



**HAL**  
open science

# Fixed points of $n$ -valued maps on surfaces and the Wecken property – a configuration space approach

Daciberg Lima Gonçalves, John Guaschi

► **To cite this version:**

Daciberg Lima Gonçalves, John Guaschi. Fixed points of  $n$ -valued maps on surfaces and the Wecken property – a configuration space approach. 2017. hal-01469497v1

**HAL Id: hal-01469497**

**<https://hal.science/hal-01469497v1>**

Preprint submitted on 16 Feb 2017 (v1), last revised 24 Apr 2017 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Fixed points of $n$ -valued maps on surfaces and the Wecken property – a configuration space approach

DACIBERG LIMA GONÇALVES

Departamento de Matemática - IME - Universidade de São Paulo,

Rua do Matão, 1010

CEP 05508-090 - São Paulo - SP - Brazil.

e-mail: [dlgoncal@ime.usp.br](mailto:dlgoncal@ime.usp.br)

JOHN GUASCHI

Normandie Université, UNICAEN,

Laboratoire de Mathématiques Nicolas Oresme UMR CNRS 6139,

CS 14032, 14032 Caen Cedex 5, France.

e-mail: [john.guaschi@unicaen.fr](mailto:john.guaschi@unicaen.fr)

15th February 2017

## Abstract

*In this paper, we explore the fixed point theory of  $n$ -valued maps using configuration spaces and braid groups, focussing on two fundamental problems, the Wecken property, and the computation of the Nielsen number. We show that the projective plane (resp. the 2-sphere  $\mathbb{S}^2$ ) has the Wecken property for  $n$ -valued maps for all  $n \in \mathbb{N}$  (resp. all  $n \geq 3$ ). In the case  $n = 2$  and  $\mathbb{S}^2$ , we prove a partial result about the Wecken property. We then describe the Nielsen number of a non-split  $n$ -valued map  $\phi: X \multimap X$  of an orientable, compact manifold without boundary in terms of the Nielsen coincidence numbers of a certain finite covering  $q: \widehat{X} \rightarrow X$  with a subset of the coordinate maps of a lift of the  $n$ -valued split map  $\phi \circ q: \widehat{X} \multimap X$ . In the final part of the paper, we classify the homotopy classes of non-split 2-valued maps of the 2-torus, and we give explicit formulæ for the Nielsen number of such 2-valued maps using our description of it.*

## 1 Introduction

Multifunctions and their fixed point theory have been widely studied, see the books [Ber, Gor] for example, where fairly general classes of multifunctions and spaces are considered. In all of what follows,  $X$  and  $Y$  will be topological spaces. Let  $\phi: X \multimap Y$  be an  $n$ -valued function *i.e.* a function that to each  $x \in X$  associates an unordered subset  $\phi(x)$  of  $Y$  of cardinality  $n$ . Recall that such an  $n$ -valued function  $\phi$  is *continuous* if for all  $x \in X$ ,  $\phi(x)$  is closed, and for any open set  $V$  in  $Y$ , the sets  $\{x \in X \mid \phi(x) \subset V\}$  and

$\{x \in X \mid \phi(x) \cap V \neq \emptyset\}$  are open in  $X$ . We will refer to a continuous  $n$ -valued function as an  $n$ -valued map. The class of  $n$ -valued maps is of particular interest, and more information about their fixed point theory on finite complexes may be found in [Bet1, Bet2, Br2, Br3, Br4, Br5, BK, BL, Sch1, Sch2, Sch3]. A homotopy between two  $n$ -valued maps  $\phi_1, \phi_2: X \multimap Y$  is an  $n$ -valued map  $H: X \times I \multimap Y$  such that  $\phi_1 = H(\cdot, 0)$  and  $\phi_2 = H(\cdot, 1)$ . Following [Sch1], an  $n$ -valued function  $\phi: X \multimap Y$  is said to be a *split  $n$ -valued map* if there exist single-valued maps  $f_1, f_2, \dots, f_n: X \rightarrow Y$  such that  $\phi(x) = \{f_1(x), \dots, f_n(x)\}$  for all  $x \in X$ . This being the case, we shall write  $\phi = \{f_1, \dots, f_n\}$ . Let  $\text{Sp}(X, Y, n)$  denote the set of split  $n$ -valued maps between  $X$  and  $Y$ . *A priori*,  $\phi: X \multimap Y$  is just an  $n$ -valued function, but if it is split then it is continuous by [GG4, Proposition 41], which justifies the use of the word ‘map’ in the definition. Partly for this reason, split  $n$ -valued maps play an important rôle in the theory. If  $\phi: X \multimap X$  is an  $n$ -valued map from  $X$  to itself, we say that  $x \in X$  is a *fixed point* of  $\phi$  if  $x \in \phi(x)$ , and we denote the set of fixed points of  $\phi$  by  $\text{Fix}(\phi)$ .

In [Sch2, Section 5], Schirmer defined the notion of Nielsen number of  $n$ -valued maps of finite complexes. Her definition is similar to that for single-valued maps, although it is a little more elaborate. As for the case of single-valued maps, for appropriate spaces, the Nielsen number  $N(\phi)$  of an  $n$ -valued map  $\phi: X \multimap X$  provides a lower bound for the number of fixed points among all  $n$ -valued maps homotopic to  $\phi$ . The computation of the Nielsen number of a self- or  $n$ -valued map is an important problem in fixed point theory, and is not easy in general. In the split case, we have the following formula for the Nielsen number of  $n$ -valued maps of polyhedra in terms of the constituent single-valued maps.

**THEOREM 1.** [Sch2, Corollary 7.2] *Let  $\phi = \{f_1, f_2, \dots, f_n\}: X \multimap X$  be a split  $n$ -valued map, where  $X$  is a compact polyhedron. Then  $N(\phi) = N(f_1) + \dots + N(f_n)$ .*

A second fundamental problem in fixed point theory is to decide whether a space  $X$  has the Wecken property. Recall that the homotopy class of an  $n$ -valued map  $\phi: X \multimap X$  is said to have the *Wecken property* if there exists an  $n$ -valued map  $\psi: X \multimap X$  homotopic to  $\phi$  that has exactly  $N(\phi)$  fixed points, and that a space  $X$  has the *Wecken property for  $n$ -valued maps* if every homotopy class of  $n$ -valued maps of  $X$  has the Wecken property. For single-valued maps, many complexes of dimension at least three have the Wecken property [Ke, W1, W2, W3]. In the case of surfaces, the 2-sphere  $\mathbb{S}^2$ , the real projective plane  $\mathbb{R}P^2$  have the Wecken property [BGZ, Ji3], as do the 2-torus  $\mathbb{T}^2$  and the Klein bottle (see [GK] for example). However, Jiang showed that no other compact surface without boundary has the Wecken property [Ji1, Ji2]. For  $n$ -valued maps, substantial progress has been made in the study of the Wecken property. The following result mirrors that for single-valued maps.

**THEOREM 2.** [Sch3, Theorem 5.2] *Let  $M$  be a compact triangulable manifold (with or without boundary) of dimension greater than or equal to 3, and let  $n \in \mathbb{N}$ . Then every  $n$ -valued map  $\phi: M \multimap M$  is homotopic to an  $n$ -valued map that has  $N(\phi)$  fixed points. In particular,  $M$  has the Wecken property for  $n$ -valued maps.*

Theorem 2 has been extended to a larger class of spaces, including some 2-dimensional complexes, as follows.

**THEOREM 3.** [Bet2, Theorem 1] *Let  $X$  be a compact polyhedron without local cut points and such that no connected component of  $X$  is a surface. Then  $X$  has the Wecken property for  $n$ -valued maps.*

In a recent paper [GG4], we studied some aspects of fixed point theory of  $n$ -valued maps from  $X$  to  $Y$  by introducing an equivalent and natural formulation in terms of single-valued maps from  $X$  to the  $n^{\text{th}}$  unordered configuration space  $D_n(Y)$  of  $X$ , where  $D_n(Y)$  is the quotient of the  $n^{\text{th}}$  (ordered) configuration space  $F_n(X)$  of  $X$ , defined by:

$$F_n(Y) = \{(y_1, \dots, y_n) \mid y_i \in Y, \text{ and } y_i \neq y_j \text{ if } i \neq j\},$$

by the free action of the symmetric group  $S_n$  given by permuting coordinates. It is well known that  $\pi_1(D_n(Y))$  (resp.  $\pi_1(F_n(Y))$ ) is the braid group  $B_n(Y)$  (resp. pure braid group  $P_n(Y)$ ) of  $Y$  on  $n$  strings, that the quotient map  $\pi: F_n(Y) \rightarrow D_n(Y)$  is a regular  $n!$ -fold covering, and that  $P_n(Y)$  is the kernel of the surjective homomorphism  $\tau: B_n(Y) \rightarrow S_n$  that to a braid associates its induced permutation. Configuration spaces play an important rôle in several branches of mathematics and have been extensively studied, see [CG, FH1] for example. As in [GG4], a map  $\Phi: X \rightarrow D_n(Y)$  will be called an  *$n$ -unordered map*, and a map  $\Psi: X \rightarrow F_n(Y)$  will be called an  *$n$ -ordered map*. For such an  $n$ -ordered map, for  $i = 1, \dots, n$ , there exist maps  $f_i: X \rightarrow Y$  such that  $\Psi(x) = (f_1(x), \dots, f_n(x))$  for all  $x \in X$ , and for which  $f_i(x) \neq f_j(x)$  for all  $1 \leq i, j \leq n$ ,  $i \neq j$ , and all  $x \in X$ . In this case, we will often write  $\Psi = (f_1, \dots, f_n)$ . There is an obvious bijection between the set of  $n$ -point subsets of  $Y$  and  $D_n(Y)$  that induces a bijection between the set of  $n$ -valued functions from  $X$  to  $Y$  and the set of functions from  $X$  to  $D_n(Y)$ . If we suppose in addition that  $X$  and  $Y$  are metric spaces, then  $D_n(Y)$  may be equipped with a certain Hausdorff metric [GG4, Appendix], in which case this bijection restricts to a bijection between the set of  $n$ -valued maps from  $X$  to  $Y$  and the set of (continuous) maps from  $X$  to  $D_n(Y)$  [GG4, Theorem 8]. If  $\phi: X \multimap Y$  is an  $n$ -valued map then we shall say that the map  $\Phi: X \rightarrow D_n(Y)$  obtained by this bijection as the map associated to  $\phi$ . If a map  $\Phi: X \rightarrow D_n(Y)$  admits a lift  $\widehat{\Phi}: X \rightarrow F_n(Y)$  via the covering map  $\pi$  then we say that  $\widehat{\Phi}$  is a *lift* of  $\phi$ . By [GG4, Section 2.1],  $\phi$  is split if and only if it admits a lift.

In [GG4], we showed that spheres and real and complex projective spaces of even dimension have the fixed point property for  $n$ -valued maps, and we expressed the problem of deforming an  $n$ -valued map to a fixed point free  $n$ -valued map in terms of an algebraic criterion involving the braid groups of  $X$ , which we then used to determine an infinite number of homotopy classes of split 2-valued maps of  $\mathbb{T}^2$  that contain a fixed point free representative. In this paper, we explore some other aspects of fixed point theory of  $n$ -valued maps using the above formulation in terms of configuration spaces. We establish an equality for the Nielsen number in the non-split case, we study the Wecken property for some surfaces, and in the case of  $\mathbb{T}^2$ , we describe its 2-valued maps in terms of pairs of elements of  $B_2(\mathbb{T}^2)$ , which enables us to give more explicit formulæ for the Nielsen number. We now describe the contents of this paper in more detail.

In Section 2, we recall some fundamental results from [GG4], and in Section 2.1, we show that the 2-disc has the Wecken property for  $n$ -valued maps for all  $n \in \mathbb{N}$ . In Sections 2.2 and 3, we analyse the Wecken problem for  $n$ -valued maps of  $\mathbb{S}^2$  and  $\mathbb{R}P^2$  respectively. In the first case, one important step is to determine the number of homotopy classes of  $n$ -valued maps of  $\mathbb{S}^2$ . In Lemma 13, we show that there is just one such class if  $n \geq 3$ , while if  $n = 2$ , we show that the set of such homotopy classes is in bijection with  $\mathbb{N}$ . If  $\phi: \mathbb{S}^2 \multimap \mathbb{S}^2$  is a 2-valued map then we refer to the non-negative integer given by this correspondence as the *degree* of  $\phi$  (or its homotopy class). Let  $A: \mathbb{S}^2 \rightarrow \mathbb{S}^2$  denote the antipodal map. In the case of  $\mathbb{R}P^2$ , in Proposition 16, we show that there are precisely two homotopy classes of  $n$ -valued maps of  $\mathbb{R}P^2$  for all  $n \geq 2$ , and we give explicit

representatives of these homotopy classes. To prove the Wecken property for  $n$ -valued maps of  $\mathbb{R}P^2$ , it then suffices to check that each of these homotopy classes has the Wecken property. We summarise the main results of Sections 2 and 3 as follows.

PROPOSITION 4.

- (a) Let  $f_0: \mathbb{S}^2 \rightarrow \mathbb{S}^2$  be the constant map at a point  $x_0 \in \mathbb{S}^2$ . Then  $f_0$  has degree 0, and  $f_0$  and  $A \circ f_0$  each have precisely one fixed point.
- (b) There exists a map  $f_1: \mathbb{S}^2 \rightarrow \mathbb{S}^2$  of degree 1 that has one fixed point and such that the map  $A \circ f_1$  is fixed point free.
- (c) The Wecken property holds for:
  - (i)  $n$ -valued maps of  $\mathbb{S}^2$  for all  $n \geq 3$ .
  - (ii) the homotopy classes of 2-valued maps of degree 0 or 1.

Proposition 4(c)(ii) has been obtained independently in [Br1]. We do not know whether the Wecken property holds for 2-valued maps of degree greater than or equal to 2.

THEOREM 5. The projective plane  $\mathbb{R}P^2$  has the Wecken property for  $n$ -valued maps for all  $n \in \mathbb{N}$ .

In Section 4, we consider the problem of the computation of the Nielsen number of  $n$ -valued maps of a compact, orientable manifold  $X$  without boundary. To do this, we shall use [GG4, Proposition 16], which given an  $n$ -valued map  $\phi: X \multimap X$ , states that there exists a finite covering  $q: \widehat{X} \rightarrow X$  such that the composition  $\phi \circ q: \widehat{X} \multimap X$  is split, and [GG4, Proposition 17], which describes the fixed points of  $\phi$  in terms of the coincidences of  $q$  with the coordinate maps of a lift of the split  $n$ -valued map  $\phi \circ q$ . This gives a partial analogue of Theorem 1 in the non-split case as follows.

THEOREM 6. Let  $X$  be an orientable, compact manifold without boundary, and let  $\phi: X \multimap X$  be a non-split  $n$ -valued map. Let  $\Phi: X \rightarrow D_n(X)$  be the map associated to  $\phi$ , let  $H$  be the kernel of the composition  $\tau \circ \Phi_{\#}: \pi_1(X) \rightarrow S_n$ , let  $L' = \text{Im}(\tau \circ \Phi_{\#})$ , and for  $i = 1, \dots, n$ , let  $L'_{i,i} = \{\alpha \in L' \mid \alpha(i) = i\}$ . Let  $q: \widehat{X} \rightarrow X$  be the finite regular covering of  $X$  that corresponds to  $H$ . Let  $\widehat{\Phi}_1 = (f_1, \dots, f_n): \widehat{X} \rightarrow F_n(X)$  be a lift of the split  $n$ -valued map  $\phi \circ q: \widehat{X} \multimap X$ , and suppose that the action of  $L'_{i,i}$  on  $\{1, \dots, n\}$  is free for all  $i \in \{1, \dots, n\}$ . Then the Nielsen number  $N(\phi)$  of  $\phi$  is equal to  $\sum_{j \in I_0} N(q, f_{i_j})$ , where  $i_j$  runs over a set  $I_0$  of representatives of the orbits of this action.

If  $n = 2$ , the condition that the action be free is necessarily satisfied, and in this case, we shall prove in Corollary 24 that  $N(\phi) = N(q, f_1) = N(q, f_2)$ .

In Section 5, we study fixed point free split 2-valued maps of the 2-torus  $\mathbb{T}^2$ . Recall from [GG4, Corollary 27] that  $P_2(\mathbb{T}^2)$  is isomorphic to the direct product of a free group  $\mathbb{F}_2(u, v)$  of rank 2 by a free Abelian group of rank 2. We may thus write the elements of  $P_2(\mathbb{T}^2)$  in the form  $(w, (a, b))$ , where  $w \in \mathbb{F}_2(u, v)$  and  $(a, b) \in \mathbb{Z}^2$ . If  $|w|_u$  (resp.  $|w|_v$ ) denotes the exponent sum of  $w$  with respect to  $u$  (resp.  $v$ ), the Abelianisation homomorphism  $\text{Ab}: \mathbb{F}_2(u, v) \rightarrow \mathbb{Z}^2$  that sends  $u$  to  $(1, 0)$  and  $v$  to  $(0, 1)$  is given by  $\text{Ab}(w) = (|w|_u, |w|_v)$ . If  $w = u^{n_1}v^{m_1} \dots u^{n_r}v^{m_r} \in \mathbb{F}_2(u, v)$  of rank 2, where  $n_1, m_1, \dots, n_r, m_r \in \mathbb{Z}$ , let  $\widehat{w} = u^{-n_1}v^{-m_1} \dots u^{-n_r}v^{-m_r}$ . The following result extends [GG4, Theorem 7], which was obtained under the hypothesis that at least one of  $p$  and  $q$  belongs to  $\{0, 1\}$ .

