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# SHARP MEASURE CONTRACTION PROPERTY FOR GENERALIZED H-TYPE CARNOT GROUPS

DAVIDE BARILARI<sup>†</sup> AND LUCA RIZZI<sup>‡</sup>

ABSTRACT. We prove that H-type Carnot groups of rank  $k$  and dimension  $n$  satisfy the MCP( $K, N$ ) if and only if  $K \leq 0$  and  $N \geq k + 3(n - k)$ . The latter integer coincides with the geodesic dimension of the Carnot group. The same result holds true for the larger class of *generalized H-type* Carnot groups introduced in this paper, and for which we compute explicitly the optimal synthesis. This extends the results of [25] and constitutes the largest class of Carnot groups for which the curvature exponent coincides with the geodesic dimension. We stress that generalized H-type Carnot groups have step 2, include all corank 1 groups and, in general, admit abnormal minimizing curves.

As a corollary, we prove the absolute continuity of the Wasserstein geodesics for the quadratic cost on all generalized H-type Carnot groups.

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## 1. INTRODUCTION

The measure contraction property (MCP) is one of the generalizations of Ricci curvature bounds to non-smooth metric measure spaces  $(X, d, \mu)$ . The condition MCP( $K, N$ ) we discuss here was introduced by Ohta [20] and controls how the measure  $\mu$  of a set in  $X$  is distorted along geodesics defined by the distance  $d$ , where  $(X, d)$  is assumed to be a length space (see also Sturm [27], for a slightly stronger version of the same condition).

When  $X$  is a complete Riemannian manifold of dimension  $N \geq 2$ ,  $d$  is the Riemannian distance, and  $\mu$  is the Riemannian measure, the condition MCP( $K, N$ ) is equivalent to require that  $X$  has Ricci curvature bounded from below by  $K$  (see [20, Thm. 3.2]). The measure contraction property is thus a synthetic replacement for Ricci curvature bounds on metric spaces. We refer to Section 4 for a precise definition. We recall that the MCP is the weakest among the synthetic curvature conditions introduced in [26, 27, 18] and developed in the subsequent literature. We stress that all these conditions are stable under pointed Gromov-Hausdorff limits.

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In this paper, we continue the investigation of measure contraction properties in sub-Riemannian geometry, started in [15, 21, 25]. More precisely we focus our attention on a class of Carnot groups that we call *generalized H-type groups*, which includes the classical H-type Carnot groups, in the sense of Kaplan [16, 11]. Here a Carnot group  $G$  is considered as a metric measure space  $(G, d, \mu)$  equipped with the Carnot-Carathéodory distance  $d$  and a left-invariant measure  $\mu$  (thus a multiple of Haar and Popp measures, see [19, 10]).

Even though Carnot Groups (and more in general sub-Riemannian manifolds) can be seen as Gromov-Hausdorff limits of sequences of Riemannian ones with the same dimension, these sequences have Ricci curvature unbounded from below (for an explicit example, see [21]). Thus, it is not clear whether these structures satisfy some MCP.

The simplest Carnot groups, that is the  $2d + 1$  dimensional Heisenberg groups  $\mathbb{H}_{2d+1}$ , do not satisfy the  $\text{MCP}(K, 2d + 1)$  for any  $K$ . Indeed, in [15], Juillet proved that  $\mathbb{H}_{2d+1}$  satisfies  $\text{MCP}(K, N)$  if and only if  $K \leq 0$  and  $N \geq 2d + 3$ . The number  $2d + 3$  is then the optimal dimension for the synthetic condition  $\text{MCP}(0, N)$ , and is larger than both the topological  $(2d + 1)$  and the Hausdorff  $(2d + 2)$  dimensions of  $\mathbb{H}_{2d+1}$ .

Juliet's results have been extended in [25], where it is proved that any corank 1 Carnot group  $(G, d, \mu)$  of rank  $k$  satisfies the  $\text{MCP}(K, N)$  if and only if  $K \leq 0$  and  $N \geq k + 3$ . In particular, they satisfy the  $\text{MCP}(0, k + 3)$ .

For what concerns more general structures, in [21], Rifford proved that if a Carnot group of rank  $k$  and dimension  $n$  is ideal (i.e. does not admit abnormal minimizing curves), then it satisfies  $\text{MCP}(0, N)$  for some finite  $N$  satisfying the inequality  $N \geq Q + n - k$ , where  $Q$  is the Hausdorff dimension of the Carnot group. For a non-ideal Carnot group, if one *assumes* that  $\text{MCP}(0, N)$  is satisfied for some  $N$ , the inequality  $N \geq Q + n - k$  still holds.

**1.1. Geodesic dimension.** Since  $\text{MCP}(K, N)$  implies  $\text{MCP}(K, N')$  for all  $N' \geq N$ , the above results raised the question of which is the optimal  $N$  such that the  $\text{MCP}(K, N)$  is satisfied on a general metric measure space. A bound is given by the concept of geodesic dimension, introduced in [3] for sub-Riemannian structures and in [25] for metric measure spaces, and which we briefly overview here (see Section 4 for details). Let  $(X, d, \mu)$  be a metric measure space, which for simplicity we assume having negligible cut loci. Let  $\Omega \subset X$  be a bounded, measurable set, with  $0 < \mu(\Omega) < +\infty$ , and consider its homothety  $\Omega_t$  with center  $x$  and ratio  $t \in [0, 1]$ , such that  $\Omega_1 = \Omega$  and  $\Omega_0 = \{x\}$ . The *geodesic dimension*  $\mathcal{N}(x)$  at the point  $x \in X$  is, roughly speaking, the largest integer such that  $\mu(\Omega_t) = O(t^{\mathcal{N}(x)})$  when  $t \rightarrow 0$ , for all such  $\Omega$ .

**Proposition 1** ([25]). *Let  $(X, d, \mu)$  be a metric measure space with negligible cut loci that satisfies the  $\text{MCP}(K, N)$ , for some  $K \in \mathbb{R}$  and  $N > 1$  or  $K \leq 0$  and  $N = 1$ . Then*

$$(1) \quad N \geq \sup\{\mathcal{N}(x) \mid x \in X\}.$$

where  $\mathcal{N}(x)$  is the geodesic dimension of  $X$  at  $x$ .

**1.2. Carnot groups.** In the case of left-invariant structures on Carnot (or, more generally, Lie) groups, the number  $\mathcal{N}(x)$  is independent on  $x \in X$ , and we denote it by  $\mathcal{N}$ . Furthermore, notice that spaces verifying  $\text{MCP}(K, N)$  with  $K > 0$  are compact, so that Carnot groups can satisfy the  $\text{MCP}(K, N)$  only if  $K \leq 0$ . On the other hand, thanks to the homogeneity properties of Carnot groups, if  $\text{MCP}(K, N)$  is satisfied for some  $K \leq 0$ , then  $\text{MCP}(\varepsilon K, N)$  is also satisfied, for all  $\varepsilon \geq 0$ . For this reason the  $\text{MCP}(0, N)$  plays a special role for Carnot groups, which motivated the definition of *curvature exponent*, as

$$N_0 := \inf\{N > 1 \mid \text{MCP}(0, N) \text{ is satisfied}\}.$$

Indeed, by Proposition 1,  $N_0 \geq \sup\{\mathcal{N}(x) \mid x \in X\}$ . The following example shows that, in general, the curvature exponent  $N_0$  can be strictly bigger than the geodesic dimension.

*Example 2* (Riemannian Heisenberg). Consider the left-invariant Riemannian structure on  $\mathbb{R}^3$  generated by the following global orthonormal vector fields:

$$(2) \quad X = \partial_x - \frac{y}{2}\partial_z, \quad Y = \partial_y + \frac{x}{2}\partial_z, \quad Z = \partial_z.$$

As for any Riemannian structure,  $k = n = \mathcal{N} = 3$ . In [21] it is proved that, when equipped with the Riemannian volume, this structure satisfies the MCP(0, 5). The same techniques proves that the MCP(0, 5- $\varepsilon$ ) is violated for any  $\varepsilon > 0$ , so its curvature exponent is  $N_0 = 5$ . We stress that this is a left-invariant structure, but is not a Carnot group.

We expect, however, that for Carnot groups the geodesic dimension coincides with the curvature exponent (cf. the conjecture formulated in [25]). Our main result is a proof of this fact for generalized H-type groups (see Section 2), which generalizes the results of [15, 25], and constitutes then the largest class of structures for which the aforementioned conjecture is verified.

**Theorem 3.** *Let  $(G, d, \mu)$  be a generalized H-type Carnot group of rank  $k$  and dimension  $n$ . Then it satisfies the MCP( $K, N$ ) if and only if  $K \leq 0$  and  $N \geq k + 3(n - k)$ , the latter integer being the geodesic dimension of the Carnot group.*

We observe that generalized H-type groups may have non-trivial abnormal minimizing curves (albeit not strictly abnormal ones). Equivalently, not all generalized H-type groups are ideal, or fat (see Definitions 6 and 7).

As a byproduct of the proof, we describe explicitly normal and abnormal geodesics of generalized H-type groups (Lemmas 16, 18), the injectivity domain of the exponential map (Lemma 21), and an explicit formula for its Jacobian determinant (Lemma 19). In particular, this yields the optimal synthesis of generalized H-type groups.

**1.3. Consequences for optimal transport.** Theorem 3 has consequences on the optimal transportation problem on generalized H-type groups, which we briefly outline here.

