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Fixed points of n -valued maps on surfaces and the Wecken property – a configuration space approach

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Abstract

In this paper, we explore the fixed point theory of n -valued maps using configuration spaces and braid groups, focussing on two fundamental problems, the Wecken property, and the computation of the Nielsen number. We show that the projective plane (resp. the 2-sphere \mathbb{S}^2) has the Wecken property for n -valued maps for all $n \in \mathbb{N}$ (resp. all $n \geq 3$). In the case $n = 2$ and \mathbb{S}^2 , we prove a partial result about the Wecken property. We then describe the Nielsen number of a non-split n -valued map $\phi: X \multimap X$ of an orientable, compact manifold without boundary in terms of the Nielsen coincidence numbers of a certain finite covering $q: \widehat{X} \rightarrow X$ with a subset of the coordinate maps of a lift of the n -valued split map $\phi \circ q: \widehat{X} \multimap X$.

1 Introduction

Multifunctions and their fixed point theory have been widely studied, see the books [1, 25] for example, where fairly general classes of multifunctions and spaces are considered. In all of what follows, X and Y will be topological spaces. Let $\phi: X \multimap Y$ be an n -valued function *i.e.* a function that to each $x \in X$ associates an unordered subset $\phi(x)$ of Y of cardinality n . Recall that such an n -valued function ϕ is *continuous* if for all $x \in X$, $\phi(x)$ is closed, and for any open set V in Y , the sets $\{x \in X \mid \phi(x) \subset V\}$ and

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$\{x \in X \mid \phi(x) \cap V \neq \emptyset\}$ are open in X . We will refer to a continuous n -valued function as an n -valued map. The class of n -valued maps is of particular interest, and more information about their fixed point theory on finite complexes may be found in [2, 3, 7, 8, 9, 10, 13, 14, 32, 33, 34]. A *homotopy* between two n -valued maps $\phi_1, \phi_2: X \multimap Y$ is an n -valued map $H: X \times I \multimap Y$ such that $\phi_1 = H(\cdot, 0)$ and $\phi_2 = H(\cdot, 1)$. Following [32], an n -valued function $\phi: X \multimap Y$ is said to be a *split n -valued map* if there exist single-valued maps $f_1, f_2, \dots, f_n: X \rightarrow Y$ such that $\phi(x) = \{f_1(x), \dots, f_n(x)\}$ for all $x \in X$. This being the case, we shall write $\phi = \{f_1, \dots, f_n\}$. Let $\text{Spl}(X, Y, n)$ denote the set of split n -valued maps between X and Y . *A priori*, $\phi: X \multimap Y$ is just an n -valued function, but if it is split then it is continuous by [22, Proposition 42], which justifies the use of the word ‘map’ in the definition. Partly for this reason, split n -valued maps play an important rôle in the theory. If $\phi: X \multimap X$ is an n -valued map from X to itself, we say that $x \in X$ is a *fixed point* of ϕ if $x \in \phi(x)$, and we denote the set of fixed points of ϕ by $\text{Fix}(\phi)$.

In [33, Section 5], Schirmer defined the notion of Nielsen number of n -valued maps of finite complexes. Her definition is similar to that for single-valued maps, although it is a little more elaborate. As for the case of single-valued maps, for appropriate spaces, the Nielsen number $N(\phi)$ of an n -valued map $\phi: X \multimap X$ provides a lower bound for the number of fixed points among all n -valued maps homotopic to ϕ . The computation of the Nielsen number of a self- or n -valued map is an important problem in fixed point theory, and is not easy in general. In the split case, we have the following formula for the Nielsen number of n -valued maps of polyhedra in terms of the constituent single-valued maps.

THEOREM 1. [33, Corollary 7.2] *Let $\phi = \{f_1, f_2, \dots, f_n\}: X \multimap X$ be a split n -valued map, where X is a compact polyhedron. Then $N(\phi) = N(f_1) + \dots + N(f_n)$.*

A second fundamental problem in fixed point theory is to decide whether a space X has the Wecken property. Recall that the homotopy class of an n -valued map $\phi: X \multimap X$ is said to have the *Wecken property* if there exists an n -valued map $\psi: X \multimap X$ homotopic to ϕ that has exactly $N(\phi)$ fixed points, and that a space X has the *Wecken property for n -valued maps* if every homotopy class of n -valued maps of X has the Wecken property. For single-valued maps, many complexes of dimension at least three have the Wecken property [30, 35, 36, 37]. In the case of surfaces, the 2-sphere \mathbb{S}^2 , the real projective plane $\mathbb{R}P^2$ have the Wecken property [5, 29], as do the 2-torus \mathbb{T}^2 and the Klein bottle (see [23] for example). However, Jiang showed that no other compact surface without boundary has the Wecken property [27, 28]. For n -valued maps, substantial progress has been made in the study of the Wecken property. The following result mirrors that for single-valued maps.

THEOREM 2. [34, Theorem 5.2] *Let M be a compact triangulable manifold (with or without boundary) of dimension greater than or equal to 3, and let $n \in \mathbb{N}$. Then every n -valued map $\phi: M \multimap M$ is homotopic to an n -valued map that has $N(\phi)$ fixed points. In particular, M has the Wecken property for n -valued maps.*

Theorem 2 has been extended to a larger class of spaces, including some 2-dimensional complexes, as follows.

THEOREM 3. [3, Theorem 1] *Let X be a compact polyhedron without local cut points and such that no connected component of X is a surface. Then X has the Wecken property for n -valued maps.*

In a recent paper [22], we studied some aspects of fixed point theory of n -valued maps from X to Y by introducing an equivalent and natural formulation in terms of single-valued maps from X to the n^{th} unordered configuration space $D_n(Y)$ of X , where $D_n(Y)$ is the quotient of the n^{th} (ordered) configuration space $F_n(X)$ of X , defined by:

$$F_n(Y) = \{(y_1, \dots, y_n) \mid y_i \in Y, \text{ and } y_i \neq y_j \text{ if } i \neq j\},$$

by the free action of the symmetric group S_n given by permuting coordinates. It is well known that $\pi_1(D_n(Y))$ (resp. $\pi_1(F_n(Y))$) is the braid group $B_n(Y)$ (resp. pure braid group $P_n(Y)$) of Y on n strings, that the quotient map $\pi: F_n(Y) \rightarrow D_n(Y)$ is a regular $n!$ -fold covering, and that $P_n(Y)$ is the kernel of the surjective homomorphism $\tau: B_n(Y) \rightarrow S_n$ that to a braid associates its induced permutation. Configuration spaces play an important rôle in several branches of mathematics and have been extensively studied, see [15, 16] for example. As in [22], a map $\Phi: X \rightarrow D_n(Y)$ will be called an *n -unordered map*, and a map $\Psi: X \rightarrow F_n(Y)$ will be called an *n -ordered map*. For such an n -ordered map, for $i = 1, \dots, n$, there exist maps $f_i: X \rightarrow Y$ such that $\Psi(x) = (f_1(x), \dots, f_n(x))$ for all $x \in X$, and for which $f_i(x) \neq f_j(x)$ for all $1 \leq i, j \leq n$, $i \neq j$, and all $x \in X$. In this case, we will often write $\Psi = (f_1, \dots, f_n)$. There is an obvious bijection between the set of n -point subsets of Y and $D_n(Y)$ that induces a bijection between the set of n -valued functions from X to Y and the set of functions from X to $D_n(Y)$. If we suppose in addition that X and Y are metric spaces, then $D_n(Y)$ may be equipped with a certain Hausdorff metric [22, Appendix], in which case this bijection restricts to a bijection between the set of n -valued maps from X to Y and the set of (continuous) maps from X to $D_n(Y)$ [22, Theorem 8]. If $\phi: X \multimap Y$ is an n -valued map then we shall say that the map $\Phi: X \rightarrow D_n(Y)$ obtained by this bijection as the map associated to ϕ . If a map $\Phi: X \rightarrow D_n(Y)$ admits a lift $\widehat{\Phi}: X \rightarrow F_n(Y)$ via the covering map π then we say that $\widehat{\Phi}$ is a *lift* of ϕ . By [22, Section 2.1], ϕ is split if and only if it admits a lift.

In [22], we showed that spheres and real and complex projective spaces of even dimension have the fixed point property for n -valued maps, and we expressed the problem of deforming an n -valued map to a fixed point free n -valued map in terms of an algebraic criterion involving the braid groups of X , which we then used to determine an infinite number of homotopy classes of split 2-valued maps of \mathbb{T}^2 that contain a fixed point free representative. In this paper, we explore some other aspects of fixed point theory of n -valued maps using the above formulation in terms of configuration spaces. We establish an equality for the Nielsen number in the non-split case, and we study the Wecken property for some surfaces. We now describe the contents of this paper in more detail.

In Section 2, we recall some fundamental results from [22], and in Section 2.1, we show that the 2-disc has the Wecken property for n -valued maps for all $n \in \mathbb{N}$. In Sections 2.2 and 3, we analyse the Wecken problem for n -valued maps of \mathbb{S}^2 and $\mathbb{R}P^2$ respectively. In the first case, one important step is to determine the number of homotopy classes of n -valued maps of \mathbb{S}^2 . In Lemma 10, we show that there is just one such class if $n \geq 3$, while if $n = 2$, we show that the set of such homotopy classes is in bijection with \mathbb{N} . If $\phi: \mathbb{S}^2 \multimap \mathbb{S}^2$ is a 2-valued map then we refer to the non-negative integer given by this correspondence as the *degree* of ϕ (or its homotopy class). Let $A: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ denote the antipodal map. In the case of $\mathbb{R}P^2$, in Proposition 13, we show that there are precisely two homotopy classes of n -valued maps of $\mathbb{R}P^2$ for all $n \geq 2$, and we give explicit representatives of these homotopy classes. To prove the Wecken property for n -valued maps of $\mathbb{R}P^2$, it then suffices

to check that each of these homotopy classes has the Wecken property. We summarise the main results of Sections 2 and 3 as follows.

PROPOSITION 4.