**THEOREM 7.** *Suppose that  $(a-1, b), (c, d-1)$  and  $(m, n)$  belong to a cyclic subgroup of  $\mathbb{Z}^2$  generated by an element of the form  $(p, q)$ , where  $p, q \in \mathbb{Z}$ , and let  $r, s \in \mathbb{Z}$ . Then there exist  $w \in \mathbb{F}_2(u, v)$ , a split fixed point free 2-valued map  $\phi: \mathbb{T}^2 \dashrightarrow \mathbb{T}^2$  and a lift  $\widehat{\Phi}: \mathbb{T}^2 \rightarrow F_2(\mathbb{T}^2)$  of  $\phi$  such that  $Ab(w) = (m, n)$ ,  $\widehat{\Phi}_\#(e_1) = (w^r, (a, b))$  and  $\widehat{\Phi}_\#(e_2) = (w^s, (c, d))$ .*

Finally, in Section 6, we obtain the following classification of the homotopy classes of non-split 2-valued maps of  $\mathbb{T}^2$ . This completes the analysis given in [GG4, Section 5.3] of the split case.

**PROPOSITION 8.** *Let  $\phi: \mathbb{T}^2 \dashrightarrow \mathbb{T}^2$  be a non-split 2-valued map. Then there exist  $\lambda \in \mathbb{F}_2(u, v)$  and  $a, b, c, d, l, m \in \mathbb{Z}$  such that  $\phi$  is defined by a pair of braids  $(\alpha, \beta)$  as follows:*

- (a) *if  $\beta = \beta'\sigma$  if  $\alpha \in P_2(\mathbb{T}^2)$  and  $\beta \in B_2(\mathbb{T}^2) \setminus P_2(\mathbb{T}^2)$ ,  $\alpha = ((\lambda\widehat{\lambda})^m, (a, b))$ , where  $\beta' = (\mu^{-l}\lambda vu^{-1}, (c, d))$ ,*
- (b) *if  $\alpha \in B_2(\mathbb{T}^2) \setminus P_2(\mathbb{T}^2)$  and  $\beta \in P_2(\mathbb{T}^2)$ ,  $\alpha = \alpha'\sigma$  and  $\beta = ((\lambda\widehat{\lambda})^m, (c, d))$ , where  $\alpha' = (\mu^{-l}\lambda vu^{-1}, (a, b))$ ,*
- (c) *if  $\alpha, \beta \in B_2(\mathbb{T}^2) \setminus P_2(\mathbb{T}^2)$ ,  $\alpha = \alpha'\sigma$  and  $\beta = \beta'\sigma$ , where  $\alpha' = (\mu^{-l}\lambda vu^{-1}, (a, b))$  and  $\beta' = (\mu^{-l}\lambda(\widehat{\lambda}\lambda)^m vu^{-1}, (c, d))$ ,*

where  $\mu$  is a generator of the centraliser of  $\lambda\widehat{\lambda}$  if  $\lambda\widehat{\lambda}$  is non trivial, and is arbitrary otherwise.

Using the framework of Section 4, and notably the set-up of Theorem 6, the description of the fixed point set of a non-split 2-valued map  $\phi$  of  $\mathbb{T}^2$  in terms of the set of coincidences of the covering map  $q: \widehat{\mathbb{T}^2} \rightarrow \mathbb{T}^2$  with  $f_1$  and  $f_2$ , where  $\phi \circ q = \{f_1, f_2\}$  and Corollary 24, Proposition 8 enables us to give formulæ for the Lefschetz coincidence numbers of  $f_1$  and  $f_2$  with  $q$ , and the Nielsen number of  $\phi$ . We also obtain necessary and sufficient conditions for the Nielsen number of  $\phi$  to be zero. Let  $(e_1, e_2)$  denote a basis of  $\pi_1(\mathbb{T}^2)$ , where  $e_1$  (resp.  $e_2$ ) is the homotopy class of the meridian (resp. longitude) of  $\mathbb{T}^2$ .

**THEOREM 9.** *Let  $\phi: \mathbb{T}^2 \dashrightarrow \mathbb{T}^2$  be a non-split 2-valued map, let  $\Phi: \mathbb{T}^2 \rightarrow D_2(\mathbb{T}^2)$  be the associated 2-unordered map that corresponds to  $\phi$ , let  $H$  be the subgroup  $\Phi_\#^{-1}(P_2(\mathbb{T}^2))$  of  $\pi_1(\mathbb{T}^2)$  of index 2, let  $q: \mathbb{T}^2 \rightarrow \mathbb{T}^2$  be the double covering of  $\mathbb{T}^2$  that corresponds to the subgroup  $H$  of  $\pi_1(\mathbb{T}^2)$ , and let  $(f_1, f_2): \mathbb{T}^2 \rightarrow F_2(\mathbb{T}^2)$  be a lift of  $\phi$ . For  $i = 1, 2$ , let  $M_i$  denote the matrix of  $(f_i)_\#: \pi_1(\mathbb{T}^2) \rightarrow \pi_1(\mathbb{T}^2)$  with respect to the basis  $(e_1, e_2)$ . Then  $M_1 = M_2$ , and for  $i = 1, 2$ ,  $M_i$  and the Lefschetz coincidence numbers of  $f_i$  with  $q$  are given as follows:*

- (a) *if  $\alpha \in P_2(\mathbb{T}^2)$  and  $\beta \in B_2(\mathbb{T}^2) \setminus P_2(\mathbb{T}^2)$ , then:*

$$M_i = \begin{cases} \begin{pmatrix} a & 2c + |\lambda|_u - 1 \\ b & 2d + |\lambda|_v + 1 \end{pmatrix} & \text{if } \lambda\widehat{\lambda} \neq 1 \\ \begin{pmatrix} a & 2c - l|\mu|_u + |\lambda|_u - 1 \\ b & 2d - l|\mu|_v + |\lambda|_v + 1 \end{pmatrix} & \text{if } \lambda\widehat{\lambda} = 1 \end{cases}$$

and

$$L(q, f_i) = \begin{cases} \begin{vmatrix} a-1 & 2c + |\lambda|_u - 1 \\ b & 2d + |\lambda|_v - 1 \end{vmatrix} & \text{if } \lambda\widehat{\lambda} \neq 1 \\ \begin{vmatrix} a-1 & 2c - l|\mu|_u + |\lambda|_u - 1 \\ b & 2d - l|\mu|_v + |\lambda|_v - 1 \end{vmatrix} & \text{if } \lambda\widehat{\lambda} = 1, \end{cases}$$



(b) if  $\alpha \in B_2(\mathbb{T}^2) \setminus P_2(\mathbb{T}^2)$  and  $\beta \in P_2(\mathbb{T}^2)$ , then

$$M_i = \begin{cases} \begin{pmatrix} 2a + |\lambda|_u - 1 & c \\ 2b + |\lambda|_v + 1 & d \end{pmatrix} & \text{if } \lambda\widehat{\lambda} \neq 1 \\ \begin{pmatrix} 2a - l|\mu|_u + |\lambda|_u - 1 & c \\ 2b - l|\mu|_v + |\lambda|_v + 1 & d \end{pmatrix} & \text{if } \lambda\widehat{\lambda} = 1 \end{cases}$$

and

$$L(q, f_i) = \begin{cases} \begin{vmatrix} 2a + |\lambda|_u - 3 & c \\ 2b + |\lambda|_v + 1 & d - 1 \end{vmatrix} & \text{if } \lambda\widehat{\lambda} \neq 1 \\ \begin{vmatrix} 2a - l|\mu|_u + |\lambda|_u - 3 & c \\ 2b - l|\mu|_v + |\lambda|_v + 1 & d - 1 \end{vmatrix} & \text{if } \lambda\widehat{\lambda} = 1, \end{cases}$$

(c) if  $\alpha, \beta \in B_2(\mathbb{T}^2) \setminus P_2(\mathbb{T}^2)$ , then

$$M_i = \begin{cases} \begin{pmatrix} 2a + |\lambda|_u - 1 & a + c + |\lambda|_u - 1 \\ 2b + |\lambda|_v + 1 & b + d + |\lambda|_v + 1 \end{pmatrix} & \text{if } \lambda\widehat{\lambda} \neq 1 \\ \begin{pmatrix} 2a - l|\mu|_u + |\lambda|_u - 1 & a + c - l|\mu|_u + |\lambda|_u - 1 \\ 2b - l|\mu|_v + |\lambda|_v + 1 & b + d - l|\mu|_v + |\lambda|_v + 1 \end{pmatrix} & \text{if } \lambda\widehat{\lambda} = 1 \end{cases}$$

and

$$L(q, f_i) = \begin{cases} \begin{vmatrix} 2a + |\lambda|_u - 3 & a + c + |\lambda|_u - 2 \\ 2b + |\lambda|_v + 1 & b + d + |\lambda|_v \end{vmatrix} & \text{if } \lambda\widehat{\lambda} \neq 1 \\ \begin{vmatrix} 2a - l|\mu|_u + |\lambda|_u - 3 & a + c - l|\mu|_u + |\lambda|_u - 2 \\ 2b - l|\mu|_v + |\lambda|_v + 1 & b + d - l|\mu|_v + |\lambda|_v \end{vmatrix} & \text{if } \lambda\widehat{\lambda} = 1, \end{cases}$$

where in all cases,  $a, b, c, d, l, \lambda$  and  $\mu$  are given by Proposition 8, and  $N(\phi) = |L(q, f_i)|$ . If  $\phi$  can be deformed to a fixed point free 2-valued map then  $N(\phi) = 0$ . Moreover, if  $N(\phi) = 0$  then both pairs of maps  $(q, f_1)$  and  $(q, f_2)$  can be deformed to coincidence free pairs.

## Acknowledgements

The first-named author was partially supported by FAPESP-Fundação de Amparo a Pesquisa do Estado de São Paulo, Projeto Temático Topologia Algébrica, Geométrica e Diferencial 2012/24454-8. The second-named author was also partially supported by the same project as well as the CNRS/FAPESP PRC project n° 275209 during his visit to the Instituto de Matemática e Estatística, Universidade de São Paulo, from the 4<sup>th</sup> to the 22<sup>nd</sup> of February 2017.

## 2 Preliminaries and the Wecken property for $n$ -valued maps of the 2-disc and 2-sphere

Let  $n \in \mathbb{N}$ . Given a topological space  $X$ , we are interested in understanding the fixed point theory of its  $n$ -valued maps. In this section, we will show that if  $n \in \mathbb{N}$  (resp.  $n \geq 3$ ), the

2-disc  $\mathbb{D}^2$  and the (resp. the 2-sphere  $\mathbb{S}^2$ ) has the Wecken property for  $n$ -valued maps. We first recall some definitions and results from [GG4]. Given an  $n$ -valued map  $\phi: X \multimap X$  as defined in Section 1, let  $\Phi: X \rightarrow D_n(X)$  denote the associated  $n$ -unordered map, and if  $\phi$  is split, let  $\widehat{\Phi}: X \rightarrow F_n(X)$  be an  $n$ -ordered map that is a lift of  $\phi$ . By a single-valued map (or simply a map)  $f: X \rightarrow Y$  between two topological spaces, we shall always mean a continuous function from  $X$  to  $Y$ , and  $[X, Y]$  will denote the set of unbased homotopy classes of maps between  $X$  and  $Y$ . We recall the following result, which shows that for a large class of spaces, (homotopy classes of)  $n$ -valued maps may be identified with (homotopy classes of) maps whose target is an unordered configuration space. Let  $I$  denote the unit interval  $[0, 1]$ .

**THEOREM 10.** *Let  $X$  and  $Y$  be metric spaces, and let  $n \in \mathbb{N}$ .*

(a) *An  $n$ -valued function  $\phi: X \multimap Y$  is continuous if and only if the corresponding function  $\Phi: X \rightarrow D_n(Y)$  is continuous.*

(b) *The set of homotopy classes of  $n$ -valued maps from  $X$  to  $Y$  is in one-to-one correspondence with the set  $[X, D_n(Y)]$  of homotopy classes of maps from  $X$  to  $D_n(Y)$ .*

*Proof.* Part (a) is a direct consequence of [GG4, Theorem 8]. For part (b), since  $X \times I$  is also a metric space, we may apply [GG4, Theorem 8] to  $n$ -valued maps between  $X \times I$  and  $Y$ . So it follows that two  $n$ -valued maps  $\phi_1, \phi_2: X \multimap Y$  are homotopic if and only if the corresponding maps  $\Phi_1, \Phi_2: X \rightarrow D_n(Y)$  are homotopic, and the result follows.  $\square$

**REMARK 11.** Theorem 10 also holds under weaker hypotheses.

(a) If we assume just that  $Y$  is a metric space, then the statement of Theorem 10(a) is [BG, Corollary 4.1], and that of Theorem 10(b) follows by applying this corollary to the spaces  $X \times I$  and  $Y$ .

(b) If  $X$  is just locally path-connected and semi-locally simply connected then so is  $X \times I$ . The statement of Theorem 10(a) is [BG, Corollary 3.1], and that of Theorem 10(b) follows by applying this corollary to the spaces  $X \times I$  and  $Y$ .

From now on, we will assume that our spaces are metric spaces, so that we may interpret an  $n$ -valued map from  $X$  to  $Y$  as a map from  $X$  to  $D_n(Y)$  using Theorem 10(a).

## 2.1 The case of $n$ -valued maps of the 2-disc

**PROPOSITION 12.** *The 2-disc  $\mathbb{D}^2$  has the Wecken property for  $n$ -valued maps for all  $n \in \mathbb{N}$ .*

*Proof.* Let  $n \in \mathbb{N}$ , and let  $(x_1, \dots, x_n) \in F_n(\mathbb{D}^2)$ . By Theorem 10(b), the set of homotopy classes of  $n$ -valued maps of the 2-disc  $\mathbb{D}^2$  may be identified with the set  $[\mathbb{D}^2, D_n(\mathbb{D}^2)]$  of homotopy classes of maps between  $\mathbb{D}^2$  and  $D_n(\mathbb{D}^2)$ . Since the domain  $\mathbb{D}^2$  is contractible, this set has only one element, which is the class of the constant map  $c = \{c_1, \dots, c_n\}: \mathbb{D}^2 \rightarrow D_n(\mathbb{D}^2)$ , where for all  $i = 1, \dots, n$ , the map  $c_i: \mathbb{D}^2 \rightarrow \mathbb{D}^2$  is the constant map at  $x_i$ . Then the map  $c$  has exactly  $n$  fixed points, and  $N(c) = n$  by Theorem 1, which proves the proposition.  $\square$

## 2.2 The case of $n$ -valued maps of the 2-sphere

We now show that the Wecken property holds for  $n$ -valued maps of  $\mathbb{S}^2$  for all  $n \neq 2$ . The case  $n = 2$  remains open, although we are able to provide some partial results. Since the



case  $n = 1$  is well known, see for example [BGZ, Proposition 2.2], we will assume that  $n \geq 2$ .

LEMMA 13. *If  $n > 2$  (resp.  $n = 2$ ), the set of homotopy classes of  $n$ -valued maps of  $\mathbb{S}^2$  possesses exactly one element (resp. is in one-to-one correspondence with  $\mathbb{N}$ ).*

*Proof.* Theorem 10(b) implies that it suffices to determine the set  $[\mathbb{S}^2, D_n(\mathbb{S}^2)]$ , which in turn is the orbit space of  $\pi_2(D_n(\mathbb{S}^2))$  under the action of  $\pi_1(D_n(\mathbb{S}^2))$ . Now  $F_n(\mathbb{S}^2)$  is a regular  $n!$ -fold covering of  $D_n(\mathbb{S}^2)$ , and if  $n = 2$  (resp.  $n \geq 3$ ), the universal covering of  $F_n(\mathbb{S}^2)$  has the homotopy type of  $\mathbb{S}^2$  (resp. of  $\mathbb{S}^3$ ) by [BCP, FZ] or [GG2, pp. 43–44]. Using standard results about homotopy groups and covering spaces, it follows that  $\pi_2(D_n(\mathbb{S}^2))$  is isomorphic to  $\mathbb{Z}$  (resp. is trivial) if  $n = 2$  (resp.  $n > 2$ ) (see also [FVB, Corollary, page 211] for the case  $n \geq 3$ ). This proves the result if  $n \geq 3$ . If  $n = 2$ , observe that the map  $\mathbb{S}^2 \rightarrow F_2(\mathbb{S}^2)$  given by  $x \mapsto (x, -x)$  is a homotopy equivalence that is  $\mathbb{Z}_2$ -equivariant with respect to the action of the antipodal map on  $\mathbb{S}^2$  and the action on  $F_2(\mathbb{S}^2)$  given by  $s: F_2(\mathbb{S}^2) \rightarrow F_2(\mathbb{S}^2)$  defined by  $s(x, y) = (y, x)$ . This gives rise to a homotopy equivalence between the corresponding orbit spaces, namely  $\mathbb{R}P^2$  and  $D_2(\mathbb{S}^2)$ . Since the action of  $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$  on  $\pi_2(\mathbb{R}P^2) \cong \mathbb{Z}$  is multiplication by  $-1$ , the same is true for the action of  $\pi_1(D_2(\mathbb{S}^2))$  on  $\pi_2(D_2(\mathbb{S}^2))$ , and the orbits are the subsets of the form  $\{m, -m\}$ , where  $m \in \mathbb{Z}$ . It follows that the set  $[\mathbb{S}^2, D_2(\mathbb{S}^2)]$  is in bijection with  $\mathbb{N}$ .  $\square$

We now study the Wecken property for  $n$ -valued maps of  $\mathbb{S}^2$ . Recall that if  $\phi: \mathbb{S}^2 \multimap \mathbb{S}^2$  is a 2-valued map, the integer given by the correspondence with  $\mathbb{N}$  of Lemma 13 is the *degree* of  $\phi$  (or of its homotopy class).