The well posedness of the Monge problem with cost given by the squared sub-Riemannian distance for classical H-type groups was proved in [24], generalizing the previous results for the Heisenberg group obtained in [6]. The same result for general sub-Riemannian structures whose squared distance is Lipschitz in charts on  $X \times X$  was obtained in [4]. The Lipschitzness assumption on the diagonal was finally removed in [14]. We refer to [22] for background on the transportation problem in sub-Riemannian geometry.

For all step 2 sub-Riemannian structures on a smooth manifold  $X$  (such as generalized H-type Carnot groups),  $d^2$  is locally Lipschitz on  $X \times X$ . Hence, according e.g. to [14, Theorem 3.2], for any pair of compactly supported probability measures  $\mu_0, \mu_1 \in \mathcal{P}_{\text{comp}}(X)$ , with  $\mu_0$  absolutely continuous with respect to a reference measure  $\mu$ , there exists a unique optimal transport map for the Monge problem with quadratic cost.

In this setting, an interesting problem is to determine whether the Wasserstein geodesic  $\mu_t \in \mathcal{P}(X)$ , interpolating between  $\mu_0$  and  $\mu_1$ , is absolutely continuous w.r.t.  $\mu$ . An affirmative answer for the  $2d + 1$  dimensional Heisenberg group was the main result of [13]. More general structures have been discussed in [14, Theorem 3.5]. There it is proved that, if the sub-Riemannian squared distance is also locally semiconcave outside of the diagonal, then  $\mu_t \ll \mu$  for all  $t \in [0, 1)$ . Unfortunately, in presence of non-trivial abnormal minimizers,  $d^2$  is not necessarily locally semiconcave outside of the diagonal. Absolute continuity of  $\mu_t$  must then be proved directly (see, e.g. [9] for the case of corank 1 Carnot groups).

As a consequence of Theorem 3, and the fact that step 2 Carnot groups do not admit branching minimizing geodesics, we can apply [12, Theorem 1.1]<sup>1</sup>. In particular, we extend the absolute continuity result of [13] to a setting with non-trivial abnormal minimizers and higher corank. We refer to [12] for definitions.

<sup>1</sup>See also the appendix of [12], where it is discussed how, under the non-branching assumption, Ohta's MCP, which we use here, is equivalent to Sturm's one, which is the one primarily used in [12].

**Corollary 4** ([12]). *Let  $(X, d, \mu)$  be a generalized  $H$ -type Carnot group. Let  $\mu_0, \mu_1 \in \mathcal{P}_2(X)$  with  $\mu_0 \ll \mu$ . Then there exists a unique  $\nu \in \text{OptGeo}(\mu_0, \mu_1)$ ; it is induced by a map  $G : X \rightarrow \text{Geo}(X)$ , and the unique Wasserstein geodesic satisfies  $(e_t)_\# \nu = \rho_t \mu \ll \mu$  for all  $t \in [0, 1)$ . Finally, if  $\mu_0, \mu_1$  have bounded support and  $\rho_0$  is  $\mu$ -essentially bounded, then*

$$(3) \quad \|\rho_t\|_{L^\infty(X, \mu)} \leq \frac{1}{(1-t)^{k+3(n-k)}} \|\rho_0\|_{L^\infty(X, \mu)}, \quad \forall t \in [0, 1).$$

## 2. PRELIMINARIES

We recall some basic facts in sub-Riemannian geometry. For details, see [2, 22, 19].

**2.1. Basic definitions.** A sub-Riemannian manifold is a triple  $(M, \mathcal{D}, g)$ , where  $M$  is a smooth, connected manifold with  $\dim M = n \geq 3$ ,  $\mathcal{D}$  is a vector distribution of constant rank  $k \leq n$  and  $g$  is a smooth metric on  $\mathcal{D}$ . We assume that the distribution is bracket-generating. A *horizontal curve*  $\gamma : [0, 1] \rightarrow M$  is a Lipschitz continuous path such that  $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$  for almost every  $t$ , and we define its *length* as follows

$$(4) \quad \ell(\gamma) = \int_0^1 \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))} dt.$$

The *sub-Riemannian (or Carnot-Carathéodory) distance* is defined by:

$$(5) \quad d(x, y) = \inf\{\ell(\gamma) \mid \gamma(0) = x, \gamma(1) = y, \gamma \text{ horizontal}\}.$$

By Chow-Rashevskii theorem, the bracket-generating condition implies that  $d : M \times M \rightarrow \mathbb{R}$  is finite and continuous. If the metric space  $(M, d)$  is complete, then for any  $x, y \in M$  the infimum in (5) is attained. In place of the length  $\ell$ , we consider the *energy functional*

$$(6) \quad J(\gamma) = \frac{1}{2} \int_0^1 g(\dot{\gamma}(t), \dot{\gamma}(t)) dt.$$

On the space of horizontal curves defined on a fixed interval and with fixed endpoints, the minimizers of  $J$  coincide with the minimizers of  $\ell$  parametrized with constant speed. Since  $\ell$  is invariant by reparametrization (and we can always reparametrize horizontal curves in such a way that they have constant speed), we do not lose generality in defining *geodesics* as horizontal curves that locally minimize the energy between their endpoints.

The *Hamiltonian* of the sub-Riemannian structure  $H : T^*M \rightarrow \mathbb{R}$  is defined by

$$(7) \quad H(\lambda) = \frac{1}{2} \sum_{i=1}^k \langle \lambda, X_i \rangle^2, \quad \lambda \in T^*M,$$

where  $X_1, \dots, X_k$  is any local orthonormal frame for  $\mathcal{D}$ . Here  $\langle \lambda, \cdot \rangle$  denotes the dual action of covectors on vectors. The Hamiltonian vector field  $\vec{H}$  is the unique vector field such that  $\sigma(\cdot, \vec{H}) = dH$ , where  $\sigma$  is the canonical symplectic form of the cotangent bundle  $\pi : T^*M \rightarrow M$ . In particular, the *Hamilton equations* are

$$(8) \quad \dot{\lambda}(t) = \vec{H}(\lambda(t)), \quad \lambda(t) \in T^*M.$$

If  $(M, d)$  is complete, any solution of (8) can be extended to a smooth curve for all times.

**2.2. End-point map and Lagrange multipliers.** Let  $\gamma_u : [0, 1] \rightarrow M$  be an horizontal curve joining  $x$  and  $y$ . Let  $X_1, \dots, X_k$  be a smooth orthonormal frame of horizontal vector fields, defined in a neighborhood of  $\gamma_u$ , and let  $u \in L^\infty([0, 1], \mathbb{R}^k)$  be a *control* such that

$$(9) \quad \dot{\gamma}_u(t) = \sum_{i=1}^k u_i(t) X_i(\gamma_u(t)), \quad \text{a.e. } t \in [0, 1].$$

Let  $\mathcal{U} \subset L^\infty([0, 1], \mathbb{R}^k)$  be the neighborhood of  $u$  such that, for  $v \in \mathcal{U}$ , the solution of

$$(10) \quad \dot{\gamma}_v(t) = \sum_{i=1}^k v_i(t) X_i(\gamma_v(t)), \quad \gamma_v(0) = x,$$

is well defined for a.e.  $t \in [0, 1]$ . We define the *end-point map* with base point  $x$  as the map  $E_x : \mathcal{U} \rightarrow M$  that sends  $v$  to  $\gamma_v(1)$ . The end-point map is smooth on  $\mathcal{U}$ .

We can consider  $J : \mathcal{U} \rightarrow \mathbb{R}$  as a smooth functional on  $\mathcal{U}$  (identifying  $\mathcal{U}$  with a neighborhood of  $\gamma_u$  in the space of horizontal curves starting from  $x$ ). A minimizing geodesic  $\gamma_u$  is a solution of the constrained minimum problem

$$(11) \quad \min\{J(v) \mid v \in \mathcal{U}, E_x(v) = y\}.$$

By the Lagrange multipliers rule, there exists a non-trivial pair  $(\lambda_1, \nu)$ , such that

$$(12) \quad \lambda_1 \circ D_u E_x = \nu D_u J, \quad \lambda_1 \in T_y^* M, \quad \nu \in \{0, 1\},$$

where  $\circ$  denotes the composition of linear maps and  $D$  the (Fréchet) differential. If  $\gamma_u : [0, 1] \rightarrow M$  with control  $u \in \mathcal{U}$  is an horizontal curve (not necessarily minimizing), we say that a non-zero pair  $(\lambda_1, \nu) \in T_y^* M \times \{0, 1\}$  is a *Lagrange multiplier* for  $\gamma_u$  if (12) is satisfied. The multiplier  $(\lambda_1, \nu)$  and the associated curve  $\gamma_u$  are called *normal* if  $\nu = 1$  and *abnormal* if  $\nu = 0$ . Observe that Lagrange multipliers are not unique, and a horizontal curve may be both normal *and* abnormal. Observe also that  $\gamma_u$  is an abnormal curve if and only if  $u$  is a critical point for  $E_x$ . In this case,  $\gamma_u$  is also called a *singular curve*. The following characterization is a consequence of the Pontryagin Maximum Principle.

**Theorem 5.** *Let  $\gamma_u : [0, 1] \rightarrow M$  be an horizontal curve joining  $x$  and  $y$ . A non-zero pair  $(\lambda_1, \nu) \in T_y^* M \times \{0, 1\}$  is a Lagrange multiplier for  $\gamma_u$  if and only if there exists a Lipschitz curve  $\lambda(t) \in T_{\gamma_u(t)}^* M$  with  $\lambda(1) = \lambda_1$ , such that*

- if  $\nu = 1$  then  $\dot{\lambda}(t) = \vec{H}(\gamma(t))$ , i.e. it is a solution of Hamilton equations,
- if  $\nu = 0$  then  $\sigma(\dot{\lambda}(t), T_{\lambda(t)} \mathcal{D}^\perp) = 0$ ,

where  $\mathcal{D}^\perp \subset T^* M$  is the sub-bundle of covectors that annihilate the distribution.