- (a) Let $f_0: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ be the constant map at a point $x_0 \in \mathbb{S}^2$. Then f_0 has degree 0, and f_0 and $A \circ f_0$ each have precisely one fixed point.
- (b) There exists a map $f_1: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ of degree 1 that has a single fixed point and such that the map $A \circ f_1$ is fixed point free.
- (c) There exists a map $f_2: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ of degree 2 such that f_2 and the map $A \circ f_2$ each possess a single fixed point.
- (d) The Wecken property holds for:
 - (i) n -valued maps of \mathbb{S}^2 for all $n \geq 3$.
 - (ii) the homotopy classes of 2-valued maps of degree 0, 1 or 2.

Proposition 4(d)(ii) has been obtained independently in [11]. Our proof is different from that given in [11], and is perhaps more direct. The paper [11] contains other results about n -valued maps of \mathbb{S}^2 , concerning for example the minimal number of fixed points in a given homotopy class of a 2-valued map (see [11, Theorem 4.1]). Apart from the three cases given in Proposition 4(d)(ii), we believe that this minimal number is not known. We do not know either whether the Wecken property holds for 2-valued maps of \mathbb{S}^2 of degree greater than or equal to 3.

THEOREM 5. *The projective plane $\mathbb{R}P^2$ has the Wecken property for n -valued maps for all $n \in \mathbb{N}$.*

In Section 4, we consider the problem of the computation of the Nielsen number of n -valued maps of a compact, orientable manifold X without boundary. To do this, we shall use [22, Proposition 16], which given an n -valued map $\phi: X \multimap X$, states that there exists a finite covering $q: \widehat{X} \rightarrow X$ such that the composition $\phi \circ q: \widehat{X} \multimap X$ is split, and [22, Proposition 17], which describes the fixed points of ϕ in terms of the coincidences of q with the coordinate maps of a lift of the split n -valued map $\phi \circ q$. We analyse the behaviour of this description with respect to the essential Nielsen classes (*i.e.* the Nielsen classes of non-zero index). This yields a partial analogue of Theorem 1 in the non-split case as follows.

THEOREM 6. *Let X be an orientable, compact manifold without boundary, and let $\phi: X \multimap X$ be a non-split n -valued map. Let $\Phi: X \rightarrow D_n(X)$ be the map associated to ϕ , let H be the kernel of the composition $\tau \circ \Phi_\#: \pi_1(X) \rightarrow S_n$, let $L' = \text{Im}(\tau \circ \Phi_\#)$, and for $i = 1, \dots, n$, let $L'_{i,i} = \{\alpha \in L' \mid \alpha(i) = i\}$. Let $q: \widehat{X} \rightarrow X$ be the finite regular covering of X that corresponds to H . Let $\widehat{\Phi}_1 = (f_1, \dots, f_n): \widehat{X} \rightarrow F_n(X)$ be a lift of the split n -valued map $\phi \circ q: \widehat{X} \multimap X$, and suppose that the action of $L'_{i,i}$ on $\{1, \dots, n\}$ is free for all $i \in \{1, \dots, n\}$. Then the Nielsen number $N(\phi)$ of ϕ is equal to $\sum_{j \in I_0} N(q, f_{i_j})$, where i_j runs over a set I_0 of representatives of the orbits of this action.*

If $n = 2$, the condition that the action be free is necessarily satisfied, and in this case, we shall prove in Corollary 23 that $N(\phi) = N(q, f_1) = N(q, f_2)$. The study of the Wecken property for multivalued maps of the torus constitutes work in progress by the authors, and some preliminary results that are closely related to this problem may be found in [22].

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2 Preliminaries and the Wecken property for n -valued maps of the 2-disc and 2-sphere

Let $n \in \mathbb{N}$. Given a topological space X , we are interested in understanding the fixed point theory of its n -valued maps. In this section, we will show that if $n \in \mathbb{N}$ (resp. $n \geq 3$), the 2-disc \mathbb{D}^2 and the (resp. the 2-sphere \mathbb{S}^2) has the Wecken property for n -valued maps. We first recall some definitions and results from [22]. Given an n -valued map $\phi: X \multimap X$ as defined in Section 1, let $\Phi: X \rightarrow D_n(X)$ denote the associated n -unordered map, and if ϕ is split, let $\widehat{\Phi}: X \rightarrow F_n(X)$ be an n -ordered map that is a lift of ϕ . By a single-valued map (or simply a map) $f: X \rightarrow Y$ between two topological spaces, we shall always mean a continuous function from X to Y , and $[X, Y]$ will denote the set of unbased homotopy classes of maps between X and Y . We recall the following result, which shows that for a large class of spaces, (homotopy classes of) n -valued maps may be identified with (homotopy classes of) maps whose target is an unordered configuration space. Let I denote the unit interval $[0, 1]$.

THEOREM 7. *Let X and Y be metric spaces, and let $n \in \mathbb{N}$.*

(a) *An n -valued function $\phi: X \multimap Y$ is continuous if and only if the corresponding function $\Phi: X \rightarrow D_n(Y)$ is continuous.*

(b) *The set of homotopy classes of n -valued maps from X to Y is in one-to-one correspondence with the set $[X, D_n(Y)]$ of homotopy classes of maps from X to $D_n(Y)$.*

Proof. Part (a) is a direct consequence of [22, Theorem 8]. For part (b), since $X \times I$ is also a metric space, we may apply [22, Theorem 8] to n -valued maps between $X \times I$ and Y . So it follows that two n -valued maps $\phi_1, \phi_2: X \multimap Y$ are homotopic if and only if the corresponding maps $\Phi_1, \Phi_2: X \rightarrow D_n(Y)$ are homotopic, and the result follows. \square

REMARK 8. Theorem 7 also holds under weaker hypotheses.

(a) If we assume just that Y is a metric space, then the statement of Theorem 7(a) is [12, Corollary 4.1], and that of Theorem 7(b) follows by applying this corollary to the spaces $X \times I$ and Y .

(b) If X is just locally path-connected and semi-locally simply connected then so is $X \times I$. The statement of Theorem 7(a) is [12, Corollary 3.1], and that of Theorem 7(b) follows by applying this corollary to the spaces $X \times I$ and Y .

From now on, we will assume that our spaces are metric spaces, so that we may interpret an n -valued map from X to Y as a map from X to $D_n(Y)$ using Theorem 7(a).

2.1 The case of n -valued maps of the 2-disc

PROPOSITION 9. *The 2-disc \mathbb{D}^2 has the Wecken property for n -valued maps for all $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$, and let $(x_1, \dots, x_n) \in F_n(\mathbb{D}^2)$. By Theorem 7(b), the set of homotopy classes of n -valued maps of the 2-disc \mathbb{D}^2 may be identified with the set $[\mathbb{D}^2, D_n(\mathbb{D}^2)]$ of homotopy classes of maps between \mathbb{D}^2 and $D_n(\mathbb{D}^2)$. Since the domain \mathbb{D}^2 is contractible, this set has only one element, which is the class of the constant map $c = \{c_1, \dots, c_n\} : \mathbb{D}^2 \rightarrow D_n(\mathbb{D}^2)$, where for all $i = 1, \dots, n$, the map $c_i : \mathbb{D}^2 \rightarrow \mathbb{D}^2$ is the constant map at x_i . Then the map c has exactly n fixed points, and $N(c) = n$ by Theorem 1, which proves the proposition. \square

2.2 The case of n -valued maps of the 2-sphere

We now show that the Wecken property holds for n -valued maps of \mathbb{S}^2 for all $n \neq 2$. The case $n = 2$ remains open, although we are able to provide some partial results (see also [11]). Since the case $n = 1$ is well known, see for example [5, Proposition 2.2], we will assume that $n \geq 2$.

LEMMA 10. *If $n > 2$ (resp. $n = 2$), the set of homotopy classes of n -valued maps of \mathbb{S}^2 possesses exactly one element (resp. is in one-to-one correspondence with \mathbb{N}).*

Proof. Theorem 7(b) implies that it suffices to determine the set $[\mathbb{S}^2, D_n(\mathbb{S}^2)]$, which in turn is the orbit space of $\pi_2(D_n(\mathbb{S}^2))$ under the action of $\pi_1(D_n(\mathbb{S}^2))$. Now $F_n(\mathbb{S}^2)$ is a regular $n!$ -fold covering of $D_n(\mathbb{S}^2)$, and if $n = 2$ (resp. $n \geq 3$), the universal covering of $F_n(\mathbb{S}^2)$ has the homotopy type of \mathbb{S}^2 (resp. of \mathbb{S}^3) by [4, 18] or [20, pp. 43–44]. Using standard results about homotopy groups and covering spaces, it follows that $\pi_2(D_n(\mathbb{S}^2))$ is isomorphic to \mathbb{Z} (resp. is trivial) if $n = 2$ (resp. $n > 2$) (see also [17, Corollary, page 211] for the case $n \geq 3$). This proves the result if $n \geq 3$. If $n = 2$, observe that the map $\mathbb{S}^2 \rightarrow F_2(\mathbb{S}^2)$ given by $x \mapsto (x, -x)$ is a homotopy equivalence that is \mathbb{Z}_2 -equivariant with respect to the action of the antipodal map on \mathbb{S}^2 and the action on $F_2(\mathbb{S}^2)$ given by $s : F_2(\mathbb{S}^2) \rightarrow F_2(\mathbb{S}^2)$ defined by $s(x, y) = (y, x)$. This gives rise to a homotopy equivalence between the corresponding orbit spaces, namely $\mathbb{R}P^2$ and $D_2(\mathbb{S}^2)$. Since the action of $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$ on $\pi_2(\mathbb{R}P^2) \cong \mathbb{Z}$ is multiplication by -1 , the same is true for the action of $\pi_1(D_2(\mathbb{S}^2))$ on $\pi_2(D_2(\mathbb{S}^2))$, and the orbits are the subsets of the form $\{m, -m\}$, where $m \in \mathbb{Z}$. It follows that the set $[\mathbb{S}^2, D_2(\mathbb{S}^2)]$ is in bijection with \mathbb{N} . \square

We now study the Wecken property for n -valued maps of \mathbb{S}^2 . Recall that if $\phi : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is a 2-valued map, the integer given by the correspondence with \mathbb{N} of Lemma 10 is the *degree* of ϕ (or of its homotopy class).