*Proof of Proposition 4.* Part (a) is clear since the constant map  $f_0: \mathbb{S}^2 \rightarrow \mathbb{S}^2$  at a point  $x_0 \in \mathbb{S}^2$  has precisely one fixed point, namely  $x_0$ , and  $-x_0$  is the unique fixed point of  $A \circ f_0$ . To prove part (b), let  $f_1$  be a self-map of the unit 2-sphere  $\mathbb{S}^2$  that is a small deformation of the identity, i.e.  $f_1$  satisfies  $|x - f_1(x)| < \pi/2$  for all  $x \in \mathbb{S}^2$ , and that has exactly one fixed point  $x_0$ . Such a map may be constructed using a vector field on  $\mathbb{S}^2$  that possesses just one singular point, and in this way, the degree of  $f_1$  is equal to 1. Then  $f_1(x) \neq -x$  for all  $x \in \mathbb{S}^2$ , and thus  $A \circ f_1$  is fixed point free, which concludes the proof of part (b). To prove part (c)(i), by Lemma 13, there is only one homotopy class of  $n$ -valued maps of  $\mathbb{S}^2$  if  $n \geq 3$ , which is that of the constant  $n$ -valued map. The result then follows from part (a). To prove part (c)(ii), the case of the homotopy class of degree 0 follows as in (c)(i). For the homotopy class of degree 1, consider the split 2-valued map  $\phi = \{f_1, A \circ f_1\}: \mathbb{S}^2 \multimap \mathbb{S}^2$ . Note that  $L(f_1) = 2$ , so  $N(f_1) \neq 0$ , and thus  $N(f_1) = 1$  because  $\mathbb{S}^2$  is simply connected. Hence by Theorem 1,  $N(\phi) = N(f_1) + N(A \circ f_1) = 1 + 0 = 1$ , and since  $\text{Fix}(\phi) = \{x_0\}$ , it follows that the map  $\phi$ , whose degree is equal to 1, has the Wecken property.  $\square$

### 3 The case of $n$ -valued maps of the projective plane

The aim of this section is to prove that the projective plane  $\mathbb{R}P^2$  has the Wecken property for  $n$ -valued maps for all  $n \in \mathbb{N}$ . Jiang showed that  $\mathbb{R}P^2$  has the Wecken property for single-valued maps [Ji3], so it will suffice to study the case  $n \geq 2$ . We start by computing the Nielsen number of an  $n$ -valued map of  $\mathbb{R}P^2$ .

LEMMA 14. *Let  $n \geq 1$ , and let  $\phi: \mathbb{R}P^2 \multimap \mathbb{R}P^2$  be an  $n$ -valued map. Then  $N(\phi) = n$ .*

*Proof.* From [GG4, Lemma 14],  $\phi$  is split, so for  $i = 1, \dots, n$ , there exist self-maps  $f_i: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$  such that  $\phi = \{f_1, \dots, f_n\}$ . Applying Theorem 1,  $N(\phi) = n$  since  $N(f_i) = 1$  for all  $i = 1, \dots, n$  by [Ji3].  $\square$

We now classify the homotopy classes of  $n$ -valued maps of  $\mathbb{R}P^2$ . If  $n = 1$ , it is well known that the set  $[\mathbb{R}P^2, \mathbb{R}P^2]$  of homotopy classes of self-maps of  $\mathbb{R}P^2$  that induce the trivial homomorphism on the level of fundamental groups has two elements, see [GS, Proposition 2.1] for example. One of these two homotopy classes is that of the constant map. We will describe the second homotopy class in terms of a representative map  $W_P: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$  that we shall now define, where  $P \in \mathbb{S}^2$ , and  $\mathbb{S}^2 \subset \mathbb{R}^3$  is the unit sphere in  $\mathbb{R}^3$ , which we equip with spherical coordinates  $(\theta, \varphi)$ , where  $\theta \in [0, 2\pi)$  and  $\varphi \in [-\pi/2, \pi/2]$ , so that  $P = (\theta, \pi/2)$ . With respect to the Cartesian coordinate system for which  $P = (0, 0, 1)$  and the point  $(1, 0, 0)$  has spherical coordinates  $(0, 0)$ , the point with spherical coordinates  $(\theta, \varphi)$  has Cartesian coordinates  $(\cos \varphi \cos \theta, \cos \varphi \sin \theta, \sin \varphi)$ . From this, one may see that with respect to the spherical coordinate system,  $(\theta, \varphi - \frac{\pi}{2})$  may be identified with  $(\theta + \pi, -\varphi - \frac{\pi}{2})$ . We regard  $\mathbb{R}P^2$  as the quotient of  $\mathbb{S}^2$  by the free action of the group generated by the antipodal map  $A$ . Let  $p: \mathbb{S}^2 \rightarrow \mathbb{R}P^2$  be the usual covering map, for all  $x \in \mathbb{S}^2$ , let  $\bar{x} = p(x)$ , and let  $H_P^+$  be the hemisphere of  $\mathbb{S}^2$  whose pole is  $P$ . Let  $U_P: \mathbb{S}^2 \rightarrow \mathbb{S}^2$  be the map defined by  $U_P(\theta, \varphi) = (\theta, 2\varphi - \frac{\pi}{2})$ . The restriction  $U_P|_{H_P^+}: H_P^+ \rightarrow \mathbb{S}^2$  sends each semi-meridian lying in  $H_P^+$  that starts at  $P$  and ends at the equator linearly to the meridian of  $\mathbb{S}^2$  that starts at  $P$ , ends at  $-P$  and contains the original semi-meridian. Since  $U_P$  sends the whole of the equator to the point  $-P$ ,  $U_P|_{H_P^+}$  induces a map  $W'_P: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$  defined by  $W'_P(\bar{x}) = U_P(x)$  for all  $x \in H_P^+$ . Let  $W_P: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$  be defined by  $W_P = p \circ W'_P$ . Now  $A(\theta, \varphi) = (\theta + \pi, -\varphi)$ , and up to the above-mentioned identification within the spherical coordinate system, for all  $(\theta, \varphi) \in \mathbb{S}^2$ , we have:

$$U_P \circ A(\theta, \varphi) = U_P(\theta + \pi, -\varphi) = (\theta + \pi, -2\varphi - \frac{\pi}{2}) = (\theta, 2\varphi - \frac{\pi}{2}) = U_P(\theta, \varphi),$$

and it follows that  $U_P$  is a lift of  $W_P$ . We thus have the following commutative diagram:

$$\begin{array}{ccccc} H_P^+ & \hookrightarrow & \mathbb{S}^2 & \xrightarrow{p} & \mathbb{R}P^2 \\ & \searrow & \downarrow U_P & & \downarrow W_P \\ & & \mathbb{S}^2 & \xrightarrow{p} & \mathbb{R}P^2, \end{array}$$

for which  $W_P = p \circ W'_P$  also. The following lemma summarises various properties of  $W_P$ .

LEMMA 15. *Let  $P \in \mathbb{S}^2$ . The map  $W_P: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$  satisfies the following properties:*

- (a)  $W_P = W_{-P}$ , and  $\text{Fix}(W_P) = \{p(P)\}$ .
- (b) The map  $W_P$  is non null-homotopic, so it belongs to the non-constant homotopy class of  $[\mathbb{R}P^2, \mathbb{R}P^2]$  that induces the trivial homomorphism on the fundamental group.
- (c) Let  $P_1, P_2 \in \mathbb{S}^2$ . If  $P_1 \neq \pm P_2$ , then  $\text{Coin}(W_{P_1}, W_{P_2})$  is empty.
- (d) If  $c_0: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$  is a constant map then the pair  $(W_P, c_0)$  cannot be deformed to a coincidence-free pair.

*Proof.*

(a) Let  $\bar{x} \in \mathbb{R}P^2$ , where  $\bar{x} = p(x)$  and we take  $x$  to belong to  $H_P^+$ . Thus  $-x \in H_{-P}^+$ ,  $p(-x) = \bar{x}$ , and so  $W'_{-P}(\bar{x}) = U_{-P}(-x)$ . With respect to the same spherical coordinate system that was used to define  $U_P$ , one may see that  $U_{-P}(\theta, \varphi) = (\theta, 2\varphi + \frac{\pi}{2})$  for all  $(\theta, \varphi) \in \mathbb{S}^2$ . From the above definitions, if  $x = (\theta, \varphi)$ , we have:

$$W_P(\bar{x}) = p \circ W'_P(\bar{x}) = p \circ U_P(x) = p(\theta, 2\varphi - \frac{\pi}{2}),$$

from which it follows that:

$$\begin{aligned} W_{-P}(\bar{x}) &= p \circ W'_{-P}(\bar{x}) = p \circ U_{-P}(-x) = p \circ U_{-P}(\theta + \pi, -\varphi) = p(\theta + \pi, -2\varphi + \frac{\pi}{2}) \\ &= p \circ A(\theta, 2\varphi - \frac{\pi}{2}) = p(\theta, 2\varphi - \frac{\pi}{2}) = W_P(\bar{x}), \end{aligned}$$

hence  $W_P = W_{-P}$  as required.

(b) Since  $W_P$  factors through  $\mathbb{S}^2$ , it induces the trivial homomorphism on the fundamental group of  $\mathbb{R}P^2$ . Further, the map  $W'_P$  is a lift of  $W_P$ , so is non null-homotopic (it represents the non-trivial element of  $[\mathbb{R}P^2, \mathbb{S}^2]$ ) because its absolute degree is congruent to 1 mod 2.

(c) Let  $P_1, P_2 \in \mathbb{S}^2$  be such that  $P_1 \neq \pm P_2$ , let  $\mathcal{C}$  be the (unique) great circle that passes through  $P_1$  and  $P_2$ , and let  $x \in \mathbb{S}^2$  be such that  $\bar{x} \in \text{Coin}(W_{P_1}, W_{P_2})$ . Then either  $x \in \text{Coin}(U_{P_1}, U_{P_2})$  or  $x \in \text{Coin}(U_{P_1}, A \circ U_{P_2})$ . Let  $i \in \{1, 2\}$ . Observe that any two great circles of  $\mathbb{S}^2$  either coincide, or intersect in exactly two (antipodal) points, and that  $U_{P_i}$  maps every great circle that passes through  $P_i$  to itself. Suppose first that  $x \notin \mathcal{C}$ , and let  $\mathcal{C}_i$  be the great circle that passes through  $P_i$  and  $x$ . Then  $\mathcal{C}_1 \cap \mathcal{C}_2 = \{x, A(x)\}$ . Since  $U_{P_i}(x) \in \mathcal{C}_i$  and  $x \in \text{Coin}(U_{P_1}, U_{P_2}) \cup \text{Coin}(U_{P_1}, A \circ U_{P_2})$ , it follows that  $U_{P_1}(x) \in \{x, A(x)\}$ , and so  $p \circ U_{P_1}(x) = W_{P_1}(\bar{x}) = \bar{x} \in \mathcal{C}$ . By part (a), this implies that  $\bar{x} = \overline{P_1}$ , which yields a contradiction since  $x \notin \mathcal{C}$ . So assume that  $x \in \mathcal{C}$ . We write the elements of  $\mathcal{C}$  in exponential form, taking  $P_1$  to be  $e^{i\pi/2}$ . Let  $\rho$  be the oriented angle  $\widehat{P_1 P_2}$ , and let  $x = e^{i\varphi}$ . Then  $U_{P_1}(x) = e^{(2\varphi - \frac{\pi}{2})i}$  and  $U_{P_2}(x) = e^{(2(\varphi - \rho) - \frac{\pi}{2})i + \rho i} = U_{P_1}(x) \cdot e^{-\rho i}$ . Since  $x \in \text{Coin}(U_{P_1}, U_{P_2}) \cup \text{Coin}(U_{P_1}, A \circ U_{P_2})$ , we see that  $\rho \in \{0, \pi\}$ , but this implies that  $P_1 \in \{P_2, -P_2\}$ , which yields a contradiction. We conclude that  $W_{P_1}$  and  $W_{P_2}$  are coincidence free.

(d) Suppose on the contrary that the pair  $(W_P, c_0)$  can be deformed to a pair of coincidence-free self-maps of  $\mathbb{R}P^2$ . By [Bro], there exists a map  $h: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$  that is homotopic to  $W_P$  such that the pair  $(h, c_0)$  is coincidence free, and hence  $h$  is non surjective. The maps  $h$  and  $c_0$  lift to maps  $\tilde{h}, \tilde{c}_0: \widetilde{\mathbb{R}P^2} \rightarrow \mathbb{S}^2$ , where  $\tilde{c}_0$  is also a constant map, and the non surjectivity of  $h$  implies that of  $\tilde{h}$ . Thus  $\tilde{h}$  is null homotopic, but then so is  $h$ , which yields a contradiction because  $h$  is homotopic to  $W_P$ , and  $W_P$  is non-null homotopic by part (b).  $\square$

In the following proposition, we describe the set  $[\mathbb{R}P^2, F_n(\mathbb{R}P^2)]$  (resp.  $[\mathbb{R}P^2, D_n(\mathbb{R}P^2)]$ ) of homotopy classes of maps between  $\mathbb{R}P^2$  and  $F_n(\mathbb{R}P^2)$  (resp.  $D_n(\mathbb{R}P^2)$ ), and the set of homotopy classes of  $n$ -valued maps of  $\mathbb{R}P^2$ , from which we will see that they each contain two elements. Let  $N = (0, 0, 1) \in \mathbb{S}^2$ . As we mentioned in the proof of Lemma 14, any  $n$ -valued map  $\phi: \mathbb{R}P^2 \multimap \mathbb{R}P^2$  is split, and so the set of homotopy classes of  $n$ -valued maps of  $\mathbb{R}P^2$  is equal to  $\text{Sp}(\mathbb{R}P^2, \mathbb{R}P^2, n)/\sim$ , where  $\sim$  denotes the homotopy equivalence relation in  $\text{Sp}(\mathbb{R}P^2, \mathbb{R}P^2, n)$ .

PROPOSITION 16. Let  $n \geq 2$ .

(a) The sets  $[\mathbb{R}P^2, F_n(\mathbb{R}P^2)]$ ,  $[\mathbb{R}P^2, D_n(\mathbb{R}P^2)]$  and  $\text{Sp}(\mathbb{R}P^2, \mathbb{R}P^2, n)/\sim$  are in bijection, and each possesses two elements.

(b) The two homotopy classes of  $n$ -valued maps of  $\mathbb{R}P^2$  may be described as follows:

(i) the first homotopy class consists of those  $n$ -valued maps  $\phi: \mathbb{R}P^2 \multimap \mathbb{R}P^2$  such that any lift  $\widehat{\Phi}: \mathbb{R}P^2 \rightarrow F_n(\mathbb{R}P^2)$  of  $\phi$  induces the trivial homomorphism on the level of fundamental groups, and is homotopic to the constant map between  $\mathbb{R}P^2$  and  $F_n(\mathbb{R}P^2)$ .

(ii) if  $\phi_n: \mathbb{R}P^2 \multimap \mathbb{R}P^2$  is an  $n$ -valued map of  $\mathbb{R}P^2$  that represents the second homotopy class, and  $\widehat{\Phi}_n: \mathbb{R}P^2 \rightarrow F_n(\mathbb{R}P^2)$  is a lift of  $\phi_n$ , then for all  $i = 1, \dots, n$ , the composition of  $\widehat{\Phi}_n$  with the projection  $p_i: F_n(\mathbb{R}P^2) \rightarrow \mathbb{R}P^2$  onto the  $i^{\text{th}}$  coordinate is homotopic to the map  $W_N: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$ . Moreover, for all  $i = 1, \dots, n+1$ , the composition of  $\widehat{\Phi}_{n+1}$  with the projection  $q_i: F_{n+1}(\mathbb{R}P^2) \rightarrow F_n(\mathbb{R}P^2)$  given by forgetting the  $i^{\text{th}}$  coordinate is homotopic to  $\widehat{\Phi}_n$ .

*Proof.* Let  $n \geq 2$ .

(a) We start by showing that the set of homotopy classes  $[\mathbb{R}P^2, F_n(\mathbb{R}P^2)]$  of  $n$ -ordered maps of  $\mathbb{R}P^2$  has two elements. Consider the following Barratt-Puppe sequence:

$$\dots \rightarrow [\mathbb{S}^2, F_n(\mathbb{R}P^2)] \rightarrow [\mathbb{R}P^2, F_n(\mathbb{R}P^2)] \rightarrow [\mathbb{S}^1, F_n(\mathbb{R}P^2)] \rightarrow [\mathbb{S}^1, F_n(\mathbb{R}P^2)] \quad (1)$$

associated with the cofibration sequence  $\mathbb{S}^1 \xrightarrow{2} \mathbb{S}^1 \rightarrow \mathbb{R}P^2 \rightarrow \mathbb{S}^2 \rightarrow \mathbb{S}^2 \rightarrow \dots$  for the space  $F_n(\mathbb{R}P^2)$ , where the map  $[\mathbb{S}^1, F_n(\mathbb{R}P^2)] \rightarrow [\mathbb{S}^1, F_n(\mathbb{R}P^2)]$  sends  $[\beta]$  to  $[\beta^2]$  for all maps  $\beta: \mathbb{S}^1 \rightarrow F_n(\mathbb{R}P^2)$ . Now  $[\mathbb{S}^2, F_n(\mathbb{R}P^2)]$  consists of a single homotopy class because  $\pi_2(F_n(\mathbb{R}P^2)) = 0$  [FVB, Corollary, p. 244]. Therefore (1) implies that the set  $[\mathbb{R}P^2, F_n(\mathbb{R}P^2)]$  is in one-to-one correspondence with the set of elements of  $\pi_1(F_n(\mathbb{R}P^2)) = P_n(\mathbb{R}P^2)$  of order less than or equal to 2, namely the trivial element and the full twist  $\Delta_n^2$  [GG1, Proposition 23]. Let  $\alpha: \mathbb{R}P^2 \rightarrow F_n(\mathbb{R}P^2)$  be an  $n$ -ordered map of  $\mathbb{R}P^2$  whose homotopy class  $[\alpha]$  corresponds to  $\Delta_n^2$ . By (1), the image of  $[\alpha]$  in  $[\mathbb{S}^1, F_n(\mathbb{R}P^2)]$  is non trivial, and so the induced homomorphism  $\alpha_{\#}: \pi_1(\mathbb{R}P^2) \rightarrow P_n(\mathbb{R}P^2)$  is non trivial (and injective). In particular,  $\alpha$  is non-homotopic to the constant map  $c: \mathbb{R}P^2 \rightarrow F_n(\mathbb{R}P^2)$ , from which we conclude that  $[\mathbb{R}P^2, F_n(\mathbb{R}P^2)] = \{[c], [\alpha]\}$  has two distinct elements.