In the first (resp. second) case,  $\lambda(t)$  is called a *normal* (resp. *abnormal*) *extremal*. Normal extremals are integral curves  $\lambda(t)$  of  $\vec{H}$ . As such, they are smooth, and characterized by their *initial covector*  $\lambda = \lambda(0)$ . A geodesic is normal (resp. abnormal) if admits a normal (resp. abnormal) extremal. On the other hand, it is well known that the projection  $\gamma_\lambda(t) = \pi(\lambda(t))$  of a normal extremal is locally minimizing, hence it is a normal geodesic. The *exponential map* at  $x \in M$  is the map

$$(13) \quad \exp_x : T_x^* M \rightarrow M,$$

which assigns to  $\lambda \in T_x^* M$  the final point  $\pi(\lambda(1))$  of the corresponding normal geodesic. The curve  $\gamma_\lambda(t) := \exp_x(t\lambda)$ , for  $t \in [0, 1]$ , is the normal geodesic corresponding to  $\lambda$ , which has constant speed  $\|\dot{\gamma}_\lambda(t)\| = \sqrt{2H(\lambda)}$  and length  $\ell(\gamma|_{[t_1, t_2]}) = \sqrt{2H(\lambda)}(t_2 - t_1)$ .

**Definition 6.** A sub-Riemannian structure  $(M, \mathcal{D}, g)$  is *ideal* if the metric space  $(M, d)$  is complete and there exists no non-trivial abnormal minimizers.

**Definition 7.** A sub-Riemannian structure  $(M, \mathcal{D}, g)$  is *fat* if for all  $x \in M$  and  $X \in \mathcal{D}$ ,  $X(x) \neq 0$ , then  $\mathcal{D}_x + [X, \mathcal{D}]_x = T_x M$ .

The definition of ideal structures appears in [21, 22], in the equivalent language of singular curves. We stress that fat sub-Riemannian structures admit no non-trivial abnormal curves (see [19, Section 5.6]). In particular, complete fat structures are ideal.



### 3. GENERALIZED H-TYPE CARNOT GROUPS

A Carnot group  $(G, \star)$  of step  $s$  is a connected, simply connected Lie group of dimension  $n$ , such that its Lie algebra  $\mathfrak{g} = T_e G$  is stratified and nilpotent of step  $s$ , that is

$$(14) \quad \mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s,$$

with

$$(15) \quad [\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{1+j}, \quad \forall 1 \leq j \leq s, \quad \mathfrak{g}_s \neq \{0\}, \quad \mathfrak{g}_{s+1} = \{0\}.$$

The group exponential map  $\exp_G : \mathfrak{g} \rightarrow G$  associates with  $V \in \mathfrak{g}$  the element  $\gamma_V(1)$ , where  $\gamma_V : [0, 1] \rightarrow G$  is the integral line, starting at  $\gamma_V(0) = e$ , of the left invariant vector field associated with  $V$ . Since  $G$  is simply connected and  $\mathfrak{g}$  is nilpotent,  $\exp_G$  is a smooth diffeomorphism. Thus, the choice of a basis of  $\mathfrak{g}$  induces coordinates on  $G \simeq \mathbb{R}^n$ , which are called *exponential coordinates*.

Let  $\mathcal{D}$  be the left-invariant distribution generated by  $\mathfrak{g}_1$ , with a left-invariant scalar product  $g$ . This defines a sub-Riemannian structure  $(G, \mathcal{D}, g)$  on the Carnot group. For  $x \in G$ , we denote with  $L_x(y) := x \star y$  the left translation. The map  $L_x : G \rightarrow G$  is a smooth isometry. Any Carnot group, equipped with the Carnot-Carathéodory distance  $d$  and the Lebesgue measure  $\mu$  of  $G = \mathbb{R}^n$  is a complete metric measure space  $(X, d, \mu)$ . Moreover, a Carnot group is fat if and only if it is ideal [25, Section 8.2].

**3.1. Generalized H-type groups.** Let  $(G, \mathcal{D}, g)$  be a step 2 Carnot group, with Lie algebra  $\mathfrak{g}$  of rank  $k$ , dimension  $n$  satisfying  $\dim \mathfrak{g}_1 = k$ ,  $\dim \mathfrak{g}_2 = n - k$  and

$$(16) \quad [\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_2, \quad [\mathfrak{g}_i, \mathfrak{g}_2] = 0, \quad i = 1, 2.$$

Any choice  $g$  of a scalar product on  $\mathfrak{g}_1$  induces a left-invariant sub-Riemannian structure  $(\mathcal{D}, g)$  on  $G$ , such that  $\mathcal{D}(p) = \mathfrak{g}_1(p)$  for all  $p \in G$ . Extend the scalar product  $g$  on  $\mathfrak{g}_1$  to a scalar product on the whole  $\mathfrak{g}$ , which we denote with the same symbol.

For any  $V \in \mathfrak{g}_2$ , the skew-symmetric operator  $J_V : \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$  is defined by

$$(17) \quad g(X, J_V Y) = g(V, [X, Y]), \quad \forall X, Y \in \mathfrak{g}_1.$$

**Definition 8.** We say that a step 2 Carnot group is of *generalized H-type* if there exists a symmetric, non-zero and non-negative operator  $S : \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$  such that

$$(18) \quad J_V J_W + J_W J_V = -2g(V, W)S^2, \quad \forall V, W \in \mathfrak{g}_2.$$

*Remark 9.* The above definition is well posed and does not depend on the choice of the extension of  $g$ . More precisely, if (18) is verified for the operators  $J_V$  defined by a choice of an extension of  $g$ , then the operators  $\tilde{J}_V$  defined by a different extension  $\tilde{g}$  will verify (18), with the *same* operator  $S$ . Moreover, by polarization, it is easy to show that a step 2 Carnot group is of generalized H-type if and only if there exists a symmetric, non-negative and non-zero operator  $S : \mathfrak{g}_1 \rightarrow \mathfrak{g}_1$  such that  $J_V^2 = -\|V\|^2 S^2$  for all  $V \in \mathfrak{g}_2$ .

*Remark 10.* The case  $S = \text{Id}_{\mathfrak{g}_1}$  corresponds to classical H-type groups, introduced by Kaplan in [16]. We refer to [11, Chapter 18] for a modern introduction.

*Remark 11.* When  $n = k + 1$ , we are in the case of corank 1 Carnot groups. If  $S$  is non-degenerate (and thus  $k = 2d$  is even and  $S > 0$ ), we are in the case of contact Carnot groups. The eigenvalues of  $S$  are also called *frequencies* of the Carnot group (see [1, 25]).

**3.2. On existence.** As discussed in [11, Chapter 18] and proved in [16, Corollary 1], as a consequence of the Clifford algebra structure enforced by (18), classical H-type Carnot groups of rank  $k$  and dimension  $n$  exist if and only if  $n - k \leq \rho(k)$ , where  $\rho$  is the Hurwitz-Radon function<sup>2</sup>. In particular, H-type groups of odd rank do not exist (indeed, when  $S$  is the identity,  $J_V$  must be at the same time skew-symmetric and orthogonal).

<sup>2</sup> $\rho(N) - 1$  gives the maximal number of independent vector fields on  $\mathbb{S}^{N-1}$ .

On the other hand, the class of generalized H-type groups is larger. We now discuss their existence. Let  $0 < \lambda_1 < \dots < \lambda_\ell$  be the non-zero eigenvalues of  $S$ , and let  $E_1, \dots, E_\ell$  be the corresponding eigenspaces. For each  $i = 1, \dots, \ell$ , the restrictions  $\frac{1}{\lambda_i} J_V|_{E_i} : E_i \rightarrow E_i$  must verify a standard H-type condition in dimension  $k_i = \dim E_i$ . Combining this observation with the classical results mentioned above, we have that there exists a generalized H-type Carnot group of dimension  $n$ , rank  $k$ , and multiplicities  $k_i$  if and only if

$$(19) \quad n - k < \min\{\rho(k_i) \mid i = 1, \dots, \ell\}.$$

Notice that the dimension of the kernel of  $S$  does not play any role in this condition. As a consequence, generalized H-type Carnot groups with odd rank do exist.