Proof of Proposition 4.

(a) Part (a) is clear since the constant map $f_0 : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ at a point $x_0 \in \mathbb{S}^2$ has precisely one fixed point, namely x_0 , and $-x_0$ is the unique fixed point of $A \circ f_0$.

(b) Let f_1 be a self-map of the unit 2-sphere \mathbb{S}^2 that is a small deformation of the identity, i.e. f_1 satisfies $|x - f_1(x)| < \pi/2$ for all $x \in \mathbb{S}^2$, and that has exactly one fixed point x_0 . Such a map may be constructed using a vector field on \mathbb{S}^2 that possesses just one singular point, and in this way, the degree of f_1 is equal to 1. Then $f_1(x) \neq -x$ for all $x \in \mathbb{S}^2$, and thus $A \circ f_1$ is fixed point free, which concludes the proof of part (b).

(c) Consider the self map $\rho: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ given by $z \mapsto z^2$. This map has one fixed point, which is $\{1\}$, and the map $A \circ \rho$ has a single fixed point, which is $\{-1\}$. Now consider the reduced suspension of \mathbb{S}^1 , i.e. $(\mathbb{S}^1 \times [0, 1]) / (\{1\} \times [0, 1] \cup \mathbb{S}^1 \times \{0, 1\}) = \mathbb{S}^2$, and let f_2 be the suspension of the map ρ . Then f_2 has only one fixed point which is the equivalence class of the point $\{1\} \times \{0\}$, and the map $A \circ f_2$ has a single fixed point, which is the equivalence class of $\{-1\} \times \{1/2\}$, so it has the desired property.

(d) To prove part (i), by Lemma 10, there is only one homotopy class of n -valued maps of \mathbb{S}^2 if $n \geq 3$, which is that of the constant n -valued map. The result then follows from part (a). To prove part (ii), the case of the homotopy class of degree 0 follows as in part (i). Now let $i \in \{1, 2\}$. For the homotopy class of degree i , consider the split 2-valued map $\phi_i = \{f_i, A \circ f_i\}: \mathbb{S}^2 \rightarrow \mathbb{S}^2$. Then $L(f_i) = 1 + i$, so $N(f_i) \neq 0$, and thus $N(f_i) = 1$ because \mathbb{S}^2 is simply connected. Hence by Theorem 1, $N(\phi_i) = N(f_i) + N(A \circ f_i)$, and so $N(\phi_1) = 1 + 0 = 1$ and $N(\phi_2) = 1 + 1 = 2$. Since $\text{Fix}(\phi_1) = \{x_0\}$, where x_0 is the fixed point of f_1 given in the proof of part (b), and $\text{Fix}(\phi_2) = \{1, -1\}$, it follows that the map ϕ_i has the Wecken property. \square

3 The case of n -valued maps of the projective plane

The aim of this section is to prove that the projective plane $\mathbb{R}P^2$ has the Wecken property for n -valued maps for all $n \in \mathbb{N}$. Jiang showed that $\mathbb{R}P^2$ has the Wecken property for single-valued maps [29], so it will suffice to study the case $n \geq 2$. We start by computing the Nielsen number of an n -valued map of $\mathbb{R}P^2$.

LEMMA 11. *Let $n \geq 1$, and let $\phi: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$ be an n -valued map. Then $N(\phi) = n$.*

Proof. From [22, Lemma 14], ϕ is split, so for $i = 1, \dots, n$, there exist self-maps $f_i: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$ such that $\phi = \{f_1, \dots, f_n\}$. Applying Theorem 1, $N(\phi) = n$ since $N(f_i) = 1$ for all $i = 1, \dots, n$ by [29]. \square

We now classify the homotopy classes of n -valued maps of $\mathbb{R}P^2$. If $n = 1$, it is well known that the set $[\mathbb{R}P^2, \mathbb{R}P^2]$ of homotopy classes of self-maps of $\mathbb{R}P^2$ that induce the trivial homomorphism on the level of fundamental groups has two elements, see [24, Proposition 2.1] for example. One of these two homotopy classes is that of the constant map. We will describe the second homotopy class in terms of a representative map $W_P: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$ that we shall now define, where $P \in \mathbb{S}^2$, and $\mathbb{S}^2 \subset \mathbb{R}^3$ is the unit sphere in \mathbb{R}^3 , which we equip with spherical coordinates (θ, φ) , where $\theta \in [0, 2\pi)$ and $\varphi \in [-\pi/2, \pi/2]$, so that $P = (\theta, \pi/2)$. With respect to the Cartesian coordinate system for which $P = (0, 0, 1)$ and the point $(1, 0, 0)$ has spherical coordinates $(0, 0)$, the point with spherical coordinates (θ, φ) has Cartesian coordinates $(\cos \varphi \cos \theta, \cos \varphi \sin \theta, \sin \varphi)$. From this, one may see that with respect to the spherical coordinate system, $(\theta, \varphi - \frac{\pi}{2})$ may be identified with $(\theta + \pi, -\varphi - \frac{\pi}{2})$. We regard $\mathbb{R}P^2$ as the quotient of \mathbb{S}^2 by the free action of the group generated by the antipodal map A . Let $p: \mathbb{S}^2 \rightarrow \mathbb{R}P^2$ be the usual covering map, for all $x \in \mathbb{S}^2$, let $\bar{x} = p(x)$, and let H_P^+ be the hemisphere of \mathbb{S}^2 whose pole is P . Let $U_P: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ be the map defined by $U_P(\theta, \varphi) = (\theta, 2\varphi - \frac{\pi}{2})$. The restriction $U_P|_{H_P^+}: H_P^+ \rightarrow \mathbb{S}^2$ sends each semi-meridian lying in H_P^+ that starts at P and ends at the equator linearly to the meridian of \mathbb{S}^2 that starts at P , ends at $-P$ and contains the original semi-meridian. Since U_P sends the whole of the equator to the point $-P$,

$U_P|_{H_P^+}$ induces a map $W'_P: \mathbb{R}P^2 \rightarrow \mathbb{S}^2$ defined by $W'_P(\bar{x}) = U_P(x)$ for all $x \in H_P^+$. Let $W_P: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$ be defined by $W_P = p \circ W'_P$. Now $A(\theta, \varphi) = (\theta + \pi, -\varphi)$, and up to the above-mentioned identification within the spherical coordinate system, for all $(\theta, \varphi) \in \mathbb{S}^2$, we have:

$$U_P \circ A(\theta, \varphi) = U_P(\theta + \pi, -\varphi) = (\theta + \pi, -2\varphi - \frac{\pi}{2}) = (\theta, 2\varphi - \frac{\pi}{2}) = U_P(\theta, \varphi),$$

and it follows that U_P is a lift of W_P . We thus have the following commutative diagram:

$$\begin{array}{ccccc} H_P^+ & \hookrightarrow & \mathbb{S}^2 & \xrightarrow{p} & \mathbb{R}P^2 \\ & \searrow & \downarrow U_P & & \downarrow W_P \\ U_P|_{H_P^+} & & \mathbb{S}^2 & \xrightarrow{p} & \mathbb{R}P^2, \end{array}$$

for which $W_P = p \circ W'_P$ also. The following lemma summarises various properties of W_P .

LEMMA 12. *Let $P \in \mathbb{S}^2$. The map $W_P: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$ satisfies the following properties:*

- (a) $W_P = W_{-P}$, and $\text{Fix}(W_P) = \{p(P)\}$.
- (b) The map W_P is non null-homotopic, so it belongs to the non-constant homotopy class of $[\mathbb{R}P^2, \mathbb{R}P^2]$ that induces the trivial homomorphism on the fundamental group.
- (c) Let $P_1, P_2 \in \mathbb{S}^2$. If $P_1 \neq \pm P_2$, then $\text{Coin}(W_{P_1}, W_{P_2})$ is empty.
- (d) If $c_0: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$ is a constant map then the pair (W_P, c_0) cannot be deformed to a coincidence-free pair.

Proof.

(a) Let $\bar{x} \in \mathbb{R}P^2$, where $\bar{x} = p(x)$ and we take x to belong to H_P^+ . Thus $-x \in H_{-P}^+$, $p(-x) = \bar{x}$, and so $W'_{-P}(\bar{x}) = U_{-P}(-x)$. With respect to the same spherical coordinate system that was used to define U_P , one may see that $U_{-P}(\theta, \varphi) = (\theta, 2\varphi + \frac{\pi}{2})$ for all $(\theta, \varphi) \in \mathbb{S}^2$. From the above definitions, if $x = (\theta, \varphi)$, we have:

$$W_P(\bar{x}) = p \circ W'_P(\bar{x}) = p \circ U_P(x) = p(\theta, 2\varphi - \frac{\pi}{2}),$$

from which it follows that:

$$\begin{aligned} W_{-P}(\bar{x}) &= p \circ W'_{-P}(\bar{x}) = p \circ U_{-P}(-x) = p \circ U_{-P}(\theta + \pi, -\varphi) = p(\theta + \pi, -2\varphi + \frac{\pi}{2}) \\ &= p \circ A(\theta, 2\varphi - \frac{\pi}{2}) = p(\theta, 2\varphi - \frac{\pi}{2}) = W_P(\bar{x}), \end{aligned}$$

hence $W_P = W_{-P}$ as required.

(b) Since W_P factors through \mathbb{S}^2 , it induces the trivial homomorphism on the fundamental group of $\mathbb{R}P^2$. Further, the map W'_P is a lift of W_P , so is non null-homotopic (it represents the non-trivial element of $[\mathbb{R}P^2, \mathbb{S}^2]$) because its absolute degree is congruent to 1 mod 2.

