\lambda_A z_{r_A}^N = \lambda_B z_{r_B}^N$  in that case. Finally if  $\text{tp } A = 1$  and  $\text{tp } B = 0$  then  $r_A \notin H(B)$  (because by (C0) and (C3),  $H(B) = P(B) \subseteq P(A)$ ), hence  $z_{r_A} = 0$  since  $\text{Supp } z = \text{Supp } S_B = H(B)$ . Thus  $\lambda_A y_{r_A}^N$  tends to  $\lambda_A z_{r_A}^N = 0$ , which proves the result because  $\lambda_B z_{r_B}^N = 0$  since  $\text{tp } B = 0$ .

■

We can recap now our construction and state the result which was the aim of this section.

**Lemma 7.10.** *Let  $\mathcal{A}$  be a cellular monoplex mod  $Q_{N, M+v(M)}^\times$  such that  $\bigcup \mathcal{A}$  is a closed subset of  $R^{m+1}$ . Let  $(\mathcal{U}, \psi)$  be a triangulation of  $\text{CB}(\mathcal{A})$  with parameters  $(n, N, e, M)$  such that  $M > v(N)$  and for every  $A \in \mathcal{A}$ ,  $\psi^{-1}(A) \in \mathcal{U}$  (let us denote it  $U_A$ ). Then there exists a simplicial complex  $\mathcal{S}$  of index  $M$ , a Cartesian morphism  $\Phi : \biguplus \mathcal{S} \rightarrow \biguplus \mathcal{U}$  and a semi-algebraic homeomorphism  $\varphi : \biguplus \mathcal{S} \rightarrow \bigcup \mathcal{A}$  such that for every  $A \in \mathcal{A}$ ,  $\varphi^{-1}(A) \in \mathcal{S}$  (let us denote it  $S_A$ ) and for every  $y \in S_A$*

$$\varphi(y) = (\psi \circ \Phi(y), c_A(\psi \circ \Phi(y)) + \pi^{-NM'} \lambda_A y_{r_A}^N)$$

where<sup>14</sup>  $r_A = \max(\text{Supp } S_A)$ .

*Proof:* Let  $(A_k)_{1 \leq k \leq r}$  be the list of minimal elements in  $\mathcal{A}$ , and for each  $k$  let  $\mathcal{A}_k$  be the family of elements in  $\mathcal{A}$  greater than  $A_k$ . This is a rooted, cellular monoplex mod  $Q_{N, M+v(N)}^\times$ . For every  $A \in \mathcal{A}_k$ ,  $\overline{A}$  is the union of the cells  $B \leq A$  in  $\mathcal{A}$  since  $\mathcal{A}$  is a cellular complex and  $\bigcup \mathcal{A}$  is closed. All these cells  $B$  belong to  $\mathcal{A}_k$  hence  $\bigcup \mathcal{A}_k$  is closed. Since  $\bigcup \mathcal{A} \setminus \bigcup \mathcal{A}_k$  is the union of the finitely many other  $\mathcal{A}_l$  it is closed, hence  $\bigcup \mathcal{A}_k$  is clopen in  $\bigcup \mathcal{A}$ . Let  $\mathcal{U}_k = \{\psi^{-1}(\widehat{A}) : A \in \mathcal{A}_k\}$ , this is a lower subset of  $\mathcal{U}$  with smallest element  $\psi^{-1}(\widehat{A}_k)$  hence a rooted simplicial complex in  $D^{M_1} R^{q_{1,k}}$  for some  $q_{1,k}$ . Finally let  $\psi_k$  be the restriction of  $\psi$  to  $\bigcup \mathcal{U}_k$ .

Claims 7.1 to 7.9 apply to  $(\mathcal{U}_k, \psi, \mathcal{A}_k)$  and give a simplicial complex  $\mathcal{S}_k$  in  $D^M R^{q_{2,k}}$  for some  $q_{2,k}$ , a Cartesian morphism  $\Phi_k : \bigcup \mathcal{S}_k \rightarrow \bigcup \mathcal{U}_k$  and a semi-algebraic homeomorphism  $\varphi_k : \biguplus \mathcal{S}_k \rightarrow \bigcup \mathcal{A}_k$  satisfying all the required properties. Since each  $\bigcup \mathcal{A}_k$  is clopen in  $\bigcup \mathcal{A}$ , and each  $\bigcup \mathcal{U}_k$  is clopen in  $\biguplus \mathcal{U}$ , the conclusion follows by taking for  $\mathcal{U}$  the family  $\{\mathcal{U}_k\}_{1 \leq k \leq r}$  and for  $\Phi$  (resp.  $\varphi$ ) the map obtained by glueing together the various  $\Phi_k$  (resp.  $\varphi_k$ ).