**3.3. Normal forms.** The choice of an orthonormal basis  $e_1, \dots, e_k$  on  $\mathfrak{g}_1$  and  $f_1, \dots, f_{n-k}$  on  $\mathfrak{g}_2$  yields, through the group exponential map, analytic coordinates  $(x, z)$  on  $G$  such that the sub-Riemannian distribution on  $G$  is generated by the following set of global orthonormal left-invariant vector fields

$$(20) \quad X_i = \partial_{x_i} - \frac{1}{2} \sum_{\alpha=1}^{n-k} \sum_{j=1}^k L_{ij}^\alpha x_j \partial_{z_\alpha}, \quad i = 1, \dots, k.$$

Here the  $L^\alpha$ , for  $\alpha = 1, \dots, n-k$ , constitute a set of linearly independent  $k \times k$  skew symmetric matrices. Observe in particular that

$$(21) \quad [X_i, X_j] = \sum_{\alpha=1}^{n-k} L_{ij}^\alpha \partial_{z_\alpha}, \quad i, j = 1, \dots, k.$$

If  $V = \sum v_\alpha f_\alpha$ , then  $J_V$  is represented by the skew-symmetric matrix  $L_v := \sum v_\alpha L^\alpha$ . Notice that  $v \mapsto L_v$  is linear. Property (18) corresponds to the following relations:

$$(22) \quad L_v L_w + L_w L_v = -2(v \cdot w) S^2, \quad \forall v, w \in \mathbb{R}^{n-k},$$

where the dot denotes the Euclidean scalar product. In particular,  $L_v$  and  $L_w$  anti-commute when  $v \perp w$ , and  $L_v^2 = -|v|^2 S^2$ . By a suitable orthogonal transformation of the  $e_i$ 's, we assume that  $S$  is diagonal. Its non-zero eigenvalues appear in pairs, that is

$$(23) \quad S = \begin{pmatrix} 0_{k-2d} & & & \\ & \alpha_1 \mathbb{1} & & \\ & & \ddots & \\ & & & \alpha_d \mathbb{1} \end{pmatrix}, \quad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

for some  $1 \leq d \leq k$ , and with  $0 < \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_d$ . As a consequence, for all  $v \in \mathbb{R}^{n-k}$ , the real normal form  $\tilde{L}_v$  of  $L_v$  is

$$(24) \quad \tilde{L}_v = \begin{pmatrix} 0_{k-2d} & & & \\ & |v| \alpha_1 \mathbb{J} & & \\ & & \ddots & \\ & & & |v| \alpha_d \mathbb{J} \end{pmatrix}, \quad \mathbb{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and its spectrum is  $\sigma(L_v) = \{0, \pm i|v|\alpha_1, \dots, \pm i|v|\alpha_d\}$ , where the zero value is not present if  $k = 2d$  (that is, when  $S > 0$ ).

#### 4. MEASURE CONTRACTION PROPERTY AND GEODESIC DIMENSION

Let  $(X, d, \mu)$  be a metric measure space, where  $(X, d)$  is a length space and  $\mu$  be a Borel measure such that  $0 < \mu(B(x, r)) < +\infty$  for any metric ball  $B(x, r)$ . Moreover let us assume that  $(X, d)$  has negligible cut loci: for any  $x \in X$  there exists a negligible set  $\mathcal{C}(x)$



and a measurable map  $\Phi^x : X \setminus \mathcal{C}(x) \times [0, 1] \rightarrow X$ , such that the curve  $\gamma(t) = \Phi^x(y, t)$  is the unique minimizing geodesic from  $x$  with  $y$ .<sup>3</sup>

For any set  $\Omega$ , we consider its geodesic homothety of center  $x \in X$  and ratio  $t \in [0, 1]$ :

$$(25) \quad \Omega_t := \{\Phi^x(y, t) \mid y \in X \setminus \mathcal{C}(x)\}.$$

For any  $K \in \mathbb{R}$ , define the function

$$(26) \quad s_K(t) := \begin{cases} (1/\sqrt{K}) \sin(\sqrt{K}t) & \text{if } K > 0, \\ t & \text{if } K = 0, \\ (1/\sqrt{-K}) \sinh(\sqrt{-K}t) & \text{if } K < 0. \end{cases}$$

**Definition 12** (Ohta [20]). Let  $K \in \mathbb{R}$  and  $N > 1$ , or  $K \leq 0$  and  $N = 1$ . We say that  $(X, d, \mu)$  satisfies the *measure contraction property* MCP( $K, N$ ) if for any  $x \in M$  and any measurable set  $\Omega$  with  $0 < \mu(\Omega) < +\infty$  (and with  $\Omega \subset B(x, \pi\sqrt{N-1/K})$  if  $K > 0$ )

$$(27) \quad \mu(\Omega_t) \geq \int_{\Omega} t \left[ \frac{s_K(td(x, z)/\sqrt{N-1})}{s_K(d(x, z)/\sqrt{N-1})} \right]^{N-1} d\mu(z), \quad \forall t \in [0, 1],$$

where we set  $0/0 = 1$  and the term in square bracket is 1 if  $K \leq 0$  and  $N = 1$ .

**4.1. The geodesic dimension.** The geodesic dimension was introduced in [3] for sub-Riemannian structures and extended in [25] in the more general setting of metric measure spaces (with negligible cut loci).

**Definition 13.** Let  $(X, d, \mu)$  be a metric measure space with negligible cut locus. For any  $x \in X$  and  $s > 0$ , define

$$(28) \quad C_s(x) := \sup \left\{ \limsup_{t \rightarrow 0^+} \frac{1}{t^s} \frac{\mu(\Omega_t)}{\mu(\Omega)} \mid \Omega \text{ measurable, bounded, } 0 < \mu(\Omega) < +\infty \right\},$$

where  $\Omega_t$  is the homothety of  $\Omega$  with center  $x$  and ratio  $t$  as in (25). The *geodesic dimension* of  $(X, d, \mu)$  at  $x \in X$  is the non-negative real number

$$(29) \quad \mathcal{N}(x) := \inf\{s > 0 \mid C_s(x) = +\infty\} = \sup\{s > 0 \mid C_s(x) = 0\},$$

with the conventions  $\inf \emptyset = +\infty$  and  $\sup \emptyset = 0$ .

Roughly speaking, the measure of  $\mu(\Omega_t)$  vanishes at least as  $t^{\mathcal{N}(x)}$  or more rapidly, for  $t \rightarrow 0$ . The two definitions in (29) are equivalent since  $s \geq s'$  implies  $C_s(x) \geq C_{s'}(x)$ .

When  $(X, d, \mu)$  is a metric measure space defined by an equiregular sub-Riemannian or Riemannian structure, equipped with a smooth measure  $\mu$ , we have

$$(30) \quad \mathcal{N}(x) \geq \dim_H(X) \geq \dim(X), \quad \forall x \in X,$$

and both equalities hold if and only if  $(X, d, \mu)$  is Riemannian (see [3, Proposition 5.49] and [25]). Here  $\dim_H(X)$  denotes the Hausdorff dimension of the metric space  $(X, d)$ . We refer to [25] for a discussion of the other properties of the geodesic dimension. We only remark that, in the case of left-invariant structure on Carnot (or, more generally, Lie) groups, thanks to left-invariance, the number  $\mathcal{N}(x)$  is independent on  $x \in X$ , and we denote it by  $\mathcal{N}$ . As a consequence of the explicit formula for  $\mathcal{N}$  on sub-Riemannian structures [25, Section 7], we have the following.

**Lemma 14.** *The geodesic dimension of generalized H-type Carnot groups is  $k + 3(n - k)$ .*

<sup>3</sup> Ohta defines the measure contraction property for general length spaces, possibly with non-negligible cut loci. In our setting, the definition we give in the following is equivalent to Ohta's, see [20, Lemma 2.3].

*Remark 15.* For simplicity, and since all step 2 sub-Riemannian structures satisfy this assumption, we defined the geodesic dimension and the measure contraction property for metric measure spaces with negligible cut loci. Nevertheless, these notions can be generalized to any metric measure space with little effort. Finally, let us recall that it is not known whether general Carnot groups have negligible cut loci (this is related with the Sard conjecture in sub-Riemannian geometry [5, 23, 17]).

## 5. GEODESICS OF GENERALIZED H-TYPE CARNOT GROUPS

In this section we compute the explicit expression of geodesics in generalized H-type Carnot groups. The unified notation permits to treat the corank 1 and the H-type cases at the same time. By left-invariance, it is sufficient to consider geodesics starting from the identity  $e = (0, 0)$ . We denote covectors  $\lambda = udx + vdz \in T_e^*G$ , with  $(u, v) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$  coordinates on  $T_e^*G$ , dual to the exponential coordinates  $(x, z) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$  of  $G$ .

**Convention.** For any analytic function  $f : \mathbb{C} \rightarrow \mathbb{C}$  and any square matrix  $M$ , the expression  $f(M)$  denotes the matrix valued function defined by the Taylor series of  $f$  at zero. For example,

$$(31) \quad \frac{\sin M}{M} := \sum_{\ell=0}^{\infty} \frac{M^{2\ell}}{(2\ell+1)!},$$

which is well defined for any  $M$ , possibly degenerate. With this convention, we routinely simplify matrix computations by reducing to functional calculus, and then evaluating on matrices. For example, for any pair  $f, g$  of analytic functions,  $f(M)g(M) = (fg)(M)$ . Furthermore, if  $M = M(v)$  depends smoothly on a parameter  $v$ , then

$$(32) \quad \partial_v f(M(v)) = f'(M(v))\partial_v M(v),$$

and similarly for integration. If  $M$  can be diagonalized and has eigenvalues  $\lambda_i \in \mathbb{C}$ , then

$$(33) \quad \det(f(M)) = \prod_i f(\lambda_i).$$

**Lemma 16** (Normal geodesics). *The exponential map  $\exp_e : T_e^*G \rightarrow G$  of a generalized H-type Carnot group is given by  $\exp_e(u, v) = (x, z)$ , where*

$$(34) \quad x = f(L_v)u,$$

$$(35) \quad z = \left( \frac{g(L_v)u \cdot u}{2|v|} \right) \frac{v}{|v|},$$

where  $|\cdot|$  is the Euclidean norm and  $f, g$  are the real-analytic functions

$$(36) \quad f(z) = \frac{1 - e^{-z}}{z}, \quad g(z) = 1 - \frac{\sinh(z)}{z}.$$

Geodesics  $\gamma_\lambda : [0, 1] \rightarrow G$  with initial covector  $\lambda = (u, v)$  are given by  $\gamma_\lambda(t) = \exp_e(tu, tv)$ .