(c) Let $P_1, P_2 \in \mathbb{S}^2$ be such that $P_1 \neq \pm P_2$, let \mathcal{C} be the (unique) great circle that passes through P_1 and P_2 , and let $x \in \mathbb{S}^2$ be such that $\bar{x} \in \text{Coin}(W_{P_1}, W_{P_2})$. Then either $x \in \text{Coin}(U_{P_1}, U_{P_2})$ or $x \in \text{Coin}(U_{P_1}, A \circ U_{P_2})$. Let $i \in \{1, 2\}$. Observe that any two great circles of \mathbb{S}^2 either coincide, or intersect in exactly two (antipodal) points, and that U_{P_i} maps every great circle that passes through P_i to itself. Suppose first that $x \notin \mathcal{C}$, and let \mathcal{C}_i be the great circle that passes through P_i and x . Then $\mathcal{C}_1 \cap \mathcal{C}_2 = \{x, A(x)\}$. Since

$U_{P_i}(x) \in \mathcal{C}_i$ and $x \in \text{Coin}(U_{P_1}, U_{P_2}) \cup \text{Coin}(U_{P_1}, A \circ U_{P_2})$, it follows that $U_{P_1}(x) \in \{x, A(x)\}$, and so $p \circ U_{P_1}(x) = W_{P_1}(\bar{x}) = \bar{x} \in \mathcal{C}$. By part (a), this implies that $\bar{x} = \overline{P_1}$, which yields a contradiction since $x \notin \mathcal{C}$. So assume that $x \in \mathcal{C}$. We write the elements of \mathcal{C} in exponential form, taking P_1 to be $e^{i\pi/2}$. Let ρ be the oriented angle $\widehat{P_1 P_2}$, and let $x = e^{i\varphi}$. Then $U_{P_1}(x) = e^{(2\varphi - \frac{\pi}{2})i}$ and $U_{P_2}(x) = e^{(2(\varphi - \rho) - \frac{\pi}{2})i + \rho i} = U_{P_1}(x) \cdot e^{-\rho i}$. Since $x \in \text{Coin}(U_{P_1}, U_{P_2}) \cup \text{Coin}(U_{P_1}, A \circ U_{P_2})$, we see that $\rho \in \{0, \pi\}$, but this implies that $P_1 \in \{P_2, -P_2\}$, which yields a contradiction. We conclude that W_{P_1} and W_{P_2} are coincidence free.

(d) Suppose on the contrary that the pair (W_P, c_0) can be deformed to a pair of coincidence-free self-maps of $\mathbb{R}P^2$. By [6], there exists a map $h: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$ that is homotopic to W_P such that the pair (h, c_0) is coincidence free, and hence h is non surjective. The maps h and c_0 lift to maps $\tilde{h}, \tilde{c}_0: \mathbb{R}P^2 \rightarrow \mathbb{S}^2$, where \tilde{c}_0 is also a constant map, and the non surjectivity of h implies that of \tilde{h} . Thus \tilde{h} is null homotopic, but then so is h , which yields a contradiction because h is homotopic to W_P , and W_P is non-null homotopic by part (b). \square

In the following proposition, we describe the set $[\mathbb{R}P^2, F_n(\mathbb{R}P^2)]$ (resp. $[\mathbb{R}P^2, D_n(\mathbb{R}P^2)]$) of homotopy classes of maps between $\mathbb{R}P^2$ and $F_n(\mathbb{R}P^2)$ (resp. $D_n(\mathbb{R}P^2)$), and the set of homotopy classes of n -valued maps of $\mathbb{R}P^2$, from which we will see that they each contain two elements. Let $N = (0, 0, 1) \in \mathbb{S}^2$. As we mentioned in the proof of Lemma 11, any n -valued map $\phi: \mathbb{R}P^2 \multimap \mathbb{R}P^2$ is split, and so the set of homotopy classes of n -valued maps of $\mathbb{R}P^2$ is equal to $\text{Spl}(\mathbb{R}P^2, \mathbb{R}P^2, n)/\sim$, where \sim denotes the homotopy equivalence relation in $\text{Spl}(\mathbb{R}P^2, \mathbb{R}P^2, n)$.

PROPOSITION 13. *Let $n \geq 2$.*

(a) *The sets $[\mathbb{R}P^2, F_n(\mathbb{R}P^2)]$, $[\mathbb{R}P^2, D_n(\mathbb{R}P^2)]$ and $\text{Spl}(\mathbb{R}P^2, \mathbb{R}P^2, n)/\sim$ are in bijection, and each possesses two elements.*

(b) *The two homotopy classes of n -valued maps of $\mathbb{R}P^2$ may be described as follows:*

(i) *the first homotopy class consists of those n -valued maps $\phi: \mathbb{R}P^2 \multimap \mathbb{R}P^2$ such that any lift $\widehat{\Phi}: \mathbb{R}P^2 \rightarrow F_n(\mathbb{R}P^2)$ of ϕ induces the trivial homomorphism on the level of fundamental groups, and is homotopic to the constant map between $\mathbb{R}P^2$ and $F_n(\mathbb{R}P^2)$.*

(ii) *if $\phi_n: \mathbb{R}P^2 \multimap \mathbb{R}P^2$ is an n -valued map of $\mathbb{R}P^2$ that represents the second homotopy class, and $\widehat{\Phi}_n: \mathbb{R}P^2 \rightarrow F_n(\mathbb{R}P^2)$ is a lift of ϕ_n , then for all $i = 1, \dots, n$, the composition of $\widehat{\Phi}_n$ with the projection $p_i: F_n(\mathbb{R}P^2) \rightarrow \mathbb{R}P^2$ onto the i^{th} coordinate is homotopic to the map $W_N: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$. Moreover, for all $i = 1, \dots, n+1$, the composition of $\widehat{\Phi}_{n+1}$ with the projection $q_i: F_{n+1}(\mathbb{R}P^2) \rightarrow F_n(\mathbb{R}P^2)$ given by forgetting the i^{th} coordinate is homotopic to $\widehat{\Phi}_n$.*

Proof. Let $n \geq 2$.

(a) We start by showing that the set of homotopy classes $[\mathbb{R}P^2, F_n(\mathbb{R}P^2)]$ of n -ordered maps of $\mathbb{R}P^2$ has two elements. Consider the following Barratt-Puppe sequence:

$$\dots \rightarrow [\mathbb{S}^2, F_n(\mathbb{R}P^2)] \rightarrow [\mathbb{R}P^2, F_n(\mathbb{R}P^2)] \rightarrow [\mathbb{S}^1, F_n(\mathbb{R}P^2)] \rightarrow [\mathbb{S}^1, F_n(\mathbb{R}P^2)] \quad (1)$$

associated with the cofibration sequence $\mathbb{S}^1 \xrightarrow{2} \mathbb{S}^1 \rightarrow \mathbb{R}P^2 \rightarrow \mathbb{S}^2 \rightarrow \mathbb{S}^2 \rightarrow \dots$ for the space $F_n(\mathbb{R}P^2)$, where the map $[\mathbb{S}^1, F_n(\mathbb{R}P^2)] \rightarrow [\mathbb{S}^1, F_n(\mathbb{R}P^2)]$ sends $[\beta]$ to $[\beta^2]$ for all maps $\beta: \mathbb{S}^1 \rightarrow F_n(\mathbb{R}P^2)$. Now $[\mathbb{S}^2, F_n(\mathbb{R}P^2)]$ consists of a single homotopy class

because $\pi_2(F_n(\mathbb{R}P^2)) = 0$ [17, Corollary, p. 244]. Therefore (1) implies that the set $[\mathbb{R}P^2, F_n(\mathbb{R}P^2)]$ is in one-to-one correspondence with the set of elements of $\pi_1(F_n(\mathbb{R}P^2)) = P_n(\mathbb{R}P^2)$ of order less than or equal to 2, namely the trivial element and the full twist Δ_n^2 [19, Proposition 23]. Let $\alpha: \mathbb{R}P^2 \rightarrow F_n(\mathbb{R}P^2)$ be an n -ordered map of $\mathbb{R}P^2$ whose homotopy class $[\alpha]$ corresponds to Δ_n^2 . By (1), the image of $[\alpha]$ in $[\mathbb{S}^1, F_n(\mathbb{R}P^2)]$ is non trivial, and so the induced homomorphism $\alpha_{\#}: \pi_1(\mathbb{R}P^2) \rightarrow P_n(\mathbb{R}P^2)$ is non trivial (and injective). In particular, α is non-homotopic to the constant map $c: \mathbb{R}P^2 \rightarrow F_n(\mathbb{R}P^2)$, from which we conclude that $[\mathbb{R}P^2, F_n(\mathbb{R}P^2)] = \{[c], [\alpha]\}$ has two distinct elements.

We now prove that there are bijections between $[\mathbb{R}P^2, F_n(\mathbb{R}P^2)]$, $[\mathbb{R}P^2, D_n(\mathbb{R}P^2)]$ and $\text{Spl}(\mathbb{R}P^2, \mathbb{R}P^2, n)/\sim$. As we pointed out in the proof of [22, Lemma 9(b)], the covering map $\pi: F_n(\mathbb{R}P^2) \rightarrow D_n(\mathbb{R}P^2)$ induces a map $\hat{\pi}: [\mathbb{R}P^2, F_n(\mathbb{R}P^2)] \rightarrow [\mathbb{R}P^2, D_n(\mathbb{R}P^2)]$ defined by $\hat{\pi}([\Phi]) = [\pi \circ \Phi]$ for all $\Phi \in F_n(\mathbb{R}P^2)^{\mathbb{R}P^2}$. If $\Psi: \mathbb{R}P^2 \rightarrow D_n(\mathbb{R}P^2)$ is a map and $\psi: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$ is the associated n -valued map, then ψ is split by [22, Lemma 14]. It follows that there exists a lift $\hat{\Psi}: \mathbb{R}P^2 \rightarrow F_n(\mathbb{R}P^2)$ of ψ , and this map satisfies $\pi \circ \hat{\Psi} = \Psi$. In particular, $\hat{\pi}$ is surjective, and hence $[\mathbb{R}P^2, D_n(\mathbb{R}P^2)] = \{[\pi \circ c], [\pi \circ \alpha]\}$ has at most two elements. Since $\pi_{\#}: P_n(\mathbb{R}P^2) \rightarrow B_n(\mathbb{R}P^2)$ is inclusion, $(\pi \circ \alpha)_{\#}: \pi_1(\mathbb{R}P^2) \rightarrow B_n(\mathbb{R}P^2)$ is injective, and thus $[\pi \circ c] \neq [\pi \circ \alpha]$. Therefore $\hat{\pi}$ is a bijection.