We now prove that there are bijections between  $[\mathbb{R}P^2, F_n(\mathbb{R}P^2)]$ ,  $[\mathbb{R}P^2, D_n(\mathbb{R}P^2)]$  and  $\text{Sp}(\mathbb{R}P^2, \mathbb{R}P^2, n)/\sim$ . As we pointed out in the proof of [GG4, Lemma 9(b)], the covering map  $\pi: F_n(\mathbb{R}P^2) \rightarrow D_n(\mathbb{R}P^2)$  induces a map  $\widehat{\pi}: [\mathbb{R}P^2, F_n(\mathbb{R}P^2)] \rightarrow [\mathbb{R}P^2, D_n(\mathbb{R}P^2)]$  defined by  $\widehat{\pi}([\Phi]) = [\pi \circ \Phi]$  for all  $\Phi \in F_n(\mathbb{R}P^2)^{\mathbb{R}P^2}$ . If  $\Psi: \mathbb{R}P^2 \rightarrow D_n(\mathbb{R}P^2)$  is a map and  $\psi: \mathbb{R}P^2 \multimap \mathbb{R}P^2$  is the associated  $n$ -valued map, then  $\psi$  is split by [GG4, Lemma 14]. It follows that there exists a lift  $\widehat{\Psi}: \mathbb{R}P^2 \rightarrow F_n(\mathbb{R}P^2)$  of  $\psi$ , and this map satisfies  $\pi \circ \widehat{\Psi} = \Psi$ . In particular,  $\widehat{\pi}$  is surjective, and hence  $[\mathbb{R}P^2, D_n(\mathbb{R}P^2)] = \{[\pi \circ c], [\pi \circ \alpha]\}$  has at most two elements. Since  $\pi_{\#}: P_n(\mathbb{R}P^2) \rightarrow B_n(\mathbb{R}P^2)$  is inclusion,  $(\pi \circ \alpha)_{\#}: \pi_1(\mathbb{R}P^2) \rightarrow B_n(\mathbb{R}P^2)$  is injective, and thus  $[\pi \circ c] \neq [\pi \circ \alpha]$ . Therefore  $\widehat{\pi}$  is a bijection.

Finally, by [GG4, Lemma 9(b)], the set  $\text{Sp}(\mathbb{R}P^2, \mathbb{R}P^2, n)/\sim$  is in one-to-one correspondence with the orbits of the set  $[\mathbb{R}P^2, F_n(\mathbb{R}P^2)]$  under the action of  $S_n$  induced by that of  $S_n$  on  $F_n(\mathbb{R}P^2)^{\mathbb{R}P^2}$ . But  $[\mathbb{R}P^2, F_n(\mathbb{R}P^2)] = \{[c], [\alpha]\}$ , and since the orbit of the homotopy class  $[c]$  of the constant map under the action of  $S_n$  must be  $\{[c]\}$ , the orbit of  $[\alpha]$  must be  $\{[\alpha]\}$ . It follows that  $\text{Sp}(\mathbb{R}P^2, \mathbb{R}P^2, n)/\sim$  has precisely two elements.

(b) (i) If  $\phi: \mathbb{R}P^2 \multimap \mathbb{R}P^2$  is an  $n$ -valued map of  $\mathbb{R}P^2$  such that the associated map  $\Phi: \mathbb{R}P^2 \rightarrow D_n(\mathbb{R}P^2)$  belongs to the homotopy class  $[\pi \circ c]$  in  $[\mathbb{R}P^2, D_n(\mathbb{R}P^2)]$  of the

constant map  $c: \mathbb{R}P^2 \rightarrow F_n(\mathbb{R}P^2)$ , then it follows from the proof of part (a) that any lift  $\widehat{\Phi}: \mathbb{R}P^2 \rightarrow F_n(\mathbb{R}P^2)$  of  $\phi$  is homotopic to the constant map, and induces the trivial homomorphism on the level of fundamental groups.

(ii) Let  $\phi_n: \mathbb{R}P^2 \multimap \mathbb{R}P^2$  be an  $n$ -valued map of  $\mathbb{R}P^2$  that represents the second (non-trivial) homotopy class, and let  $\widehat{\Phi}_n: \mathbb{R}P^2 \rightarrow F_n(\mathbb{R}P^2)$  be a lift of  $\phi_n$ , so that  $\widehat{\Phi}_n$  is homotopic to the map  $\alpha$  given in the proof of part (a). For all  $1 \leq i \leq n$ , the image of  $\Delta_n^2$  under the homomorphism induced by the projection  $p_i: F_n(\mathbb{R}P^2) \rightarrow \mathbb{R}P^2$  is trivial (this follows from the proof of [GG3, Proposition 8] using the fact that  $\Delta_n^2$  may be written as a product of the generators  $(A_{i,j})_{1 \leq i,j \leq n}$  given in that paper). Since the image of the induced homomorphism  $\alpha_{\#}: \pi_1(\mathbb{R}P^2) \rightarrow P_n(\mathbb{R}P^2)$  is equal to  $\langle \Delta_n^2 \rangle$ , the composition  $p_i \circ \widehat{\Phi}_n: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$  induces the trivial homomorphism on the level of fundamental groups. Now let  $P_1, \dots, P_n$  be  $n$  distinct points of  $\mathbb{S}^2$  that lie on the geodesic arc between  $(1, 0, 0)$  and  $(0, 0, 1)$ , and consider the map  $(W_{P_1}, \dots, W_{P_n}): \mathbb{R}P^2 \rightarrow F_n(\mathbb{R}P^2)$ . The fact that this map is well defined is a consequence of Lemma 15(c). Since  $p_i \circ (W_{P_1}, \dots, W_{P_n}) = W_{P_i}$  for all  $i = 1, \dots, n$ , the map  $(W_{P_1}, \dots, W_{P_n})$  is not homotopic to the constant map  $c$ , and so it is homotopic to  $\alpha$  by the proof of part (a). In particular,  $\widehat{\Phi}_n$  is homotopic to  $(W_{P_1}, \dots, W_{P_n})$ , and the statements of part (b)(ii) then follow.  $\square$

We are now able to prove the Wecken property for  $\mathbb{R}P^2$ .

*Proof of Theorem 5.* As we mentioned previously,  $\mathbb{R}P^2$  has the Wecken property for self-maps. So suppose that  $n > 1$ . From Proposition 16, there are two homotopy classes of  $n$ -valued maps of  $\mathbb{R}P^2$ , and so by Lemma 14, it suffices to show that each of these classes admits a representative for which the number of fixed points is equal to  $n$ . Let  $P_1, \dots, P_n$  be as in the proof of Proposition 16(b)(ii), and let  $c_{P_i}: \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$  be the constant map at  $P_i$ . From Proposition 16, the two homotopy classes of  $n$ -valued maps of  $\mathbb{R}P^2$  contain an  $n$ -valued map  $\phi: \mathbb{R}P^2 \multimap \mathbb{R}P^2$  that is split and admits a lift  $\widehat{\Phi} = (\phi_1, \dots, \phi_n): \mathbb{R}P^2 \rightarrow F_n(\mathbb{R}P^2)$ , where either  $\phi_i = c_{\overline{P_i}}$  is the constant map at  $\overline{P_i}$  for all  $i = 1, \dots, n$ , or  $\phi_i = W_{P_i}$  for all  $i = 1, \dots, n$ . Using Lemma 15(a),  $\text{Fix}(\phi_i) = \{\overline{P_i}\}$  for all  $i = 1, \dots, n$ , and hence  $\phi$  has exactly  $n$  fixed points. Moreover, by Lemma 14 we have  $N(\phi) = n$ . So each of the two homotopy classes of  $n$ -valued maps contains a representative  $\phi$  for which  $N(\phi) = n$ , and hence  $\mathbb{R}P^2$  has the Wecken property for  $n$ -valued maps.  $\square$

## 4 Nielsen numbers of $n$ -valued maps

The Nielsen number for  $n$ -valued maps of a compact polyhedron  $X$  was defined in [Sch2], and may be determined for split  $n$ -valued maps using Theorem 1. The aim of this section is to prove Theorem 6, where we give a formula for the Nielsen number of non-split  $n$ -valued maps of a space  $X$  that is in the same spirit as Theorem 1. For this theorem, we shall require  $X$  to be a compact, orientable manifold without boundary, in order to have the notions of Lefschetz and Nielsen numbers for coincidences at our disposal. However, many of the results that lead to Theorem 6 are valid under weaker hypotheses on  $X$ , namely those of Proposition 17 below.

We start by recalling some notation and results from [GG4, Section 3.2] that will be used throughout the rest of the paper. Given an  $n$ -valued map  $\phi: X \multimap X$  of a topological space  $X$  that is locally path-connected and semi-locally simply connected, we consider the corresponding map  $\Phi: X \rightarrow D_n(X)$ , and the induced homomorphism  $\Phi_{\#}: \pi_1(X) \rightarrow$



$\pi_1(D_n(X))$  on the level of fundamental groups, where  $\pi_1(D_n(X)) = B_n(X)$ . By the short exact sequence

$$1 \longrightarrow P_n(X) \longrightarrow B_n(X) \xrightarrow{\tau} S_n \longrightarrow 1,$$

$P_n(X)$  is a normal subgroup of  $B_n(X)$  of finite index  $n!$ , so the subgroup  $H = \Phi_{\#}^{-1}(P_n(X))$  is a normal subgroup of  $\pi_1(X)$  of finite index. Let  $L$  be the finite quotient group  $\pi_1(X)/H$ , and let  $q: \widehat{X} \rightarrow X$  be the covering of  $X$  that corresponds to the subgroup  $H$ . Such a covering exists due to the hypotheses on  $X$ . As the following proposition shows, the fixed points of  $\phi$  may be described in terms of the coincidences of  $q$  with the coordinate maps  $f_1, \dots, f_n: \widehat{X} \rightarrow X$  of a lift of the  $n$ -valued map  $\phi \circ q: \widehat{X} \rightarrow X$ .

**PROPOSITION 17.** [GG4, Propositions 16 and 17] *Let  $n \in \mathbb{N}$ , and suppose that  $X$  is a connected, locally arcwise-connected metric space.*

(a) *With the above notation, the  $n$ -valued map  $\phi_1 = \phi \circ q: \widehat{X} \rightarrow X$  admits exactly  $n!$  lifts, which are  $n$ -ordered maps from  $\widehat{X}$  to  $F_n(X)$ . If one such lift  $\widehat{\Phi}_1: \widehat{X} \rightarrow F_n(X)$  is given by  $\widehat{\Phi}_1 = (f_1, \dots, f_n)$ , where for  $i = 1, \dots, n$ ,  $f_i$  is a map from  $\widehat{X}$  to  $X$ , then the other lifts are of the form  $(f_{\rho(1)}, \dots, f_{\rho(n)})$ , where  $\rho \in S_n$ .*

(b) *if the lift  $\widehat{\Phi}_1 = (f_1, \dots, f_n)$  is as in part (a) then the restriction of  $q: \widehat{X} \rightarrow X$  to  $\bigcup_{i=1}^n \text{Coin}(q, f_i) \rightarrow \text{Fix}(\phi)$  is surjective. Furthermore, the pre-image of a point  $x \in \text{Fix}(\phi)$  by this map is precisely  $q^{-1}(x)$ , namely the fibre over  $x \in X$  of the covering map  $q$ .*

Although the lift  $\widehat{\Phi}_1$  is not unique, the set  $\{f_1, \dots, f_n\}$  is, and so the set  $\bigcup_{i=1}^n \text{Coin}(q, f_i)$  is independent of the choice of lift of  $\phi$ . In what follows, we aim to describe the Nielsen classes of  $\phi$  in terms of the Nielsen coincidence classes of the pairs  $(q, f_i)$ , where  $i = 1, \dots, n$ , which will lead to a formula for  $N(\phi)$  similar in spirit to that of Theorem 1. Observe that the composition  $\pi_1(X) \xrightarrow{\Phi_{\#}} B_n(X) \xrightarrow{\tau} S_n$  is a homomorphism whose kernel is  $H$ , so it induces an injective homomorphism  $\Gamma: L \rightarrow S_n$ . Let  $L' = \text{Im}(\Gamma) = \text{Im}(\tau \circ \Phi_{\#})$ , and for  $i, j = 1, \dots, n$ , let  $L'_{i,j} = \{\rho \in L' \mid \rho(i) = j\}$ . The subset  $L'_{i,i}$  is a subgroup of  $L'$ , and if  $i, j \in \{1, \dots, n\}$ , the subset  $L'_{i,j}$  is either empty or is a left coset of  $L'_{i,i}$  in  $L'$ .

In the rest of this section, we will suppose without further comment that  $X$  satisfies the hypotheses of Proposition 17, so that it is a connected, locally arcwise-connected metric space. If  $\phi: X \rightarrow X$  is an  $n$ -valued map, we recall the Nielsen relation on  $\text{Fix}(\phi)$  from [Sch2, Section 5]. First note that by [GG4, Lemma 12], if  $\lambda: I \rightarrow X$  is a path then the  $n$ -valued map  $\phi \circ \lambda: I \rightarrow X$  is split. Let  $x, x' \in \text{Fix}(\phi)$ . We say that  $x$  and  $x'$  are Nielsen equivalent if there exist maps  $g_1, g_2, \dots, g_n: I \rightarrow X$ , a path  $\lambda: I \rightarrow X$  from  $\lambda(0) = x$  to  $\lambda(1) = x'$  and  $j \in \{1, \dots, n\}$  such that  $\phi \circ \lambda = \{g_1, g_2, \dots, g_n\}$ , and  $g_j$  is a path from  $g_j(0) = x$  to  $g_j(1) = x'$  that is homotopic to  $\lambda$  relative to their endpoints. This defines an equivalence relation on  $\text{Fix}(\phi)$ , and the resulting equivalence classes are called Nielsen fixed point classes for the  $n$ -valued map  $\phi$ .

**REMARK 18.** Within our framework, the maps  $g_1, g_2, \dots, g_n$  may be chosen as follows: given  $x_0, x'_0 \in \text{Fix}(\phi)$ , a point  $\tilde{x}_0 \in \widehat{X}$  such that  $q(\tilde{x}_0) = x_0$ , and a path  $\lambda: I \rightarrow X$  from  $x_0$  to  $x'_0$ , let  $\tilde{\lambda}: I \rightarrow \widehat{X}$  be the unique lift of  $\lambda$  to  $\widehat{X}$  for which  $\tilde{\lambda}(0) = \tilde{x}_0$ . Consider the  $n$ -ordered map  $\widehat{\Phi}_1 = (f_1, \dots, f_n): \widehat{X} \rightarrow F_n(X)$  given by Proposition 17(a). Then  $\phi \circ \lambda = \{g_1, \dots, g_n\}$ , where for  $i = 1, \dots, n$ ,  $g_i: I \rightarrow X$  is the map defined by  $g_i = f_i \circ \tilde{\lambda}$ . So  $x_0$  and  $x'_0$  are Nielsen equivalent if there is a path  $\lambda$  as above and  $j \in \{1, \dots, n\}$  such that  $\lambda(0) = g_j(0) = f_j \circ \tilde{\lambda}(0)$ ,  $\lambda(1) = g_j(1) = f_j \circ \tilde{\lambda}(1)$  and  $\lambda$  is homotopic to  $g_j$  relative to their endpoints.



In the following lemmas, we will compare the coincidences of  $q$  and the  $f_i$  with the fixed points of  $\phi$ .

LEMMA 19. *With the above notation, let  $x_0 \in \text{Fix}(\phi)$ , let  $\tilde{x}_0 \in \widehat{X}$  and  $i \in \{1, \dots, n\}$  be such that  $q(\tilde{x}_0) = x_0$  and  $\tilde{x}_0 \in \text{Coin}(q, f_i)$ . If  $y \in q^{-1}(x_0)$  and  $j \in \{1, \dots, n\}$  then  $y \in \text{Coin}(q, f_j)$  if and only if  $\Gamma(\alpha) \in L'_{i,j}$ , where  $\alpha$  is the  $H$ -coset of  $[q(\gamma)]$ ,  $\gamma$  being any path from  $\tilde{x}_0$  to  $y$ . In particular, the points of  $q^{-1}(x_0)$  that belong to  $\text{Coin}(q, f_i)$  are in one-to-one correspondence with the subgroup  $L'_{i,i}$ .*

Since  $L'_{i,i}$  does not depend on  $x_0$ , Lemma 19 implies that for all  $x \in \text{Fix}(\phi)$ , the set  $q^{-1}(x) \cap \text{Coin}(q, f_i)$  has  $|L'_{i,i}|$  elements.

*Proof of Lemma 19.* The proof makes use of basic covering space theory. Let  $\alpha \in L$  be the unique deck transformation for which  $y$  is equal to the element  $\alpha.\tilde{x}_0$  of  $\widehat{X}$  that arises from the action of deck transformation group  $L$  on  $\widehat{X}$ . Then  $\Gamma(\alpha) \in L'$  defines a deck transformation of the covering  $\pi: F_n(X) \rightarrow D_n(X)$ . Using the fact that  $q$  and  $\pi$  are covering maps and  $\widehat{\Phi}_1$  is a lift of  $\Phi$ , we have:

$$\widehat{\Phi}_1(\alpha.\tilde{x}) = \Gamma(\alpha).\widehat{\Phi}_1(\tilde{x}) \quad (2)$$

for all  $\tilde{x} \in q^{-1}(x_0)$ , where  $\Gamma(\alpha).\widehat{\Phi}_1(\tilde{x})$  is the element of  $F_n(X)$  arising from the action of  $S_n$  on  $F_n(X)$ . Since  $\tilde{x}_0 \in \text{Coin}(q, f_i)$ ,  $\widehat{\Phi}_1(\tilde{x}_0)$  is an element of  $F_n(X)$  whose  $i^{\text{th}}$  coordinate is  $x_0$ , and  $\Gamma(\alpha).\widehat{\Phi}_1(\tilde{x}_0)$  is an element of  $F_n(X)$  whose  $\Gamma(\alpha)(i)^{\text{th}}$  coordinate is  $x_0$ . So if  $y \in \text{Coin}(q, f_j)$ , the  $j^{\text{th}}$  coordinate of  $\widehat{\Phi}_1(y)$  is  $x_0$ , and thus  $\Gamma(\alpha) \in L'_{i,j}$ . Conversely, if  $\Gamma(\alpha) \in L'_{i,j}$  then  $\Gamma(\alpha)(i) = j$ , so the  $j^{\text{th}}$  coordinate of  $\widehat{\Phi}_1(y)$  is  $x_0$ , and hence  $y \in \text{Coin}(q, f_j)$ . For the last part of the statement, since  $L'_{i,i}$  is a subgroup of  $L'$ , it suffices to take  $j = i$ .  $\square$

With Lemma 19 in mind, we define the following notation. For each  $i \in \{1, \dots, n\}$ , let  $\mathbb{O}_i$  be the orbit of  $i$  by the action of the subgroup  $L'$  of  $S_n$  on the set  $\{1, \dots, n\}$ , and let  $I_0 = \{i_1, \dots, i_s\}$  be such that the sets  $\{\mathbb{O}_i\}_{i \in I_0}$  form a partition of  $\{1, \dots, n\}$ . As examples, if  $H = \pi_1(X)$  (resp.  $L' = S_n$ ) then  $\mathbb{O}_i = \{i\}$  (resp.  $\mathbb{O}_i = S_n$ ) for all  $i \in \{1, \dots, n\}$ . The following result underlines the relevance of these orbits.