■

## 8 Triangulation

We have come up to the moment when we can show that  $\mathbf{T}_m \Rightarrow \mathbf{T}_{m+1}$ . As  $\mathbf{T}_0$  is rather obvious, this will finish the proof of  $\mathbf{T}_m$  for every  $m$ .

**Theorem 8.1.** *Assume  $\mathbf{T}_m$ . Let  $(\theta_i : A_i \subseteq K^{m+1} \rightarrow K)_{i \in I}$  be a finite family of semi-algebraic functions, and  $n, N \geq 1$  be any integers. Then for some integers  $e, M \geq 1$  which can be chosen arbitrarily large (in the sense of footnote 1), there exists a simplicial complex  $\mathcal{T}$  of index  $M$  and a semi-algebraic homeomorphism  $\varphi : \biguplus \mathcal{T} \rightarrow \bigcup_{i \in I} A_i$  such that for every  $i$  in  $I$ :*

1.  $\{\varphi(T) : T \in \mathcal{T} \text{ and } \varphi(T) \subseteq A_i\}$  is a partition of  $A_i$ .
2.  $\forall T \in \mathcal{T}$  such that  $\varphi(T) \subseteq A_i$ ,  $\theta_i \circ \varphi|_T$  is  $N$ -monomial mod  $U_{e,n}$ .

*Proof:* By using the same partition of  $K^{m+1}$  as in the proof of Lemma 3.3 we are reduced to the case where each  $A_i$  is contained in  $R^{m+1}$ . We can also extend each  $\theta_i$  to  $R^{m+1}$  by an arbitrary value, and add to this family the indicator functions of each  $A_i$  inside  $R^{m+1}$ , hence assume that all these functions have domain  $R^{m+1}$ , which is closed and bounded. Let  $e_* \geq 1$  and  $M_* \geq 1$  be any integers.

Theorem 4.6 applies to  $(\theta_i)_{i \in I}$ . It gives an integer  $e_0 \geq 1$ , a tuple  $\eta \in R^m$ , a linear automorphism  $u_\eta(x, t) = (x + t\eta, t)$  of  $K^{m+1}$  (note that  $u_\eta(R^{m+1}) = R^{m+1}$  since  $\eta \in R^{m+1}$ ), a pair of integers  $N_0 \geq 1$  and  $M_0 > 2v(e_0)$  such that  $e_0 N$  divides  $N_0$ , and a finite family  $\mathcal{A}$  of largely continuous cells mod  $Q_{N_0, M_0}^\times$  partitioning  $u_\eta^{-1}(R^{m+1}) = R^{m+1}$  such that for every  $i \in I$ , every  $A \in \mathcal{A}$  and every  $(x, t) \in A$

$$\theta_i \circ u_\eta(x, t) = \mathcal{U}_{e_0, n}(x, t) h_{i, A}(x) \left[ \lambda_A^{-1}(t - c_A(x)) \right]^{\frac{\alpha_{i, A}}{e_0}} \quad (31)$$

<sup>14</sup>If  $\text{Supp } S_A = \emptyset$  then  $r_A$  is not defined but in that case  $S_A$  is a point, hence so is  $A$  so  $\lambda_A = 0$  and we can let  $\lambda_A y_{r_A}^N = 0$  by convention.



where  $h_{i,A} : \widehat{A} \rightarrow K$  is a semi-algebraic function and  $\alpha_{i,A} \in \mathbf{Z}$ .