Finally, by [22, Lemma 9(b)], the set $\text{Spl}(\mathbb{R}P^2, \mathbb{R}P^2, n)/\sim$ is in one-to-one correspondence with the orbits of the set $[\mathbb{R}P^2, F_n(\mathbb{R}P^2)]$ under the action of S_n induced by that of S_n on $F_n(\mathbb{R}P^2)^{\mathbb{R}P^2}$. But $[\mathbb{R}P^2, F_n(\mathbb{R}P^2)] = \{[c], [\alpha]\}$, and since the orbit of the homotopy class $[c]$ of the constant map under the action of S_n must be $\{[c]\}$, the orbit of $[\alpha]$ must be $\{[\alpha]\}$. It follows that $\text{Spl}(\mathbb{R}P^2, \mathbb{R}P^2, n)/\sim$ has precisely two elements.

(b) (i) If $\phi: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$ is an n -valued map of $\mathbb{R}P^2$ such that the associated map $\Phi: \mathbb{R}P^2 \rightarrow D_n(\mathbb{R}P^2)$ belongs to the homotopy class $[\pi \circ c]$ in $[\mathbb{R}P^2, D_n(\mathbb{R}P^2)]$ of the constant map $c: \mathbb{R}P^2 \rightarrow F_n(\mathbb{R}P^2)$, then it follows from the proof of part (a) that any lift $\hat{\Phi}: \mathbb{R}P^2 \rightarrow F_n(\mathbb{R}P^2)$ of ϕ is homotopic to the constant map, and induces the trivial homomorphism on the level of fundamental groups.

(ii) Let $\phi_n: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$ be an n -valued map of $\mathbb{R}P^2$ that represents the second (non-trivial) homotopy class, and let $\hat{\Phi}_n: \mathbb{R}P^2 \rightarrow F_n(\mathbb{R}P^2)$ be a lift of ϕ_n , so that $\hat{\Phi}_n$ is homotopic to the map α given in the proof of part (a). For all $1 \leq i \leq n$, the image of Δ_n^2 under the homomorphism induced by the projection $p_i: F_n(\mathbb{R}P^2) \rightarrow \mathbb{R}P^2$ is trivial (this follows from the proof of [21, Proposition 8] using the fact that Δ_n^2 may be written as a product of the generators $(A_{i,j})_{1 \leq i,j \leq n}$ given in that paper). Since the image of the induced homomorphism $\alpha_{\#}: \pi_1(\mathbb{R}P^2) \rightarrow P_n(\mathbb{R}P^2)$ is equal to $\langle \Delta_n^2 \rangle$, the composition $p_i \circ \hat{\Phi}_n: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$ induces the trivial homomorphism on the level of fundamental groups. Now let P_1, \dots, P_n be n distinct points of \mathbb{S}^2 that lie on the geodesic arc between $(1, 0, 0)$ and $(0, 0, 1)$, and consider the map $(W_{P_1}, \dots, W_{P_n}): \mathbb{R}P^2 \rightarrow F_n(\mathbb{R}P^2)$. The fact that this map is well defined is a consequence of Lemma 12(c). Since $p_i \circ (W_{P_1}, \dots, W_{P_n}) = W_{P_i}$ for all $i = 1, \dots, n$, the map $(W_{P_1}, \dots, W_{P_n})$ is not homotopic to the constant map c , and so it is homotopic to α by the proof of part (a). In particular, $\hat{\Phi}_n$ is homotopic to $(W_{P_1}, \dots, W_{P_n})$, and the statements of part (b)(ii) then follow. \square

We are now able to prove the Wecken property for $\mathbb{R}P^2$.

Proof of Theorem 5. As we mentioned previously, $\mathbb{R}P^2$ has the Wecken property for self-maps. So suppose that $n > 1$. From Proposition 13, there are two homotopy classes of n -valued maps of $\mathbb{R}P^2$, and so by Lemma 11, it suffices to show that each of these

classes admits a representative for which the number of fixed points is equal to n . Let P_1, \dots, P_n be as in the proof of Proposition 13(b)(ii), and let $c_{P_i}: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$ be the constant map at P_i . From Proposition 13, the two homotopy classes of n -valued maps of $\mathbb{R}P^2$ contain an n -valued map $\phi: \mathbb{R}P^2 \multimap \mathbb{R}P^2$ that is split and admits a lift $\widehat{\Phi} = (\phi_1, \dots, \phi_n): \mathbb{R}P^2 \rightarrow F_n(\mathbb{R}P^2)$, where either $\phi_i = c_{\overline{P_i}}$ is the constant map at $\overline{P_i}$ for all $i = 1, \dots, n$, or $\phi_i = W_{P_i}$ for all $i = 1, \dots, n$. Using Lemma 12(a), $\text{Fix}(\phi_i) = \{\overline{P_i}\}$ for all $i = 1, \dots, n$, and hence ϕ has exactly n fixed points. Moreover, by Lemma 11 we have $N(\phi) = n$. So each of the two homotopy classes of n -valued maps contains a representative ϕ for which $N(\phi) = n$, and hence $\mathbb{R}P^2$ has the Wecken property for n -valued maps. \square

4 Nielsen numbers of n -valued maps

The Nielsen number for n -valued maps of a compact polyhedron X was defined in [33], and may be determined for split n -valued maps using Theorem 1. The aim of this section is to prove Theorem 6, where we give a formula for the Nielsen number of non-split n -valued maps of a space X that is in the same spirit as that of Theorem 1. For Theorem 6, we shall require X to be a compact, orientable manifold without boundary, in order to have the notions of index, Lefschetz and Nielsen numbers for coincidences of pairs of maps from finite coverings of X to X at our disposal. However, many of the results that lead to Theorem 6 are valid under weaker hypotheses on X , namely those of Proposition 14 below.

We start by recalling some notation and results from [22, Section 3.2] that will be used throughout the rest of the paper. Given an n -valued map $\phi: X \multimap X$ of a topological space X that is locally path-connected and semi-locally simply connected, we consider the corresponding map $\Phi: X \rightarrow D_n(X)$, and the induced homomorphism $\Phi_{\#}: \pi_1(X) \rightarrow \pi_1(D_n(X))$ on the level of fundamental groups, where $\pi_1(D_n(X)) = B_n(X)$. By the short exact sequence

$$1 \longrightarrow P_n(X) \longrightarrow B_n(X) \xrightarrow{\tau} S_n \longrightarrow 1,$$

$P_n(X)$ is a normal subgroup of $B_n(X)$ of finite index $n!$, so the subgroup $H = \Phi_{\#}^{-1}(P_n(X))$ is a normal subgroup of $\pi_1(X)$ of finite index. Let L be the finite quotient group $\pi_1(X)/H$, and let $q: \widehat{X} \rightarrow X$ be the covering of X that corresponds to the subgroup H . Such a covering exists due to the hypotheses on X . As the following proposition shows, the fixed points of ϕ may be described in terms of the coincidences of q with the coordinate maps $f_1, \dots, f_n: \widehat{X} \rightarrow X$ of a lift of the n -valued map $\phi \circ q: \widehat{X} \multimap X$.

PROPOSITION 14. [22, Propositions 16 and 17] *Let $n \in \mathbb{N}$, and suppose that X is a connected, locally arcwise-connected metric space.*

(a) *With the above notation, the n -valued map $\phi_1 = \phi \circ q: \widehat{X} \multimap X$ admits exactly $n!$ lifts, which are n -ordered maps from \widehat{X} to $F_n(X)$. If one such lift $\widehat{\Phi}_1: \widehat{X} \rightarrow F_n(X)$ is given by $\widehat{\Phi}_1 = (f_1, \dots, f_n)$, where for $i = 1, \dots, n$, f_i is a map from \widehat{X} to X , then the other lifts are of the form $(f_{\rho(1)}, \dots, f_{\rho(n)})$, where $\rho \in S_n$.*

(b) *if the lift $\widehat{\Phi}_1 = (f_1, \dots, f_n)$ is as in part (a) then the restriction of $q: \widehat{X} \rightarrow X$ to $\bigcup_{i=1}^n \text{Coin}(q, f_i) \rightarrow \text{Fix}(\phi)$ is surjective. Furthermore, the pre-image of a point $x \in \text{Fix}(\phi)$ by this map is precisely $q^{-1}(x)$, namely the fibre over $x \in X$ of the covering map q .*

Although the lift $\widehat{\Phi}_1$ is not unique, the set $\{f_1, \dots, f_n\}$ is, and so the set $\bigcup_{i=1}^n \text{Coin}(q, f_i)$ is independent of the choice of lift of ϕ . In what follows, we aim to describe the Nielsen

classes of ϕ in terms of the Nielsen coincidence classes of the pairs (q, f_i) , where $i = 1, \dots, n$, which will lead to a formula for $N(\phi)$ similar in spirit to that of Theorem 1. Observe that the composition $\pi_1(X) \xrightarrow{\Phi_{\#}} B_n(X) \xrightarrow{\tau} S_n$ is a homomorphism whose kernel is H , so it induces an injective homomorphism $\Gamma: L \rightarrow S_n$. Let $L' = \text{Im}(\Gamma) = \text{Im}(\tau \circ \Phi_{\#})$, and for $i, j = 1, \dots, n$, let $L'_{i,j} = \{\rho \in L' \mid \rho(i) = j\}$. The subset $L'_{i,i}$ is a subgroup of L' , and if $i, j \in \{1, \dots, n\}$, the subset $L'_{i,j}$ is either empty or is a left coset of $L'_{i,i}$ in L' .