LEMMA 20. *With the above notation, let  $x_0 \in \text{Fix}(\phi)$ . Then there exists  $i \in \{1, \dots, n\}$  such that:*

$$\{j \in \{1, \dots, n\} \mid q^{-1}(x_0) \cap \text{Coin}(q, f_j) \neq \emptyset\} = \mathbb{O}_i.$$

*Proof.* The proof uses arguments similar to those of Lemma 19. Let  $x_0 \in \text{Fix}(\phi)$ , and let  $\tilde{x}_0 \in \widehat{X}$  be a lift of  $x_0$ . By Proposition 17(b),  $\tilde{x}_0$  belongs to  $\text{Coin}(q, f_i)$  for some  $i \in \{1, \dots, n\}$ . First, suppose that  $y \in q^{-1}(x_0) \cap \text{Coin}(q, f_j)$  for some  $j \in \{1, \dots, n\}$ , and let  $\alpha \in L$  be such that  $\alpha.\tilde{x}_0 = y$ . From (2),  $\Gamma(\alpha)(i) = j$ , so  $j \in \mathbb{O}_i$ . Conversely, suppose that  $j \in \mathbb{O}_i$ . Then there exists  $\alpha \in L$  such that  $\Gamma(\alpha)(i) = j$ , and taking  $y = \alpha.\tilde{x}_0$ , we have  $y \in q^{-1}(x_0) \cap \text{Coin}(q, f_j)$ .  $\square$

Note that by Proposition 17(b) and Lemma 20,  $\text{Fix}(\phi) = q(\bigcup_{j \in I_0} \text{Coin}(q, f_j))$ . Since we wish to express the Nielsen number of  $\phi$  in terms of the Nielsen coincidence numbers of the pairs  $(q, f_i)$ , for the values of  $i$  belonging to  $I_0$ , we shall compare the Nielsen coincidence relation and the Nielsen coincidence number of the pairs  $(q, f_i)$  with the Nielsen relation and the Nielsen number for  $\phi$  respectively.

LEMMA 21. *With the above notation, let  $i \in \{1, \dots, n\}$ , and let  $y_1$  and  $y_2$  be elements of  $\text{Coin}(q, f_i)$  that belong to the same Nielsen coincidence class of the pair  $(q, f_i)$ . Then  $q(y_1)$  and  $q(y_2)$  are elements of  $\text{Fix}(\phi)$  that belong to the same Nielsen fixed point class of the  $n$ -valued map  $\phi$ . Further,  $q$  sends each Nielsen coincidence class of the pair  $(q, f_i)$  surjectively onto a Nielsen fixed point class of  $\phi$ .*

*Proof.* Let  $y_1$  and  $y_2$  be elements of  $\text{Coin}(q, f_i)$  for some  $i \in \{1, \dots, n\}$  that belong to the same Nielsen coincidence class of the pair  $(q, f_i)$ . By Proposition 17(b),  $q(y_1)$  and  $q(y_2)$  are fixed points of  $\phi$ . Since  $y_1$  and  $y_2$  belong to the same Nielsen coincidence class of  $(q, f_i)$ , there exists a path  $\tilde{\lambda}: I \rightarrow \widehat{X}$  from  $y_1$  to  $y_2$  such that the path  $f_i \circ \tilde{\lambda}$  is homotopic in  $X$  to  $\lambda$  relative to the endpoints  $q(y_1)$  and  $q(y_2)$ , where  $\lambda: I \rightarrow X$  is the path defined by  $\lambda = q \circ \tilde{\lambda}$ . Now the  $n$ -valued map  $\phi \circ \lambda: I \rightarrow X$  is split by [GG4, Lemma 12], and by Remark 18,  $\phi \circ \lambda = \{g_1, \dots, g_n\}$ , where for  $j = 1, \dots, n$ ,  $g_j = f_j \circ \tilde{\lambda}$ . So  $g_i(0) = f_i \circ \tilde{\lambda}(0) = f_i(y_1) = q(y_1)$ ,  $g_i(1) = f_i \circ \tilde{\lambda}(1) = f_i(y_2) = q(y_2)$ , and  $g_i$  is homotopic to  $\lambda$  relative to the endpoints  $q(y_1)$  and  $q(y_2)$ , from which we deduce that  $q(y_1)$  and  $q(y_2)$  belong to the same Nielsen fixed point class of  $\phi$ . To prove the second part of the statement, by the first part, it suffices to show that if  $x$  is a fixed point of  $\phi$  that belongs to the same Nielsen fixed point class of  $\phi$  as  $q(y_1)$  then there exists  $y \in \text{Coin}(q, f_i)$  such that  $q(y) = x$ , and  $y$  and  $y_1$  belong to the same Nielsen coincidence class of the pair  $(q, f_i)$ . To see this, note that by Remark 18, there exist a path  $\lambda: I \rightarrow X$  from  $q(y_1)$  to  $x$ , a lift  $\tilde{\lambda}: I \rightarrow \widehat{X}$  such that  $\tilde{\lambda}(0) = y_1$  and  $j \in \{1, \dots, n\}$  such that  $\lambda(0) = g_j(0) = f_j \circ \tilde{\lambda}(0)$ ,  $\lambda(1) = g_j(1) = f_j \circ \tilde{\lambda}(1)$ , and  $\lambda$  is homotopic to  $g_j$  relative to their endpoints. In particular,  $f_j(y_1) = q(y_1)$ , and so  $j = i$  because  $y_1 \in \text{Coin}(q, f_i)$ . Further, if  $y = \lambda(1)$  then  $q(y) = x$  because  $\tilde{\lambda}$  is a lift of  $\lambda$ , and  $x = \lambda(1) = f_i(y)$ , so  $y \in \text{Coin}(q, f_i)$ . Finally, the paths  $\lambda$  and  $g_i = f_i \circ \tilde{\lambda}$  are homotopic in  $X$  relative to their endpoints, and hence  $y_1$  and  $y$  belong to the same Nielsen coincidence class of the pair  $(q, f_i)$  as required.  $\square$

In order to obtain a formula for  $N(\phi)$ , another ingredient that we require is the number of points of  $q^{-1}(x_0) \cap \text{Coin}(q, f_i)$  that belong to the same Nielsen coincidence class of the pair  $(q, f_i)$ , where  $x_0 \in \text{Fix}(\phi)$  and  $i \in I_0$ . Suppose that  $q^{-1}(x_0) \cap \text{Coin}(q, f_i) \neq \emptyset$ , and let  $\tilde{x}_0, y \in \widehat{X}$  be elements of this intersection. There exists a unique  $\mu \in L = \pi_1(X)/H$  such that  $y = \mu \cdot \tilde{x}_0$ . Let  $L_i$  be the subset of  $L$  consisting of such  $\mu$  as  $y$  runs over the elements of  $q^{-1}(x_0) \cap \text{Coin}(q, f_i)$ . Note that  $L_i = \{\mu \in L \mid \Gamma(\mu)(i) = i\}$ , in particular,  $L_i$  is independent of  $\tilde{x}_0$ ,  $L_i$  is a subgroup of  $L$ , the order of  $L_i$  is equal to the cardinality of  $q^{-1}(x_0) \cap \text{Coin}(q, f_i)$ , and  $L'_{i,i} = \Gamma(L_i)$ , so  $L'_{i,i} \cong L_i$ . If  $\mu \in L_i$ , consider the corresponding element  $y \in q^{-1}(x_0) \cap \text{Coin}(q, f_i)$  defined by  $y = \mu \cdot \tilde{x}_0$ , and let  $\gamma: I \rightarrow \widehat{X}$  be a path from  $\tilde{x}_0$  to  $y$ . Then  $f_i \circ \gamma$  and  $q \circ \gamma$  are loops in  $X$  based at  $x_0$ . Let  $W_{\tilde{x}_0}(\mu)$  be the subset of  $\pi_1(X)$  of loop classes of the form  $[(q \circ \gamma) * (f_i \circ \gamma)^{-1}]$ , where  $\gamma$  runs over the set of paths from  $\tilde{x}_0$  to  $y$ . Observe that  $W_{\tilde{x}_0}(\mu)$  contains the trivial element of  $\pi_1(X)$  if and only if  $\tilde{x}_0$  and  $y$  belong to the same Nielsen coincidence class for the pair  $(q, f_i)$ . With this in mind, let  $K_i(\tilde{x}_0)$  be the set of elements  $y \in q^{-1}(x_0) \cap \text{Coin}(q, f_i)$  for which  $W_{\tilde{x}_0}(\mu)$  contains the trivial element, where  $\mu \in L_i$  is such that  $y = \mu \cdot \tilde{x}_0$ . Then  $K_i(\tilde{x}_0)$  is the subset of elements  $q^{-1}(x_0) \cap \text{Coin}(q, f_i)$  that belong to the same Nielsen coincidence class of the pair  $(q, f_i)$  as  $\tilde{x}_0$ . Let  $|K_i(\tilde{x}_0)|$  denote the cardinality of  $K_i(\tilde{x}_0)$ .

LEMMA 22. *With the above notation, let  $x_0 \in \text{Fix}(\phi)$ , and let  $\tilde{x}_0 \in q^{-1}(x_0) \cap \text{Coin}(q, f_i)$ .*

(a) The number of coincidence points of the pair  $(q, f_i)$  that belong to  $q^{-1}(x_0)$  is equal to the order of the subgroup  $L_i$ .

(b) If  $\tilde{z} \in q^{-1}(x_0) \cap \text{Coin}(q, f_i)$ , the set  $K_i(\tilde{z})$  of points of  $q^{-1}(x_0)$  that belong to the same Nielsen coincidence class for the pair  $(q, f_i)$  as  $\tilde{z}$  is in one-to-one correspondence with the set  $K_i(\tilde{x}_0)$ .

(c) Let  $\tilde{z} \in \text{Coin}(q, f_i)$  be such that  $\tilde{x}_0$  and  $\tilde{z}$  belong to the same Nielsen coincidence class for the pair  $(q, f_i)$ . Then  $K_i(\tilde{z})$  is in one-to-one correspondence with the set  $K_i(\tilde{x}_0)$ .

*Proof.*

(a) This follows from Lemma 19 and the isomorphism  $L_i \cong L'_{i,i}$ .

(b) Let us construct an injective map from  $K_i(\tilde{x}_0)$  to  $K_i(\tilde{z})$ . Given  $y \in K_i(\tilde{x}_0)$ , by definition, there exists a path  $\gamma: I \rightarrow \widehat{X}$  such that  $\gamma(0) = \tilde{x}_0$ ,  $\gamma(1) = \tilde{y}$  and  $q \circ \gamma$  is homotopic to  $f_i \circ \gamma$  relative to their endpoints. Let  $\gamma_1$  be the unique lift of  $q \circ \gamma$  for which  $\gamma_1(0) = \tilde{z}$ . Since  $q \circ \gamma$  is a loop in  $X$  based at  $x_0$ ,  $\gamma_1(1) \in q^{-1}(x_0)$ . We claim that  $\gamma_1(1) \in K_i(\tilde{z})$ . To see this, let us show that  $q \circ \gamma_1 = q \circ \gamma$  is homotopic to  $f_i \circ \gamma_1$  relative to their endpoints. It suffices to observe that  $f_i \circ \gamma = f_i \circ \gamma_1$ . Since the two compositions  $\widehat{X} \xrightarrow{q} X \xrightarrow{\Phi} D_n(X)$  and  $\widehat{X} \xrightarrow{\widehat{\Phi}_1} F_n(X) \xrightarrow{\pi} D_n(X)$  are equal, we have  $\pi(\widehat{\Phi} \circ \gamma_1) = \pi(\widehat{\Phi} \circ \gamma)$ , in other words, the  $n$ -tuple  $(f_1 \circ \gamma, \dots, f_n \circ \gamma)$  of paths is a permutation of the  $n$ -tuple  $(f_1 \circ \gamma_1, \dots, f_n \circ \gamma_1)$  of paths. Since these two  $n$ -tuples are paths in  $F_n(X)$  and  $f_i \circ \gamma(0) = f_i \circ \gamma_1(0) = x_0$ , it follows that  $f_i \circ \gamma = f_i \circ \gamma_1$ , and hence  $f_i \circ \gamma_1$  is homotopic to  $q \circ \gamma_1$  relative to their endpoints. In particular,  $f_i \circ \gamma_1(1) = f_i \circ \gamma(1) = q \circ \gamma(1) = q \circ \gamma_1(1)$ , so  $\gamma_1(1) \in q^{-1}(x_0) \cap \text{Coin}(q, f_i)$ , and thus  $\gamma_1(1) \in K_i(\tilde{z})$ , which proves the claim. By construction, the map from  $K_i(\tilde{x}_0)$  to  $K_i(\tilde{z})$  that to  $y$  associates  $\gamma_1(1)$  is injective. Exchanging the rôles of  $\tilde{x}_0$  and  $z$ , we see that  $K_i(\tilde{x}_0)$  and  $K_i(\tilde{z})$  have the same number of elements, and that this map is a bijection.

(c) Let  $\lambda: I \rightarrow \widehat{X}$  be a path such that  $\lambda(0) = \tilde{x}_0$ ,  $\lambda(1) = \tilde{z}$  and  $q \circ \lambda$  is homotopic to  $f_i \circ \lambda$  relative to their endpoints. Let  $y \in q^{-1}(x_0) \cap \text{Coin}(q, f_i)$  be a point that belongs to the same Nielsen coincidence class of  $(q, f_i)$  as  $\tilde{x}_0$ , and let  $\gamma: I \rightarrow \widehat{X}$  be a path such that  $\gamma(0) = \tilde{x}_0$ ,  $\gamma(1) = y$  and  $q \circ \gamma$  is homotopic to  $f_i \circ \gamma$  relative to their endpoints. Let  $\lambda': I \rightarrow \widehat{X}$  be the unique lift of  $q \circ \lambda$  for which  $\lambda'(0) = y$ , and let  $\gamma': I \rightarrow \widehat{X}$  be the path defined by  $\gamma' = \lambda^{-1} * \gamma * \lambda'$ . Then  $\gamma'(0) = \tilde{z}$ ,  $q \circ \gamma'(1) = q \circ \lambda'(1) = q \circ \lambda(1) = q(\tilde{z})$ , so  $\gamma'(1) \in q^{-1}(q(\tilde{z}))$ . As in the proof of part (b),  $q \circ \lambda'$  is homotopic to  $f_i \circ \lambda'$  relative to their endpoints, and it follows that  $q \circ \gamma'$  is homotopic to  $f_i \circ \gamma'$  relative to their endpoints, so  $\gamma'(1) \in K_i(\tilde{z})$ . We thus obtain an injective map from  $K_i(\tilde{x}_0)$  and  $K_i(\tilde{z})$  that to  $y$  associates  $\gamma'(1)$ . By exchanging the rôles of  $\tilde{x}_0$  and  $\tilde{z}$ , we see that this map is a bijection.  $\square$

If  $i \in \{1, \dots, n\}$ , by Lemma 22, the set  $q^{-1}(x_0) \cap \text{Coin}(q, f_i)$  contains  $|L_i|$  points, and is partitioned into Nielsen coincidence classes of the pair  $(q, f_i)$  that each contain  $|K_i(\tilde{x}_0)|$  points. So this set is partitioned into  $|L_i|/|K_i(\tilde{x}_0)|$  disjoint subsets each of which is contained in a Nielsen coincidence class of  $(q, f_i)$ . If  $W_1, W_2$  are Nielsen coincidence classes of  $(q, f_i)$ , we say that they are *related* if  $q(W_1) \cap q(W_2) \neq \emptyset$ . This defines an equivalence relation. Let  $C_1, \dots, C_r$  denote the corresponding set of equivalence classes of the Nielsen coincidence classes of  $(q, f_i)$ .