Let  $n_1 = \max(1 + 2v(e_0), n + v(e_0))$ , Lemma 6.1 applied to  $\mathcal{A}$  and the family  $\mathcal{F}_0 = \{h_{i,A} : i \in I, A \in \mathcal{A}\}$  gives a pair of integers  $e_1 \geq 1$  and  $M_1 > 2v(e_1)$ , a cellular monoplex  $\mathcal{B} \bmod Q_{N_0, M_0}$  refining  $\mathcal{A}$  such that  $\mathcal{B} \triangleleft^{n_1} \mathcal{A}$ , a  $\triangleleft^{n_1}$ -system  $\mathcal{F}_1$  for  $(\mathcal{B}, \mathcal{A})$ , and a triangulation  $(\mathcal{U}, \psi)$  of  $\mathcal{F}_0 \cup \mathcal{F}_1 \cup \text{CB}(\mathcal{B})$  with parameters  $(n_1, N_0, e_1, M_1)$ . Moreover  $e_1, M_1$  can be chosen arbitrarily large, in the sense of footnote 1, so we can require that  $e_*$  divides  $e_1$  and  $M_1 \geq M_*$ , and that  $M_1 \geq M_0 - v(N_0)$  and  $M_1 > v(N_0) \geq v(e_0)$ .

$Q_{N_0, M_1 + v(N_0)}^\times$  is a subgroup of  $Q_{N_0, M_0}^\times$  (because  $M_1 + v(N_0) \geq M_0$ ) with finite index. Hence every cell in  $\mathcal{B}$  is the disjoint union of finitely many cells  $C \bmod Q_{N_0, M_1 + v(N_0)}^\times$  with the same socle and bounds as  $B$ . Since  $vQ_{N_0, M_1 + v(N_0)}^\times = N_0\mathcal{Z} = vQ_{N_0, M_0}^\times$ , these cells  $C$  are still fitting cells by Proposition 5.5. One easily sees that they form a cellular monoplex  $\mathcal{C}$  refining  $\mathcal{A}$  such that  $\mathcal{C} \triangleleft^{n_1} \mathcal{A}$  and  $\mathcal{F}_1$  is a  $\triangleleft^{n_1}$ -system for  $(\mathcal{C}, \mathcal{A})$ . Moreover  $\text{CB}(\mathcal{C}) = \text{CB}(\mathcal{B})$  and  $\widehat{\mathcal{C}} = \widehat{\mathcal{B}}$  so  $(\mathcal{U}, \psi)$  is a triangulation of  $\mathcal{B}(\mathcal{C})$  with parameters  $(n_1, N_0, e_1, M_1)$  such that  $\psi^{-1}(\widehat{\mathcal{C}}) \in \mathcal{U}$  for every  $C \in \mathcal{C}$ .

Since  $M_1 > v(N_0)$ , Lemma 7.10 applies to  $\mathcal{C}$  and  $(\mathcal{U}, \psi)$ . It gives a simplicial complex  $\mathcal{T}$  of index  $M_1$ , a Cartesian morphism  $\Phi : \bigsqcup \mathcal{T} \rightarrow \bigsqcup \mathcal{U}$  and a semi-algebraic homeomorphism  $\varphi : \bigsqcup \mathcal{T} \rightarrow \bigcup \mathcal{C}$  such that  $\varphi^{-1}$  maps each  $C$  in  $\mathcal{C}$  onto some  $T$  in  $\mathcal{T}$ , and for every  $y$  in  $T$

$$\varphi(y) = (\psi \circ \Phi(y), c_C(\psi \circ \Phi(y)) + \pi^{-NM'} \lambda_C y_{r_C}^{N_0}) \quad (32)$$

where<sup>15</sup>  $r_C = \max(\text{Supp } T)$ . Let  $\varphi_\eta = u_\eta \circ \varphi$ , this is a semi-algebraic homeomorphism from  $\bigsqcup \mathcal{T}$  to  $R^{m+1}$ . We are going to check that  $\theta_i \circ \varphi_\eta|_T$  is  $N$ -monomial mod  $U_{e_0 e_1, n}$  for every  $i \in I$  and every  $T \in \mathcal{T}$ . This will prove the result, with  $e = e_0 e_1$  and  $M = M_1$ .