In the rest of this section, we will suppose without further comment that X satisfies the hypotheses of Proposition 14, so that it is a connected, locally arcwise-connected metric space. If $\phi: X \multimap X$ is an n -valued map, we recall the Nielsen relation on $\text{Fix}(\phi)$, the index of a Nielsen fixed point class of ϕ and the definition of the Nielsen number $N(\phi)$ of ϕ from [33, Section 5]. For the definition of index, using [32, Theorem 6] and the homotopy invariance of the Nielsen number [32, Theorem 6.5], without loss of generality, we may restrict ourselves to the case where $\text{Fix}(\phi)$ is finite. First note that by [22, Lemma 12], if $\lambda: I \rightarrow X$ is a path then the n -valued map $\phi \circ \lambda: I \multimap X$ is split. Let $x, x' \in \text{Fix}(\phi)$. We say that x and x' are Nielsen equivalent if there exist maps $g_1, g_2, \dots, g_n: I \rightarrow X$, a path $\lambda: I \rightarrow X$ from $\lambda(0) = x$ to $\lambda(1) = x'$ and $j \in \{1, \dots, n\}$ such that $\phi \circ \lambda = \{g_1, g_2, \dots, g_n\}$, and g_j is a path from $g_j(0) = x$ to $g_j(1) = x'$ that is homotopic to λ relative to their endpoints. This defines an equivalence relation on $\text{Fix}(\phi)$, and the resulting equivalence classes are called *Nielsen fixed point classes* of the n -valued map ϕ .

To define the index of an isolated point x in $\text{Fix}(\phi)$, we suppose that X is a compact polyhedron. Following [33, Section 3], let x be in the interior of a maximal simplex $\bar{\sigma}$. By [33, Splitting Lemma 2.1] $\phi|_{\bar{\sigma}}$ is split, so may be written in the form $\phi|_{\bar{\sigma}} = \{f_1, \dots, f_n\}$, where $x \in \text{Fix}(f_j)$ for some (unique) $1 \leq j \leq n$. We then define $\text{Ind}(\phi, x) = \text{Ind}(f_j, x)$, where the right-hand side is the usual fixed point index (see [33, Sections 3 and 5] for more details). As in the single-valued case, the *index* of a Nielsen fixed point class of ϕ that contains a finite number of points is the sum of the indices of these fixed points, and such a fixed point class is said to be *essential* if its index is non zero. The *Nielsen number* $N(\phi)$ is defined to be the number of essential Nielsen fixed point classes of ϕ .

REMARK 15. Within our framework, the maps g_1, g_2, \dots, g_n may be chosen as follows: given $x_0, x'_0 \in \text{Fix}(\phi)$, a point $\tilde{x}_0 \in \widehat{X}$ such that $q(\tilde{x}_0) = x_0$, and a path $\lambda: I \rightarrow X$ from x_0 to x'_0 , let $\tilde{\lambda}: I \rightarrow \widehat{X}$ be the unique lift of λ to \widehat{X} for which $\tilde{\lambda}(0) = \tilde{x}_0$. Consider the n -ordered map $\widehat{\Phi}_1 = (f_1, \dots, f_n): \widehat{X} \rightarrow F_n(X)$ given by Proposition 14(a). Then $\phi \circ \lambda = \{g_1, \dots, g_n\}$, where for $i = 1, \dots, n$, $g_i: I \rightarrow X$ is the map defined by $g_i = f_i \circ \tilde{\lambda}$. So x_0 and x'_0 are Nielsen equivalent if there is a path λ as above and $j \in \{1, \dots, n\}$ such that $\lambda(0) = g_j(0) = f_j \circ \tilde{\lambda}(0)$, $\lambda(1) = g_j(1) = f_j \circ \tilde{\lambda}(1)$ and λ is homotopic to g_j relative to their endpoints.

In the following lemmas, we will compare the coincidences of q and the f_i with the fixed points of ϕ .

LEMMA 16. *With the above notation, let $x_0 \in \text{Fix}(\phi)$, let $\tilde{x}_0 \in \widehat{X}$ and $i \in \{1, \dots, n\}$ be such that $q(\tilde{x}_0) = x_0$ and $\tilde{x}_0 \in \text{Coin}(q, f_i)$. If $y \in q^{-1}(x_0)$ and $j \in \{1, \dots, n\}$ then $y \in \text{Coin}(q, f_j)$ if and only if $\Gamma(\alpha) \in L'_{i,j}$, where α is the H -coset of $[q(\gamma)]$, γ being any path from \tilde{x}_0 to y . In particular, the points of $q^{-1}(x_0)$ that belong to $\text{Coin}(q, f_i)$ are in one-to-one correspondence with the subgroup $L'_{i,i}$.*

Since $L'_{i,i}$ does not depend on x_0 , Lemma 16 implies that for all $x \in \text{Fix}(\phi)$, the set $q^{-1}(x) \cap \text{Coin}(q, f_i)$ has $|L'_{i,i}|$ elements.

Proof of Lemma 16. The proof makes use of basic covering space theory. Let $\alpha \in L$ be the unique deck transformation for which y is equal to the element $\alpha.\tilde{x}_0$ of \widehat{X} that arises from the action of deck transformation group L on \widehat{X} . Then $\Gamma(\alpha) \in L'$ defines a deck transformation of the covering $\pi: F_n(X) \rightarrow D_n(X)$. Using the fact that q and π are covering maps and $\widehat{\Phi}_1$ is a lift of Φ , we have:

$$\widehat{\Phi}_1(\alpha.\tilde{x}) = \Gamma(\alpha).\widehat{\Phi}_1(\tilde{x}) \quad (2)$$

for all $\tilde{x} \in q^{-1}(x_0)$, where $\Gamma(\alpha).\widehat{\Phi}_1(\tilde{x})$ is the element of $F_n(X)$ arising from the action of S_n on $F_n(X)$. Since $\tilde{x}_0 \in \text{Coin}(q, f_i)$, $\widehat{\Phi}_1(\tilde{x}_0)$ is an element of $F_n(X)$ whose i^{th} coordinate is x_0 , and $\Gamma(\alpha).\widehat{\Phi}_1(\tilde{x}_0)$ is an element of $F_n(X)$ whose $\Gamma(\alpha)(i)^{\text{th}}$ coordinate is x_0 . So if $y \in \text{Coin}(q, f_j)$, the j^{th} coordinate of $\widehat{\Phi}_1(y)$ is x_0 , and thus $\Gamma(\alpha) \in L'_{i,j}$. Conversely, if $\Gamma(\alpha) \in L'_{i,j}$ then $\Gamma(\alpha)(i) = j$, so the j^{th} coordinate of $\widehat{\Phi}_1(y)$ is x_0 , and hence $y \in \text{Coin}(q, f_j)$. For the last part of the statement, since $L'_{i,i}$ is a subgroup of L' , it suffices to take $j = i$. \square

With Lemma 16 in mind, we define the following notation. For each $i \in \{1, \dots, n\}$, let \mathbb{O}_i be the orbit of i by the action of the subgroup L' of S_n on the set $\{1, \dots, n\}$, and let $I_0 = \{i_1, \dots, i_s\}$ be such that the sets $\{\mathbb{O}_i\}_{i \in I_0}$ form a partition of $\{1, \dots, n\}$. As examples, if $H = \pi_1(X)$ (resp. $L' = S_n$) then $\mathbb{O}_i = \{i\}$ (resp. $\mathbb{O}_i = S_n$) for all $i \in \{1, \dots, n\}$. The following result underlines the relevance of these orbits.

LEMMA 17. *With the above notation, let $x_0 \in \text{Fix}(\phi)$. Then there exists $i \in \{1, \dots, n\}$ such that:*

$$\{j \in \{1, \dots, n\} \mid q^{-1}(x_0) \cap \text{Coin}(q, f_j) \neq \emptyset\} = \mathbb{O}_i.$$

Proof. The proof uses arguments similar to those of Lemma 16. Let $x_0 \in \text{Fix}(\phi)$, and let $\tilde{x}_0 \in \widehat{X}$ be a lift of x_0 . By Proposition 14(b), \tilde{x}_0 belongs to $\text{Coin}(q, f_i)$ for some $i \in \{1, \dots, n\}$. First, suppose that $y \in q^{-1}(x_0) \cap \text{Coin}(q, f_j)$ for some $j \in \{1, \dots, n\}$, and let $\alpha \in L$ be such that $\alpha.\tilde{x}_0 = y$. From (2), $\Gamma(\alpha)(i) = j$, so $j \in \mathbb{O}_i$. Conversely, suppose that $j \in \mathbb{O}_i$. Then there exists $\alpha \in L$ such that $\Gamma(\alpha)(i) = j$, and taking $y = \alpha.\tilde{x}_0$, we have $y \in q^{-1}(x_0) \cap \text{Coin}(q, f_j)$. \square

Note that by Proposition 14(b) and Lemma 17, $\text{Fix}(\phi) = q(\bigcup_{j \in I_0} \text{Coin}(q, f_j))$. Since we wish to express the Nielsen number of ϕ in terms of the Nielsen coincidence numbers of the pairs (q, f_i) , for the values of i belonging to I_0 , we shall compare the Nielsen coincidence relation and the Nielsen coincidence number of the pairs (q, f_i) with the Nielsen relation and the Nielsen number for ϕ respectively.