LEMMA 23. *With the above notation, for  $j = 0, 1$ , let  $x_j \in \text{Fix}(\phi)$ , and suppose that  $\tilde{x}_j \in q^{-1}(x_j) \cap \text{Coin}(q, f_i)$  for some  $i \in \{1, \dots, n\}$ . If  $\tilde{x}_0$  and  $\tilde{x}_1$  belong to related Nielsen coincidence classes of  $(q, f_i)$  then  $|K_i(\tilde{x}_0)| = |K_i(\tilde{x}_1)|$ .*

*Proof.* We shall construct a bijection between  $K_i(\tilde{x}_0)$  and  $K_i(\tilde{x}_1)$ . First, we claim that there is an element  $\tilde{x}'_1 \in q^{-1}(q(\tilde{x}_1))$  that belongs to the same Nielsen coincidence class of  $(q, f_i)$  as  $\tilde{x}_0$ . To see this, let  $\tilde{x}_2, \tilde{x}'_2 \in \text{Coin}(q, f_i)$  be such that  $q(\tilde{x}_2) = q(\tilde{x}'_2)$ , and for which  $\tilde{x}_2$  (resp.  $\tilde{x}'_2$ ) belongs to the same Nielsen coincidence class of  $(q, f_i)$  as  $\tilde{x}_0$  (resp. as  $\tilde{x}_1$ ). Let  $\lambda: I \rightarrow \widehat{X}$  be a path such that  $\lambda(0) = \tilde{x}'_2$ ,  $\lambda(1) = \tilde{x}_1$  and  $q \circ \lambda$  is homotopic to  $f_i \circ \lambda$  relative to their endpoints, and let  $\lambda': I \rightarrow \widehat{X}$  be the unique lift of  $q \circ \lambda$  for which  $\lambda'(0) = \tilde{x}_2$ , and let  $\tilde{x}'_1 = \lambda'(1)$ . Then  $q(\tilde{x}'_1) = q(\tilde{x}_1)$ , and arguing as in the proof of Lemma 22, it follows that  $q \circ \lambda'$  is homotopic to  $f_i \circ \lambda'$  relative to their endpoints. Thus  $\tilde{x}_2$  and  $\tilde{x}'_1$  belong to the same Nielsen coincidence class of  $(q, f_i)$ , which proves the claim since  $\tilde{x}_2$  and  $\tilde{x}_0$  belong to the same Nielsen coincidence class of  $(q, f_i)$ . Then by Lemma 22(b) and (c), we have  $|K_i(\tilde{x}_0)| = |K_i(\tilde{x}_2)| = |K_i(\tilde{x}'_2)| = |K_i(\tilde{x}'_1)|$  as required.  $\square$

If  $s \in \{1, \dots, r\}$  and  $i \in I_0$ , let  $m_{i,s} = |L_i|/|K_i(\tilde{x}_s)|$ , where  $\tilde{x}_s$  is an element of one of the Nielsen coincidence classes of  $(q, f_i)$  that belongs to the equivalence class  $C_s$ . This quantity is the number of Nielsen coincidence classes of  $(q, f_i)$  in  $C_s$  that are sent to the same Nielsen fixed point class of  $\phi$  under the map  $q$ . Observe that if  $|L_i| = 1$  then  $m_{i,s} = 1$  for all  $s \in \{1, \dots, r\}$ .

We now come to the proof of the main result of this section.

*Proof of Theorem 6.* Let  $i_j \in \mathbb{O}_j$ , where  $j$  runs over the set  $I_0$  defined just before Lemma 20. From Lemma 19, the image by  $q$  of  $\text{Coin}(q, f_{i_j})$  is independent of the choice of representative  $i_j$  in  $\mathbb{O}_j$ . Since the action of  $L'_{i,i}$  on  $\{1, \dots, n\}$  is free,  $|L'_{i,i}| = |L_i| = 1$ , which implies that the map induced by  $q$  between the Nielsen coincidence classes of  $(q, f_i)$  and the Nielsen fixed point classes of  $\phi$  is injective for all  $i \in \{1, \dots, n\}$ . Further, if  $j, j' \in I_0$  are distinct then  $q(\text{Coin}(q, f_{i_j})) \cap q(\text{Coin}(q, f_{i_{j'}})) = \emptyset$  by Lemma 19, and we conclude that the map between the union of the Nielsen coincidence classes of the pairs  $(q, f_{i_j})$ , where  $i_j \in I_0$ , and the Nielsen fixed point classes of  $\phi$  is injective too. This map is also surjective by Proposition 17(b) and Lemma 21. From this, it follows that  $N(\phi)$  is equal to  $\sum_{j \in I_0} N(q, f_{i_j})$ .  $\square$

In the case  $n = 2$ , the hypothesis on the action of  $L_i$  is always satisfied.

**COROLLARY 24.** *Let  $X$  be an orientable compact manifold without boundary and  $\phi: X \rightarrow X$  a non-split 2-valued map. Then  $N(\phi) = N(q, f_1) = N(q, f_2)$ .*

*Proof.* Since  $\phi$  is non-split,  $L \cong L' \cong \mathbb{Z}_2$ , there is a single orbit  $\mathbb{O}_1 = \{1, 2\}$ , and  $|L_1| = |L_2| = 1$ . Then by Theorem 6,  $N(\phi) = N(q, f_j)$  for all  $j \in \{1, 2\}$ .  $\square$

## 5 Fixed point free split 2-valued maps of the 2-torus

In [GG4, Theorem 7], we gave some examples of split 2-valued maps of the 2-torus  $\mathbb{T}^2$  that may be deformed to fixed point free  $n$ -valued maps. The classification of such 2-valued maps is still an open problem. For example, we do not know whether there exists a 2-valued map of  $\mathbb{T}^2$  whose Nielsen number is zero but that cannot be deformed to a fixed point 2-valued free map. The aim of this section is to prove Theorem 7, which extends [GG4, Theorem 7] by exhibiting more families of 2-valued maps of  $\mathbb{T}^2$  that may be deformed to fixed point free  $n$ -valued maps. As we saw in that paper, the root setting appears to provide a more suitable framework in which to study the algebraic formulation of this problem, and

we begin by recalling the description of the fixed point problem for  $n$ -valued maps of  $\mathbb{T}^2$  in terms of roots that was given in [GG4, Section 5.3.2]. If  $\phi = \{f_1, f_2, \dots, f_n\} : \mathbb{T}^2 \multimap \mathbb{T}^2$  is a split  $n$ -valued map, there is a split  $n$ -valued map  $\varphi = \{g_1, \dots, g_n\} : \mathbb{T}^2 \multimap \mathbb{T}^2$  defined by  $g_i(x) = f_i(x).x^{-1}$  for all  $x \in \mathbb{T}^2$  and  $i \in \{1, \dots, n\}$ , and the fixed points of  $\phi$  are the roots of  $\varphi$ .

Before proving Theorem 7, we recall some notation that was defined in [GG4, Section 5.1] and illustrated in [GG4, Figure 1], as well as the structure of certain braid groups of the torus. Let  $1 \in \mathbb{T}^2$  be a basepoint, and let  $(e_1, e_2)$  be the basis of  $\pi_1(\mathbb{T}^2)$  described in the introduction. Let  $(x_1, x_2)$  be a basepoint of  $F_2(\mathbb{T}^2 \setminus \{1\})$  that we also take to be a basepoint of  $F_2(\mathbb{T}^2)$  and of  $\mathbb{T}^2 \times \mathbb{T}^2$  via the inclusions  $\iota_0 : F_2(\mathbb{T}^2 \setminus \{1\}) \rightarrow F_2(\mathbb{T}^2)$  and  $\iota_1 : F_2(\mathbb{T}^2) \rightarrow \mathbb{T}^2 \times \mathbb{T}^2$ . If  $i \in \{1, 2\}$ , the pair  $(\rho_{i,1}, \rho_{i,2})$  of elements of  $F_2(\mathbb{T}^2 \setminus \{1\})$  and of  $P_2(\mathbb{T}^2)$  projects to  $(e_1, e_2)$  under the homomorphism induced by the projection  $p_i : F_2(\mathbb{T}^2) \rightarrow \mathbb{T}^2$  onto the  $i^{\text{th}}$  coordinate. Let  $B_{1,2}$  denote the square of the standard Artin generator  $\sigma = \sigma_1$  that swaps  $x_1$  and  $x_2$ . By [GG4, Proposition 25],  $P_2(\mathbb{T}^2 \setminus \{1\})$  is isomorphic to a semi-direct product  $\mathbb{F}_3(u, v, B) \rtimes \mathbb{F}_2(x, y)$  of free groups, where from [GG4, Proposition 24 and equation (8)],  $u, v, x, y$  and  $B$  are described in terms of the elements of  $\{\rho_{i,j}\}_{1 \leq i, j \leq 2} \cup \{B_{1,2}\}$  by:

$$u = \rho_{1,1}, v = \rho_{1,2}, x = \rho_{1,1}B_{1,2}^{-1}\rho_{2,1}, y = \rho_{1,2}B_{1,2}^{-1}\rho_{2,2} \text{ and } B = [\rho_{1,1}, \rho_{1,2}^{-1}]B_{1,2}^{-1}, \quad (3)$$

and by [GG4, Proposition 25 and equation (9)],

$$\begin{cases} xux^{-1} = u, & xvx^{-1} = uvu^{-1}B_{1,2}, & x^{-1}vx = u^{-1}vB_{1,2}^{-1}u, \\ yvy^{-1} = v, & yuy^{-1} = vB_{1,2}^{-1}uv^{-1}, & y^{-1}uy = B_{1,2}v^{-1}uv. \end{cases} \quad (4)$$

Further, from [GG4, Corollary 27],  $P_2(\mathbb{T}^2)$  is isomorphic to the direct product of  $\mathbb{F}_2(u, v)$  with the free Abelian group generated by  $(x, y)$ , and we will often write the elements of  $P_2(\mathbb{T}^2)$  in the form  $(w, (a, b))$ , where  $w \in \mathbb{F}_2(u, v)$  and  $(a, b) \in \mathbb{Z}^2$ .

*Proof of Theorem 7.* The case  $(p, q) = (0, 0)$  was treated in [GG4, Theorem 7]. So suppose that  $(p, q) \neq (0, 0)$ . The groups  $\pi_1(\mathbb{T}^2)$  and  $H_1(\mathbb{T}^2; \mathbb{Z})$  will be identified via the Hurewicz isomorphism where necessary, and via this identification,  $e_1$  and  $e_2$  will also denote the corresponding elements of  $H_1(\mathbb{T}^2; \mathbb{Z})$ . Without loss of generality, we may assume that  $p$  and  $q$  are coprime, so there exists a linear toral homeomorphism  $h : \mathbb{T}^2 \rightarrow \mathbb{T}^2$  such that  $h(1) = 1$ , and for which the induced automorphisms  $h_{\#} : \pi_1(\mathbb{T}^2) \rightarrow \pi_1(\mathbb{T}^2)$  and  $h_* : H_1(\mathbb{T}^2; \mathbb{Z}) \rightarrow H_1(\mathbb{T}^2; \mathbb{Z})$  satisfy  $h_{\#}(e_1) = pe_1 + qe_2$  and  $h_*(e_1) = pe_1 + qe_2$ . By relaxing the condition that  $h$  is a linear toral automorphism, we may suppose that  $h$  fixes  $x_1$  and  $x_2$  as well as  $1$ . The homeomorphism  $h \times h : \mathbb{T}^2 \times \mathbb{T}^2 \rightarrow \mathbb{T}^2 \times \mathbb{T}^2$  restricts to a homeomorphism  $h \times h|_{F_2(\mathbb{T}^2)} : F_2(\mathbb{T}^2) \rightarrow F_2(\mathbb{T}^2)$ , and since  $h(1) = 1$ , to a homeomorphism  $f_h : F_2(\mathbb{T}^2 \setminus \{1\}) \rightarrow F_2(\mathbb{T}^2 \setminus \{1\})$  given by  $f_h(y_1, y_2) = (h(y_1), h(y_2))$  for all  $(y_1, y_2) \in F_2(\mathbb{T}^2 \setminus \{1\})$ . We thus have the following commutative diagram:

$$\begin{array}{ccccc} F_2(\mathbb{T}^2 \setminus \{1\}) & \xrightarrow{\iota_0} & F_2(\mathbb{T}^2) & \xrightarrow{\iota_1} & \mathbb{T}^2 \times \mathbb{T}^2 \\ \downarrow f_h & & \downarrow h \times h|_{F_2(\mathbb{T}^2)} & & \downarrow h \times h \\ F_2(\mathbb{T}^2 \setminus \{1\}) & \xrightarrow{\iota_0} & F_2(\mathbb{T}^2) & \xrightarrow{\iota_1} & \mathbb{T}^2 \times \mathbb{T}^2. \end{array} \quad (5)$$



On the level of the fundamental groups, this gives rise to the following commutative diagram:

$$\begin{array}{ccccc}
P_2(\mathbb{T}^2 \setminus \{1\}) & \xrightarrow{\iota_{0\#}} & P_2(\mathbb{T}^2) & \xrightarrow{\iota_{1\#}} & \pi_1(\mathbb{T}^2 \times \mathbb{T}^2) \\
\downarrow f_{h\#} & & \downarrow (h \times h|_{F_2(\mathbb{T}^2)})_{\#} & & \downarrow (h \times h)_{\#} \\
P_2(\mathbb{T}^2 \setminus \{1\}) & \xrightarrow{\iota_{0\#}} & P_2(\mathbb{T}^2) & \xrightarrow{\iota_{1\#}} & \pi_1(\mathbb{T}^2 \times \mathbb{T}^2),
\end{array} \tag{6}$$

the basepoint being  $(x_1, x_2)$  in each case. With respect to the bases chosen above, if  $(z, (\delta, \gamma)) \in P_2(\mathbb{T}^2)$ , where  $z \in \mathbb{F}_2(u, v)$  and  $(\delta, \gamma) \in \mathbb{Z}^2$ , then:

$$\iota_{1\#}(z, (\delta, \gamma)) = (|z|_u + \delta, |z|_v + \gamma, \delta, \gamma) = (\text{Ab}(z) + (\delta, \gamma), \delta, \gamma). \tag{7}$$

by (3). By (7) and the commutativity of (6), considering  $u$  and  $x$  to be elements of  $P_2(\mathbb{T}^2)$ , we have:

$$\begin{aligned}
\iota_{1\#} \circ (h \times h|_{F_2(\mathbb{T}^2)})_{\#}(u) &= (h \times h)_{\#} \circ \iota_{1\#}(u) = (h \times h)_{\#}(1, 0, 0, 0) = (p, q, 0, 0) \\
\iota_{1\#} \circ (h \times h|_{F_2(\mathbb{T}^2)})_{\#}(x) &= (h \times h)_{\#} \circ \iota_{1\#}(x) = (h \times h)_{\#}(1, 0, 1, 0) = (p, q, p, q),
\end{aligned}$$

from which we see that  $(h \times h|_{F_2(\mathbb{T}^2)})_{\#}(u)$  (resp.  $(h \times h|_{F_2(\mathbb{T}^2)})_{\#}(x)$ ) is of the form  $(w, (0, 0))$  (resp.  $(w', (p, q))$ ), where  $w, w' \in \mathbb{F}_2(u, v)$  and  $\text{Ab}(w) = (p, q)$  (resp.  $\text{Ab}(w') = (0, 0)$ ). On the other hand,  $(h \times h|_{F_2(\mathbb{T}^2)})_{\#}(u)$  (resp.  $(h \times h|_{F_2(\mathbb{T}^2)})_{\#}(x)$ ) is the image of the element  $u$  (resp.  $x$ ), considered as an element of  $P_2(\mathbb{T}^2 \setminus \{1\})$ , by  $\iota_{0\#} \circ (h \times h|_{F_2(\mathbb{T}^2)})_{\#}$ . But as elements of  $P_2(\mathbb{T}^2 \setminus \{1\})$ ,  $u$  and  $x$  commute by (4), so  $(h \times h|_{F_2(\mathbb{T}^2)})_{\#}(u)$  and  $(h \times h|_{F_2(\mathbb{T}^2)})_{\#}(x)$  do too, where in these expressions,  $u$  and  $x$  are considered as elements of  $P_2(\mathbb{T}^2)$ . We conclude that  $w$  and  $w'$  commute also, and so  $w'$  belongs to the centraliser of  $w$  in  $\mathbb{F}_2(u, v)$ . But since  $\text{Ab}(w') = (0, 0)$  and  $\text{Ab}(w) \neq (0, 0)$ , it follows that  $w' = 1$ . The rest of the proof is similar to that of [GG4, Theorem 7] and is as follows. Since  $(a-1, b)$ ,  $(c, d-1)$  and  $(m, n)$  belong to the cyclic subgroup generated by  $(p, q)$ , there exist  $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{Z}$  such that  $(a-1, b) = \lambda_1(p, q)$ ,  $(c, d-1) = \lambda_2(p, q)$  and  $(m, n) = \lambda_3(p, q)$ . We define the homomorphism  $\varphi: \pi_1(\mathbb{T}^2) \rightarrow P_2(\mathbb{T}^2 \setminus \{1\})$  by  $\varphi(e_1) = (f_{h\#}(u))^{\lambda_3 r} (f_{h\#}(x))^{\lambda_1}$  and  $\varphi(e_2) = (f_{h\#}(u))^{\lambda_3 s} (f_{h\#}(x))^{\lambda_2}$ . It is well defined because  $u$  and  $x$  commute in  $P_2(\mathbb{T}^2 \setminus \{1\})$ , so their images under  $f_{h\#}$  commute also. Let  $\phi$  be a split 2-valued map that admits a lift  $\widehat{\Phi}: \mathbb{T}^2 \rightarrow F_2(\mathbb{T}^2)$  for which  $\widehat{\Phi}_{\#}(e_1) = (w^{\lambda_3 r}, \lambda_1(p, q))$  and  $\widehat{\Phi}_{\#}(e_2) = (w^{\lambda_3 s}, \lambda_2(p, q))$ . Then the result follows from [GG4, Proposition 39] and the relation between the fixed point and root problems mentioned at the beginning of this section.  $\square$

REMARKS 25.

(a) Given two coprime integers  $p$  and  $q$ , we do not know in general how to obtain the two elements of  $P_2(\mathbb{T}^2 \setminus \{1\}) = \mathbb{F}_3 \rtimes \mathbb{F}_2$  that commute and that are sent to  $(p, q, 0, 0)$  and  $(p, q, p, q)$  respectively by the homomorphism  $\iota_{1\#} \circ \iota_{0\#}: \mathbb{F}_3 \rtimes \mathbb{F}_2 \rightarrow \mathbb{Z}^4$ . By ad hoc computations, one can check that the following pairs of elements commute:

- (i)  $xyxyx, uvB_{1,2}^{-1}uvB_{1,2}^{-1}u$   $((p, q) = (3, 2))$ .
- (ii)  $yxxyxy, vB_{1,2}^{-1}uvB_{1,2}^{-1}uv$   $((p, q) = (2, 3))$ .
- (iii)  $x^2yx^2yxy, u^2vu^2vuv$   $((p, q) = (5, 3))$ .