So pick any  $T \in \mathcal{T}$ , let  $C = \varphi(T)$  and  $r_C$  be as above. There is a unique  $B \in \mathcal{B}$  containing  $C$ , a unique  $A \in \mathcal{A}$  containing  $B$ . For every  $(x, t) \in C$  let  $\delta_C(x, t) = t - c_C(x)$ . Let  $\delta_A$  and  $\delta_B$  be defined accordingly. Note that  $\delta_C = \delta_B$  on  $C$  because  $C$  has the same center as  $B$  by construction. For every  $y \in T$ , by (31) and (32) we have

$$\theta_i \circ \varphi_\eta(y) = \mathcal{U}_{e_0, n}(\varphi(y)) h_{i, A}(\psi \circ \Phi(y)) [\lambda_A^{-1} \delta_A(\varphi(y))]^{\frac{\alpha_{i, A}}{e_0}}.$$

We have  $\mathcal{U}_{e_0, n}(\varphi(y)) \in U_{e_0, n} \subseteq U_{e_0 e_1, n}$  so the factor  $\mathcal{U}_{e_0, n}(\varphi(y))$  can be replaced by  $\mathcal{U}_{e_0 e_1, n}(y)$ . Recalling that  $(\mathcal{V}, \psi)$  is a triangulation of  $\mathcal{F}_0 \cup \mathcal{F}_1$  with parameters  $(n_1, N_0, e_1, M_1)$ , that  $\Phi$  is a Cartesian morphism and  $h_{i, A} \in \mathcal{F}_0$ , we get that the second factor  $h_{i, A}(\psi \circ \Phi(y)) = h_{i, A} \circ \psi(\Phi(y))$  is  $N_0$ -monomial mod  $U_{e_1, n_1}$  hence *a fortiori*  $N$ -monomial mod  $U_{e_0 e_1, n}$  since  $N$  divides  $N_0$  and  $n_1 \geq n$ . So it only remains to prove that the last factor  $[\lambda_A^{-1} \delta_A \circ \varphi|_T]^{\alpha_{i, A}/e_0}$  is  $N$ -monomial mod  $U_{e_0 e_1, n}$ . It suffices to prove it for  $[\lambda_A^{-1} \delta_A \circ \varphi]^{1/e_0}$ .

We can assume that  $\text{tp } A = 1$  otherwise  $\lambda_A^{-1} \delta_A = 1$  and the result is trivial (see Remark 4.7). Recall that  $\mathcal{C} \triangleleft^{n_1} \mathcal{A}$  and  $\mathcal{F}_1$  is a  $\triangleleft^n$ -system for  $(\mathcal{C}, \mathcal{A})$ . For every  $(x, t) \in C$  we then have

$$t - c_A(x) = \mathcal{U}_{n_1}(x, t) h_{C, A}(x)^\beta (t - c_C(x))^{1-\beta}$$

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<sup>15</sup>See footnote 14.

with  $h_{C,A} \in \mathcal{F}_1$  and  $\beta \in \{0, 1\}$  (depending on  $A, C$ ). So by (32) we have

$$\delta_A(\varphi(y)) = \mathcal{U}_{n_1}(\varphi(y))h_{C,A}(\psi \circ \Phi(y))^\beta (\pi^{-NM} \lambda_C y_{r_C}^{N_0})^{1-\beta}. \quad (33)$$

$(\mathcal{V}, \psi)$  is a triangulation of  $\mathcal{F}_1$  with parameters  $(n_1, N_0, e_1, M_1)$  hence  $h_{C,A}(\psi \circ \Phi(y))$  is  $N_0$ -monomial mod  $U_{e_1, n_1}$ . So (33) implies that  $\delta_A \circ \varphi|_T$  is  $N_0$ -monomial mod  $U_{e_1, n_1}$ , hence so is  $\lambda_A^{-1} \delta_A \circ \varphi|_T$ . Let  $\chi : T \rightarrow \mathbf{U}_{e_1}$  and  $g : T \rightarrow K$  be semi-algebraic functions that for every  $y \in T$

$$\lambda_A^{-1} \delta_A \circ \varphi(y) = \chi(y) \mathcal{U}_{n_1}(y) \zeta g(y) \quad \text{and} \quad g(y) = \prod_{1 \leq i \leq q} y_i^{\alpha_i N_0}$$