LEMMA 18. *With the above notation, let $i \in \{1, \dots, n\}$, and let y_1 and y_2 be elements of $\text{Coin}(q, f_i)$ that belong to the same Nielsen coincidence class of the pair (q, f_i) . Then $q(y_1)$ and $q(y_2)$ are elements of $\text{Fix}(\phi)$ that belong to the same Nielsen fixed point class of the n -valued map ϕ . Further, q sends each Nielsen coincidence class of the pair (q, f_i) surjectively onto a Nielsen fixed point class of ϕ .*

Proof. Let y_1 and y_2 be elements of $\text{Coin}(q, f_i)$ for some $i \in \{1, \dots, n\}$ that belong to the same Nielsen coincidence class of the pair (q, f_i) . By Proposition 14(b), $q(y_1)$ and

$q(y_2)$ are fixed points of ϕ . Since y_1 and y_2 belong to the same Nielsen coincidence class of (q, f_i) , there exists a path $\tilde{\lambda}: I \rightarrow \widehat{X}$ from y_1 to y_2 such that the path $f_i \circ \tilde{\lambda}$ is homotopic in X to λ relative to the endpoints $q(y_1)$ and $q(y_2)$, where $\lambda: I \rightarrow X$ is the path defined by $\lambda = q \circ \tilde{\lambda}$. Now the n -valued map $\phi \circ \lambda: I \rightarrow X$ is split by [22, Lemma 12], and by Remark 15, $\phi \circ \lambda = \{g_1, \dots, g_n\}$, where for $j = 1, \dots, n$, $g_j = f_j \circ \tilde{\lambda}$. So $g_i(0) = f_i \circ \tilde{\lambda}(0) = f_i(y_1) = q(y_1)$, $g_i(1) = f_i \circ \tilde{\lambda}(1) = f_i(y_2) = q(y_2)$, and g_i is homotopic to λ relative to the endpoints $q(y_1)$ and $q(y_2)$, from which we deduce that $q(y_1)$ and $q(y_2)$ belong to the same Nielsen fixed point class of ϕ . To prove the second part of the statement, by the first part, it suffices to show that if x is a fixed point of ϕ that belongs to the same Nielsen fixed point class of ϕ as $q(y_1)$ then there exists $y \in \text{Coin}(q, f_i)$ such that $q(y) = x$, and y and y_1 belong to the same Nielsen coincidence class of the pair (q, f_i) . To see this, note that by Remark 15, there exist a path $\lambda: I \rightarrow X$ from $q(y_1)$ to x , a lift $\tilde{\lambda}: I \rightarrow \widehat{X}$ such that $\tilde{\lambda}(0) = y_1$ and $j \in \{1, \dots, n\}$ such that $\lambda(0) = g_j(0) = f_j \circ \tilde{\lambda}(0)$, $\lambda(1) = g_j(1) = f_j \circ \tilde{\lambda}(1)$, and λ is homotopic to g_j relative to their endpoints. In particular, $f_j(y_1) = q(y_1)$, and so $j = i$ because $y_1 \in \text{Coin}(q, f_i)$. Further, if $y = \lambda(1)$ then $q(y) = x$ because $\tilde{\lambda}$ is a lift of λ , and $x = \lambda(1) = f_i(y)$, so $y \in \text{Coin}(q, f_i)$. Finally, the paths λ and $g_i = f_i \circ \tilde{\lambda}$ are homotopic in X relative to their endpoints, and hence y_1 and y belong to the same Nielsen coincidence class of the pair (q, f_i) as required. \square

In order to obtain a formula for $N(\phi)$, another ingredient that we require is the number of points of $q^{-1}(x_0) \cap \text{Coin}(q, f_i)$ that belong to the same Nielsen coincidence class of the pair (q, f_i) , where $x_0 \in \text{Fix}(\phi)$ and $i \in I_0$. Suppose that $q^{-1}(x_0) \cap \text{Coin}(q, f_i) \neq \emptyset$, and let $\tilde{x}_0, y \in \widehat{X}$ be elements of this intersection. There exists a unique $\mu \in L = \pi_1(X)/H$ such that $y = \mu \cdot \tilde{x}_0$. Let L_i be the subset of L consisting of such μ as y runs over the elements of $q^{-1}(x_0) \cap \text{Coin}(q, f_i)$. Note that $L_i = \{\mu \in L \mid \Gamma(\mu)(i) = i\}$, in particular, L_i is independent of \tilde{x}_0 , L_i is a subgroup of L , the order of L_i is equal to the cardinality of $q^{-1}(x_0) \cap \text{Coin}(q, f_i)$, and $L'_{i,i} = \Gamma(L_i)$, so $L'_{i,i} \cong L_i$. If $\mu \in L_i$, consider the corresponding element $y \in q^{-1}(x_0) \cap \text{Coin}(q, f_i)$ defined by $y = \mu \cdot \tilde{x}_0$, and let $\gamma: I \rightarrow \widehat{X}$ be a path from \tilde{x}_0 to y . Then $f_i \circ \gamma$ and $q \circ \gamma$ are loops in X based at x_0 . Let $W_{\tilde{x}_0}(\mu)$ be the subset of $\pi_1(X)$ of loop classes of the form $[(q \circ \gamma) * (f_i \circ \gamma)^{-1}]$, where γ runs over the set of paths from \tilde{x}_0 to y . Observe that $W_{\tilde{x}_0}(\mu)$ contains the trivial element of $\pi_1(X)$ if and only if \tilde{x}_0 and y belong to the same Nielsen coincidence class for the pair (q, f_i) . With this in mind, let $K_i(\tilde{x}_0)$ be the set of elements $y \in q^{-1}(x_0) \cap \text{Coin}(q, f_i)$ for which $W_{\tilde{x}_0}(\mu)$ contains the trivial element, where $\mu \in L_i$ is such that $y = \mu \cdot \tilde{x}_0$. Then $K_i(\tilde{x}_0)$ is the subset of elements $q^{-1}(x_0) \cap \text{Coin}(q, f_i)$ that belong to the same Nielsen coincidence class of the pair (q, f_i) as \tilde{x}_0 . Let $|K_i(\tilde{x}_0)|$ denote the cardinality of $K_i(\tilde{x}_0)$.

LEMMA 19. *With the above notation, let $x_0 \in \text{Fix}(\phi)$, and let $\tilde{x}_0 \in q^{-1}(x_0) \cap \text{Coin}(q, f_i)$.*

(a) *The number of coincidence points of the pair (q, f_i) that belong to $q^{-1}(x_0)$ is equal to the order of the subgroup L_i .*

(b) *If $\tilde{z} \in q^{-1}(x_0) \cap \text{Coin}(q, f_i)$, the set $K_i(\tilde{z})$ of points of $q^{-1}(x_0)$ that belong to the same Nielsen coincidence class for the pair (q, f_i) as \tilde{z} is in one-to-one correspondence with the set $K_i(\tilde{x}_0)$.*

(c) *Let $\tilde{z} \in \text{Coin}(q, f_i)$ be such that \tilde{x}_0 and \tilde{z} belong to the same Nielsen coincidence class for the pair (q, f_i) . Then $K_i(\tilde{z})$ is in one-to-one correspondence with the set $K_i(\tilde{x}_0)$.*

Proof.

(a) This follows from Lemma 16 and the isomorphism $L_i \cong L'_{i,i}$.

(b) Let us construct an injective map from $K_i(\tilde{x}_0)$ to $K_i(\tilde{z})$. Given $y \in K_i(\tilde{x}_0)$, by definition, there exists a path $\gamma: I \rightarrow \widehat{X}$ such that $\gamma(0) = \tilde{x}_0$, $\gamma(1) = \tilde{y}$ and $q \circ \gamma$ is homotopic to $f_i \circ \gamma$ relative to their endpoints. Let γ_1 be the unique lift of $q \circ \gamma$ for which $\gamma_1(0) = \tilde{z}$. Since $q \circ \gamma$ is a loop in X based at x_0 , $\gamma_1(1) \in q^{-1}(x_0)$. We claim that $\gamma_1(1) \in K_i(\tilde{z})$. To see this, let us show that $q \circ \gamma_1 = q \circ \gamma$ is homotopic to $f_i \circ \gamma_1$ relative to their endpoints. It suffices to observe that $f_i \circ \gamma = f_i \circ \gamma_1$. Since the two compositions $\widehat{X} \xrightarrow{q} X \xrightarrow{\Phi} D_n(X)$ and $\widehat{X} \xrightarrow{\widehat{\Phi}_1} F_n(X) \xrightarrow{\pi} D_n(X)$ are equal, we have $\pi(\widehat{\Phi} \circ \gamma_1) = \pi(\widehat{\Phi} \circ \gamma)$, in other words, the n -tuple $(f_1 \circ \gamma, \dots, f_n \circ \gamma)$ of paths is a permutation of the n -tuple $(f_1 \circ \gamma_1, \dots, f_n \circ \gamma_1)$ of paths. Since these two n -tuples are paths in $F_n(X)$ and $f_i \circ \gamma(0) = f_i \circ \gamma_1(0) = x_0$, it follows that $f_i \circ \gamma = f_i \circ \gamma_1$, and hence $f_i \circ \gamma_1$ is homotopic to $q \circ \gamma_1$ relative to their endpoints. In particular, $f_i \circ \gamma_1(1) = f_i \circ \gamma(1) = q \circ \gamma(1) = q \circ \gamma_1(1)$, so $\gamma_1(1) \in q^{-1}(x_0) \cap \text{Coin}(q, f_i)$, and thus $\gamma_1(1) \in K_i(\tilde{z})$, which proves the claim. By construction, the map from $K_i(\tilde{x}_0)$ to $K_i(\tilde{z})$ that to y associates $\gamma_1(1)$ is injective. Exchanging the rôles of \tilde{x}_0 and z , we see that $K_i(\tilde{x}_0)$ and $K_i(\tilde{z})$ have the same number of elements, and that this map is a bijection.