*Remark 17.* For classical H-type groups,  $L_v^2 = -|v|^2\mathbb{1}$ , and the above expressions reduce to

$$(37) \quad x = \left( \frac{\sin |v|}{|v|} \mathbb{1} + \frac{\cos |v| - 1}{|v|^2} L_v \right) u,$$

$$(38) \quad z = |u|^2 \left( \frac{|v| - \sin |v|}{2|v|^2} \right) \frac{v}{|v|}.$$

If  $v = 0$ , our convention implies  $\exp_e(u, 0) = (u, 0)$ .

*Proof.* Let  $h_x = (h_x^1, \dots, h_x^k) : T^*G \rightarrow \mathbb{R}^k$  and  $h_z = (h_z^1, \dots, h_z^{n-k}) : T^*G \rightarrow \mathbb{R}^{n-k}$ , where  $h_x^i(\lambda) := \langle \lambda, X_i \rangle$ , for  $i = 1, \dots, k$  and  $h_z^\alpha(\lambda) := \langle \lambda, \partial_{z_\alpha} \rangle$  for all  $\alpha = 1, \dots, n-k$ . Thus,  $H = \frac{1}{2}|h_x|^2$ . Hamilton equations are

$$(39) \quad \dot{h}_z = 0, \quad \dot{h}_x = -L_{h_z} h_x, \quad \dot{x} = h_x, \quad \dot{z}^\alpha = -\frac{1}{2} h_x \cdot L^\alpha x,$$

the dot denoting the derivative with respect to  $t$ . Set  $h_z(0) = v$  and  $h_x(0) = u$ . Then,

$$(40) \quad h_z(t) = v, \quad h_x(t) = e^{-tL_v} u.$$

The equations for  $(x, z)$  can be easily integrated. Exploiting our convention, we obtain

$$(41) \quad x(t) = \int_0^t e^{-sL_v} u \, ds = \frac{\mathbb{1} - e^{-tL_v}}{L_v} u = f(tL_v)tu.$$

To deal with the  $z$  coordinate, observe that for any  $w \perp v \in \mathbb{R}^{n-k}$ , we have

$$(42) \quad 2w \cdot \dot{z} = -h_x \cdot L_w x = \sum_{m,n=0}^{\infty} c_{m,n} t^{n+m} (L_v^m u) \cdot (L_w L_v^n u),$$

for some coefficients  $c_{m,n}$ . We claim that each term  $(L_v^m u) \cdot (L_w L_v^n u) = 0$ , and thus the above expression vanishes identically. To prove this claim, assume without loss of generality that  $m \geq n$ . The argument is different depending on whether the difference  $m - n$  is even or odd. If  $m = n + 2\ell$ , we have

$$(43) \quad (L_v^{n+2\ell} u) \cdot (L_w L_v^n u) = (-1)^\ell (L_v^{n+\ell} u) \cdot (L_v^\ell L_w L_v^n u) = (L_v^{n+\ell} u) \cdot (L_w L_v^{n+\ell} u) = 0.$$

If  $m = n + 2\ell + 1$ , on the other hand, we have

$$(44) \quad (L_v^{n+2\ell+1} u) \cdot (L_w L_v^n u) = -(L_v^n u) \cdot (L_v^{2\ell+1} L_w L_v^n u) = (L_v^n u) \cdot (L_w L_v^{n+2\ell+1} u) = \\ = -(L_w L_v^n u) \cdot (L_v^{n+2\ell+1} u) = -(L_v^{n+2\ell+1} u) \cdot (L_w L_v^n u),$$

which implies  $(L_v^{n+2\ell+1} u) \cdot (L_w L_v^n u) = 0$ , and proves the claim.

As a consequence of the claim,  $z(t)$  is parallel to  $v$ . Then, we have

$$(45) \quad \dot{z}(t) = \frac{\dot{z}(t) \cdot v}{|v|} \frac{v}{|v|} = -\frac{h_x(t) \cdot L_v x(t)}{2|v|} \frac{v}{|v|}$$

$$(46) \quad = -\frac{tL_v f(tL_v) e^{tL_v} u \cdot u}{2|v|} \frac{v}{|v|}$$

$$(47) \quad = \frac{[\mathbb{1} - \cosh(tL_v)] u \cdot u}{2|v|} \frac{v}{|v|},$$

where in the second line we used that  $L_v$  is skew-symmetric, and in the third line we replaced the matrix  $tL_v f(tL_v) e^{tL_v}$  with its symmetric part, which can be easily computed from the explicit form of  $f$ . Integration on the time interval  $[0, 1]$  concludes the proof.  $\square$

Consider a non-zero covector  $\lambda = (u, v)$ , with  $u \in \ker S$ . Since  $Su = 0$  implies  $L_v u = 0$ , by Lemma 16, these curves are straight lines of the form

$$(48) \quad \exp_e(tu, tv) = (tu, 0), \quad \forall v \in \mathbb{R}.$$

In particular, there is an infinite number of initial covectors yielding the same geodesic, which is thus abnormal. All abnormal geodesics are of this type.

**Lemma 18** (Abnormal curves and abnormal geodesics). *Abnormal curves of generalized  $H$ -type groups are the Lipschitz curves  $\gamma_u(t) : [0, 1] \rightarrow G$  with control  $u \in L^\infty([0, 1], \mathbb{R}^k)$  such that  $u(t) \in \ker S$ . The curve  $\gamma_u(t)$  is an abnormal geodesic if and only if it is a normal geodesic with initial covector  $\lambda = (u, v)$  with  $u \in \ker S$ , i.e.*

$$(49) \quad \exp_e(tu, tv) = (tu, 0), \quad \forall v \in \mathbb{R}.$$

*Proof.* We use the explicit characterization of abnormal curves of Theorem 5. Let  $h_x = (h_x^1, \dots, h_x^k) : T^*G \rightarrow \mathbb{R}^k$  and  $h_z = (h_z^1, \dots, h_z^{n-k}) : T^*G \rightarrow \mathbb{R}^{n-k}$ , where  $h_x^i(\lambda) := \langle \lambda, X_i \rangle$ , for  $i = 1, \dots, k$  and  $h_z^\alpha(\lambda) := \langle \lambda, \partial_{z_\alpha} \rangle$  for all  $\alpha = 1, \dots, n - k$ . In these coordinates,

$$(50) \quad \mathcal{D}^\perp = \{(x, z, h_x, h_z) \in T^*G \mid h_x = 0\}.$$

Let  $\gamma_u(t)$  be an abnormal geodesic, with  $u \in L^\infty([0, 1], \mathbb{R}^k)$ . Let  $\lambda(t) = (x(t), z(t), 0, h_z(t))$  be a Lipschitz curve in  $\mathcal{D}^\perp$  such that  $\pi(\lambda(t)) = \gamma_u(t)$ . In particular

$$(51) \quad \dot{\lambda}(t) = \sum_{i=1}^k u_i(t) X_i + \sum_{\alpha=1}^{n-k} \dot{h}_z(t) \partial_{h_{z_\alpha}},$$

where the vector field  $X_i$  in the r.h.s. is regarded as a vector field on  $T^*G$ . We now demand that  $\sigma_{\lambda(t)}(\dot{\lambda}, T_\lambda \mathcal{D}^\perp) = 0$ . First, since  $\partial_{h_{z_\alpha}}|_{\lambda(t)} \in T_{\lambda(t)} \mathcal{D}^\perp$ , we obtain the condition

$$(52) \quad 0 = \sigma_{\lambda(t)}(\dot{\lambda}, \partial_{h_{z_\alpha}}) = \dot{h}_{z_\alpha}(t), \quad \forall \alpha = 1, \dots, n - k, \quad \text{a.e. } t \in [0, 1].$$

Hence  $h_z(t)$  is constant. This constant must be  $h_z(t) = v \neq 0$ , otherwise the associated abnormal Lagrange multiplier  $\lambda_1 = \lambda(1) \in D_{(x,z)}^\perp \subset T_{(x,z)}^*G$  would be the zero covector, which is impossible by Theorem 5. Furthermore, since  $X_j|_{\lambda(t)} \in T_{\lambda(t)} \mathcal{D}^\perp$ , we have

$$(53) \quad 0 = \sigma_{\lambda(t)}(\dot{\lambda}, X_j) = \sum_{i=1}^k \sum_{\alpha=1}^{n-k} u_i(t) L_{ij}^\alpha v_\alpha, \quad \forall j = 1, \dots, k, \quad \text{a.e. } t \in [0, 1],$$

which implies  $u(t) \in \ker L_v$ . Since  $v \neq 0$  and  $L_v^2 = -|v|^2 S^2$ , this condition is equivalent to  $u(t) \in \ker S$ . Thus, an horizontal curve  $\gamma_u(t)$  is abnormal if and only if it satisfies

$$(54) \quad \dot{\gamma}_u(t) = \sum_{i=1}^k u_i(t) X_i(\gamma_u(t)), \quad u(t) \in \ker S, \quad u \in L^\infty([0, 1], \mathbb{R}^k).$$

Using the explicit form of the vector fields  $X_i$ , this implies  $x(t) \in \ker S$  and, in turn,  $z(t) = 0$ . In particular  $\gamma_u(t) = \int_0^t u(s) ds$  lies in the plane  $\mathbb{R}^k \subset \mathbb{R}^k \oplus \mathbb{R}^{n-k}$  of  $G$ , and its length (resp. energy) is equal to its Euclidean length (resp. energy) in  $\mathbb{R}^k$ . If we demand  $\gamma_u(t)$  to be a geodesic, i.e. a locally energy minimizing curve, we have that,  $x(t)$  must be a straight line of the form  $x(t) = tu$ , in particular  $u(t) = u \in \ker S$ . From Lemma 16, we observe that the abnormal geodesics  $x(t) = (tu, 0)$ , with  $u \in \ker S$ , are also normal, with initial covectors  $\lambda = (u, v)$ , for any  $v \in \mathbb{R}^{n-k}$ .  $\square$

## 6. JACOBIAN DETERMINANT AND COTANGENT INJECTIVITY DOMAIN

The most important steps in the proof of Theorem 3 are the computation of the Jacobian determinant of the exponential map, and the characterization of the cotangent injectivity domain of generalized H-type Carnot groups. These results have independent interests. For example, the analogous formulas for the Heisenberg group and, more generally, corank 1 Carnot groups have been employed in [8, 9] to obtain Jacobian determinant inequalities.