To see this, it was shown in the proof of [GG4, Proposition 25] that  $B_{1,2}$  commutes with  $x$  and  $y$ . So using (4), we obtain:

$$\begin{aligned} x(uvB_{1,2}^{-1}uvB_{1,2}^{-1}u)x^{-1} &= u. uvu^{-1}B_{1,2}. B_{1,2}^{-1}u. uvu^{-1}B_{1,2}. B_{1,2}^{-1}u = u(uv)^2 \\ xyu(xy)^{-1} &= x. vB_{1,2}^{-1}uv^{-1}. x^{-1} = uvu^{-1}B_{1,2}. B_{1,2}^{-1}u. B_{1,2}^{-1}uv^{-1}u^{-1} = uvB_{1,2}^{-1}u(uv)^{-1} \\ xyv(xy)^{-1} &= xv x^{-1} = uvu^{-1}B_{1,2}. \end{aligned}$$

By [GG4, Lemma 38],  $xy$  commutes with  $uv$ , and so:

$$\begin{aligned} xyxyx. (uvB_{1,2}^{-1}uvB_{1,2}^{-1}u). (xyxyx)^{-1} &= (xy)^2u(uv)^2(xy)^{-2} = xy. uvB_{1,2}^{-1}u(uv)^{-1}(xy)^{-1}. (uv)^2 \\ &= uvB_{1,2}^{-1}. uvB_{1,2}^{-1}u(uv)^{-1}. (uv)^{-1}. (uv)^2 \\ &= uvB_{1,2}^{-1}uvB_{1,2}^{-1}u \text{ and} \\ yxyxy. vB_{1,2}^{-1}uvB_{1,2}^{-1}uv. (yxyxy)^{-1} &= yxy. uvu^{-1}B_{1,2}. B_{1,2}^{-1}uvB_{1,2}^{-1}uv(yxy)^{-1} \\ &= yxy. uvvB_{1,2}^{-1}uv(yxy)^{-1} = y. uv. uvu^{-1}B_{1,2}. B_{1,2}^{-1}uv. y^{-1} \\ &= y. (uv)^2v. y^{-1} = (vB_{1,2}^{-1}u)^2v = vB_{1,2}^{-1}uvB_{1,2}^{-1}uv, \end{aligned}$$

from which it follows that the pairs of elements in (i) and in (ii) commute. For (iii), we have:

$$\begin{aligned} x^2yuy^{-1}x^{-2} &= x(uvB_{1,2}^{-1}u(uv)^{-1})x^{-1} = u. uvu^{-1}B_{1,2}. B_{1,2}^{-1}u. B_{1,2}^{-1}uv^{-1}u^{-1}. u^{-1} \\ &= u^2vB_{1,2}^{-1}uv^{-1}u^{-2} \\ x^2yvy^{-1}x^{-2} &= x(uvu^{-1}B_{1,2})x^{-1} = u. uvu^{-1}B_{1,2}. u^{-1}B_{1,2} = u^2vu^{-1}B_{1,2}u^{-1}B_{1,2}. \end{aligned}$$

and so:

$$x^2yuvy^{-1}x^{-2} = u^2vB_{1,2}^{-1}uv^{-1}u^{-2}. u^2vu^{-1}B_{1,2}u^{-1}B_{1,2} = u^2vu^{-1}B_{1,2}$$

By [GG4, Lemma 38],  $x^2y$  commutes with  $u^2y$ , hence:

$$\begin{aligned} (x^2y)^2xy(u^2v)^2uv(xy)^{-1}(x^2y)^{-2} &= (x^2y)^2(uvB_{1,2}^{-1}u(uv)^{-1}. uv)^2uv(x^2y)^{-2} \\ &= x^2y(u^2vu^{-1}B_{1,2}. B_{1,2}^{-1}. u^2vB_{1,2}^{-1}uv^{-1}u^{-2})^2u^2vu^{-1}B_{1,2}(x^2y)^{-1} \\ &= x^2yu^2vu^{-1}u^2vB_{1,2}^{-1}u^2vB_{1,2}^{-1}uv^{-1}u^{-2}u^2vu^{-1}B_{1,2}(x^2y)^{-1} \\ &= x^2yu^2vuvB_{1,2}^{-1}u^2v(x^2y)^{-1} = u^2v. u^2vu^{-1}B_{1,2}. B_{1,2}^{-1}. u^2v \\ &= (u^2v)^2uv, \end{aligned}$$

which shows that the pair of elements in (iii) commute.

(b) If  $a, b, c, d, m, n, p, q \in \mathbb{Z}$  are integers that satisfy the hypotheses of Theorem 7, then there exists a split fixed point free 2-valued map  $\phi: \mathbb{T}^2 \multimap \mathbb{T}^2$  for which the conclusion of the theorem holds, in particular  $N(\phi) = 0$ . However, if  $\phi': \mathbb{T}^2 \multimap \mathbb{T}^2$  is a split 2-valued map,  $w' \in \mathbb{F}_2(u, v)$  and  $\widehat{\Phi}': \mathbb{T}^2 \rightarrow F_2(\mathbb{T}^2)$  is a lift of  $\phi'$  such that  $\text{Ab}(w') = (m, n)$ ,  $\widehat{\Phi}'_{\#}(e_1) = (w'^r, (a, b))$  and  $\widehat{\Phi}'_{\#}(e_2) = (w'^s, (c, d))$ , we do not know whether  $\phi'$  can be deformed to a fixed point free 2-valued map, even though  $N(\phi') = 0$ .

## 6 The classification of the homotopy classes of two-valued maps of $\mathbb{T}^2$ and some fixed point free 2-valued maps

In this section, we start by describing the homotopy classes of 2-valued maps of  $\mathbb{T}^2$  in Section 6.1, extending [GG4, Proposition 29]. Given a 2-valued map  $\phi: \mathbb{T}^2 \dashrightarrow \mathbb{T}^2$ , a necessary condition for it to be deformable to a fixed point free 2-valued map is that  $N(\phi) = 0$ . We then give a formula in terms of the braids that determine  $\phi$ , which enables us to characterise those 2-valued maps for which  $N(\phi) = 0$ , extending [GG4, Proposition 5].

In [GG4], the problem of deforming a 2-valued map was considered for the family of split 2-valued maps of  $\mathbb{T}^2$ . In this section, we shall study non-split 2-valued maps. The classification of those 2-valued maps (not necessarily split) that can be deformed to fixed point free 2-valued maps may be carried out in a manner similar to that of split 2-valued maps, either by making use of the mixed braid group  $B_{1,2}(\mathbb{T}^2)$  and [GG4, Proposition 19], or by applying [GG4, Theorem 23] to the braid group  $B_2(\mathbb{T}^2 \setminus \{1\})$ . We will pursue the question of the classification of those 2-valued maps that can be deformed to fixed point free 2-valued maps elsewhere.

### 6.1 The classification of 2-valued maps of $\mathbb{T}^2$

In [GG4], we exhibited the following presentation of  $B_2(\mathbb{T}^2)$ .

PROPOSITION 26 ([GG4, Proposition 28]). *The group  $B_2(\mathbb{T}^2, (x_1, x_2))$  admits the following presentation:*

*generators:  $u, v, x, y$  and  $\sigma$ .*

*relations:*

- (a)  $xux^{-1} = u$  and  $yuy^{-1} = v$ .
- (b)  $xvx^{-1} = v$  and  $yvy^{-1} = v$ .
- (c)  $xyx^{-1} = y$ .
- (d)  $\sigma^2 = [u, v^{-1}]$ .
- (e)  $\sigma x \sigma^{-1} = x$  and  $\sigma y \sigma^{-1} = y$ .
- (f)  $\sigma u \sigma^{-1} = [u, v^{-1}]u^{-1}x$  and  $\sigma v \sigma^{-1} = [u, v^{-1}]v^{-1}y$ .

REMARK 27. The following notation and facts will be used in what follows.

(a) As we mentioned at the beginning of Section 5,  $P_2(\mathbb{T}^2)$  is isomorphic to the direct product of  $\mathbb{F}_2(u, v)$  with the free Abelian group of rank 2 generated by  $(x, y)$ . The elements  $u, v, x$  and  $y$  of Proposition 26 belong to  $P_2(\mathbb{T}^2)$ , and correspond to the images of the corresponding generators of  $P_2(\mathbb{T}^2 \setminus \{1\})$  by the homomorphism  $\iota_{0\#}$  of (6). From now on, if  $w \in P_2(\mathbb{T}^2)$  then it will be written in terms of the direct product  $\mathbb{F}_2(u, v) \times \mathbb{Z}^2$ , namely  $w = (w_1, w_2)$ , where  $w_1 \in \mathbb{F}_2(u, v)$  and  $w_2 = (a, b)$ , where  $(a, b) \in \mathbb{Z}^2$ .

(b) If  $x \in \mathbb{F}_2(u, v)$ , recall that  $\widehat{w}$ ,  $|w|_u$  and  $|w|_v$  were defined in the introduction. Note that  $\widehat{w}$  is the image of  $w$  under the homomorphism defined in the paragraph preceding [GG4, Lemma 32].

(c) The centraliser of a non-trivial element  $w$  of  $\mathbb{F}_2(u, v)$  is isomorphic to  $\mathbb{Z}$ , and so there are two choices for the generator of this centraliser. Further, if  $w \in \mathbb{F}_2(u, v)$  and  $m \in \mathbb{Z} \setminus \{0\}$  then the centraliser of  $w^m$  in  $\mathbb{F}_2(u, v)$  is equal to that of  $w$ .

The following result, which is basically contained in the proof of [GG4, Lemma 32], characterises the solutions of a certain equation in  $\mathbb{F}_2(u, v)$ , and will be used in the description of 2-valued maps in Proposition 8.

LEMMA 28. *In the group  $\mathbb{F}_2(u, v)$ , the solutions of the equation  $\widehat{\theta} = \xi\theta\xi^{-1}$  are given by the set of pairs  $(\theta, \xi)$ , where  $\theta = (\lambda\widehat{\lambda})^m$  and  $\xi = \lambda^{-1}\mu^l$ , where  $\lambda \in \mathbb{F}_2(u, v)$  and  $m, l \in \mathbb{Z}$  are arbitrary, and  $\mu$  is a generator of the centraliser of  $\lambda\widehat{\lambda}$  if  $\lambda\widehat{\lambda}$  is non trivial, and is arbitrary if  $\lambda\widehat{\lambda} = 1$ .*

*Proof.* One may check in a straightforward manner that the given pairs do indeed constitute solutions of the equation. Conversely, suppose that the pair  $(\theta, \xi)$  is a solution of the equation  $\widehat{\theta} = \xi\theta\xi^{-1}$ . By [GG4, Lemma 32(c)], there exist  $\theta, \lambda \in \mathbb{F}_2(u, v)$  and  $m \in \mathbb{Z}$  such that  $\theta = (\lambda\widehat{\lambda})^m$ . If  $\xi_0 = \lambda^{-1}$  then the pair  $(\theta, \xi_0)$  is a solution of the equation  $\widehat{\theta} = \xi\theta\xi^{-1}$  because  $\lambda^{-1}(\lambda\widehat{\lambda})^m\lambda = (\lambda^{-1}.\lambda\widehat{\lambda}.\lambda)^m = (\widehat{\lambda}\lambda)^m$ . So  $\xi\theta\xi^{-1} = \widehat{\theta} = \xi_0\theta\xi_0^{-1} = \lambda^{-1}\theta\lambda$ , hence  $(\lambda\xi)\theta(\lambda\xi)^{-1} = \theta$ , and thus  $\lambda\xi$  belongs to the centraliser of  $(\lambda\widehat{\lambda})^m$ . The result is then a consequence of Remark 27(c).  $\square$

Using Lemma 28, we may prove the following result, which does not seem obvious, and is interesting in its own right.

PROPOSITION 29. *Let  $\lambda \in \mathbb{F}_2(u, v)$  be such that  $\lambda\widehat{\lambda}$  is non trivial, and let  $\mu$  be a generator of the centraliser of  $\lambda\widehat{\lambda}$ . Then there exists  $\alpha \in \mathbb{F}_2(u, v)$  such that  $\mu = \alpha\widehat{\alpha}$ .*

*Proof.* Let  $\lambda$  and  $\mu$  be non-trivial elements of  $\mathbb{F}_2(u, v)$  as in the statement of the proposition. By Remark 27(c),  $\mu$  is primitive *i.e.* if  $\rho \in \mathbb{F}_2(u, v)$  and  $p \in \mathbb{Z}$  are such that  $\mu = \rho^p$  then  $p \in \{1, -1\}$ , and there exists  $l \in \mathbb{Z}$  such that  $\lambda\widehat{\lambda} = \mu^l$ . So  $\lambda^{-1}\mu^l\lambda = \widehat{\lambda}\lambda = \widehat{\mu}^l$  in the free group  $\mathbb{F}_2(u, v)$ , from which we conclude that  $\lambda^{-1}\mu\lambda = \widehat{\mu}$ . Taking  $\xi = \lambda^{-1}$  and  $\theta = \mu$  in the statement of Lemma 28, it follows that there exist  $\delta \in \mathbb{F}_2(u, v)$  and  $m \in \mathbb{Z}$  such that  $\mu = (\delta\widehat{\delta})^m$ . But  $\mu$  is primitive, so  $m \in \{1, -1\}$ . If  $m = 1$  (resp.  $m = -1$ ), we may take  $\alpha = \delta$  (resp.  $\alpha = \widehat{\delta}^{-1}$ ).  $\square$

REMARK 30. With the notation and hypotheses of Proposition 29, let  $l \in \mathbb{Z}$  be such that  $\lambda\widehat{\lambda} = \mu^l$ , and let  $\alpha \in \mathbb{F}_2(u, v)$  be such that  $\mu = \alpha\widehat{\alpha}$ . If  $\beta$  belongs to the centraliser of  $\lambda\widehat{\lambda}$  in  $\mathbb{F}_2(u, v)$  then there exists  $m \in \mathbb{Z}$  such that  $\beta = \mu^m$ . Replacing  $\mu$  by  $\mu^{-1}$  if necessary, we may suppose that  $m \geq 0$ . If  $m$  is odd then  $\beta$  is of the form  $\delta\widehat{\delta}$ , where  $\delta = (\alpha\widehat{\alpha})^{m-1}\alpha$ . If  $m$  is even (and non zero) then it is probable that  $\beta$  cannot be written in the form  $\delta\widehat{\delta}$ .

*Proof of Proposition 8.* Let  $\phi: \mathbb{T}^2 \multimap \mathbb{T}^2$  be a 2-valued map of  $\mathbb{T}^2$ , and let  $\alpha$  and  $\beta$  be as in the statement of the proposition. Since  $\beta \in B_2(\mathbb{T}^2) \setminus P_2(\mathbb{T}^2)$  (resp.  $\alpha \in B_2(\mathbb{T}^2) \setminus P_2(\mathbb{T}^2)$ ,  $\alpha, \beta \in B_2(\mathbb{T}^2) \setminus P_2(\mathbb{T}^2)$ ) in part (a) (resp. in part (b), in part (c)), there exist  $w, z \in \mathbb{F}_2(u, v)$  and  $a, b, c, d \in \mathbb{Z}$  such that  $\alpha = (w(u, v), (a, b))$  and  $\beta = \beta'\sigma$ , where  $\beta' = (z(u, v), (c, d))$  (resp.  $\alpha = \alpha'\sigma$  and  $\beta = (z(u, v), (c, d))$ , where  $\alpha' = (w(u, v), (a, b))$ ,  $\alpha = \alpha'\sigma$  and  $\beta = \beta'\sigma$ , where  $\alpha' = (w(u, v), (a, b))$  and  $\beta' = (z(u, v), (c, d))$ ). We now treat the three cases separately.

(a) The equation  $\alpha\beta = \beta\alpha$ , which is the condition given in [GG4, Proposition 29] for the pair  $(\alpha, \beta)$  of braids to define a 2-valued map, is equivalent to the equation  $\sigma\alpha\sigma^{-1} = \beta'^{-1}\alpha\beta'$ , and in turn to:

$$\sigma(w, 0, 0)\sigma^{-1} = (z^{-1}wz, 0, 0). \quad (8)$$

On the other hand, by [GG4, equation (16)], we have:

$$\sigma(w, 0, 0)\sigma^{-1} = (uv^{-1}\widehat{w}vu^{-1}, |w|_u, |w|_v). \quad (9)$$

Comparing equations (8) and (9), we see that  $\widehat{w} = (vu^{-1}z^{-1})w(zuv^{-1})$ , and applying Lemma 28, there exist  $l, m \in \mathbb{Z}$  and  $\lambda \in \mathbb{F}_2(u, v)$  such that  $w = (\lambda\widehat{\lambda})^m$  and  $vu^{-1}z^{-1} = \lambda^{-1}\mu^l$ , where  $\mu$  is a generator of the centraliser of  $\lambda\widehat{\lambda}$ , or is arbitrary if  $\lambda\widehat{\lambda} = 1$ . Hence  $z = \mu^{-l}\lambda vu^{-1}$ , which proves part (a).

(b) follows by exchanging the rôles of  $\alpha$  and  $\beta$  in part (a).