(c) Let $\lambda: I \rightarrow \widehat{X}$ be a path such that $\lambda(0) = \tilde{x}_0$, $\lambda(1) = \tilde{z}$ and $q \circ \lambda$ is homotopic to $f_i \circ \lambda$ relative to their endpoints. Let $y \in q^{-1}(x_0) \cap \text{Coin}(q, f_i)$ be a point that belongs to the same Nielsen coincidence class of (q, f_i) as \tilde{x}_0 , and let $\gamma: I \rightarrow \widehat{X}$ be a path such that $\gamma(0) = \tilde{x}_0$, $\gamma(1) = y$ and $q \circ \gamma$ is homotopic to $f_i \circ \gamma$ relative to their endpoints. Let $\lambda': I \rightarrow \widehat{X}$ be the unique lift of $q \circ \lambda$ for which $\lambda'(0) = y$, and let $\gamma': I \rightarrow \widehat{X}$ be the path defined by $\gamma' = \lambda^{-1} * \gamma * \lambda'$. Then $\gamma'(0) = \tilde{z}$, $q \circ \gamma'(1) = q \circ \lambda'(1) = q \circ \lambda(1) = q(\tilde{z})$, so $\gamma'(1) \in q^{-1}(q(\tilde{z}))$. As in the proof of part (b), $q \circ \lambda'$ is homotopic to $f_i \circ \lambda'$ relative to their endpoints, and it follows that $q \circ \gamma'$ is homotopic to $f_i \circ \gamma'$ relative to their endpoints, so $\gamma'(1) \in K_i(\tilde{z})$. We thus obtain an injective map from $K_i(\tilde{x}_0)$ and $K_i(\tilde{z})$ that to y associates $\gamma'(1)$. By exchanging the rôles of \tilde{x}_0 and \tilde{z} , we see that this map is a bijection. \square

If $i \in \{1, \dots, n\}$, by Lemma 19, the set $q^{-1}(x_0) \cap \text{Coin}(q, f_i)$ contains $|L_i|$ points, and is partitioned into Nielsen coincidence classes of the pair (q, f_i) that each contain $|K_i(\tilde{x}_0)|$ points. So this set is partitioned into $|L_i|/|K_i(\tilde{x}_0)|$ disjoint subsets each of which is contained in a Nielsen coincidence class of (q, f_i) . If W_1, W_2 are Nielsen coincidence classes of (q, f_i) , we say that they are *related* if $q(W_1) \cap q(W_2) \neq \emptyset$. This defines an equivalence relation. Let C_1, \dots, C_r denote the corresponding set of equivalence classes of the Nielsen coincidence classes of (q, f_i) .

LEMMA 20. *With the above notation, for $j = 0, 1$, let $x_j \in \text{Fix}(\phi)$, and suppose that $\tilde{x}_j \in q^{-1}(x_j) \cap \text{Coin}(q, f_i)$ for some $i \in \{1, \dots, n\}$. If \tilde{x}_0 and \tilde{x}_1 belong to related Nielsen coincidence classes of (q, f_i) then $|K_i(\tilde{x}_0)| = |K_i(\tilde{x}_1)|$.*

Proof. We shall construct a bijection between $K_i(\tilde{x}_0)$ and $K_i(\tilde{x}_1)$. First, we claim that there is an element $\tilde{x}'_1 \in q^{-1}(q(\tilde{x}_1))$ that belongs to the same Nielsen coincidence class of (q, f_i) as \tilde{x}_0 . To see this, let $\tilde{x}_2, \tilde{x}'_2 \in \text{Coin}(q, f_i)$ be such that $q(\tilde{x}_2) = q(\tilde{x}'_2)$, and for which \tilde{x}_2 (resp. \tilde{x}'_2) belongs to the same Nielsen coincidence class of (q, f_i) as \tilde{x}_0 (resp. as \tilde{x}_1). Let $\lambda: I \rightarrow \widehat{X}$ be a path such that $\lambda(0) = \tilde{x}'_2$, $\lambda(1) = \tilde{x}_1$ and $q \circ \lambda$ is homotopic to $f_i \circ \lambda$ relative to their endpoints, and let $\lambda': I \rightarrow \widehat{X}$ be the unique lift of $q \circ \lambda$ for which $\lambda'(0) = \tilde{x}_2$, and let $\tilde{x}'_1 = \lambda'(1)$. Then $q(\tilde{x}'_1) = x_1$, and arguing as in the proof of Lemma 19, it follows that $q \circ \lambda'$ is homotopic to $f_i \circ \lambda'$ relative to their endpoints. Thus \tilde{x}_2 and \tilde{x}'_1

belong to the same Nielsen coincidence class of (q, f_i) , which proves the claim since \tilde{x}_2 and \tilde{x}_0 belong to the same Nielsen coincidence class of (q, f_i) . Then by Lemma 19(b) and (c), we have $|K_i(\tilde{x}_0)| = |K_i(\tilde{x}_2)| = |K_i(\tilde{x}'_2)| = |K_i(\tilde{x}_1)|$ as required. \square

The following lemma is a key ingredient in the process of giving a computable formula for $N(\phi)$, which is the number of essential Nielsen classes. We fix once and for all an orientation of the manifold X , and we choose the unique orientation of \widehat{X} for which q preserves orientation. Further, the orientation on \widehat{X} (resp. X) induces an orientation on any open subset of \widehat{X} (resp. of X), and hence a local orientation system on \widehat{X} (resp. on X). The map q carries the local orientation of \widehat{X} to that of X . The fixed point index of maps defined on open sets of X as well as the coincidence index of maps from open sets of \widehat{X} to X , are computed with respect to these orientations.

LEMMA 21. *Let X be an orientable manifold, $x_0 \in \text{Fix}(\phi)$ be an isolated fixed point of ϕ , and let $\tilde{x}_0 \in \widehat{X}$ and $i \in \{1, \dots, n\}$ be such that $q(\tilde{x}_0) = x_0$ and $\tilde{x}_0 \in \text{Coin}(q, f_i)$. Then the fixed point index of ϕ at x_0 is equal to the coincidence index of the pair (q, f_i) at \tilde{x}_0 .*

Proof. Since X is a manifold and \widehat{X} is a finite covering of X , there exists an open, contractible neighbourhood U of x_0 such that the restriction $q|_{\tilde{U}}$ of q to the component \tilde{U} of $q^{-1}(U)$ that contains \tilde{x}_0 is a homeomorphism. The restriction of the n -valued map ϕ to U is split, and a splitting is given by $\{\bar{f}_1, \dots, \bar{f}_n\}$, where $\bar{f}_j = f_j \circ (q|_{\tilde{U}})^{-1}: U \rightarrow X$ for $j = 1, \dots, n$. So x_0 is a fixed point of the map \bar{f}_j for some $j \in \{1, \dots, n\}$, and by definition, $\text{Ind}(\phi, x_0)$ is equal to $\text{Ind}(\bar{f}_j, x_0)$. Since $q|_{\tilde{U}}$ is an orientation-preserving homeomorphism, $\text{Ind}(\bar{f}_j, x_0)$ is equal to the coincidence index of the pair (q, f_j) at the coincidence \tilde{x}_0 , where we use the local orientation in a neighbourhood of \tilde{x}_0 determined by the local homeomorphism q , and this proves the lemma. \square

One consequence of Lemma 21 is that if a Nielsen coincidence class of (q, f_i) is sent to a Nielsen fixed point class of ϕ then one of these Nielsen classes is essential if and only if the other is.

COROLLARY 22. *Under the hypothesis of Lemma 21, if C is a Nielsen coincidence class of the pair (q, f_i) and $\tilde{x}_0 \in C$, then the coincidence index of C is equal to $|K_i(\tilde{x}_0)|$ times the index of the fixed point class $q(C)$ of ϕ .*

Proof. Let C be a Nielsen coincidence class of the pair (q, f_i) for some $i \in \{1, \dots, n\}$. By Lemma 21, the points of C that lie in $q^{-1}(q(\tilde{x}_0))$ all have the same coincidence index for the pair (q, f_i) , this index being equal to $\text{Ind}(\phi, x_0)$. By Lemma 20, the cardinality of the set $C \cap q^{-1}(q(\tilde{x}_0))$ is equal to $|K_i(\tilde{x}_0)|$. If \tilde{x}_1 is another point of C (so is Nielsen equivalent to \tilde{x}_0), the same conclusions hold, and similarly the cardinality of the set $C \cap q^{-1}(q(\tilde{x}_1))$ is equal to $|K_i(\tilde{x}_0)|$, and all of the points of this set have the same coincidence index for the pair (q, f_i) . Since the index of C is the sum of the indices over the elements of the class C , which we can assume to be finite, the result follows. \square

For results related with Lemma 21 and Corollary 22 and more results of similar nature, see [26, 31].

If $s \in \{1, \dots, r\}$ and $i \in I_0$, let $m_{i,s} = |L_i|/|K_i(\tilde{x}_s)|$, where \tilde{x}_s is an element of one of the Nielsen coincidence classes of (q, f_i) that belongs to the equivalence class C_s . This quantity is the number of Nielsen coincidence classes of (q, f_i) in C_s that are sent to the

same Nielsen fixed point class of ϕ under the map q . Observe that if $|L_i| = 1$ then $m_{i,s} = 1$ for all $s \in \{1, \dots, r\}$.

We now come to the proof of the main result of this section.

Proof of Theorem 6. Let $i_j \in \mathbb{O}_j$, where j runs over the set I_0 defined just before Lemma 17. From Lemma 16, the image by q of $\text{Coin}(q, f_{i_j})$ is independent of the choice of representative i_j in \mathbb{O}_j . Since the action of $L'_{i,i}$ on $\{1, \dots, n\}$ is free, $|L'_{i,i}| = |L_i| = 1$, which implies that the map induced by q between the Nielsen coincidence classes of (q, f_i) and the Nielsen fixed point classes of ϕ is injective for all $i \in \{1, \dots, n\}$. Further, if $j, j' \in I_0$ are distinct then $q(\text{Coin}(q, f_{i_j})) \cap q(\text{Coin}(q, f_{i_{j'}})) = \emptyset$ by Lemma 16, and we conclude that the map between the union of the Nielsen coincidence classes of the pairs (q, f_{i_j}) , where $i_j \in I_0$, and the Nielsen fixed point classes of ϕ is injective too. This map is also surjective by Proposition 14(b) and Lemma 18. By Corollary 22, a coincidence class of (q, f_{i_j}) is essential if and only if its image under q is essential. From this, it follows that $N(\phi)$ is equal to $\sum_{j \in I_0} N(q, f_{i_j})$. \square

If the space X is a non-orientable manifold, the situation is more complex, and it is not clear for the moment how to obtain a formula similar to that of Theorem 6 in this case. In the case $n = 2$, the hypothesis of that theorem on the action of L_i is always satisfied.

COROLLARY 23. *Let X be an orientable compact manifold without boundary and $\phi: X \multimap X$ a non-split 2-valued map. Then $N(\phi) = N(q, f_1) = N(q, f_2)$.*

Proof. Since ϕ is non-split, $L \cong L' \cong \mathbb{Z}_2$, there is a single orbit $\mathbb{O}_1 = \{1, 2\}$, and $|L_1| = |L_2| = 1$. Then by Theorem 6, $N(\phi) = N(q, f_j)$ for all $j \in \{1, 2\}$. \square

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