**Notation.** For the statement of the next Lemma 19, we recall the notation of Section 3.3. For each  $v \in \mathbb{R}^{n-k}$ , the skew-symmetric matrix  $L_v$  has real normal form given by (24), with non-zero singular values  $|v|\alpha_i$ , with  $i = 1, \dots, d$ , with  $0 < \alpha_1 \leq \dots \leq \alpha_d$ , repeated according to their multiplicity. To each  $\alpha_j$  corresponds a pair of non-zero eigenvalues of the symmetric matrix  $S$  defining the H-type group, see (23). If  $v \neq 0$ , after applying an orthogonal transformation putting  $L_v$  in its real normal form (24), we denote  $u = (u_0, u_1, \dots, u_d)$ , where  $u_0 \in \mathbb{R}^{n-k}$  is the component in the kernel of  $L_v$ , while each  $u_i \in \mathbb{R}^2$ , for  $i = 1, \dots, d$ , is the component of  $u$  in the real eigenspace associated with the singular value  $\alpha_i$ . Notice that this decomposition is non-unique. The forthcoming lemma can be expressed in a more intrinsic way by introducing multiplicities of singular values, but for further manipulations we prefer to present it in its actual form.

**Lemma 19** (Jacobian determinant). *The Jacobian determinant of the exponential map  $\exp_e(u, v) = (x, z)$  of a generalized H-type group is*

$$(55) \quad J(u, v) = \frac{2^{2d}}{\alpha^2 |v|^{2d+2}} \left( \sum_{i=1}^d |u_i|^2 \frac{|v| \alpha_i - \sin(|v| \alpha_i)}{2 |v|^3 \alpha_i} \right)^{n-k-1} \times \\ \times \sum_{i=1}^d |u_i|^2 \prod_{j \neq i} \sin \left( \frac{\alpha_j |v|}{2} \right)^2 \sin \left( \frac{\alpha_i |v|}{2} \right) \left( \sin \left( \frac{|v| \alpha_j}{2} \right) - \frac{|v| \alpha_j}{2} \cos \left( \frac{|v| \alpha_j}{2} \right) \right),$$

where  $\alpha = \prod_{j=1}^d \alpha_j$  is the product of the non-zero singular values of  $L_v/|v|$ . If  $v = 0$ , the formula must be taken in the limit  $v \rightarrow 0$ . In particular,

$$(56) \quad J(u, 0) = \left( \sum_{i=1}^d \frac{(\alpha_i |u_i|)^2}{12} \right)^{n-k} = \left( \frac{|Su|^2}{12} \right)^{n-k}.$$

*Remark 20.* For H-type groups,  $L_v^2 = -|v|^2 \mathbb{1}$ , and the above expression reduces to

$$(57) \quad J(u, v) = \frac{2^k |u|^{2(n-k)}}{|v|^{k+2}} \left( \frac{|v| - \sin |v|}{2 |v|^3} \right)^{n-k-1} \sin \left( \frac{|v|}{2} \right)^{k-1} \left( \sin \left( \frac{|v|}{2} \right) - \frac{|v|}{2} \cos \left( \frac{|v|}{2} \right) \right).$$

*Proof.* It is convenient to rewrite the exponential map  $\exp_e(u, v) = (x, z)$  using polar coordinates on the  $u$  and  $z$  components. Namely,  $v \mapsto (|v|, \hat{v} = v/|v|) \in \mathbb{R}_+ \times \mathbb{S}^{n-k-1}$ , and  $z \mapsto (|z|, \hat{z} = z/|z|) \in \mathbb{R}_+ \times \mathbb{S}^{n-k-1}$ . One can check that  $g(L_v)u \cdot u \geq 0$ . Lemma 16 reads

$$(58) \quad x = f(L_v)u, \quad |z| = \frac{g(L_v)u \cdot u}{2|v|}, \quad \hat{z} = \hat{v}.$$

Thus, passing in polar coordinates, we have

$$(59) \quad J(u, v) = \frac{|z|^{n-k-1}}{|v|^{n-k-1}} \tilde{J}(u, |v|, \hat{v}),$$

where  $\tilde{J}$  is the Jacobian determinant in polar coordinates,

$$(60) \quad \tilde{J}(u, |v|, \hat{v}) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial |v|} & \frac{\partial x}{\partial \hat{v}} \\ \frac{\partial |z|}{\partial u} & \frac{\partial |z|}{\partial |v|} & \frac{\partial |z|}{\partial \hat{v}} \\ \frac{\partial \hat{z}}{\partial u} & \frac{\partial \hat{z}}{\partial |v|} & \frac{\partial \hat{z}}{\partial \hat{v}} \end{pmatrix} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial |v|} & \frac{\partial x}{\partial \hat{v}} \\ \frac{\partial |z|}{\partial u} & \frac{\partial |z|}{\partial |v|} & \frac{\partial |z|}{\partial \hat{v}} \\ 0 & 0 & \mathbb{1} \end{pmatrix} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial |v|} \\ \frac{\partial |z|}{\partial u} & \frac{\partial |z|}{\partial |v|} \end{pmatrix}.$$

Using the explicit expressions (58), and recalling that  $L_v = |v|L_{\hat{v}}$ , we obtain

$$(61) \quad \frac{\partial x}{\partial u} = f(L_v), \quad \frac{\partial x}{\partial |v|} = \frac{f'(L_v)L_v}{|v|}u,$$

$$(62) \quad \frac{\partial |z|}{\partial u} = \frac{u^* g(L_v)}{|v|}, \quad \frac{\partial |z|}{\partial |v|} = \frac{[g'(L_v)L_v - g(L_v)]u \cdot u}{2|v|^2},$$

where in the third equation we used the fact that  $g(L_v)^* = g(L_v^*) = g(-L_v) = g(L_v)$ , and where the prime denotes the derivative of  $f$  and  $g$  defined in (36).

The matrix on the right hand side of (60) has the following block structure

$$(63) \quad \Xi = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with  $A = f(L_v)$  a  $k \times k$  matrix. If  $A$  is invertible,  $\det(\Xi) = \det(A) \det(D - CA^{-1}B)$ . If  $A$  is not invertible, we can still apply this formula replacing  $A$  with a sequence of invertible matrices  $A_n \rightarrow A$ , and then taking the limit. We omit this standard density argument,

and in the following, we will formally manipulate all the expression as if  $A$  were invertible. Thus, using the fact that  $f(L_v)^{-1} = (1/f)(L_v)$ , we obtain

$$(64) \quad \tilde{J}(u, |v|, \hat{v}) = \frac{\det(f(L_v))}{2|v|^2} u^* \underbrace{\left[ g'(L_v)L_v - g(L_v) - \frac{2g(L_v)f'(L_v)L_v}{f(L_v)} \right]}_{\Theta(L_v)} u.$$

To compute the non trivial matrix  $\Theta(L_v)$ , we first notice that we can replace it with its symmetric part  $(\Theta(L_v) + \Theta(L_v^*))/2$  without changing the result. Moreover,  $\Theta(L_v)^* = \Theta(L_v^*) = \Theta(-L_v)$ . Exploiting our convention, it is sufficient to perform this computation for the corresponding analytic function of one variable  $\Theta(z)$ . We obtain

$$(65) \quad \frac{\Theta(z) + \Theta(-z)}{2} = 2 \frac{\sinh\left(\frac{z}{2}\right) - \frac{z}{2} \cosh\left(\frac{z}{2}\right)}{\sinh\left(\frac{z}{2}\right)}.$$

Thanks to this expression, we obtain

$$(66) \quad \tilde{J}(u, |v|, \hat{v}) = \frac{\det(f(L_v))}{|v|^2} u^* \left[ \frac{\sinh\left(\frac{L_v}{2}\right) - \frac{L_v}{2} \cosh\left(\frac{L_v}{2}\right)}{\sinh\left(\frac{L_v}{2}\right)} \right] u.$$