(c) Arguing as in case (a), the equation  $\alpha\beta = \beta\alpha$  gives rise to the following equation:

$$\widehat{z}\widehat{w}^{-1} = (vu^{-1})w^{-1}z(uv^{-1}) = (vu^{-1}w^{-1})zw^{-1}(wuv^{-1}),$$

and applying Lemma 28, there exist  $\lambda \in \mathbb{F}_2(u, v)$  and  $l, m \in \mathbb{Z}$  such that  $zw^{-1} = (\lambda\widehat{\lambda})^m$  and  $vu^{-1}w^{-1} = \lambda^{-1}\mu^l$ , where  $\mu$  is a generator of the centraliser of  $\lambda\widehat{\lambda}$  if  $\lambda\widehat{\lambda} \neq 1$  and is arbitrary if  $\lambda\widehat{\lambda} = 1$ . So  $w = \mu^{-l}\lambda vu^{-1}$  and  $z = (\lambda\widehat{\lambda})^m \mu^{-l} \lambda vu^{-1} = \mu^{-l} \lambda (\lambda\widehat{\lambda})^m vu^{-1}$ . If  $\lambda\widehat{\lambda} = 1$  then  $z = w$  is arbitrary.  $\square$

## 6.2 A necessary condition for a non-split 2-valued map to be deformed to a fixed point free 2-valued map

Let  $\phi: \mathbb{T}^2 \multimap \mathbb{T}^2$  be a non-split 2-valued map. An obvious necessary condition for  $\phi$  to be deformable to a fixed point free 2-valued map, which is that in the split case, is that  $N(\phi) = 0$ . In this section, we will express  $N(\phi)$  in terms of the braids associated to  $\phi$ , and in particular, we will obtain necessary and sufficient conditions for  $N(\phi)$  to be zero. Let  $\Phi: \mathbb{T}^2 \rightarrow D_2(\mathbb{T}^2)$  be the 2-unordered map that corresponds to  $\phi$ , and as in Section 4, let  $H$  be the subgroup  $\Phi_{\#}^{-1}(P_2(\mathbb{T}^2))$  of  $\pi_1(\mathbb{T}^2)$  of index 2. Taking  $n = 2$  in Proposition 17(a), the 2-valued map  $\phi \circ q: \mathbb{T}^2 \multimap \mathbb{T}^2$  has exactly two lifts of the form  $(f_1, f_2)$  and  $(f_2, f_1)$ , where for  $i = 1, 2$ ,  $f_i: \mathbb{T}^2 \rightarrow \mathbb{T}^2$  is a self-map of  $\mathbb{T}^2$ . We now prove Theorem 9, which gives the matrices of  $(f_1)_{\#}$  and  $(f_2)_{\#}$  with respect to the basis  $(e_1, e_2)$  of  $\pi_1(\mathbb{T}^2)$ , the Lefschetz coincidence numbers of  $f_1$  and  $f_2$  with  $q$  and the Nielsen number of  $\phi$ .

*Proof of Theorem 9.* The three cases of the statement correspond to those of Proposition 8. We start by calculating  $M_1$  and  $M_2$ . If  $\phi_1 = \phi \circ q$  and  $\widehat{\Phi}_1 = (f_1, f_2)$  is a lift of  $\phi_1$ , then from the proof of Proposition 5 in [GG4, Section 5.3.1], we have  $f_{i\#} = p_{i\#} \circ (\widehat{\Phi}_1)_{\#}$  for  $i = 1, 2$ . By construction,  $\widehat{\Phi}_{1\#}(e_1) = \alpha$ ,  $\widehat{\Phi}_{1\#}(e_2) = \beta$  and  $H = \text{Ker}(\tau \circ \Phi_{\#})$ , from which it follows that  $H = \langle e_1, 2e_2 \rangle$  in case (a), and we take  $q_{\#}(e_1) = e_1$  and  $q_{\#}(e_2) = 2e_2$ ,  $H = \langle 2e_1, e_2 \rangle$  in case (b), and we take  $q_{\#}(e_1) = 2e_1$  and  $q_{\#}(e_2) = e_2$ , and  $H = \langle 2e_1, e_1 + e_2 \rangle$  in case (c), and we take  $q_{\#}(e_1) = 2e_1$  and  $q_{\#}(e_2) = e_1 + e_2$ . We now compute  $f_{i\#}(e_j)$  in the three cases, where  $i, j \in \{1, 2\}$ .

(a) Suppose that  $\alpha \in P_2(\mathbb{T}^2)$  and  $\beta \in B_2(\mathbb{T}^2) \setminus P_2(\mathbb{T}^2)$ . Then from Proposition 8(a),  $(\widehat{\Phi}_1)_{\#}(e_1) = \alpha = ((\lambda\widehat{\lambda})^m, (a, b))$  and  $(\widehat{\Phi}_1)_{\#}(e_2) = \beta^2 = ((\mu^{-l}\lambda vu^{-1}, (c, d))\sigma)^2$ , where  $l, m, a, b, c, d \in \mathbb{Z}$ ,  $\lambda \in \mathbb{F}_2(u, v)$ , and  $\mu$  is a generator of the centraliser of  $\lambda\widehat{\lambda}$  if  $\lambda\widehat{\lambda} \neq 1$ , or

is arbitrary if  $\lambda\widehat{\lambda} = 1$ . By Proposition 26 and [GG4, equation (16)], we obtain:

$$\begin{aligned}\beta^2 &= (\mu^{-l}\lambda v u^{-1}, (c, d)) \cdot \sigma(\mu^{-l}\lambda v u^{-1}, (c, d))\sigma^{-1} \cdot \sigma^2 \\ &= (\mu^{-l}\lambda v u^{-1}, (c, d))(uv^{-1}\widehat{\mu}^{-l}\widehat{\lambda}v^{-1}uvu^{-1}, (c - l|\mu|_u + |\lambda|_u - 1, d - l|\mu|_v + |\lambda|_v + 1)) \cdot \\ &\quad ([u, v^{-1}], (0, 0)) \\ &= (\mu^{-l}\lambda\widehat{\mu}^{-l}\widehat{\lambda}, (2c - l|\mu|_u + |\lambda|_u - 1, 2d - l|\mu|_v + |\lambda|_v + 1)).\end{aligned}$$

Using (3) and the relations

$$p_{i\#}(\rho_{k,j}) = \begin{cases} \rho_{i,j} & \text{if } k = i \\ 1 & \text{if } k \neq i, \end{cases}$$

it follows as in the proof of [GG4, Proposition 5] that:

$$\begin{aligned}f_{1\#}(e_1) &= \rho_{1,1}^a \rho_{1,2}^b, \quad f_{1\#}(e_2) = \rho_{1,1}^{2c-l|\mu|_u+|\lambda|_u-1} \rho_{1,2}^{2d-l|\mu|_v+|\lambda|_v+1} \\ f_{2\#}(e_1) &= \rho_{2,1}^a \rho_{2,2}^b, \quad \text{and } f_{2\#}(e_2) = \rho_{2,1}^{2c-l|\mu|_u+|\lambda|_u-1} \rho_{2,2}^{2d-l|\mu|_v+|\lambda|_v+1},\end{aligned}$$

hence  $M_1 = M_2 = \begin{pmatrix} a & 2c - l|\mu|_u + |\lambda|_u - 1 \\ b & 2d - l|\mu|_v + |\lambda|_v + 1 \end{pmatrix}$ . If  $\lambda\widehat{\lambda} \neq 1$ , Proposition 29 implies that  $|\mu|_u = |\mu|_v = 0$ .

(b) The formula for  $M_1$  and  $M_2$  follows in a similar way for that of part (a) by exchanging the rôles of  $\alpha$  and  $\beta$  in the computations and using Proposition 8(b).

(c) Suppose that  $\alpha, \beta \in B_2(\mathbb{T}^2) \setminus P_2(\mathbb{T}^2)$ . From Proposition 8(c),  $(\widehat{\Phi}_1)_{\#}(e_1) = \alpha^2$  and  $(\widehat{\Phi}_1)_{\#}(e_2) = \alpha\beta$ , where  $\alpha = (\mu^{-l}\lambda v u^{-1}, (a, b))\sigma$  and  $\beta = ((\lambda\widehat{\lambda})^m \mu^{-l}\lambda v u^{-1}, (c, d))\sigma$ ,  $l, m, a, b, c, d \in \mathbb{Z}$ ,  $\lambda \in \mathbb{F}_2(u, v)$ , and  $\mu$  is a generator of the centraliser of  $\lambda\widehat{\lambda}$  if  $\lambda\widehat{\lambda} \neq 1$ , or is arbitrary if  $\lambda\widehat{\lambda} = 1$ . The computation of  $\alpha^2$  is the same as that of part (b), and yields  $f_{1\#}(e_1) = \rho_{1,1}^{2a-l|\mu|_u+|\lambda|_u-1} \rho_{1,2}^{2b-l|\mu|_v+|\lambda|_v+1}$  and  $f_{2\#}(e_1) = \rho_{2,1}^{2a-l|\mu|_u+|\lambda|_u-1} \rho_{2,2}^{2b-l|\mu|_v+|\lambda|_v+1}$ .

It remains to calculate  $(\widehat{\Phi}_1)_{\#}(e_2)$ . We have:

$$\begin{aligned}(\widehat{\Phi}_1)_{\#}(e_2) &= \alpha\beta = (\mu^{-l}\lambda v u^{-1}, (a, b)) \cdot \sigma((\lambda\widehat{\lambda})^m \mu^{-l}\lambda v u^{-1}, (c, d))\sigma^{-1} \cdot \sigma^2 \\ &= (\mu^{-l}\lambda v u^{-1}, (a, b)) \cdot \\ &\quad (uv^{-1}(\widehat{\lambda}\lambda)^m \widehat{\mu}^{-l}\widehat{\lambda}v^{-1}u \cdot vu^{-1} \cdot [u, v^{-1}], (c - l|\mu|_u + |\lambda|_u - 1, d - l|\mu|_v + |\lambda|_v + 1)) \\ &= (\mu^{-l}\lambda(\widehat{\lambda}\lambda)^m \widehat{\mu}^{-l}\widehat{\lambda}, (a + c - l|\mu|_u + |\lambda|_u - 1, b + d - l|\mu|_v + |\lambda|_v + 1)).\end{aligned}$$

As in part (a), we obtain:

$$f_{1\#}(e_2) = \rho_{1,1}^{a+c-l|\mu|_u+|\lambda|_u-1} \rho_{1,2}^{b+d-l|\mu|_v+|\lambda|_v+1} \quad \text{and} \quad f_{2\#}(e_2) = \rho_{2,1}^{a+c-l|\mu|_u+|\lambda|_u-1} \rho_{2,2}^{b+d-l|\mu|_v+|\lambda|_v+1}.$$

It follows that  $M_1 = M_2 = \begin{pmatrix} 2a - l|\mu|_u + |\lambda|_u - 1 & a + c - l|\mu|_u + |\lambda|_u - 1 \\ 2b - l|\mu|_v + |\lambda|_v + 1 & b + d - l|\mu|_v + |\lambda|_v + 1 \end{pmatrix}$ . Once more from Proposition 29, we have  $|\mu|_u = |\mu|_v = 0$  if  $\lambda\widehat{\lambda} = 1$ .

If  $i = 1, 2$ , then in all three cases,  $L(q, f_i)$  is equal to the determinant of the matrix obtained by taking the difference of  $M_i$  and the matrix of  $q_{\#}$  with respect to the basis  $(e_1, e_2)$  (see [BBPT] or [DO, Corollary 1.2]). The formula for  $N(\phi)$  is then a consequence of Corollary 24. If  $\phi$  can be deformed to a fixed point free 2-valued map then  $N(\phi) = 0$  using [Sch2, Theorem 6.5]. If  $N(\phi) = 0$  then for  $i = 1, 2$ ,  $L(q, f_i) = 0$ , and so the pair can be deformed to a coincidence free pair.  $\square$

REMARK 31. Suppose that  $\lambda\widehat{\lambda} = 1$ , and assume that we are in case (a) of Theorem 9. Then  $\alpha = (1, (a, b))$ , and  $\beta = (w, (c, d))\sigma$  is an arbitrary element of  $B_2(\mathbb{T}^2) \setminus P_2(\mathbb{T}^2)$ . By a calculation similar to those in the proof of Theorem 9,

$$(w, (c, d))\sigma)^2 = (w, (c, d))\sigma(w, (c, d))\sigma^{-1}\sigma^2 = (wuv^{-1}\widehat{w}u^{-1}v, (2c + |w|_u, 2d + |w|_v)),$$

so  $M_1 = M_2 = \begin{pmatrix} a & 2c + |w|_u \\ b & 2d + |w|_v \end{pmatrix}$ , and  $L(q, f_i) = \begin{vmatrix} a - 1 & 2c + |w|_u \\ b & 2d + |w|_v - 2 \end{vmatrix}$ . Similar computations may be made in cases (b) and (c) of Theorem 9.

## References

- [Ber] C. Berge, *Topological Spaces*, Olivier & Boyd, Edinburgh and London, 1963.
- [Bet1] J. Better, Equivariant Nielsen fixed point theory for  $n$ -valued maps, *Top. Appl.* **157** (2010), 1804–1814.
- [Bet2] J. Better, A Wecken theorem for  $n$ -valued maps, *Top. Appl.* **159** (2012), 3707–3715.
- [BCP] C.-F. Bödigheimer, F. R. Cohen and M. D. Peim, Mapping class groups and function spaces, Homotopy methods in algebraic topology (Boulder, CO, 1999), *Contemp. Math.* **271**, 17–39, Amer. Math. Soc., Providence, RI, 2001.
- [BGZ] S. Bogatyı, D. L. Gonçalves and H. Zieschang, The minimal number of roots of surface mappings and quadratic equations in free groups, *Math. Z.* **236** (2001), 419–452.
- [Bro] R. B. S. Brooks, On removing coincidences of two maps when only one, rather than both, of them may be deformed by a homotopy, *Pac. J. Math.* **40** (1972), 45–52.
- [BBPT] R. Brooks, R. F. Brown, J. Pak and D. Taylor, Nielsen numbers of maps of tori, *Proc. Amer. Math. Soc.* **52** (1975), 398–400.
- [Br1] R. F. Brown, private communication.
- [Br2] R. F. Brown, Fixed points of  $n$ -valued maps of the circle, *Bull. Pol. Acad. Sci. Math.* **54** (2006), 153–162.
- [Br3] R. F. Brown, The Lefschetz number of an  $n$ -valued multimap, *J. Fixed Point Th. Appl.* **2** (2007), 53–60.
- [Br4] R. F. Brown, Nielsen numbers of  $n$ -valued fibre maps, *J. Fixed Point Th. Appl.* **4** (2008), 183–201.
- [Br5] R. F. Brown, Construction of multiply fixed  $n$ -valued maps, *Top. Appl.* **196** (2015), 249–259.
- [BG] R. F. Brown and D. L. Gonçalves, On the topology of  $n$ -valued maps, preprint, 2016, submitted.
- [BK] R. F. Brown and K. Kolahi, Nielsen coincidence, fixed point and root theories of  $n$ -valued maps, *J. Fixed Point Theory Appl.* **14** (2013), 309–324.
- [BL] R. F. Brown and J. T. Lo Kim Lin, Coincidences of projections and linear  $n$ -valued maps of tori, *Top. Appl.* **157** (2010), 1990–1998.



- [CG] F. R. Cohen and S. Gitler, On loop spaces of configuration spaces, *Trans. Amer. Math. Soc.* **354** (2002), 1705–1748.
- [DO] E. de Oliveira, Some applications of coincidence theory. *Manuscripta Math.* **86** (1995), 159–167.
- [FH1] E. Fadell and S. Husseini, The Nielsen number on surfaces. Topological methods in nonlinear functional analysis (Toronto, Ont., 1982), *Contemp. Math.* **21**, Amer. Math. Soc., Providence, RI, 1983, 59–98.
- [FVB] E. Fadell and J. Van Buskirk, The braid groups of  $\mathbb{D}^2$  and  $\mathbb{S}^2$ , *Duke Math. J.* **29** (1962), 243–257.
- [FZ] E. M. Feichtner and G. M. Ziegler, The integral cohomology algebras of ordered configuration spaces of spheres, *Doc. Math.* **5** (2000), 115–139.
- [GG1] D. L. Gonçalves and J. Guaschi, The braid groups of the projective plane, *Alg. Geom. Topol.* **4** (2004), 757–780.
- [GG2] D. L. Gonçalves and J. Guaschi, The classification of the virtually cyclic subgroups of the sphere braid groups, SpringerBriefs in Mathematics (2013).
- [GG3] D. L. Gonçalves and J. Guaschi, Inclusion of configuration spaces in Cartesian products, and the virtual cohomological dimension of the braid groups of  $\mathbb{S}^2$  and  $\mathbb{R}P^2$ , *Pac. J. Math.* **287** (2017), 71–99.
- [GG4] D. L. Gonçalves and J. Guaschi, Fixed points of multimaps, the fixed point property and the case of surfaces – a braid approach, preprint, November 2016, submitted for publication.
- [GK] D. L. Gonçalves and J. C. de Souza Kiihl, Teoria do índice, 14o Colóquio Brasileiro de Matemática, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 1983.
- [GS] D. L. Gonçalves and M. Spreafico, The fundamental group of the space of maps from a surface into the projective plane, *Math. Scand.* **104** (2009), 161–181.
- [Gor] L. Górniewicz, Topological fixed point theory of multivalued mappings, Second edition, Topological Fixed Point Theory and Its Applications **4**, Springer, Dordrecht, 2006.
- [Ji1] B. Jiang, Fixed points and braids, *Invent. Math.* **75** (1984), 69–74.
- [Ji2] B. Jiang, Fixed points and braids II, *Math. Ann.* **272** (1985), 249–256.
- [Ji3] B. Jiang, The Wecken property of the projective plane, *Banach Center Publ.*, Polish Acad. Sci. **49** (1999), 223–225.
- [Ke] M. R. Kelly, Minimizing the number of fixed points for self-maps of compact surfaces, *Pac. J. Math.* **126** (1987), 81–123.
- [Sch1] H. Schirmer, Fix-finite approximations of  $n$ -valued multifunctions, *Fund. Math.* **121** (1984), 73–80.
- [Sch2] H. Schirmer, An index and a Nielsen number for  $n$ -valued multifunctions, *Fund. Math.* **124** (1984), 207–219.

- [Sch3] H. Schirmer, A minimum theorem for  $n$ -valued multifunctions, *Fund. Math.* **126** (1985), 83–92.
- [W1] F. Wecken, Fixpunktclassen I, *Math. Ann.* **117** (1941), 659–671.
- [W2] F. Wecken, Fixpunktclassen II, Homotopieinvarianten der Fixpunkttheorie, *Math. Ann.* **118** (1941), 216–234.
- [W3] F. Wecken, Fixpunktclassen III, Mindestzahlen von Fixpunkten *Math. Ann.* **118** (1942), 544–577.