with  $\zeta \in K$ ,  $\alpha_1, \dots, \alpha_q \in \mathbf{Z}$ . Let  $k = N_0/(e_0 N)$ , by construction  $e_0 N$  divides  $N_0$  hence  $k \in \mathbf{N}^*$ . Since  $T \subseteq D^{M_1} R^{q'}$ , each  $y_i \in D^{M_1} R \subseteq Q_{1, M_1} \subseteq Q_{1, v(e_0)+1}$  (because  $M_1 > v(e_0)$ ) hence  $y_i^{e_0} \in Q_{e_0, 2v(e_0)+1}$ . *A fortiori*  $y^{\alpha_i N_0} = y^{e_0 N k \alpha_i}$  belongs to  $Q_{e_0, 2v(e_0)+1}$  hence  $g$  takes values in  $Q_{e_0, 2v(e_0)+1}$  and  $g^{1/e_0}$  is  $N$ -monomial:

$$(g(y))^{1/e_0} = \left( \prod_{1 \leq i \leq q} y_i^{e_0 N k \alpha_i} \right)^{1/e_0} = \prod_{1 \leq i \leq q} y_i^{N k \alpha_i}$$

But  $\lambda_A^{-1} \delta_A$  also takes values in  $Q_{e_0, 2v(e_0)+1}$  because  $\delta_A(x, t) \in \lambda_A Q_{N_0, M_0}$  for every  $(x, t) \in A$ , and  $Q_{N_0, M_0} \subseteq Q_{e_0, 2v(e_0)+1}$  since  $e_0$  divides  $N_0$  and  $M_0 > 2v(e_0)$ . Thus  $(\lambda_A^{-1} \delta_A \circ \varphi|_T)/g = \mathcal{U}_{n_1} \zeta \chi$  takes values in  $Q_{e_0, 2v(e_0)+1}$  as well. So does the factor  $\mathcal{U}_{n_1}$  since  $n_1 > 2v(e_0)$ . Hence finally  $\zeta \chi(y) \in Q_{e_0, 2v(e_0)+1}$  for every  $y \in T$ , so  $(\zeta \chi)^{1/e_0}$  is well defined. Note that  $\zeta^{e_1} = \zeta^{e_1} \chi^{e_1} = [(\zeta \chi)^{1/e_0}]^{e_0 e_1}$  hence  $\zeta^{e_1} \in P_{e_0 e_1}$ . Pick any  $\eta \in K$  such that  $\zeta^{e_1} = \eta^{e_0 e_1}$ , and for every  $y \in T$  let  $\chi'(y) = (\zeta \chi(y))^{1/e_0} / \eta$ . This is a semi-algebraic function taking values in  $\mathbf{U}_{e_0 e_1}$  because

$$[(\zeta \chi)^{1/e_0}]^{e_0 e_1} = \zeta^{e_1} = \eta^{e_0 e_1}.$$

By Remark 2.10,  $\mathcal{U}_{n_1}^{1/e_0} = \mathcal{U}_{n_1 - v(e_0)}$  because  $n_1 > 2v(e_0)$ , and by definition  $\chi' \mathcal{U}_{n_1 - v(e_0)} = \mathcal{U}_{e_0 e_1, n_1 - v(e_0)}$ . Altogether this gives that

$$\begin{aligned} [\lambda_A^{-1} \delta_A \circ \varphi|_T]^{1/e_0} &= \mathcal{U}_{n_1}^{1/e_0} (\zeta \chi)^{1/e_0} g^{1/e_0} \\ &= \chi' \mathcal{U}_{n_1 - v(e_0)} ((\zeta \chi)^{1/e_0} / \chi') g^{1/e_0} \\ &= \mathcal{U}_{e_0 e_1, n_1 - v(e_0)} \eta g^{1/e_0} \end{aligned}$$

Thus  $[\lambda_A^{-1} \delta_A \circ \varphi]^{1/e_0}$  is  $N$ -monomial mod  $U_{e_0 e_1, n_1 - v(e_0)}$  (because  $g^{1/e_0}$  is so). It is *a fortiori*  $N$ -monomial mod  $U_{e_0 e_1, n}$  since  $n_1 - v(e_0) \geq n$  by construction.  $\blacksquare$

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