We now perform an orthogonal change of basis in the  $u$  space, in such a way that  $L_v$  is in normal form (24). Accordingly, we decompose  $u = (u_0, u_1, \dots, u_d)$  according to the real eigenspaces corresponding to the kernel of  $L_v$  and the pairs of eigenvalues  $\pm i|v|\alpha_j$ , for  $j = 1, \dots, d$  of  $L_v$ . In particular,  $u_0 \in \mathbb{R}^{k-2d}$ , and  $u_i \in \mathbb{R}^2$ . Then, we get

$$(67) \quad \tilde{J}(u, |v|, \hat{v}) = \frac{f(0)^{k-2d}}{|v|^2} \prod_{j=1}^d f(i|v|\alpha_j) f(-i|v|\alpha_j) \sum_{i=1}^d |u_i|^2 \frac{\sin\left(\frac{|v|\alpha_i}{2}\right) - \frac{|v|\alpha_i}{2} \cos\left(\frac{|v|\alpha_i}{2}\right)}{\sin\left(\frac{|v|\alpha_i}{2}\right)}$$

$$(68) \quad = \frac{1}{|v|^2} \prod_{j=1}^d \left( \frac{2}{|v|\alpha_j} \sin\left(\frac{|v|\alpha_j}{2}\right) \right)^2 \sum_{i=1}^d |u_i|^2 \frac{\sin\left(\frac{|v|\alpha_i}{2}\right) - \frac{|v|\alpha_i}{2} \cos\left(\frac{|v|\alpha_i}{2}\right)}{\sin\left(\frac{|v|\alpha_i}{2}\right)}$$

$$(69) \quad = \frac{2^{2d}}{\alpha^2 |v|^{2d+2}} \sum_{i=1}^d |u_i|^2 \prod_{j \neq i} \sin\left(\frac{\alpha_j |v|}{2}\right)^2 \sin\left(\frac{\alpha_i |v|}{2}\right) \times$$

$$(70) \quad \times \left( \sin\left(\frac{|v|\alpha_j}{2}\right) - \frac{|v|\alpha_j}{2} \cos\left(\frac{|v|\alpha_j}{2}\right) \right),$$

where in the second line we used the fact that  $f(0) = 1$ , and  $\alpha = \prod_{i=1}^d \alpha_i$ . From this expression, we recover the expression for the Jacobian determinant in the original Cartesian coordinates  $(u, v)$  through (59), and observing that

$$(71) \quad |z| = \frac{g(L_v)u \cdot u}{2|v|} = \sum_{i=1}^d |u_i|^2 \frac{|v|\alpha_i - \sin(|v|\alpha_i)}{2|v|^2 \alpha_i}. \quad \square$$

**Lemma 21** (Cotangent injectivity domain). *Let  $\alpha_d$  be the largest eigenvalue of  $S$ , and*

$$(72) \quad D := \left\{ \lambda = (u, v) \in T_e^*G \text{ such that } |v| < \frac{2\pi}{\alpha_d}, \text{ and } Su \neq 0 \right\} \subset T_e^*G.$$

*Then  $\exp_e : D \rightarrow \exp_e(D)$  is a smooth diffeomorphism and  $\mathcal{C}(e) := G \setminus \exp_e(D)$  is a closed set with zero measure. Moreover, for all  $\lambda \in D$ , the curve  $\gamma_\lambda(t) = \exp_e(t\lambda)$ , with  $t \in [0, 1]$ , is the unique minimizing geodesic joining its endpoints.*

*Remark 22.* By Lemma 16, geodesics  $\gamma_\lambda$  with  $\lambda = (u, v)$  satisfying  $Su = 0$  or  $v = 0$  are Euclidean straight lines  $(tu, 0)$ , which are optimal for all times. This observation and



Lemma 21 imply that the cut time, that is the first time of loss of optimality,

$$(73) \quad t_{\text{cut}}(u, v) = \begin{cases} \frac{2\pi}{\alpha_d|v|} & v \neq 0 \text{ and } Su \neq 0, \\ +\infty & \text{otherwise.} \end{cases}$$

For standard H-type groups  $S = \mathbb{1}_{2d}$ , and Lemma 16 yields  $\text{Cut}_e = \{\exp(t_{\text{cut}}(\lambda)\lambda) \mid \lambda \in T_e^*G\} = \{(0, z) \mid z \in \mathbb{R}^{n-k}\}$  (the center of  $G$ ). This fact was the main result proven in [7].

*Proof.* Since all geodesic are normal and  $(G, d)$  is complete, each point of  $G$  is reached by at least one minimizing normal geodesic  $\gamma_\lambda : [0, 1] \rightarrow G$ , with  $\lambda = (u, v) \in T_e^*G$ . If  $|v| > 2\pi/\alpha_d$ , and  $Su \neq 0$ , then  $\gamma_\lambda$  is a strictly normal geodesic (i.e. not abnormal) with a conjugate time at  $t_* = 2\pi|v|/\alpha_d < 1$ . Since these structures are real-analytic,  $\gamma_\lambda$  does not contain any abnormal segment, hence it loses optimality after its first conjugate time (see [2, Chapter 8]). Thus  $\gamma_\lambda$  is not minimizing on  $[0, 1]$ . On the other hand, if  $Su = 0$ , for any value of  $v$  we obtain the same abnormal geodesic (see Lemma 18). It follows that  $\exp_e : \bar{D} \rightarrow G$  is onto (where  $\bar{D}$  denotes the closure of  $D$ ). Since also  $\exp_e : D \rightarrow \exp_e(D)$  is trivially onto,  $\mathcal{C}(e) = G \setminus \exp_e(D) \subseteq \exp_e(\partial D)$  has zero measure.

We now prove that, if  $\lambda \in D$ , then  $\gamma_\lambda : [0, 1] \rightarrow G$  is the unique minimizing geodesic joining its endpoints. Assume by contradiction that  $\lambda = (u, v)$  and  $\bar{\lambda} = (\bar{u}, \bar{v}) \in D$ , with  $\lambda \neq \bar{\lambda}$ , are such that  $\exp_e(\lambda) = \exp_e(\bar{\lambda})$ . Observe that the spectrum of  $L_v$  is given by

$$(74) \quad \sigma(L_v) = \{0, \pm i|v|\alpha_1, \dots, \pm i|v|\alpha_d\}.$$

Using the  $z$  part of Lemma 16, and (74), we have  $g(L_v)u \cdot u/|v|^2 \geq 0$ , with equality if and only if  $v = 0$ . This readily implies that  $\bar{v} = v$ . From the  $x$  equation of Lemma 16, we have that  $\exp_e(u, v) = \exp_e(\bar{u}, v)$  is equivalent to

$$(75) \quad f(L_v)(u - \bar{u}) = 0.$$

If  $v = 0$ , and since  $f(0) = 1$ , (75) implies  $u = \bar{u}$ , which is a contradiction. Hence we proceed assuming  $v \neq 0$ . Since  $|v| < 2\pi/\alpha_d$ , we have that  $|v|\alpha_j < 2\pi$  for all  $j = 1, \dots, d$ . It is then easy to check that  $f(\pm i|v|\alpha_j) \neq 0$ . This observation, (75) and (74) imply that  $u - \bar{u} \in \ker L_v$ . Using again that  $f(0) = 1$ , (75) implies  $u = \bar{u}$ , which is a contradiction. Thus  $\exp_e : D \rightarrow \exp_e(D)$  is a bijection.

To conclude, no  $\lambda \in D$  can be a critical point for  $\exp_e$ , since the Jacobian determinant of Lemma 19 is strictly positive for  $|v| < 2\pi/\alpha_d$  and  $Su \neq 0$ .  $\square$

The next corollary formalizes the fact that Definition 12 for the measure contraction property makes sense in the setting of generalized H-type groups.

**Corollary 23.** *For any  $x \in G$ , let  $\mathcal{C}(x) := L_x\mathcal{C}(e)$ , where  $L_x : G \rightarrow G$  is the left-translation. There exists a measurable map  $\Phi^x : G \setminus \mathcal{C}(x) \times [0, 1] \rightarrow G$ , given by*

$$(76) \quad \Phi^x(y, t) = L_x \exp_e(t \exp_e^{-1}(L_x^{-1}y)),$$

such that  $\Phi^x(y, t)$  is the unique minimizing geodesic joining  $x$  with  $y$ .

## 7. PROOF OF THE MAIN RESULT

To prove Theorem 3, we begin with a simplified version of [15, Lemma 2.6]. The argument will be useful to prove further inequalities appearing in generalized H-type groups.

**Lemma 24.** *Let  $g(x) := \sin(x) - x \cos(x)$ . Then, for all  $x \in (0, \pi)$  and  $t \in [0, 1]$ ,*

$$(77) \quad g(tx) \geq t^N g(x), \quad \forall N \geq 3.$$

*Proof.* The condition  $N \geq 3$  is necessary, as  $g(x) = x^3/3 + O(x^4)$ . It is sufficient to prove the statement for  $N = 3$ . The cases  $t = 0$  and  $t = 1$  are trivial, hence we assume  $t \in (0, 1)$ . By Gronwall's Lemma the above statement is implied by the differential inequality

$$(78) \quad g'(s) \leq 3g(s)/s, \quad s \in (0, \pi).$$

In fact, it is sufficient to integrate the above inequality on  $[tx, x] \subset (0, \pi)$  to prove our claim. The above inequality reads

$$(79) \quad f(s) := (3 - s^2) \sin(s) - 3s \cos(s) \geq 0, \quad s \in (0, \pi).$$

To prove it, we observe that  $f(0) = 0$  and  $f'(s) = s(\sin(s) - s \cos(s)) \geq 0$  on  $[0, \pi]$ .  $\square$

The next inequality is required to deal with the extra terms appearing in the case of generalized H-type groups, due to the fact that they have corank larger than one.

**Lemma 25.** *Let  $f(x) := x - \sin(x)$ . Then, for all  $x \in (0, 2\pi)$  and  $t \in [0, 1]$ ,*

$$(80) \quad f(tx) \geq t^N f(x), \quad \forall N \geq 3.$$

*Proof.* Similarly as in the proof of Lemma 24, and by Gronwall's Lemma applied to the interval  $[tx, x] \subset (0, 2\pi)$ , it is sufficient to prove the differential inequality

$$(81) \quad f'(s) \leq 3f(s)/s, \quad s \in (0, 2\pi).$$

The above inequality reads

$$(82) \quad 2s - 3 \sin(s) + s \cos(s) \geq 0, \quad s \in (0, 2\pi).$$

It is sufficient to observe that  $f'(0) = f''(0) = 0$  and  $f'''(s) = s \sin(s) \geq 0$  on  $[0, 2\pi]$ .  $\square$

**Corollary 26.** *For all  $(u, v) \in D$ , we have the following inequality*

$$(83) \quad \frac{J(tu, tv)}{J(u, v)} \geq t^{2(n-k)}, \quad \forall t \in [0, 1].$$

*Proof.* Apply Lemmas 24 and 25 to the explicit expression of  $J$  from Lemma 19, and then use the standard inequality  $\sin(tx) \geq t \sin(x)$ , valid for all  $x \in [0, \pi]$  and  $t \in [0, 1]$ .  $\square$

**7.1. Proof of Theorem 3.** We are now ready to prove Theorem 3. Observe that the integer  $k + 3(n - k)$  coincides with the geodesic dimension of the Carnot group, by Lemma 14.

**Step 1.** We first prove that the MCP(0,  $N$ ) holds for  $N \geq k + 3$ . By left-translation, it is sufficient to prove the inequality (27) for the homothety with center equal to the identity  $e = (0, 0)$ . Let  $\Omega$  be a measurable set with  $0 < \mu(\Omega) < +\infty$ .

By Lemma 21, up to removing a set of zero measure,  $\Omega = \exp_e(A)$  for some  $A \subset D \subset T_e^*G$ . On the other hand, by Corollary 23, we have

$$(84) \quad \Omega_t = \exp_e(tA), \quad \forall t \in [0, 1],$$

where  $tA$  denotes the set obtained by multiplying by  $t$  any point of the set  $A \subset T_e^*G$  (an Euclidean homothety). Thus, for all  $t \in [0, 1]$  we have

$$(85) \quad \mu(\Omega_t) = \int_{\Omega_t} d\mu = \int_{tA} J(u, v) dudv$$

$$(86) \quad = t^n \int_A J(tu, tv) dudv \geq t^{k+3(n-k)} \int_A J(u, v) dudv = t^{k+3(n-k)} \mu(\Omega),$$

where we used Corollary 26. In particular  $\mu(\Omega_t) \geq t^N \mu(\Omega)$  for all  $N \geq k + 3(n - k)$ .

**Step 2.** Fix  $\varepsilon > 0$ . Let  $N = k + 3(n - k)$ . We prove that the MCP(0,  $N - \varepsilon$ ) does not hold. Let  $\lambda = (u, 0) \in D$ . By Lemma 19, and recalling that  $J > 0$  on  $D$ , we have

$$(87) \quad J(tu, 0) = t^{2(n-k)} J(u, 0) < t^{2(n-k)-\varepsilon} J(u, 0), \quad \forall t \in [0, 1].$$

By continuity of  $J$  and compactness of  $[0, 1]$ , we find an open neighborhood  $A \subset D$  such that  $J(t\lambda) < t^{2(n-k)-\varepsilon} J(\lambda)$ , for all  $t \in [0, 1]$  and  $\lambda \in A$ . Arguing as in the first step, for  $\Omega = \exp_e(A)$ , we obtain

$$(88) \quad \mu(\Omega_t) < t^{n+2(n-k)-\varepsilon} \mu(\Omega) = t^{N-\varepsilon} \mu(\Omega), \quad t \in [0, 1].$$

**Step 3.** To prove that MCP( $K, N$ ) does not hold for  $K > 0$  and any  $N > 1$ , we observe that spaces verifying this condition are bounded, while  $G \simeq \mathbb{R}^n$  is not. Finally, assume

that  $(G, d, \mu)$  satisfies MCP( $K, N$ ) for some  $K < 0$  and  $N < k + 3(n - k)$ . Then the scaled space  $(G, \varepsilon^{-1}d, \varepsilon^{-Q}\mu)$  (where  $\varepsilon > 0$  and  $Q = k + 2(n - k)$  is the Hausdorff dimension of  $(G, d)$ ) verifies MCP( $\varepsilon^2 K, N$ ) for [20, Lemma 2.4]. But the two spaces  $(G, d, \mu)$  and  $(G, \varepsilon^{-1}d, \varepsilon^{-Q}\mu)$  are isometric through the dilation  $\delta_\varepsilon(x, z) := (\varepsilon x, \varepsilon^2 z)$ . In particular  $(G, d, \mu)$  satisfies the MCP( $\varepsilon K, N$ ) for all  $\varepsilon > 0$ , that is

$$(89) \quad \mu(\Omega_t) \geq \int_{\Omega} t \left[ \frac{s_{\varepsilon K}(td(x, z)/\sqrt{N-1})}{s_{\varepsilon K}(d(x, z)/\sqrt{N-1})} \right]^{N-1} d\mu(z), \quad \forall t \in [0, 1].$$

Taking the limit for  $\varepsilon \rightarrow 0^+$ , we obtain that  $(G, d, \mu)$  satisfies the MCP( $0, N$ ) with  $N < k + 3(n - k)$ , but this is false as a result of the previous step.  $\square$

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## REFERENCES

- [1] A. Agrachev, D. Barilari, and U. Boscain. On the Hausdorff volume in sub-Riemannian geometry. *Calc. Var. Partial Differential Equations*, 43(3-4):355–388, 2012.
- [2] A. Agrachev, D. Barilari, and U. Boscain. Introduction to Riemannian and sub-Riemannian geometry (Lecture Notes). <http://webusers.imj-prg.fr/~davide.barilari/notes.php>. v20/11/16.
- [3] A. Agrachev, D. Barilari, and L. Rizzi. Curvature: a variational approach. *Memoirs of the AMS (in press)*.
- [4] A. Agrachev and P. Lee. Optimal transportation under nonholonomic constraints. *Trans. Amer. Math. Soc.*, 361(11):6019–6047, 2009.
- [5] A. A. Agrachev. Some open problems. In *Geometric control theory and sub-Riemannian geometry*, volume 5 of *Springer INdAM Ser.*, pages 1–13. Springer, Cham, 2014.
- [6] L. Ambrosio and S. Rigot. Optimal mass transportation in the Heisenberg group. *J. Funct. Anal.*, 208(2):261–301, 2004.
- [7] C. Autenried and M. G. Molina. The sub-Riemannian cut locus of  $H$ -type groups. *Math. Nachr.*, 289(1):4–12, 2016.
- [8] Z. M. Balogh, A. Kristály, and K. Sipos. Geometric inequalities on Heisenberg groups. *ArXiv e-prints*, May 2016.
- [9] Z. M. Balogh, A. Kristály, and K. Sipos. Jacobian determinant inequality on Corank 1 Carnot groups with applications. *ArXiv e-prints*, Jan. 2017.
- [10] D. Barilari and L. Rizzi. A formula for Popp’s volume in sub-Riemannian geometry. *Anal. Geom. Metr. Spaces*, 1:42–57, 2013.
- [11] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni. *Stratified Lie groups and potential theory for their sub-Laplacians*. Springer Monographs in Mathematics. Springer, Berlin, 2007.
- [12] F. Cavalletti and A. Mondino. Optimal maps in essentially non-branching spaces. *Commun. Contemp. Math.*, 2016.
- [13] A. Figalli and N. Juillet. Absolute continuity of Wasserstein geodesics in the Heisenberg group. *J. Funct. Anal.*, 255(1):133–141, 2008.
- [14] A. Figalli and L. Rifford. Mass transportation on sub-Riemannian manifolds. *Geom. Funct. Anal.*, 20(1):124–159, 2010.
- [15] N. Juillet. Geometric inequalities and generalized Ricci bounds in the Heisenberg group. *Int. Math. Res. Not. IMRN*, (13):2347–2373, 2009.
- [16] A. Kaplan. Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms. *Trans. Amer. Math. Soc.*, 258(1):147–153, 1980.
- [17] E. Le Donne, R. Montgomery, A. Ottazzi, P. Pansu, and D. Vittone. Sard property for the endpoint map on some Carnot groups. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(6):1639–1666, 2016.
- [18] J. Lott and C. Villani. Ricci curvature for metric-measure spaces via optimal transport. *Ann. of Math. (2)*, 169(3):903–991, 2009.
- [19] R. Montgomery. *A tour of subriemannian geometries, their geodesics and applications*, volume 91 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.
- [20] S.-i. Ohta. On the measure contraction property of metric measure spaces. *Comment. Math. Helv.*, 82(4):805–828, 2007.
- [21] L. Rifford. Ricci curvatures in Carnot groups. *Math. Control Relat. Fields*, 3(4):467–487, 2013.

- [22] L. Rifford. *Sub-Riemannian Geometry and Optimal Transport*. SpringerBriefs in Mathematics. Springer, 2014.
- [23] L. Rifford and E. Trélat. Morse-Sard type results in sub-Riemannian geometry. *Math. Ann.*, 332(1):145–159, 2005.
- [24] S. Rigot. Mass transportation in groups of type  $H$ . *Commun. Contemp. Math.*, 7(4):509–537, 2005.
- [25] L. Rizzi. Measure contraction properties of Carnot groups. *Calc. Var. Partial Differential Equations*, 55(3):Art. 60, 20, 2016.
- [26] K.-T. Sturm. On the geometry of metric measure spaces. I. *Acta Math.*, 196(1):65–131, 2006.
- [27] K.-T. Sturm. On the geometry of metric measure spaces. II. *Acta Math.*, 196(1):133–177, 2006.